Prestrained Elasticity: Curvature Constraints and Differential Geometry with Low Regularity

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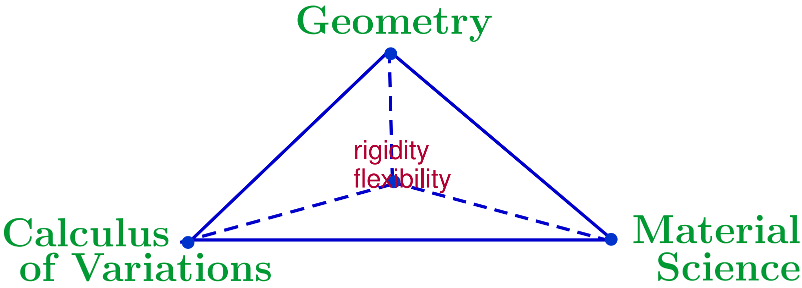


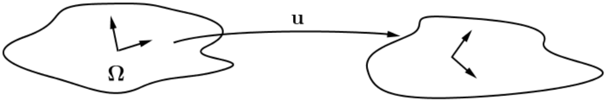
Key connection: Rigidity and flexibility of solutions to nonlinear problems at low regularity

Key role: Energy functional in the description of elastic materials with residual stress at free equilibria

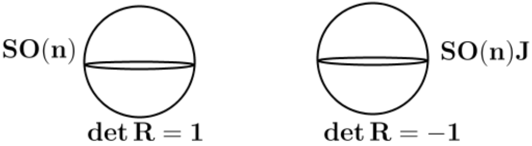
An old story: isometric immersions (equidimensional)

Assume that $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$ satisfies: $\nabla u(x)^T \nabla u(x) = Id_n$

- Equation of isometric immersion: $\langle \partial_i u, \partial_j u \rangle = \delta_{ij} = \langle e_i, e_j \rangle$
  (For $u \in \mathcal{C}^1$, this is equivalent to u preserving length of curves)



- Equivalent to: $\nabla u \in O(n) = \left\{ R;\ R^T R = Id \right\} = SO(n) \cup SO(n)J$



$J = \mathrm{diag}\{-1, 1, \ldots, 1\}$

- Liouville (1850), Reshetnyak (1967): $u \in W^{1,\infty}$ and $\nabla u \in SO(n)$ a.e. in $\Omega \Rightarrow \nabla u \equiv const \ \Rightarrow \ u(x) = Rx + b$ rigid motion
- $u \in W^{1,\infty}$ and $\nabla u \in SO(n)J$ a.e. in $\Omega \Rightarrow \nabla u \equiv const$

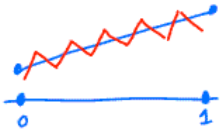
An old story: isometric immersions (equidimensional)

- Friesecke-James-Muller (2006): Rigidity estimate:

$$\forall u \in W^{1,2} \;\; \exists R \in SO(n) \quad \int_\Omega |\nabla u - R|^2 \le C_\Omega \int_\Omega \text{dist}^2(\nabla u, SO(n))$$

- Gromov (1973): Convex integration:
$\exists u \in W^{1,\infty}$ such that $(\nabla u)^T \nabla u = Id$ a.e. in Ω, and ∇u takes values in $SO(n)$ and in $SO(n)J$, in every open $U \subset \Omega$.

Even more: $\exists u$ arbitrarily close to any u_0 with $0 < (\nabla u_0)^T \nabla u_0 < Id$



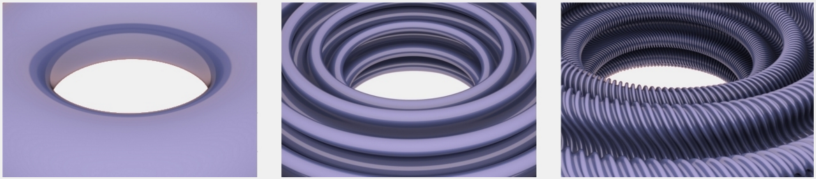
Example:

Given $u_0 : (0,1) \to \mathbb{R}$ with $(u_0')^2 < 1$

want: $u_k \xrightarrow{\text{uniformly}} u_0$ with $(u_k')^2 = 1$

more oscillations as $k \to \infty$

Isometric immersions of Riemann manifold (Ω, G)



Hevea project: Inst. Camille Jordan, Lab J. Kuntzmann, Gipsa-Lab (France)

Let $G \in \mathcal{C}^\infty(\Omega, \mathbb{R}^{n\times n}_{sym,+})$. Look for $u : \Omega \to \mathbb{R}^n$ so that $(\nabla u)^T \nabla u = G$ in Ω

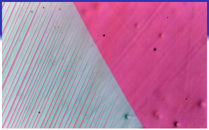
Theorem (Gromov 1986)

Let $u_0 : \Omega \to \mathbb{R}^n$ be smooth short immersion, i.e.: $0 < (\nabla u_0)^T \nabla u_0 < G$ in Ω. Then: $\forall \varepsilon > 0 \quad \exists u \in W^{1,\infty} \quad \|u - u_0\|_{\mathcal{C}^0} < \varepsilon$ and $(\nabla u)^T \nabla u = G$.

Theorem (Myers-Steenrod 1939, Calabi-Hartman 1970)

Let $u \in W^{1,\infty}$ satisfy $(\nabla u)^T \nabla u = G$ and $\det \nabla u > 0$ a.e. in Ω.
(For example, $u \in \mathcal{C}^1$ enough). Then $\Delta_G u = 0$ and so u is smooth.
In fact, u is unique up to rigid motions, and: $\exists u \Leftrightarrow Riem(G) \equiv 0$ in Ω.

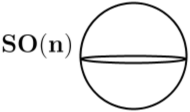
Crystal microstructure



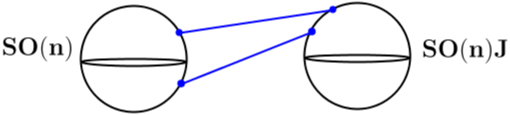
Energy $E(u) = \int_\Omega W(\nabla u(x), \theta)\, dx$

$u : \mathbb{R}^3 \supset \Omega \to \mathbb{R}^3$ deformation; $\theta \in \mathbb{R}$ temperature; W energy density

- Given θ, find u minimizing E, under some boundary conditions
  First attempt: look for $\nabla u \in$ energy well $\mathcal{K}(\theta)$ (change of shape in crystal lattice)
- Ball-James (1987): $u \in W^{1,\infty}$ and $\nabla u \in \mathcal{K} = \{A, B\}$ a.e. in Ω.
  Then: (i) $rank(B - A) \geq 2 \Rightarrow \nabla u \equiv A$ or $\nabla u \equiv B$
  (ii) $B - A = \mathbf{a} \otimes \mathbf{n} \Rightarrow$ laminate pattern of the form:
  $u(x) = Ax + f(\langle x, \mathbf{n} \rangle)\mathbf{a} + c$ where $f' \in \{0, 1\}$ a.e.



no rank-1 connections!

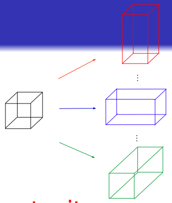


many rank-1 connections!

Remark: 4 *matrices, no rank-1 connections* $\Rightarrow$ *rigidity BUT:* 5 *matrices, no rank-1 connections may admit flexible solutions! (Chlebik, Kirchheim, Preiss)*

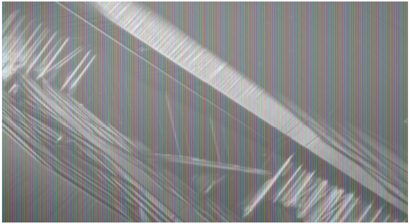
Martensitic phase transformation

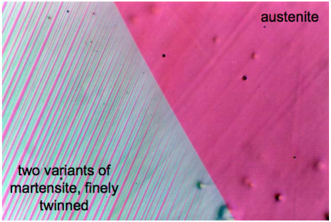
Energy $E(u) = \int_\Omega W(\nabla u(x), \theta)\, dx$



Energy well structure in $Zn_{45}Au_{30}Cu_{25}$: $W(F,\theta) = \min \Leftrightarrow$

$$F \in \mathcal{K}(\theta) = \begin{cases} \alpha(\theta)SO(3) & \theta > \theta_c \quad \text{austenite} \\ SO(3) \cup \bigcup_{i=1}^{N} SO(3)A_i(\theta_c) & \theta = \theta_c \\ \bigcup_{i=1}^{N} SO(3)A_i(\theta) & \theta < \theta_c \quad \text{martensite} \end{cases}$$





Successive heating/cooling cycles. $\theta_c \sim -37C$
The total width of the sample $\sim 0.5mm$
Courtesy of R. James.

Critical temperature $\theta_c \sim 40C$.
Alloy: $Cu_{82}Al_{14}Ni_4$
Courtesy of C. Chu.

Non-Euclidean elasticity

$$E(u) = \int_\Omega W((\nabla u)A^{-1}(x))\, dx$$

$W(F) \sim \mathrm{dist}^2(F, SO(3))$
$A \in \mathcal{C}^\infty(\Omega, \mathbb{R}^{3\times 3}_{sym,+})$ incompatibility tensor

- $E(u) = 0 \Leftrightarrow \nabla u(x) \in K(x) = SO(3)A(x) \ \ \forall a.e.\ x$
  $\Leftrightarrow (\nabla u)^T \nabla u = A^2 = G$ and $\det \nabla u > 0$

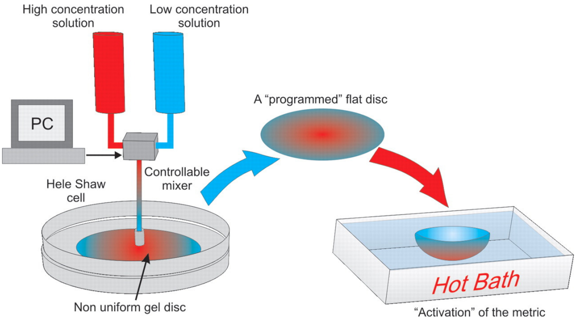
Lemma (L-Pakzad '09)

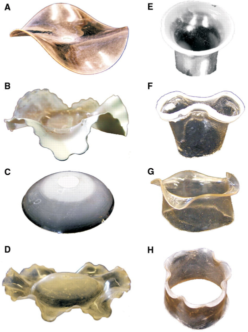
$\inf_{u \in W^{1,2}} E(u) > 0 \Leftrightarrow Riem(G) \not\equiv 0.$

Thin non-Euclidean plates: $\Omega = \Omega^h = \omega \times (-h/2, h/2), \quad \omega \subset \mathbb{R}^2$

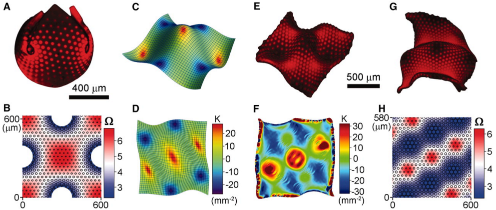
- As $h \to 0$: Scaling of: $\inf E^h \sim h^\beta$? $\mathrm{argmin} E^h \to \mathrm{argmin}\, \mathcal{I}_\beta$?
- Hierarchy of theories $\mathcal{I}_\beta$, where β depends on $Riem(A^h)^2$
  Bhattacharya, Li, L., Mahadevan, Pakzad, Raoult, Schaffner
- When $A = Id$: dimension reduction in nonlinear elasticity
  seminal analysis by LeDret-Raoult 1995, Friesecke-James-Muller 2006

Manufacturing residually-strained thin films

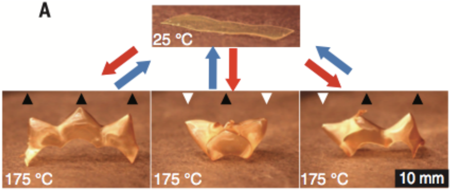




- *Shaping of elastic sheets by prescription of Non-Euclidean metrics* (Klein, Efrati, Sharon) Science, 2007



- *Half-tone gel lithography* (Kim, Hanna, Byun, Santangelo, Hayward) Science, 2012
- *Defect-activated liquid crystal elastomers* (Ware, McConney, Wie, Tondiglia, White) Science, 2015



Dimension reduction

$$G(x', x_3) = G(x') \qquad \Omega^h = \omega \times (-\tfrac{h}{2}, \tfrac{h}{2})$$

$$E^h(u^h) = \frac{1}{h}\int_{\Omega^h} W((\nabla u^h) G^{-1/2}(x))\, dx$$

Theorem (L-Pakzad 2009, Bhattacharya-L-Schaffner 2014)

If $E^h(u^h) \leq Ch^2$*, then* $\exists c^h \in \mathbb{R}^3$ *such that the following holds for:*

$$y^h(x', x_3) := u^h(x', hx_3) - c^h \in W^{1,2}(\Omega^1, \mathbb{R}^3).$$

- $y^h(x', x_3) \to y(x')$ *in* $W^{1,2}$
- $y \in W^{2,2}(\omega, \mathbb{R}^3)$ *and* $(\nabla y)^T \nabla y = G_{2\times 2}$ *on the midplate* ω
- $\frac{1}{h^2} E^h(u^h) \xrightarrow{\Gamma} \mathcal{I}_2(y) = \frac{1}{24}\int_\omega \left| sym\big((\nabla y)^T \nabla \mathbf{b}\big)\right| dx'$

  Cosserat vector $\mathbf{b} \in W^{1,2} \cap L^\infty(\omega, \mathbb{R}^3)$ *so that:*

  $$\begin{bmatrix} \partial_1 y & \partial_2 y & \mathbf{b} \end{bmatrix}^T \begin{bmatrix} \partial_1 y & \partial_2 y & \mathbf{b} \end{bmatrix} = G$$

- DeGiorgi (1975): Γ-convergence is a "variational" convergence, which "implies" that: $\quad Limits\left(\operatorname{argmin} \frac{1}{h^2} E^h\right) = \operatorname{argmin} \mathcal{I}_2$

3d energy upper bound and the isometric immersions

Corollary

$$\inf E^h \leq Ch^2 \;\Leftrightarrow\; \exists y \in W^{2,2}(\omega, \mathbb{R}^3) \quad (\nabla y)^T \nabla y = G_{2\times 2}$$

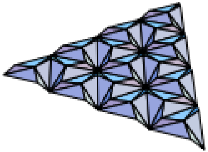
- Nirenberg (1953): $\forall G_{2\times 2},\ \kappa > 0\ \ \exists$ smooth isometr. embed. in $\mathbb{R}^3$
- Poznyak-Shikin (1995): Same true for $\kappa < 0$ on bounded $\omega \subset \mathbb{R}^2$.
- Nash-Kuiper (1956): $\forall n-$dim $G\ \ \exists \mathcal{C}^{1,\alpha}$ isometr. embed. in $\mathbb{R}^{n+1}$

Case $G_{2\times 2}$: Borisov (2004), Conti-Delellis-Szekelyhidi (2010) $\alpha < \frac{1}{7}$
Delellis-Inaunen-Szekelyhidi (2015) $\alpha < \frac{1}{5}$.

Corollary

$\bullet\ \forall G_{2\times 2} : \inf E^h \leq Ch^{2/3}$. $\bullet\ |\kappa(G_{2\times 2})| > 0 \Rightarrow \inf E^h \leq Ch^2$.

- Best to expect from convex integration: $\alpha < \frac{1}{3} \Rightarrow \inf E^h \leq Ch$.
- Conti-Maggi (2008): Example of origami-like folding pattern with $E^h \leq Ch^{5/3}$



The lower bound and the energy quantisation

Assume: $\exists y \in W^{2,2}\ \ (\nabla y)^T \nabla y = G_{2\times 2}$, or equivalently: $\inf E^h \leq Ch^2$.
Then only 3 scenarios are possible:

- $\inf E^h \sim Ch^2$
- $\inf E^h \sim Ch^4$
- $\min E^h = 0\ \ \forall h$

Theorem (L-Raoult-Ricciotti 2015)

(i) Assume that $\frac{1}{h^2} \inf E^h \to 0$. *Then:*

- $\inf E^h \leq Ch^4$, *and:* $\exists !\ y_0\ \ (\nabla y_0)^T \nabla y_0 = G_{2\times 2}$ *with* $\mathcal{I}_2(y_0) = 0$.
- $R_{1212} = R_{1213} = R_{1223} \equiv 0$ *in* Ω.
- $$\frac{1}{h^4} E^h \xrightarrow{\Gamma} \mathcal{I}_4 = \int_\omega \left| \begin{matrix} \text{change in metric} \\ \text{departing from } y_0 \end{matrix} \right|^2 + \int_\omega \left| \begin{matrix} \text{change in curvature} \\ \text{departing from } y_0 \end{matrix} \right|^2 + \int_\omega \left| \begin{bmatrix} R_{1313} & R_{1323} \\ R_{1323} & R_{2323} \end{bmatrix} \right|^2$$

(ii) If $\frac{1}{h^4} \inf E^h \to 0$, *then* $Riem(G) \equiv 0$ *so in fact:* $\min E^h = 0$.

The Monge-Ampere constrained energy

$$\text{Energy} \quad E^h(u^h) = \frac{1}{h}\int_{\Omega^h} W\big((\nabla u^h)(A^h)^{-1}(x)\big)\,\mathrm{d}x$$

Theorem (L-Ochoa-Pakzad 2014)

Let: $A^h(x', x_3) = Id_3 + h^\gamma S(x')$ *and* $\gamma \in (0,2)$. *Then:*

- $\inf E^h \leq Ch^{\gamma+2} \;\Leftrightarrow\; \exists v \in W^{2,2}(\Omega), \quad \det \nabla^2 v = -\mathit{curl}\ \mathit{curl}\ S_{2\times 2}$
- $\frac{1}{h^{\gamma+2}} E^h \xrightarrow{\Gamma} I$, *where* I *is the 2-d energy:*

$$I(v) = \int_\Omega |\nabla^2 v|^2 \text{ for } v \in W^{2,2}(\Omega), \quad \det \nabla^2 v = -\mathit{curl}\ \mathit{curl}\ S_{2\times 2}$$

Structure of minimizers to E^h: $\quad u^h(x', 0) = x' + h^{\gamma/2} v e_3$

- $\kappa\big(\nabla(id + h^{\gamma/2} v e_3)^T \nabla(id + h^{\gamma/2} v e_3)\big) = \kappa\big(Id_2 + h^\gamma \nabla v \otimes \nabla v\big)$

$$= -\tfrac{1}{2} h^\gamma \text{curl curl } (\nabla v \otimes \nabla v) + \mathcal{O}(h^{2\gamma}) = h^\gamma \det \nabla^2 v + \mathcal{O}(h^{2\gamma})$$

- Gauss curvature: $\kappa(Id_2 + 2h^\gamma S_{2\times 2}) = -h^\gamma \text{curl curl } S_{2\times 2} + \mathcal{O}(h^{2\gamma})$

The Monge-Ampère constraint = existence of second order infinitesimal isometry $id + \varepsilon v e_3 + \varepsilon^2 w$ of the metric $Id_2 + 2\varepsilon^2 S_{2\times 2}$

Back to convex integration: the Monge-Ampère equation

$\det \nabla^2 v = f$ • existence of $W^{2,2}$ solutions is not guaranteed

$$\boxed{\mathrm{Det}\, \nabla^2 v = -\frac{1}{2}\,\mathrm{curl}\,\mathrm{curl}\big(\nabla v \otimes \nabla v\big)} \qquad v \in W^{1,2}(\Omega)$$

Need to solve: $\mathrm{curl}\,\mathrm{curl}\,(\nabla v \otimes \nabla v) = \mathrm{curl}\,\mathrm{curl}\, S_{2\times 2}$

where $S_{2\times 2} = \lambda \mathrm{Id}_2$ with $\Delta\lambda = -2f$ in Ω.

Equivalently: $\boxed{\nabla v \otimes \nabla v + \mathrm{sym}\nabla w = S_{2\times 2}}$

- $v, w \in W^{1,\infty}$ and $(\nabla v, \mathrm{sym}\nabla w) \in \mathcal{K} = \big\{(\mathbf{a}, A), (\mathbf{b}, B)\big\}$ a.e. in ω.
  Let $\mathbf{n} = \mathbf{b} - \mathbf{a} \neq \mathbf{0}$. Then:
  (i) $\langle (B-A)\mathbf{n}^\perp, \mathbf{n}^\perp\rangle \neq 0 \Rightarrow \nabla v$ and $\mathrm{sym}\nabla w$ are constant.
  (ii) $\langle (B-A)\mathbf{n}^\perp, \mathbf{n}^\perp\rangle = 0 \Rightarrow$ laminate structure possible in v, w.
  (iii) $\mathbf{a}\otimes\mathbf{a} + A = \mathbf{b}\otimes\mathbf{b} + B \Rightarrow$ automatically: $\langle (B-A)\mathbf{n}^\perp, \mathbf{n}^\perp\rangle = 0$.
- Many "laminate compatible connections" available!
  Case similar to $O(2,3)$...

Flexibility of the Monge-Ampère equation

Theorem (L-Pakzad 2015)

Let $(v_0, w_0) : \omega \to \mathbb{R} \times \mathbb{R}^2$ *be a smooth short infinitesimal, i.e.:*

$$\nabla v_0 \otimes \nabla v_0 + sym\nabla w_0 < S_{2\times 2}.$$

Then $\exists (v_n, w_n) \in \mathcal{C}^{1,\frac{1}{7}-}$ $(v_n, w_n) \xrightarrow{uniformly} (v_0, w_0)$ *and*

$$\nabla v_n \otimes \nabla v_n + sym\nabla w_n = S_{2\times 2}.$$

- Counterintuitive: 3 equations in 3 unknowns. Low regularity serves as a replacement for “higher dimensionality”.

Corollary (“Ultimate flexibility”)

Let $f \in L^2(\omega)$ *and* $\alpha < \frac{1}{7}$. *The set of* $\mathcal{C}^{1,\alpha}(\bar{\omega})$ *solutions to the Monge - Ampère equation:* $\mathrm{Det}\, \nabla^2 v = f$ *is dense in the space* $\mathcal{C}^0(\bar{\omega})$.

- For $f \in L^p(\omega)$ and $p \in (1, \frac{7}{6}]$, the density holds for any $\alpha < 1 - \frac{1}{p}$.
- $\mathrm{Det}\nabla^2$ is weakly discontinuous everywhere in $W^{1,2}(\omega)$.

Rigidity of the Monge-Ampère equation

- Consequences for the energy scaling: flexibility at $\mathcal{C}^{1,\frac{1}{7}-} \Rightarrow$ $\inf E^h \leq Ch^{\frac{7}{4}\gamma+\frac{1}{2}}$. (If we had flexibility at $\mathcal{C}^{1,\frac{1}{3}-}$ which is optimal using Nash-Kuiper technique, then $\inf E^h \leq Ch^{\frac{3}{2}\gamma+1}$).
- MA eqn.: fully nonlinear, 2nd order PDE, ellipticity $\Leftrightarrow$ convexity
  - Alexandrov (1958), Bakelman (1957): existence, uniqueness of generalized (convex) solutions for $f > 0$, convex boundary data.
  - Heinz (1961): $\mathcal{C}^{2,\alpha}$ interior estimates for $f \in \mathcal{C}^{0,\alpha}$ in 2 dimensions.
  - Cheng-Yau (1977), Pogorelov (1978): general regularity results
  - regularity of convex generalized solutions in higher dimensions: Caffarelli, Caffarelli-Nirenberg-Spruck, Krylov, Trudinger-Wang.

Rigidity at Hölder regularity (very weak solutions, no convexity assumpt.):

Theorem (L-Pakzad 2015)

Let $v \in \mathcal{C}^{1,\alpha}$, $\alpha > 2/3$. If $\mathrm{Det}\nabla^2 v = 0$, then v is developable.
If $\mathrm{Det}\nabla^2 v \geq c > 0$ is Dini continuous, then v is locally convex and an Alexandrov solution in ω.

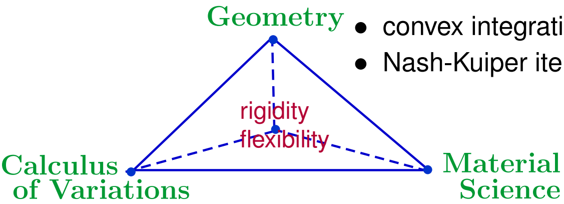
The flexibility-rigidity dichotomy

- Monge-Ampère eq: flexibility below $\mathcal{C}^{1,1/7}$; rigidity beyond $\mathcal{C}^{1,2/3}$
  - rigidity of $W^{2,2}$ solutions in the developable $f = 0$ (Pakzad) and convex $f > c > 0$ (Sverak, L-Mahadevan-Pakzad) cases.

- Isometric immersions of $G_{2\times 2}$ in $\mathbb{R}^3$: flexibility below $\mathcal{C}^{1,1/5}$ (Delellis-Inaunen-Szekelyhidi); rigidity beyond $\mathcal{C}^{1,2/3}$ (Borisov);
  - rigidity of $W^{2,2}$ immersions in the developable $\kappa = 0$ (Pakzad) and convex $\kappa > c > 0$ (Hornung-Velcic) cases.
  - Expected threshold: $\frac{1}{3}$ or $\frac{1}{2}$ or $\frac{2}{3}$.

- 3d incompressible Euler equations: flexibility below $\mathcal{C}^{0,1/5}$ (Delellis-Szekelyhidi, Isett: existence of $L^\infty(0,T;\mathcal{C}^\alpha(\mathbb{T}^3))$ solutions compactly supported in time)
  (Buckmaster-Delellis-Isett-Szekelyhidi: existence of solutions with arbitrary temporal kinetic energy profile);
  rigidity beyond $\mathcal{C}^{0,1/3}$ (Constantin-E-Titi, Eyink: every $L^\infty(0,T;\mathcal{C}^\alpha(\mathbb{T}^3))$ solution is energy conserving);
  - Expected threshold: $\frac{1}{3}$ (Onsager's conjecture)

Summary



- isometric immersions
- convex integration
- Nash-Kuiper iteration

- non-Euclidean elasticity
- dimension reduction
- Γ-convergence
- Effective constraints in the form of Monge-Ampère eqn

- microstructure formation
- matrensitic phase transition
- prestrain-activated materials

Thank you for your attention.