Motion of curves by singular interfacial energy and related topics

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It is very important to study evolving interfaces both in mathematics and other disciplines for example in materials science and in image processing. One important class of evolution laws is an interface controlled model or a local model where the speed of evolution depends only on the geometry of interfaces. A typical example is the mean curvature flow equations, which was proposed by Mullins (1956) to describe motion of anti-phase boundaries in annealing metals. Its mathematical analysis was started by Brakke (1978) and since then there has been enormous progress. Its anisotropic version was proposed in the theory of crystal growth to describe the motion of growing crystals in 1970s. Later, a model having a strong anisotropy was proposed by Angenent-Gurtin (1989) and Taylor (1991). It is an evolution law by singular interfacial energy. The resulting equation is formally a (degenerate) parabolic equation. However, its diffusivity is so strong that the solution becomes to have a flat part called a facet. Because of singular nature of diffusivity the problem has a nonlocal effect. Thus, our fundamental understanding of this model is still incomplete and it is an active field of researches.

In this series of lectures we recall various definitions of solutions of such a motion by singular interfacial energy by restricting curve evolution. This includes a viscosity approach developed by Giga-Giga (1998), (2001) as well as an original approach developed by Taylor and Angenent-Gurtin for evolving polygons. It turns out that the solution of this unusual problem can be approximated by motion by smoothed energy if the energy is suitably approximated as proved in Giga-Giga (1999), (2001).

The viscosity approach gives a widest class of solutions so far known so it is interesting to study such a solution which is not included for a class of evolving polygons. We give in what way solution behaves if one starts with a general polygon not necessarily “admissible” in the sense of Angenent-Gurtin and Taylor for crystalline interfacial energy [1]. We further discuss the case when there is a driving force term so that the evolving shape actually grows to a whole plane as time goes to the infinity. This problem is very related to snow crystal growth where facet often appears. We briefly mention the case when the driving force is not spatially homogeneous so that it causes a facet bending [2].

In materials science we often encounter the problem where the motion is by surface diffusion not by curvature. Now the problem is fourth order and viscosity theory does not apply. We also discuss this problem where the curve is graph-like [3].

The singular diffusion has many applications not only to problems in materials science but also image processing as a total variation flow. As an unusual application we give an example that it is useful to describe shock phenomena in partial differential equations [4].

[1] M.-H. Giga, Y. Giga and H. Hontani, Self-similar expanding solutions in a sector for a crystalline flow. *SIAM J. Math. Anal.* **37** (2005), 1207–1226.

[2] M.-H. Giga, Y. Giga and P. Rybka, A comparison principle for singular diffusion equations with spatially inhomogeneous driving force for graphs. *Hokkaido University Preprint Series in Math.*, **#981** (2011).

[3] M.-H. Giga and Y. Giga, Very singular diffusion equations: second and fourth order problems. *Jpn. J. Ind. Appl. Math.* **27** (2010), 323–345.

[4] M.-H. Giga and Y. Giga, Minimal vertical singular diffusion preventing overturning for the Burgers equation. *Recent advances in scientific computing and partial differential equations*, *Contemp. Math.,* **330** (2003), 73–88.