Random turn games and the infinity Laplacian

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In a random-turn game [4], the right to move in each step is decided by a coin toss. We will describe some attractive random turn games, including the connection between Hex and Percolation. Determining the expected duration of these games is an open problem.

The infinity Laplacian (informally, the "second derivative in the gradient direction") is a simple yet mysterious operator with many applications, introduced by Aronsson [1] in 1967. "Tug of war" is a two player random turn game played in a d-dimensional domain as follows: Fix a target set T in the domain, a continuous terminal payoff function F on T, a starting position x, and a constant ϵ. Place the game token at x. In each move, a fair coin is tossed and the player who wins the coin toss moves the game token up to ϵ units in the direction of his or her choice. Repeat the above until the token reaches T. If the target set is first reached at y, then player 2 pays player 1 the sum $F(y)$.

It is proved in [5] that as ϵ tends to zero, the expected payoff to player 1 under optimal play (viewed as a function of the starting point x) converges to the infinity harmonic function with boundary condition F on T. We will discuss extensions to the p-Laplacian [6], biased coins [7] and pure Neumann boundary conditions [8].

These lectures are based on joint works of the speaker with: Oded Schramm, Scott Sheffield, David Wilson, Gabor Pete, Stephanie Somersille and Tonci Antunovic, as well as related papers by Robert Jensen, L.C. Evans, Adam Oberman, Juan Manfredi, Charles Smart, Scott Armstrong and many others. No prior familiarity with PDE or game theory will be assumed.

References

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