Evolution Equations with Very Singular Diffusivity

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There is a class of nonlinear evolution equations with very (strong) singular diffusivity, so that quantity like speed in evolution laws is nonlocal. Equations with very singular diffusivity are applied to describe several phenomena in the applied sciences, and to provide several devices in technology, especially image processing. A typical example is a gradient flow of the total variation of a function, which arises in image processing, as well as in materials science to describe the motion of grain boundaries. In the theory of crystal growth the motion of a crystal surface is often described by an anisotropic curvature flow equation with a driving force term. At low temperature the equation includes a singular diffusivity, since the interfacial energy is not smooth.

Because of singular diffusivity, the notion of solution is not a priori clear even for one-dimensional total variation flow. The problem is that one has to define the notion of solution so that its initial value problem is uniquely solvable and it is approximated by a smoother problem. There are two systematic approaches. One is based on the theory of monotone operators initiated by Kömura (1967) and developed by Brezis and others, ant it applies to divergence type equations. However, there are many equations like curvature flow equations which are not exactly of divergence type. Fortunately, the second approach (a viscosity approach) based comparison principles turns to be successful in several interesting problems.

In this series of lectures we focus on well-posedness problems for equations with very singular diffusivity by emphasizing a viscosity approach. Although one dimensional problem itself is nontrivial and necessary theory has been developed more than a decade ago by Giga-Giga, higher dimensional problem is substantially tougher since the expected speed on a flat portion called a facet may not be a constant. This phenomenon called a facet splitting or breaking phenomenon is first found by Bellettini-Novaga-Paolini (1999). Because of this phenomenon a direct extension of one-dimensional method is impossible. It is very recent that the theory of viscosity solution is extended to higher dimension problems at least for total variation flow type problems [1].

In this series of lectures we start with notion of viscosity solutions for a standard parabolic equations [2], [3] and also briefly review the abstract theory of monotone operators which is a key to understand the evolution at least when it has a divergence structure. We then derive a key comparison principle of curvature like quantity which is fundamental to have a comparison principle in the viscosity approach. We emphasize that this curvature like quantity on a facet may not be constant in general. Nevertheless, we are able to show the comparison principle.

Compared with second order problems, mathematical analysis on fourth order problems is in baby stage. Our analysis is limited to equations of divergence type. We also touch this type of problem like the fourth order total variation flow in this lecture [4].

References

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[4] Y. Giga and R. V. Kohn, Scale-invariant extinction time estimates for some singular diffusion equations. Discrete Contin. Dyn. Syst. 30 (2011), 509-535