Research Statement of Marta Lewicka December 1, 2020

My main line of research concerns **Non-convex Calculus of Variations**, **Nonlinear Partial Differential Equations**, and **Continuum Mechanics**, where the effects of **Geometry** play a significant role. In this context, I have been mostly engaged with problems at the borderline of Analysis and Riemannian Geometry arising from the mathematical description of prestrained materials. These include: dimension reduction and the hierarchy of singular limits (Γ-limits) in non-Euclidean elasticity; the study of rigidity and flexibility in nonlinear PDEs such as the Monge-Ampère equation via convex integration; matching of isometries on surfaces and dimension reduction in elasticity of thin shells; regularity in the isometric immersions problems; mathematical analysis of growth; Korn's inequality and applications in various other domains such as fluid dynamics.

I have also worked on Tug-of-War games and the recently discovered relation of **Probability** and **Nonlinear Potential Theory**, where I obtained results concerning p-Laplacian and games with random noise, in the context of: Dirichlet problems, Robin problems, obstacle problems, nonlocal geometric p-Laplacian, and the subriemannian geometry. I have written a graduate textbook "A course on Tug-of-War games with random noise", and explored relations of the related techniques in the context of graph-Laplacians and data analysis.

My early works concerned well-posedness of hyperbolic systems of conservation laws in presence of large waves (large Total Variation initial data), reaction-diffusion equations, shock waves and combustion. My very early works regarded the usage of topological methods (degree and topological index) in **Nonlinear Analysis**.

Contents

1 Curvature-driven shape formation. Scaling laws and thin film models. Geometry and design of materials. 2

2 Convex integration for the Monge-Ampère equation. Rigidity and flexibility. 9

3 Nonlinear PDEs of p-laplacian type and Tug-of-War games. 10

4 Calculus of variations on thin elastic shells. 16

5 The Korn inequality. Dimension reduction in fluid dynamics. 19

6 Topics in viscoelasticity. 19

7 Topics in combustion. Traveling fronts in Boussinesq equations. 20

8 Well posedness of systems of conservation laws. 22

9 Multiplicity results for forced oscillations on manifolds. 23

10 Current research interests 24

1 Curvature-driven shape formation. Scaling laws and thin film models. Geometry and design of materials.

Elastic materials exhibit qualitatively different responses to different kinematic boundary conditions or body forces. Recently, there has been a growing interest in the study of prestrained elastica. A criterion which singles out the quality of prestraining in a body is the fact that it assumes non-trivial configurations in the absence of exterior forces or imposed boundary conditions. This phenomenon has been observed in different contexts: growing leaves, torn plastic sheets, nematic glass sheets and polymer gels. In all these situations, the shape of the lamina arises as a consequence of inelastic effects associated with growth, swelling or shrinkage, plasticity, etc., resulting in a local and heterogeneous incompatibility of strains.

1.1. The growth formalism and non-Euclidean elasticity. An analytical set-up that allows to analyze the dependence of the residual energy and deformations, on the prestrain incompatibility, is as follows. Let $G : \mathcal{U} \to \mathbb{R}^{3\times 3}$ be a smooth Riemannian metric, given on an open, bounded domain $\mathcal{U} \subset \mathbb{R}^3$. Since the matrix $G(x)$ is symmetric and positive definite, it possesses a unique symmetric, positive definite square root $A(x) = \sqrt{G(x)} \in \mathbb{R}^{3\times 3}$. We consider the following energy functional:

$$E(u) = \int_{\mathcal{U}} W\left((\nabla u)A^{-1}\right)\,\mathrm{d}x \qquad \forall u \in W^{1,2}(\mathcal{U}, \mathbb{R}^3), \tag{1}$$

where the energy density $W : \mathbb{R}^{3\times 3} \to [0, \infty]$ obeys the principles of material frame invariance (with respect to the special orthogonal group of proper rotations $SO(3)$), material consistency, normalisation, and non-degeneracy [1], valid for all $F \in \mathbb{R}^{3\times 3}$ and all $R \in SO(3)$:

$$W(RF) = W(F), \qquad W(\mathrm{Id}) = 0, \qquad W(F) \geq c\ \mathrm{dist}^2(F, SO(3)), \tag{2}$$

$$W(F) \to +\infty \quad \text{as } \det F \to 0+, \qquad \text{and} \quad \forall \det F \leq 0 \quad W(F) = +\infty. \tag{3}$$

The model in (1) assumes that the 3d elastic body $\mathcal{U}$ seeks to realize a configuration with a prescribed Riemannian metric G, through minimizing the energy, determined by the elastic part $F_e = (\nabla u)A^{-1}$ of its deformation gradient ∇u. Since $W(F_e) = 0$ if and only if $(\nabla u)^T \nabla u = G$ and $\det \nabla u > 0$, it is clear that $E(u) = 0$ if and only if u is an orientation preserving isometric immersion of G into $\mathbb{R}^3$. Such immersion exists (and is automatically smooth) when the Riemann curvature tensor R_G of G vanishes identically in $\mathcal{U}$. On the other hand, in [LewP2] we proved that E has strictly positive infimum for all non-immersable metrics G:

$$R_G \not\equiv 0 \quad \Leftrightarrow \quad \inf\left\{E(u);\ u \in W^{1,2}(\mathcal{U}, \mathbb{R}^3)\right\} > 0. \tag{4}$$

1.2. Some experimental connection. In view of (4), the quantity $\mathit{inf}E$ measures residual stresses at free equilibria in the absence of external forces or boundary conditions. We note that (1) postulates the validity of the decomposition $\nabla u = F_e A$; this formalism [95] requires that it is possible to separate out a reference configuration. It is thus relevant for the description of laminae, under inhomogeneous growth, plastic deformation, swelling or shrinkage driven by solvent absorption, or opto-thermal stimuli in liquid nematic glass sheets.

One of the first efforts to reproduce the effect of the prestrain on the shape of thin films in an artificial setting was reported in [55]. The authors manufactured thin gel films that underwent nonuniform shrinkage when activated in a hot bath according to the prescribed radially symmetric prestrain. Both large-scale buckling, multi-scale wrinkling structures and symmetry-breaking patterns appeared in the sheets, depending on the "programmed in" metrics. Another approach to controlling of shape through prestrain was suggested in [54], where a method of photopatterning polymer films yielded the temperature-responsive flat gel sheets that transformed into prescribed curved surfaces when the in-built metric was activated.

For other experimental results see [100, 99, 56, 4, 84, 107]. Notably, in a recent paper [107], the authors reported new methods to write arbitrary and spatially complex patterns of directors into liquid crystal elastomers through using photo-alignment materials. The liquid crystal director controls the inherent prestrain within the material and hence thermal or chemical stimuli transform flat sheets into surfaces, whose shapes depend on the prestrain. In several experiments, such programmed shapes were attained by actuated sheets of elastomers through writing topological defects (singular prestrains) or by introducing nonuniform director profiles through the thickness.

[1]Simple examples of W satisfying these conditions are: $W_1(F) = |(F^TF)^{1/2} - \mathrm{Id}|^2 + |\log\det F|^q$, or $W_2(F) = |(F^TF)^{1/2} - \mathrm{Id}|^2 + \left|(\det F)^{-1} - 1\right|^q$ for $\det F > 0$, where $q > 1$ and $W_{1,2}$ equal $+\infty$ if $\det F \leq 0$.

1.3. Thin films and dimension reduction. Pursuing our interest in understanding the mechanical responses of laminae to strain incompatibility, we want to reduce the complexity of the minimization problem (1) through dimension reduction. Consider thin films $\Omega^h = \omega \times (-\frac{h}{2}, \frac{h}{2})$ with mid-plate $\Omega \subset \mathbb{R}^2$, where the incompatibility tensors are determined by a family of Riemann metrics $G^h = (A^h)^2$. As in (1), we are concerned with the infima of the following energies, in the singular limit as $h \to 0$:

$$E^h(u^h) = \frac{1}{h}\int_{\Omega^h} W((\nabla u^h)(A^h)^{-1})\ \mathrm{d}x \qquad \forall u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3). \tag{5}$$

In the context of standard nonlinear elasticity for thin plates and shells (i.e. when $G = \mathrm{Id}$), this research direction has been initiated, using formal asymptotic expansions in [20, 92, 93, 35]) (see [21] for further references and background), and using the rigorous approach of Γ-convergence in the fundamental papers [63, 64, 39, 40] (and [1] in the content of 1-dimensional structures), furthered in [26, 38, LewMP3, LewMP2, LewMP1, HLewP] for thin plates and shells, in [25, 65] for incompressible materials, in [97] for heterogeneous materials, in [37] for inextensible ribbons, in [76, 77, 79, 78, Lew9, LewL] through convergence of equilibria rather than strict minimizers, and in [LewMaP3] for shallow shells.

In case when $G = G^h$ is constant along the thickness, the limit theories have been thoroughly discussed in [LewP2, LewP1, BLewS, LewRaR]. The general case of $G = G(x', x_3)$ and any admissible scaling $\inf E^h \sim h^\beta$, $\beta \geq 2$ has been resolved in [Lew13]. The paper [LewLu] dealt with a more general class of incompatibilities, where the transversal dependence of the lower order terms is nonlinear (the "oscillatory" case). When G^h is a thickness-dependent perturbation of Id_3, versions of the small-slope von Kármán theory (first postulated in [69, 33]) were rigorously derived in [LewMaP1], while a linearized Kirchhoff theory with Monge-Ampère constraints (17) and prestrained shallow shell theories [LewMaP2], each valid in their own range of growth parameters for the 3d thin sheet, were derived in [LewOP, LewMaP3]. See also the review papers [Lew11, LewP4].

1.4. The energy scaling quantisation. To describe our results in this domain, let us recall that a useful variational thin limit theory should comprise three essential ingredients. These are: a compactness result which identifies the asymptotic behavior of the minimizing sequences; two energy comparison results (in terms of liminf and limsup of energies of converging sequences) which allows to deduce that any converging minimizing sequence converges to the minimizer of the limiting theory; and a scaling analysis which identifies the range of validity of the corresponding energy scaling. In what follows, we will detail the results of [LewP2, BLewS, LewRaR, Lew13]. These results relate the context of dimension reduction in non-Euclidean elasticity with the analysis of quantitative immersability of Riemann metrics.

1. (Compactness). Let $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ be a sequence of deformations such that $E^h(u^h) \leq Ch^2$. Then, there exist constants $c^h \in \mathbb{R}^3$ and $Q^h \in SO(3)$ such that the rescaled deformations $y^h(x', x_3) := Q^h u^h(x', hx_3) - c^h$ converge to some $y \in W^{2,2}(\Omega^1, \mathbb{R}^3)$. Moreover, y depends only on the tangential variable x' and it is necessarily an isometric immersion of the midplate metric: $(\nabla y)^T \nabla y = G(x', 0)_{2\times 2}$.

2. (Liminf inequality). Let u^h and y be as above. We then have the lower bound:

$$\liminf_{h\to 0} \frac{1}{h^2} E^h(u^h) \geq \mathcal{I}_2(y) := \frac{1}{24}\int_\omega \mathcal{Q}_2\big(x', (\nabla y)^T \nabla \vec{b}_1 - \frac{1}{2}\partial_3 G(x', 0)_{2\times 2}\big)\,\mathrm{d}x', \tag{6}$$

   where $\mathcal{Q}_2(x', \cdot)$ are nonnegative quadratic forms, defined explicitly in terms of W (see [39, BLewS]), and where $\vec{b}_1$ satisfies: $\big[\partial_1 y, \partial_2 y, \vec{b}_1\big] \in SO(3) G^{1/2}(x', 0)_{2\times 2}$. Equivalently, $\vec{b}_1$ is the Cosserat vector comprising the appropriate nonzero sheer with respect to the vector $\vec{N}$ that is normal to the immersed surface $y(\omega)$:

$$\vec{b}_1 = (\nabla y) G_{2\times 2}^{-1} \begin{bmatrix} G_{13} \\ G_{23} \end{bmatrix} + \frac{\sqrt{\det G}}{\sqrt{\det G_{2\times 2}}}\vec{N}, \quad \text{with: } \ \vec{N} = \frac{\partial_1 y \times \partial_2 y}{|\partial_1 y \times \partial_2 y|}.$$

3. (Limsup inequality). For all $y \in W^{2,2}(\omega, \mathbb{R}^3)$ satisfying $(\nabla y)^T \nabla y = G(x', 0)_{2\times 2}$, there exists a sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ for which the convergence as in the compactness statement above holds true with $c_h = 0$, $Q^h = \mathrm{Id}_3$ and moreover:

$$\lim_{h\to 0} \frac{1}{h^2} E^h(u^h) = \mathcal{I}_2(y).$$

4. (Energy scaling). We have: $\inf E^h \sim h^2$ if and only if the following two conditions hold simultaneously: (a) There exists a $W^{2,2}$ isometric immersion of $(\omega, G(x',0)_{2\times 2})$ into $\mathbb{R}^3$, (b) At least one of the three Riemann curvatures $R_{12,12}, R_{12,13}$ or $R_{12,23}$ does not vanish identically on the midplate $\omega \times \{0\}$.

In [LewRaR, Lew13] we studied the scaling of E^h of order less than h^2. We discovered a gap phenomenon, i.e. the only scaling possible after the non-zero energy drops below h^2, is that of order h^4 and the Γ-limit of $\frac{1}{h^4}E^h$ in this case is a von Kàrmàn-like theory given in terms of the infinitesimal isometries and admissible strains on the surface isometrically immersing $G(x',0)_{2\times 2}$, plus an extra curvature term. More precisely:

5. (Energy scaling). If $\frac{1}{h^2}\inf E^h \to 0$ as $h \to 0$ then in fact: $\inf E^h \leq Ch^4$. In this case, there exists unique (up to rigid motions), automatically smooth immersion $y_0 : \omega \to \mathbb{R}^3$ such that $(\nabla y_0)^T \nabla y_0 = G(x',0)_{2\times 2}$ and $\big((\nabla y_0)^T \nabla \vec{b}_1\big)_{\rm sym} = \frac{1}{2}\partial_3 G(x',0)_{2\times 2}$. Further, if $\frac{1}{h^4}\inf E^h \to 0$ then there must be $Riem(G) = 0$ on $\omega \times \{0\}$. In particular, when $G = G(x')$ is constant along the thickness, then this last condition reduces to G being immersible and hence we then have: $\min E^h = 0$ for every h.

6. (Compactness and Γ-limit). Let $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ be a sequence of deformations satisfying $E^h(u^h) \leq Ch^4$. Then statements similar in nature to points 1.-3. above, hold for the rescaled deformations y^h, with the limiting 2d energy [LewRaR, Lew13] given by:

$$\mathcal{I}_4 = \quad \frac{1}{2}\int_\Omega \mathcal{Q}_2(x', \text{stretching of order } h^2)\,\mathrm{d}x' + \frac{1}{24}\int_\Omega \mathcal{Q}_2(x', \text{bending of order } h)\,\mathrm{d}x' + \frac{1}{1440}\int_\Omega \mathcal{Q}_2\Big(x', \begin{bmatrix} R_{13,13} & R_{13,23} \\ R_{13,23} & R_{23,23}\end{bmatrix}\Big)\,\mathrm{d}x'.$$

The functional $\mathcal{I}_4$ is a von Kármán-like energy, consisting of stretching and bending (with respect to the unique isometric immersion y_0 that gives the zero energy in the prior Γ-limit (6)) plus a new term, which quantifies the remaining three Riemann curvatures: $R_{13,13}, R_{13,23}, R_{23,23}$ on $\omega \times \{0\}$.

1.5. Sobolev isometric immersions of Riemannian metrics. As a corollary, we obtained new necessary and sufficient conditions for existence of $W^{2,2}$ isometric immersions of $(\omega, G_{2\times 2})$. In [LewP2] we showed that $G_{2\times 2}$ has an isometric immersion $y \in W^{2,2}(\omega, \mathbb{R}^3)$ iff $h^{-2}\inf E^h \leq C$, for a uniform constant C. In particular, if the Gaussian curvature $\kappa(G_{2\times 2}) \not\equiv 0$ in Ω then $h^{-2}\inf E^h \geq c > 0$.

Existence of local or global isometric immersions of a given 2d Riemannian manifold into $\mathbb{R}^3$ is a longstanding problem in differential geometry, its main challenge being the optimal regularity. By a classical result of Kuiper [62], a $\mathcal{C}^1$ isometric embedding into $\mathbb{R}^3$ can be obtained by means of convex integration. This regularity is far from $W^{2,2}$, where information about the second derivatives is also available. On the other hand, a smooth isometry exists for some special cases, e.g. for smooth metrics with uniformly positive or negative Gaussian curvatures on bounded domains in $\mathbb{R}^2$ [41]. Counterexamples are largely unexplored. The best result is due to Pogorelov [88]: there exists a $\mathcal{C}^{2,1}$ metric with nonnegative Gaussian curvature on the unit ball in $\mathbb{R}^2$ such that no neighborhood of the origin admits a $\mathcal{C}^2$ isometric embedding.

1.6. The complete hierarchy in the non-wrinkling regimes. The above analysis has been concluded in [Lew13], where we derived all the remaining thin limit theories, i.e. corresponding to $\inf E^h \sim h^\beta$ with $\beta > 4$.

1. (Energy scaling). For every $n \geq 2$, if $\frac{1}{h^{2n}}\inf E^h \to 0$ as $h \to 0$, then in fact: $\inf E^h \leq Ch^{2(n+1)}$. Further, the following three statements are then equivalent:

(i) $\inf E^h \leq Ch^{2(n+1)}$.

(ii) $R_{12,12}(x',0) = R_{12,13}(x',0) = R_{12,23}(x',0) = 0$ and $\partial_3^{(k)} R_{i3,j3}(x',0) = 0$ for all $x' \in \omega$, all $k = 0\ldots n-2$ and all $i,j = 1\ldots 2$.

(iii) There exist smooth fields $y_0, \{\vec{b}_k\}_{k=1}^{n+1} : \bar\omega \to \mathbb{R}^3$ giving raise to frames $\big\{B_k = \big[\partial_1\vec{b}_k,\ \partial_2\vec{b}_k,\ \vec{b}_{k+1}\big]\big\}_{k=1}^n$, and $B_0 = \big[\partial_1 y_0,\ \partial_2 y_0,\ \vec{b}_1\big]$ satisfying $\det B_0 > 0$, such that: $\sum_{k=0}^m \binom{m}{k} B_k^T B_{m-k} - \partial_3^{(m)} G(x',0) = 0$ for all $m = 0\ldots n$. Equivalently: $\Big(\sum_{k=0}^n \frac{x_3^k}{k!}B_k\Big)^T \Big(\sum_{k=0}^n \frac{x_3^k}{k!}B_k\Big) = G(x',x_3) + \mathcal{O}(h^{n+1})$ on Ω^h as $h \to 0$. The field y_0 is the unique (up to rigid motions), automatically smooth isometric immersion of $\omega, G(x',0)_{2\times 2})$ for which $\mathcal{I}_2(y_0) = 0$.

2. (Compactness and Γ-limit). Let $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ be a sequence of deformations satisfying $E^h(u^h) \leq Ch^{2(n+1)}$. Then, there exist constants $c^h \in \mathbb{R}^3$ and $Q^h \in SO(3)$ such that the displacements:

$$V^h(x') = \frac{1}{h^n} \fint_{-h/2}^{h/2} (\bar{R}^h)^T \big(u^h(x', x_3) - c^h\big) - \Big(y_0(x') + \sum_{k=1}^{n} \frac{x_3^k}{k!} \vec{b}_k(x')\Big) \, \mathrm{d}x_3$$

converge as $h \to 0$, strongly in $W^{1,2}(\omega, \mathbb{R}^3)$, to the limiting displacement: $V \in \mathcal{V}_{y_0}$. The space of first order isometries on the surface $y_0(\omega)$ is defined by: $\mathcal{V}_{y_0} = \Big\{V \in W^{2,2}(\omega, \mathbb{R}^3);\ \big((\nabla y_0)^T \nabla V\big)_{\mathrm{sym}} = 0\Big\}$. The above condition automatically yields existence of $\vec{p} \in W^{1,2}(\omega, \mathbb{R}^3)$ such that $\big(B_0^T \big[\nabla V,\ \vec{p}\big]\big)_{\mathrm{sym}} = 0$. Then, statements as in points 2.-3. in 1.4 hold, with the limiting energy of $\frac{1}{h^{2(n+1)}} E^h$ given by:

$$\begin{aligned} \mathcal{I}_{2(n+1)}(V) = \frac{1}{24} \cdot \int_\omega \mathcal{Q}_2\Big(x', (\nabla y_0)^T \nabla \vec{p} + (\nabla V)^T \nabla \vec{b}_1 + \alpha_n \big[\partial_3^{(n-1)} R_{i3,j3}\big]_{i,j=1\ldots 2}\Big) \, \mathrm{d}x' \qquad (7) \\ + \beta_n \cdot \int_\omega \mathcal{Q}_2\Big(x', \mathbb{P}_{\mathcal{S}_{y_0}^\perp}\big(\big[\partial_3^{(n-1)} R_{i3,j3}\big]_{i,j=1\ldots 2}\big)\Big) \, \mathrm{d}x' + \gamma_n \cdot \int_\omega \mathcal{Q}_2\Big(x', \mathbb{P}_{\mathcal{S}_{y_0}}\big(\big[\partial_3^{(n-1)} R_{i3,j3}\big]_{i,j=1\ldots 2}\big)\Big) \, \mathrm{d}x', \end{aligned}$$

where $\mathcal{S}_{y_0}$ stands for the space of finite strains in: $\mathcal{S}_{y_0} = \mathrm{closure}_{L^2}\Big\{\big((\nabla y_0)^T \nabla w\big)_{\mathrm{sym}};\ w \in W^{1,2}(\omega, \mathbb{R}^3)\Big\}$, whereas $\mathbb{P}_{\mathcal{S}_{y_0}}$ and $\mathbb{P}_{\mathcal{S}_{y_0}^\perp}$ denote, respectively, the orthogonal projections onto $\mathcal{S}_{y_0}$ and its orthogonal complement $\mathcal{S}_{y_0}^\perp$. The coefficients $\alpha_n, \beta_n, \gamma_n \geq 0$ are calculated explicitly in [Lew13].

3. (Identification of terms in $\mathcal{I}_{2(n+1)}$). When $G = Id_3$, then each functional in (7) reduces to the classical linear elasticity: $\mathcal{I}_{2(n+1)}(V) = \dfrac{1}{24} \displaystyle\int_\omega \mathcal{Q}_2\big(\nabla^2 v\big) \, \mathrm{d}x'$, which yields the biharmonic energy in function of the out-of-plane scalar displacement in $V = (\alpha x^\perp + \beta, v)$.

In the present geometric context, the bending term $(\nabla y_0)^T \nabla \vec{p} + (\nabla V)^T \nabla \vec{b}_1$ in (7) is of order $h^n x_3$ and it interacts with the curvature $\big[\partial_3^{(n-1)} R_{i3,j3}(\cdot, 0)\big]_{i,j=1\ldots 2}$, which is of order x_3^{n+1}. The interaction occurs only when the two terms have the same parity in x_3, namely at even n, so that $\alpha_n = 0$ for all n odd. The two remaining terms in (7) measure the (squared) L^2 norm of $\big[\partial_3^{(n-1)} R_{i3,j3}(\cdot, 0)\big]_{i,j=1\ldots 2}$, with distinct weights assigned to the $\mathcal{S}_{y_0}$ and $\big(\mathcal{S}_{y_0}\big)^\perp$ projections, again according to the parity of n. The quantity $\inf_{\mathcal{V}_{y_0}} \mathcal{I}_{2(n+1)}$ is precisely the square of a weighted L^2 norm of $\big[\nabla^{(n-1)} R_{ab,cd}\big]$ on ω, namely:

$$\inf_{\mathcal{V}_{y_0}} \mathcal{I}_{2(n+1)} \sim \big\| \big[\partial_3^{(n-1)} R_{i3,j3}(\cdot, 0)\big]_{i,j=1\ldots 2} \big\|^2_{L^2(\omega)}.$$

The finite strain space $\mathcal{S}_{y_0}$ can be identified, in particular, in the following two cases. When $y_0 = id_2$, then $\mathcal{S}_{y_0} = \{\mathbb{S} \in L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}});\ \mathrm{curl}^T \mathrm{curl}\, \mathbb{S} = 0\}$. When the Gauss curvature $\kappa((\nabla y_0)^T \nabla y_0) = \kappa\big(G_{2\times 2}\big) > 0$ on $\bar{\omega}$, then $\mathcal{S}_{y_0} = L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, as shown in [LewMP3].

4. (Viability of all energies $\mathcal{I}_{2(n+1)}$). We note that if $Riem(G) = 0$ on $\omega \times \{0\}$ and for some $n \geq 2$ there holds: $\partial_3^{(m)} \big[R_{i3j3}(\cdot, 0)\big]_{i,j=1\ldots 2} = 0$ on ω, for all $m = 0 \ldots n-2$, but $\partial_3^{(n-1)} \big[R_{i3j3}(\cdot, 0)\big]_{i,j=1\ldots 2} \not\equiv 0$, then:

$$ch^{2(n+1)} \leq \inf \mathcal{E}^h \leq Ch^{2(n+1)}, \quad \text{for some } \ c, C > 0$$

Further, the conformal metrics of the form: $G(x', x_3) = e^{2\phi(x_3)} Id_3$ provide a class of examples for the viability of all scalings: $\inf E^h \sim h^{2n}$ if and only if $\phi^{(k)}(0) = 0$ for $k = 1 \ldots n-1$ and $\phi^{(n)}(0) \neq 0$.

1.7. Coercivity. In [Lew13, LewLu] we showed that the kernel of $\mathcal{I}_2$ consists of the rigid motions of a single smooth deformation y_0 that solves: $(\nabla y_0) \nabla y_0 = G(x', 0)_{2\times 2}$, $\big((\nabla y_0)^T \nabla \vec{b}_1\big)_{\mathrm{sym}} = \frac{1}{2} \partial_3 G(x', 0)_{2\times 2}$. Further, $\mathcal{I}_2(y)$ bounds from above the squared distance of an arbitrary $W^{2,2}$ isometric immersion y of the midplate metric $G(x', 0)_{2\times 2}$ from the indicated kernel of $\mathcal{I}_2$. The parallel statement holds true for $\mathcal{I}_{2(n+1)}$ and $n > 1$, where the corresponding kernel consists of the linearised rigid motions of y_0.

For the case of $\mathcal{I}_4$, we first identify the zero-energy displacement-strain couples $(V, \mathbb{S})$; in particular, the minimizing displacements are the linearised rigid motions of the referential y_0. We then prove that the bending term in $\mathcal{I}_4$, which is solely a function of V, bounds from above the squared distance of an arbitrary $W^{2,2}$ displacement obeying $\big((\nabla y_0)^T \nabla V\big)_{\rm sym} = 0$, from the minimizing set in V. On the other hand, the full coercivity involving both V and $\mathbb{S}$ does not hold. We exhibit an example in the setting of the classical von Kármán functional, where $\mathcal{I}_4(V_n, \mathbb{S}_n) \to 0$ as $n \to \infty$, but the distance of $(V_n, \mathbb{S}_n)$ from the kernel of $\mathcal{I}_4$ remains uniformly bounded away from 0. We note that this lack of coercivity is not prevented by the fact that the kernel is finite dimensional.

1.8. Dimension reduction for thin films with transversally oscillatory prestrain. In [LewLu] we considered the "oscillatory case" where $G^h = (A^h)^2$ in (11) satisfies the following structure assumption:

$$G^h(x', x_3) = \mathcal{G}^h(x', \frac{x_3}{h}) = \bar{\mathcal{G}}(x') + h\mathcal{G}_1(x', \frac{x_3}{h}) + \frac{h^2}{2}\mathcal{G}_2(x', \frac{x_3}{h}) + \dots \qquad \text{for all } \ x = (x', x_3) \in \Omega^h.$$

Note that this set-up includes the subcase of $G^h = G$ ("non-oscillatory case"), upon taking: $\bar{\mathcal{G}}_1(x') = G(x', 0)$, $\mathcal{G}_1(x', t) = t\partial G_3(x', 0)$, $\mathcal{G}_2(x', t) = t^2\partial_{33}G(x', 0)$, etc. We exhibited connections between the two cases via projections of appropriate curvature forms on the polynomial tensor spaces and reduction to the "effective non-oscillatory cases". Statements like 1.-4. of 1.4. are then still valid, with the Γ-limits at various scaling exponents in the hierarchy indicated in 1.6, that consists of energies in (7) written for the effective metrics, plus the new "excess term" that measures the deviation of the given oscillatory metric from the effective non-oscillatory one, at an appropriate order.

1.9. The von Kármán equations with growth. Another goal of our studies has been to write the equilibrium equations of a thin elastic body subject to growth-induced finite displacements, as it bifurcates away from the flat sheet. This set-up relates as well to a small time-step in the dynamic problem of growth. One expects a hierarchy of limiting theories corresponding to the order of magnitude of the target strain tensor. In [LewMaP1, LewMaP2] we studied the general situation of weakly and strongly curved shells, with:

$$A^h - \mathrm{Id} \sim h^2.$$

We assumed that the reference configuration is given as a shell S^h_γ of small thickness h, around the midsurface S_γ that is the graph of the function γv_0 over a domain $\Omega \subset \mathbb{R}^2$. Given $\gamma = h^\alpha$ and $v_0 : \Omega \to \mathbb{R}$, we define $S_\gamma = \{(x', \gamma v_0(x')) : \ x' \in \Omega\}$ and study the corresponding 3d prestrained energy $E^{\gamma,h}$. We established that under suitable curvature constraints on $G^h = (A^h)^T A^h$, its infimum scales like h^4. Four different regimes for the Γ-limit of $h^{-4}E^{\gamma,h}$ were distinguished:

1. Case $\alpha > 1$. The Γ-limit for all values of $\alpha > 1$, i.e. when $\lim_{h\to 0} \dfrac{\gamma(h)}{h} = 0$, coincides with the zero thickness limit of the degenerate case $\gamma = 0$, which is the prestrained von Kármán model. The same energy can be obtained by taking the consecutive limits in $h^{-4}E^{\gamma,h}$, first in γ and then in h. The resulting Euler-Lagrange equations are those proposed and experimentally validated in [69]:

$$\Delta^2\Phi = -s\left(-\frac{1}{2}[v, v] + \lambda_g\right) \qquad \text{and} \qquad B\Delta^2 v = [v, \Phi] - B\Omega_g. \tag{8}$$

   Above, Φ is the Airy stress potential, v the out-of-plate displacement, and $[\cdot,\cdot]$ is the Airy's bracket [21]. Further, s stands for the Young's modulus, $-1/2[v, v] = \det \nabla^2 v$ is the Gaussian curvature of the deformation, B the bending stiffness, and ν Poisson's ratio. Finally, $\lambda_g = \mathrm{curl}^T\mathrm{curl}\,(\epsilon_g)$ and $\Omega_g = \mathrm{div}^T\mathrm{div}(\kappa_g + \nu \,\mathrm{cof}\, \kappa_g)$ for the growth tensors ϵ_g, κ_g in: $A^h(x', x_3) = \mathrm{Id} + h^2\epsilon_g(x') + hx_3\kappa_g(x')$.

2. Case $\alpha = 1$. This corresponds to $\lim_{h\to 0} \dfrac{\gamma(h)}{h} = 1$. The limit model is an unconstrained energy minimization, reflecting both the effect of shallowness and that of the prestrain. It corresponds to a simultaneous passing to the limit $(0, 0)$ of the pair (γ, h) in $h^{-4}E^{\gamma,h}$. The Euler-Lagrange equations (9) of this limit model were suggested in [69] for the description of the deployment of petals during the blooming of a flower:

$$\Delta^2\Phi = -s(\det \nabla^2 v - \det \nabla^2 v_0 + \lambda_g) \qquad \text{and} \qquad B(\Delta^2 v - \Delta^2 v_0) = [v, \Phi] - B\Omega_g\ , \tag{9}$$

3. Case $0 < \alpha < 1$. This corresponds to the flat limit $\gamma \to 0$ when the energy can be conceived as a limit of the von Kármán models $\mathcal{I}_4$ for shallow shells S_γ. In other words, this limiting model corresponds to the

case when: $\lim_{h\to 0}\frac{\gamma(h)}{h} = \infty$, and it can be also identified as the limit of $h^{-4}E^{\gamma,h}$ obtained by choosing the distinguished sequence of limits, first as the faster variable $h \to 0$ and then when $\gamma \to 0$. The Γ-limit is formulated for displacements of a plate but it inherits certain geometric properties of shallow shells S_γ, such as the first-order infinitesimal isometry constraint.

4. Case $\alpha = 0$. The 3d model is that of the prestrained non-linear elastic shell of arbitrarily large curvature (no shallowness involved). The Γ-limit of $h^{-4}E^{\gamma,h}$ in this case leads to a prestrained von Kármán model $\mathcal{I}_4$ for the 2d mid-surface.

1.10. The general von Kármán-like growth tensor. In [JLew] we further provided a complete analysis of the prestrain tensor that corresponds to a family of Riemann metrics with weak curvatures, where:

$$A^h(x', x_3) = \mathrm{Id} + h^{\alpha/2}\epsilon_g(x') + h^{\gamma/2}x_3\kappa_g(x').$$

There are essentially three new contributions in this context:

1. In the regime $\alpha \geq 4, \gamma \geq 2$, we found the Γ-limits of the rescaled energies, identified the optimal energy scaling laws, and displayed the equivalent conditions for optimality in terms of both the prestrain components and the curvatures of the related Riemannian metrics. Similarly to the case of large prestrain [LewRaR], we observeed that one such condition is the non-vanishing of the lowest order terms in the curvatures $R_{12,12}, R_{12,13}, R_{13,23}$ of $G^h = (A^h)^T A^h$ along the midplate.

2. In the larger prestrain regime $\alpha \in (0, 4)$, $\gamma > 0$, we proposed new energy upper bounds, based on the construction of a sequence of deformations via the Kirchhoff-Love extension of the highly perturbative, Hölder-regular solutions to the Monge-Ampere equation obtained by convex integration in [LewP3].

3. When the stretching-inducing prestrain is of order lower than that allowed in **1.**, but carries no in-plane modes, i.e. $\alpha, \gamma \geq 2$ and $S_{2\times 2} \equiv 0$ we still perform the full dimension-reduction analysis and discover similarities with both the theories in **1.** and the shallow shell models of [LewMaP2].

1.11. The biharmonic energy with Monge-Ampére constrains. In [LewOP], the prestrained 3d plate model E^h was studied, under the incompatibility scaling:

$$A^h - \mathrm{Id} \sim h^\theta, \quad 0 < \theta < 2. \tag{10}$$

The results of this paper are multifold and open the way for posing a range of challenging questions in the analysis of nonlinear geometric PDEs such as the Monge-Ampère equation. We derive the Γ-limit of $h^{-(\theta+2)}E^h$ where, as before, under suitable non-vanishing curvature conditions, $h^{\theta+2}$ is proved to be the optimal scaling of the infimum of E^h. The limit model, corresponding to $1 < \theta < 2$ consists of a biharmonic energy subject to the Monge-Ampère constraint, i.e. the minimizers of E^h in this regime approach asymptotically the out-of-plate displacements $v : \Omega \to \mathbb{R}$, which are minimizers of:

$$\mathcal{I}_f(v) = \int_\Omega \mathcal{Q}_2(\nabla^2 v)\,\mathrm{d}x' \quad \text{where} \quad v \in \mathcal{A}_f = \{v \in W^{2,2}(\Omega) : \ \det\nabla^2 v = f \text{ a.e. in } \Omega\}.$$

Here, $f : \Omega \to \mathbb{R}$ is a given function which asymptotically depends on the choice of the perturbation $A^h - \mathrm{Id}$. For the scaling regime $0 < \theta < 1$, we expect the thin limit to be still generically (i.e. for generic A^h) the same model. Some steps were already taken in [LewOP] to show this result. The analysis is based on observations:

1. If $f \geq 0$ and given $v \in \mathcal{A}_f$ of sufficient regularity, it is possible to isometrically parametrize the graph of v, modulo suitable uniformly controlled in-plane perturbations of the domain variable. This reparameterization provides a precise way of approximating the energy $\mathcal{I}_f(v)$ by $h^{-(\theta+2)}E^h$ computed along the recovery sequence $u^h : \Omega^h \to \mathbb{R}^3$. Indeed, for smaller values of θ, one deals with smaller values of error in the approximation.

2. If $f \equiv c > 0$, then any $v \in \mathcal{A}_f$ can be approximated by a sequence $v_k \in \mathcal{A}_f \cap C^\infty(\bar\Omega)$.

Combining 1 and 2, the remaining obstacle is to show that smooth functions are dense in $\mathcal{A}_f$ for any f. Note that, in [LewMaP3] it is proved that $W^{2,2}$ solutions of the Monge-Ampère equation $\det \nabla^2 v = f$, are locally convex and indeed coincide with the classical Alexandrov solutions. This implies that the main difficulty in the density problem is the mollification of the solution at the boundary while keeping the Hessian intact.

Finally, a partial study of the thin limit $\mathcal{I}_f$ over the admissible function space $\mathcal{A}_f$ was undertaken in [LewOP]. In particular, we were concerned with the question of multiplicity of solutions in the radially symmetric case. While the question of the Γ-limit for the scaling (10) is not yet fully settled, many open problems regarding multiplicity, regularity and even the derivation Euler Lagrange equations of the limiting model stay open.

1.12. A design problem. In [ALewP], we studied a class of design problems in solid mechanics, leading to a variation on the classical question of equi-dimensional embeddability of Riemannian manifolds. Given two smooth positive definite matrix fields $\tilde{G}, G$ on $\Omega \subset \mathbb{R}^n$, one can seek an isometry ξ between the Riemannian manifolds $(\Omega, \tilde{G})$ and $(\xi(\Omega), G \circ \xi^{-1})$. What distinguishes our problem from the classical isometric immersion problem, where one looks for an isometric mapping between two given manifolds $(\Omega, \tilde{G})$ and $(U, \mathcal{G})$, is that the target manifold $U = \xi(\Omega)$ and its metric $\mathcal{G} = G \circ \xi^{-1}$ are only given a posteriori, after the solution is found. In this context, we derived a necessary and sufficient existence condition, given through a system of total differential equations, and discussed its integrability. In the classical context, the same approach yields conditions of immersibility of a given metric in terms of the Riemann curvatures. In the present case the equations do not close, and successive differentiation of the compatibility conditions leads to a new algebraic description of integrability. Taking into account that the non-existence situations could be generic, we also recast the problem in a variational setting and analyze the infimum of the appropriate incompatibility energy:

$$E(\xi) = \int_\Omega \mathrm{dist}^2\big(G^{1/2}(\nabla\xi)\tilde{G}^{-1/2}, SO(n)\big)\,\mathrm{d}x \qquad \forall \xi \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n). \tag{11}$$

which resembles the non-Euclidean elasticity (1). We then derived a Γ-convergence result for the dimension reduction from 3d to 2d in the Kirchhoff energy scaling regime.

1.13. Discrete approximation. In paper [LewO] we studied the asymptotic behaviour of discrete elastic energies in presence of the prestrain metric G, assigned on the continuum reference configuration Ω:

$$E_\epsilon(u_\epsilon) = \sum_{\xi\in\mathbb{Z}^n} \sum_{\alpha\in R_\epsilon^\xi(\Omega)} \epsilon^n \psi(|\xi|) \Big| \frac{|u_\epsilon(\alpha+\epsilon\xi) - u_\epsilon(\alpha)|}{\epsilon|A(\alpha)\xi|} - 1\Big|^2, \tag{12}$$

where $R_\epsilon^\xi(\Omega) = \{\alpha \in \epsilon\mathbb{Z}^n : [\alpha, \alpha+\epsilon\xi] \subset \Omega\}$ denotes the set of lattice points interacting with the node α, and where a smooth cut-off function $\psi : \mathbb{R}_+ \to \mathbb{R}$ allows only for interactions with finite range:

$$\psi(0) = 0 \quad \text{ and } \quad \exists M > 0 \quad \forall n \geq M \quad \psi(n) = 0.$$

The energy in (12) measures the discrepancy between lengths of the actual displacements between the nodes $x = \alpha + \epsilon\xi$ and $y = \alpha$ due to the deformation u_ϵ, and the ideal displacement length $\langle G(\alpha)(x-y), (x-y)\rangle^{1/2} = \epsilon|A(\alpha)\xi|$. When the mesh size of the discrete lattice in Ω goes to zero, we obtain the variational bounds on the limiting (in the sense of Γ-limit) energy. In case of the nearest-neighbour and next-to-nearest-neighbour interactions, we derive asymptotic formulas, and compare them with the non-Euclidean energy relative to G.

1.14. A model of controlled growth. In paper [BrLew3] we considered an evolutionary free boundary problem for a system of PDEs, modeling the growth of a biological tissue. In this model, the morphogen with concentration u, controlling volume growth, is produced by specific cells (with concentration w) and then diffused and absorbed throughout the time-varying domain $\Omega(t) \subset \mathbb{R}^3$:

$$\text{minimize:} \quad J(u) \doteq \int_{\Omega(t)} \Big(\frac{|\nabla u|^2}{2} + \frac{u^2}{2} - wu\Big)\,\mathrm{d}x, \tag{13}$$

$$\begin{cases} w_t + \mathrm{div}(wv) = 0 & x \in \Omega(t), \\ w(0,x) = w_0(x) & x \in \Omega(0) = \Omega_0. \end{cases} \tag{14}$$

Then, the geometric shape of the growing tissue is determined by the instantaneous minimization of an elastic deformation energy, subject to a constraint on the volumetric growth:

$$\text{minimize:}\quad E(v) \;\doteq\; \frac{1}{2}\int_{\Omega(t)} |\mathrm{sym}\,\nabla v|^2 \,\mathrm{d}x \qquad \text{subject to:}\quad \mathrm{div}\, v \;=\; g(u), \tag{15}$$

$$\Omega(t) = \Big\{x(t)\,;\quad x(0) = x_0 \in \Omega_0 \;\text{ and }\; x'(s) = v(s, x(s)) \;\;\forall s \in [0,t]\Big\}. \tag{16}$$

The main goal of our analysis was to prove that, given an initial set Ω_0 and an initial density $w_0(x)$ for $x \in \Omega_0$, the equations (13-14-15-16) determine a unique evolution, up to rigid motions. Indeed, for Ω_0 with regularity $C^{2,\alpha}$, our main result established such local existence and uniqueness of a classical solution.

2 Convex integration for the Monge-Ampère equation. Rigidity and flexibility.

The work [LewP3] concerns the dichotomy of "rigidity vs. flexibility" for $\mathcal{C}^{1,\alpha}$ solutions to the Monge-Ampère equation. Let $\Omega \subset \mathbb{R}^2$ be an open set. Given a real valued function $v \in W^{1,2}_{loc}(\Omega)$ we study its very weak Hessian, understood in the sense of distributions and given in the following form:

$$\mathcal{D}et\, \nabla^2 v := -\frac{1}{2}\mathrm{curl}\;\mathrm{curl}\,(\nabla v \otimes \nabla v) = f \qquad \text{in } \Omega \subset \mathbb{R}^2. \tag{17}$$

A straightforward approximation argument shows that if $v \in W^{2,2}_{loc}$ then (17) coincides with the determinant of the matrix of second derivatives of v. Let us point out that there are other notions of a weak Hessian than (17), including the well known notion of the distributional Hessian $\mathcal{H}$. Different notions have different features, for example contrary to $\mathcal{H}$, the operator $\mathcal{D}et\,\nabla^2$ is not continuous with respect to the weak topology. Indeed, one consequence of our results below is that it is actually weakly discontinuous everywhere in $W^{1,2}(\Omega)$. Each of these notions relates to some analytical context; as we shall see (17) arises naturally in the context of the isometric immersion problem and its connection to models of elastic prestrained plates.

2.1. The flexibility results. We prove the following. Let $f \in L^{7/6}(\Omega)$ and fix an exponent: $\alpha < \frac{1}{7}$. Then the set of $\mathcal{C}^{1,\alpha}(\bar\Omega)$ solutions to (17) is dense in $\mathcal{C}^0(\bar\Omega)$. That is, for every $v_0 \in \mathcal{C}^0(\bar\Omega)$ there exists a sequence $v_n \in \mathcal{C}^{1,\alpha}(\bar\Omega)$, converging uniformly to v_0 and satisfying: $\mathcal{D}et\,\nabla^2 v_n = f$. When $f \in L^p(\Omega)$ and $p \in (1, \frac{7}{6})$, the same result is true for any $\alpha < 1 - \frac{1}{p}$. This density result is a consequence of the following statement whose proof relies on convex integration techniques applied to (17). Let $v_0 \in \mathcal{C}^1(\bar\Omega)$, $w_0 \in \mathcal{C}^1(\bar\Omega, \mathbb{R}^2)$ and $A_0 \in \mathcal{C}^{0,\beta}(\bar\Omega, \mathbb{R}^{2\times 2}_{sym})$, for some $\beta \in (0,1)$ and assume that:

$$A_0 - \big(\frac{1}{2}\nabla v_0 \otimes \nabla v_0 + \mathrm{sym}\,\nabla w_0\big) > \mathrm{Id}_2 \qquad \text{in } \bar\Omega. \tag{18}$$

Then, for every exponent α in the range: $0 < \alpha < \min\Big\{\frac{1}{7}, \frac{\beta}{2}\Big\}$, there exist sequences $v_n \in \mathcal{C}^{1,\alpha}(\bar\Omega)$ and $w_n \in \mathcal{C}^{1,\alpha}(\bar\Omega, \mathbb{R}^2)$ which converge uniformly to v_0 and w_0, respectively, and which satisfy:

$$A_0 = \frac{1}{2}\nabla v_n \otimes \nabla v_n + \mathrm{sym}\,\nabla w_n \qquad \text{in } \bar\Omega. \tag{19}$$

2.2. Rigidity versus flexibility. Flexibility results as above, that are obtained in view of the convex integration h-principle, are usually coupled with the rigidity results for more regular solutions. For the Monge-Ampère equations, we recall two recent statements regarding solutions with Sobolev regularity: following the well known unpublished work by Šverák [103], we proved in [LewMaP3] that if $v \in W^{2,2}(\Omega)$ is a solution to (17) with $f \in L^1(\Omega)$ and $f \geq c > 0$ in Ω, then in fact v must be $\mathcal{C}^1$ and globally convex. On the other hand, if $f = 0$ then [85] likewise $v \in \mathcal{C}^1(\Omega)$ and v must be developable (see also [51, 52, 53]). A clear statement of rigidity is still lacking for the general f, as is the case for isometric immersions, where rigidity results are usually formulated only for elliptic [24] or Euclidean metrics [85, 68, 53].

Our results for (17) are as follows. Assume that $\frac{2}{3} < \alpha < 1$. If $v \in \mathcal{C}^{1,\alpha}(\bar\Omega)$ is a solution to $\mathcal{D}et\,\nabla^2 v = 0$ in $\bar\Omega$, then v must be developable. More precisely, for all $x \in \Omega$ either v is affine in a neighbourhood of x, or there exists a segment l_x joining $\partial\Omega$ on its both ends, such that ∇v is constant on l_x. Likewise, when f is positive

Dini continuous, then v is convex and, in fact, it is also an Alexandrov solution to $\det \nabla^2 v = f$ in Ω. In proving the above results, we used a commutator estimate for deriving a degree formula; similar commutator have been used in [22] for the Euler equations and in [24] for the isometric immersion problem. This is not surprising, since the presence of a quadratic term plays a major role in all three cases, allowing for the efficiency of convex integration and iteration. Let us also mention that it is still an open problem which α is the critical value for the rigidity-flexibility dichotomy, and it is conjectured to be $1/3, 1/2$ or $2/3$.

2.3. Connection to the isometric immersion problem. Deformations and displacements. In order to better understand the results in [LewP3], we point out a connection between the solutions to (17) and the isometric immersions of Riemannian metrics, motivated by a study of nonlinear elastic plates. Since on a simply connected domain Ω, the kernel of the differential operator curl curl consists of the fields of the form $\mathrm{sym}\,\nabla w$, a solution to (17) with the vanishing right hand side $f \equiv 0$ can be characterized by:

$$\exists w : \Omega \to \mathbb{R}^2 \qquad \frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\,\nabla w = 0 \quad \text{in } \Omega. \tag{20}$$

The equation in (20) is an equivalent condition for the following 1-parameter family of deformations, given through the out-of-plane displacement v and the in-plane displacement w in: $\phi_\varepsilon = id + \varepsilon v e_3 + \varepsilon^2 w : \Omega \to \mathbb{R}^3$, to form a 2nd order infinitesimal isometry (bending), i.e. to induce the change of metric on the plate Ω whose 2nd order terms in ε disappear: $(\nabla\phi_\varepsilon)^T \nabla\phi_\varepsilon - \mathrm{Id}_2 = o(\varepsilon^2)$.

In this context, the celebrated work of Nash and Kuiper [81, 62] shows the density of co-dimension one $\mathcal{C}^1$ isometric immersions of Riemannian manifolds in the set of short mappings. Since we are now dealing with the 2nd order infinitesimal isometries rather than the exact isometries, the classical metric pull-back equation: $y^* g_e = h$, for a mapping y from (Ω, h) into $(\mathbb{R}^3, g_e)$ is replaced by the compatibility equation of the tensor $\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\,\nabla w$ with a matrix field A_0 that satisfies: $-\mathrm{curl}\,\mathrm{curl}\, A_0 = f$. This compatibility equation states precisely that the metric $(\nabla\phi_\varepsilon)^T \nabla\phi_\varepsilon$ agrees with the given metric $h = \mathrm{Id}_2 + 2\varepsilon^2 A_0$ on Ω, up to terms of order ε^2. The Gauss curvature κ of the metric h satisfies: $\kappa(h) = -\varepsilon^2 \mathrm{curl}\,\mathrm{curl}\, A_0 + o(\varepsilon^2)$, while $\kappa((\nabla\phi_\varepsilon)^T \nabla\phi_\varepsilon) = -\varepsilon^2 \mathrm{curl}\,\mathrm{curl}\,\big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\, w\big) + o(\varepsilon^2)$, so the problem (17) can also be interpreted as seeking for all appropriately regular out-of-plane displacements v that can be matched, by an in-plane displacement perturbation w, to achieve the prescribed Gauss curvature f of Ω, at its highest order term.

2.4. Relation to other convex integration results in nonlinear PDEs. The flexibility result in [LewP3] is the Monge-Ampère analogue of the isometric immersion problem in [24, Theorem 1], where the authors improved on the Nash-Kuiper methods and obtained higher regularity within the flexibility regime. On the other hand, rigidity of isometric immersions of elliptic metrics has been shown for $\mathcal{C}^{1,\alpha}$ isometries in [11, 24] with $\alpha > 2/3$. Recently, these methods were applied as well in the context of fluid dynamics and yielded many interesting results for the Euler equations: in [31] existence of weak solutions with bounded velocity and pressure has been proved together with their non-uniqueness and the existence of energy-decreasing solutions; in [32] existence of continuous periodic solutions of the 3d incompressible Euler equations, which dissipate the total kinetic energy has been proved; the stationary incompressible Euler equation has been studied in [19] where existence of bounded anomalous solutions was shown.

These results are to be contrasted with [22, 34], where it was shown that $\mathcal{C}^{0,\alpha}$ solutions of the Euler equations are energy conservative if $\alpha > 1/3$. There have been several improvements of [31, 32] since, linked with the Onsager's conjecture which puts the Hölder regularity threshold for the energy conservation of the weak solutions to the Euler equations at $\mathcal{C}^{0,1/3}$ [48, 49, 16, 17, 18, 19].

3 Nonlinear PDEs of p-laplacian type and Tug-of-War games.

Nonlinear PDEs, mean value properties, and stochastic differential games are intrinsically connected. In this section I report on my recent results regarding the random walk representations of the Dirichlet, Robin, and obstacle problems for the p-Laplace equation. Generally speaking, solutions to certain nonlinear PDEs can be interpreted as limits of values of specific Tug-of-War games, when the step-size ϵ determining the allowed length of move of a token, decreases to 0. This observation allows replacing some classical techniques by relying instead on suitable choices of strategies for the competing players; indeed it has inspired further studies in different directions, such as: asymptotic mean value properties, a new proof of Harnack's inequality for p-harmonic functions, a new proof of Hölder regularity, connections with the optimal Lipschitz extension problem, control theory and economic modeling, or semi-supervised machine learning. The approach we follow originated in

[86, 87] and was furthered in [70, 71]; for the case of deterministic games see the review [57] and [58, 59]. Some of the basic concepts have also been explained in a short review paper [LewM1], and more recently in the graduate-level textbook "A Course on Tug-of-War Games with Random Noise" by M. Lewicka [Lew14].

3.1. Tug-of-War with noise, case $1 < p < \infty$. It is a well known fact that for $u \in \mathcal{C}^2(\mathbb{R}^N)$ there holds:

$$\fint_{B_\epsilon(x)} u(y)\mathrm{d}y = u(x) + \frac{\epsilon^2}{2(N+2)}\Delta u(x) + o(\epsilon^2) \qquad \text{as } \epsilon \to 0+. \tag{21}$$

Indeed, since an equivalent condition for harmonicity $\Delta u = 0$ is the mean value property: $\fint_{B_\epsilon(x)} u(y)\mathrm{d}y = u(x)$, the coefficient $\Delta u(x)$ above measures the second-order error from the satisfaction of this property.

The next observation is that a similar expansion and its resulting probabilistic interpretation can be also derived, with appropriate modifications, for nonlinear operators. Note first that when we replace $B_\epsilon(x)$ by the ellipsoid E with radius r, aspect ratio $\alpha > 0$ and oriented along a unit vector ν, we obtain:

$$\fint_{E(x,\epsilon;\alpha,\nu)} u(y)\ \mathrm{d}y = u(x) + \frac{\epsilon^2}{2(N+2)}\Big(\Delta u(x) + (\alpha^2-1)\langle\nabla^2 u(x) : \nu^{\otimes 2}\rangle\Big) + o(\epsilon^2).$$

Recalling the interpolation of the normalised (so called game-theoretical) p-Laplacian Δ_p^G in:

$$\Delta_p^G u = |\nabla u|^{2-p}\Delta_p u = \Delta u + (p-2)\Delta_\infty u,$$

the mean value expansion becomes: $\fint_{E(x,\epsilon;\alpha,\nu)} u(y)\ \mathrm{d}y = u(x) + \frac{\epsilon^2}{2(N+2)}\Delta_p^G u(x) + o(\epsilon^2)$, for the choice $\alpha = \sqrt{p-1}$ and $\nu = \frac{\nabla u(x)}{|\nabla u(x)|}$. To obtain an expansion where the left hand side averaging does not require the knowledge of $\nabla u(x)$ and allows for the identification of a p-harmonic function that is a priori only bounded, one needs to average over orientations ν. This is done by superposing "the deterministic average $\frac{1}{2}(\inf + \sup)$" with "the stochastic average $\fint$", as derived in [Lew11]:

$$\frac{1}{2}\Big(\inf_{z\in B_\epsilon(x)} + \sup_{z\in B_\epsilon(x)}\Big)\fint_{E\big(z,\gamma_p\epsilon,\alpha_p(\left|\frac{z-x}{\epsilon}\right|),\frac{z-x}{|z-x|}\big)} u(y)\ \mathrm{d}y = u(x) + \frac{\gamma_p^2\epsilon^2}{2(N+2)}\Delta_p^G u(x) + o(\epsilon^2). \tag{22}$$

The above expansion is valid with γ_p a fixed stochastic sampling radius factor, and α_p the aspect ratio in radial function of the position $z \in B_\epsilon(x)$. The value of α_p varies quadratically from 1 at the center of $B_\epsilon(x)$ to ρ_p at its boundary, where ρ_p and γ_p satisfy the appropriate compatibility condition, depending on N and p. As in the linear case, one can then show that an equivalent condition for p-harmonicity $\Delta_p u = 0$ is the asymptotic satisfaction of the mean value: $\frac{1}{2}\big(\inf_{z\in B_\epsilon(x)} + \sup_{z\in B_\epsilon(x)}\big)\fint_{E\left(z,\gamma_p\epsilon,\alpha_p,\frac{z-x}{|z-x|}\right)} u(y)\ \mathrm{d}y = u(x) + o(\epsilon^2)$.

The discrete stochastic process modelled on this equation is the two-player Tug-of-War with noise. In this process, the token is initially placed at a point x_0 within the domain $\Omega \subset \mathbb{R}^N$, and at each step it is advanced according to the following rule. First, either of the two players (each acting with probability $\frac{1}{2}$) shifts the token by a chosen vector $y = z - x$ of length at most ϵ; second, the token is further shifted within the ellipsoid $E\big(z, \gamma_p\epsilon, 1 + (\rho_p - 1)\frac{|y|^2}{\epsilon^2}, \frac{y}{|y|}\big)$. The game is terminated, whenever the token reaches the ϵ-neighbourhood of $\partial\Omega$. The value $u^\epsilon(x_0)$ is defined as the expectation of the boundary function F (extended continuously on $\mathbb{R}^N$) at the stopping position x_τ, subject to both players playing optimally. The optimality criterion is based on the rule that Player II pays to Player I the value $F(x_\tau)$, thus giving Player I the incentive to maximize the gain by pulling towards portions of $\partial\Omega$ with high values of F, whereas Player II will likely try to minimize the loss by pulling towards the low values. Due to the min-max property, the optimality is well posed, i.e. the order of supremizing over strategies of the first player and infimizing over strategies of the opponent, is immaterial. We point out that the validity of this property has been posed as an open question in the context of the game first proposed in [87], where the regularity (even measurability) of the possibly distinct game values was likewise not clear. Here, u^ϵ is proved to be automatically as regular as F is (continuous / Hölder / Lipschitz).

It is expected that the family $\{u^\epsilon\}_{\epsilon\to 0}$ converges pointwise in Ω to the Perron solution u of the Dirichlet problem for Δ_p with any given continuous boundary data F. Further, it is natural to expect that this convergence is uniform for regular boundary, to the effect that $u = F$ on $\partial\Omega$. While the former result is not yet available (for exponents $p \neq 2$) at the time of this Research Statement, the latter assertions are proved to hold true.

More precisely, in [Lew11] we address the question of convergence of $\{u^\epsilon\}_{\epsilon\to 0}$: in view of its equiboundedness, it suffices to prove equicontinuity. We first observe, that this property is equivalent to the seemingly weaker property of equicontinuity at the boundary. Our argument is analytical rather than probabilistic, based on the translation and well-posedness of the mean value equation modeled on (22). We then define the game regularity of the boundary points, which turns out to be a notion equivalent to the aforementioned boundary equicontinuity. We prove that any limit of a converging sequence of u^ϵ-s must be the viscosity solution to the p-harmonic equation with boundary data F. By uniqueness of such solutions, we obtain the uniform convergence of the entire family in case of the game regular boundary. We finally check that domains that satisfy the exterior corkscrew condition are game regular. One can similarly show (see [Lew12]) that game regularity holds when $p > N$ and for any p in case of $N = 2$-dimensional domains that are simply connected.

3.2. Random walks and random Tug-of-War in the Heisenberg group. In paper [LewMR], we studied the mean value properties of p-harmonic functions on the Heisenberg group $\mathbb{H}^1$, in connection to the dynamic programming principles of stochastic processes. We thus carried out the program described in 3.1. Firstly, we developed the mean value expansions of the type (22), where the domain of averaging has been one of the following: the 3-dimensional Koràny ball in $\mathbb{H}^1$; the 2-dimensional ellipse contained in the horizontal plane; the 1-dimensional boundary of such ellipse; or the 3-dimensional Koràny ellipsoid that is the image of the ball under a suitable linear map. Then, we identified solutions u^ϵ of the related mean value equations, as values of corresponding processes with, in general, both random and deterministic components. Finally, we examined convergence of the family $\{u^\epsilon\}_{\epsilon\to 0}$ and for domains with game-regular boundary, we showed its uniform convergence to the viscosity solution of the Dirichlet problem.

3.3. The obstacle problem via optimal stopping and Tug-of-War. In paper [LewM2] we were concerned with the solutions to the obstacle problem for the p-Laplace operator Δ_p in the nonsingular range $p \geq 2$:

$$\left\{\begin{array}{rcll} -\Delta_p u & \geq & 0 & \text{in } \Omega, \\ u & \geq & \Psi & \text{in } \Omega, \\ -\Delta_p u & = & 0 & \text{in } \{x\in\Omega;\ u(x) > \Psi(x)\}, \\ u & = & F & \text{on } \partial\Omega, \end{array}\right. \tag{23}$$

In (23), $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a bounded, Lipschitz function, which we assume to be compatible with the boundary data: $F(x) \geq \Psi(x)$ for $x \in \partial\Omega$. The function Ψ is interpreted as the obstacle and in (23) we want to find a p-superharmonic function u taking boundary values F, which is above the obstacle Ψ, and which is actually p-harmonic in the complement of the contact set $\{x \in \overline{\Omega}\colon u(x) = \Psi(x)\}$. The problem (23) has been extensively studied from the variational point of view; in particular regularity requirements for the domain Ω, the boundary data F and the obstacle Ψ can be vastly generalized. It is also classical that the solution to (23) exists, it is unique, and it is the pointwise infimum of all p-superharmonic functions that are above the obstacle.

Our results show how to solve the obstacle problem in the context of the program described in 3.1. The dynamic programing principle (24) below is similar to the Wald-Bellman equations of optimal stopping. Namely, let $\alpha = \frac{p-2}{p+N}$ and $\beta = 1-\alpha$. Let $F : \bar\Gamma \to \mathbb{R}$ and $\Psi : \mathbb{R}^N \to \mathbb{R}$ be bounded, Borel functions such that $\Psi \leq F$ in an open neighbourhood Γ of $\partial\Omega$ in $\mathbb{R}^n \setminus \Omega$. Then there exists a unique u_ϵ, satisfying the mean value equation:

$$u_\epsilon(x) = \left\{\begin{array}{ll} \max\left\{\Psi(x), \dfrac{\alpha}{2}\sup\limits_{B_\epsilon(x)} u_\epsilon + \dfrac{\alpha}{2}\inf\limits_{B_\epsilon(x)} u_\epsilon + \beta \fint_{B_\epsilon(x)} u_\epsilon\right\} & \text{for } x \in \Omega \\ F(x) & \text{for } x \in \Gamma. \end{array}\right. \tag{24}$$

Then family $\{u_\epsilon\}_{\epsilon\to 0}$ converge as $\epsilon \to 0$, uniformly in $\bar\Omega$, to a continuous function u which is the unique viscosity solution to the obstacle problem (23). Since solutions u_ϵ are, in general, discontinuous, the key estimate in [LewM2] bounds the size of discontinuities and oscillations; it uses probabilistic techniques to write down the representation formulas for u_ϵ. For the case of linear equations (that correspond to $p = 2$) with variable coefficients, a similar version of the representation formula specified below is due to Pham and Øksendal-Reikvam. Namely, for the return function $G = \chi_\Gamma F + \chi_\Omega \Psi$, we define the two values: $u_I(x_0) = \sup_{\tau,\sigma_I}\inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{II}}[G\circ x_\tau]$ and $u_{II}(x_0) = \inf_{\sigma_{II}}\sup_{\tau,\sigma_I} \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{II}}[G\circ x_\tau]$, where sup and inf are taken over all strategies σ_I, σ_{II} and stopping times τ that keep the process inside Ω. Then: $u_I = u_\epsilon = u_{II}$ where u_ϵ satisfies (24).

3.4. The double obstacle problem. In [CLewM] we extended the results in 3.3 to the problem:

$$\begin{cases} -\Delta_p u \geq 0 & \text{in } \{x \in \Omega;\ u(x) < \Psi_2(x)\} \\ -\Delta_p u \leq 0 & \text{in } \{x \in \Omega;\ u(x) > \Psi_1(x)\} \\ \Psi_1 \leq u \leq \Psi_2 & \text{in } \Omega \\ u = F & \text{on } \partial\Omega. \end{cases} \tag{25}$$

Here, $\Psi_1, \Psi_2 : \mathbb{R}^N \to \mathbb{R}$ are given Lipschitz functions such that $\Psi_1 \leq \Psi_2$ in $\bar{\Omega}$ and $\Psi_1 \leq F \leq \Psi_2$ on $\partial\Omega$.

We proved that the viscosity solution to (25) is unique and it coincides both with the variational solution and with the uniform limit of solutions to the localised discrete min-max problems, that can be interpreted as the dynamic programming principle for a version of the Tug-of-War game with noise. In this game, both players in addition to choosing their strategies, are also allowed to choose stopping times. We further proposed a numerical scheme and tested the Mathlab code results on some chosen examples of obstacles and boundary data.

3.5. A random walk approach to the Robin boundary value problem. In two papers [LewPe2, LewPe3], we studied the following mean value equations (called the Robin mean value equations):

$$u_\epsilon(x) = \big(1 - \gamma s_\epsilon(x)\big) \fint_{B_\epsilon(x)\cap\mathcal{D}} u_\epsilon(y)\ \mathrm{d}y + \frac{\epsilon^2}{2(N+2)} f(x), \tag{26}$$

posed on a bounded $\mathcal{C}^{1,1}$ domain $\Omega \subset \mathbb{R}^N$, with a bounded Borel function f, a constant $\gamma > 0$, and where:

$$s_\epsilon(x) = \frac{|B_1^{N-1}|}{(N+1)|B^N_{1,d_\epsilon(x)}|} \cdot \epsilon\big(1 - d_\epsilon(x)^2\big)^{\frac{N+1}{2}}, \ \text{ with } \ B^k_{1,d} = B^k_1 \cap \{y_k < d\} \ \text{ and } \ d_\epsilon(x) = \min\big\{1, \frac{1}{\epsilon}\mathrm{dist}(x, \partial\Omega)\big\}.$$

The significance of the factor $s_\epsilon(x) \sim O(\epsilon)$ will be explained below. We view (26) as the approximation to the Robin-Laplace problem:

$$-\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial\vec{n}} + \gamma u = 0 \quad \text{on } \partial\Omega. \tag{27}$$

The analysis of (26) relies on its probabilistic interpretation as the dynamic programming principle along a discrete process $\{X_n^\epsilon\}_{n=0}^\infty$, which samples uniformly on truncated balls $B_\epsilon(X_n^\epsilon)\cap\Omega$, and stops with probability $\gamma s_\epsilon(X_n^\epsilon)$ at each X_n^ϵ. The process accumulates values of f until the stopping time τ^ϵ, whereas we define:

$$u^\epsilon(x) = \frac{\epsilon^2}{2(N+2)}\mathbb{E}\Big[\sum_{n=0}^{\tau^{\epsilon,x}-1}\big(f \circ X_n^{\epsilon,x}\big)\Big]. \tag{28}$$

In [LewPe2, LewPe3], we related the three individual problems (26), (27) and (28), combining the analytical and probabilistic techniques in their study. We now describe the main results of these manuscripts.

1. (The role of the coefficient s_ϵ). To motivate the formula on $s_\epsilon(x)$, we average the Taylor expansion of u on the truncated ball $B_\epsilon(x)\cap\Omega$. When $d = \mathrm{dist}(x, \partial\Omega) \geq \epsilon$, this procedure leads to the familiar formula (21), coinciding with (26) upon replacing $-\Delta u$ with f and setting $s_\epsilon(x) = 0$. In case of $d < \epsilon$ when $x \approx \bar{x} \in \partial\Omega$, the same reasoning requires calculating the possibly nonzero average $\fint_{B_\epsilon(x)\cap\Omega} y - x\ \mathrm{d}y$. With sufficient regularity, one can approximate this term by the average on the ball $B_\epsilon(x)$ truncated with the tangent plane to $\partial\Omega$ at $\bar{x}$, rather than by the surface $\partial\Omega$. This simpler average may be then directly computed as: $-s_\epsilon(x)\vec{n}(\bar{x}) \sim -\epsilon\big(1 - \big(\frac{d}{\epsilon}\big)^2\big)^{\frac{N+1}{2}}\vec{n}(\bar{x})$. Under the boundary condition $u(\bar{x}) + \gamma\frac{\partial u}{\partial\vec{n}}(\bar{x}) = 0$, the first two terms of Taylor's expansion thus become:

$$u(x) - \big\langle \nabla u(x), s_\epsilon(x)\vec{n}(\bar{x})\big\rangle = u(x) - s_\epsilon(x)\frac{\partial u}{\partial\vec{n}}(\bar{x}) + O(\epsilon s_\epsilon(x)) = u(x) + \gamma s_\epsilon(x)u(x) + O(\epsilon s_\epsilon(x)).$$

   Since $(1 + \gamma s_\epsilon)^{-1} = (1 - \gamma s_\epsilon) + O(s_\epsilon^2)$, we conclude (26) at its leading order terms.

2. (Well posedness and the limiting behaviour of (26)). The first main result in [LewPe2] is that each problem (26) has a unique solution $u_\epsilon = u^\epsilon$, coinciding with the value of (28), that is Borel, bounded with a bound independent of ϵ, and obeys the comparison principle. For f continuous / Hölder continuous / Lipschitz,

u_ϵ inherits the same regularity. Further, when $f \in \mathcal{C}(\bar\Omega)$, then $\{u_\epsilon\}_{\epsilon\to 0}$ converges uniformly on $\bar\Omega$ to $u \in \mathcal{C}(\bar\Omega)$ that is the unique viscosity solution to (27). In fact, u coincides with the unique $W^{2,p}(\Omega)$ solution to (27). Since the range of p covers $(1,\infty)$, it follows that $u \in \mathcal{C}^{1,\alpha}(\bar\Omega)$ for any $\alpha \in (0,1)$. In the companion paper [LewPe3] we showed that $\{u_\epsilon\}_{\epsilon\to 0}$ converges uniformly on $\bar\Omega$ to the unique $W^{2,p}(\Omega)$ solution to (27), for any bounded Borel right hand side f. To this end, we used probability techniques involving various couplings of random walks and yielding approximate Hölder regularity of u^ϵ in (28) (Lipschitz in the interior and $\mathcal{C}^{0,\alpha}$ up to the boundary of Ω, for any $\alpha \in (0,1)$).

3. (The lower bound). By further martingale techniques we deduced the lower bound on u^ϵ in the general case of nonnegative bounded f, in function of γ and the radius r of the inner supporting balls at $\partial\Omega$:

$$u^\epsilon(x) \geq \frac{\bar r}{\gamma N}\cdot \inf_{\bar\Omega} f \qquad \text{for all } \ x\in\bar\Omega.$$

Clearly, uniform convergence of $\{u_\epsilon\}_{\epsilon\to 0}$ to u implies that $u \geq \frac{r}{\gamma N}\inf_{\bar\Omega} f$. This bound is optimal and for (27), it may be obtained directly via the maximum principle.

3.6. Non-local Tug-of-War with noise for the geometric fractional p-Laplacian. In the recent papers [DELew, Lew15], we were concerned with the following fractional operator introduced in [10]. For $p\geq 2$, $s\in(\frac12,1)$, and for a given bounded function $u:\mathbb{R}^n\to\mathbb{R}$ of regularity $C^{1,1}(x)$ with $\nabla u(x)\neq 0$, one defines:

$$\Delta_p^s u(x) \doteq C_{n,p,s}\int_{T_p^{0,\infty}(\frac{\nabla u(x)}{|\nabla u(x)|})} \frac{u(x+z)+u(x-z)-2u(z)}{|z|^{n+2s}}\,\mathrm{d}z. \tag{29}$$

Above, $C_{n,p,s}$ is a specific constant depending on n,p,s, whereas the integration occurs on the infinite cone $T_p^{0,\infty}(\frac{\nabla u(x)}{|\nabla u(x)|})\subset\mathbb{R}^n$ whose centerline is aligned with the vector $\frac{\nabla u(x)}{|\nabla u(x)|}$ and whose aperture angle α depends on N,p. In particular, for $p=2$ we have $\alpha=\frac{\pi}{2}$ so that the said cone becomes the half-space and (29) is consistent with the familiar formula: $-(-\Delta)^s u(x) = C_{n,s}\int_{\mathbb{R}^n}\frac{u(z)-u(x)}{|x-z|^{n+2s}}\,\mathrm{d}z$. On the other hand, when $p\to\infty$ then $\alpha\to 0$ and the cone reduces to a line, consistently with the parallel definition for fractional infinity Laplacian $\Delta_\infty^s u(x)$ in [9]. We now describe the main results of [Lew15]; some of them were extended to the case $p=\infty$ in [DELew].

Define the following non-local and non-linear averaging operator:

$$\mathcal{A}_\epsilon u(x) \doteq \frac12\Big(\sup_{|y|=1}⨍_{T_p^{\epsilon,\infty}(y)}\frac{u(x+z)}{|z|^{n+2s}}\,\mathrm{d}z + \inf_{|y|=1}⨍_{T_p^{\epsilon,\infty}(y)}\frac{u(x+z)}{|z|^{n+2s}}\,\mathrm{d}z\Big),$$

where the integration takes place on the truncated infinite cones $T_p^{\epsilon,\infty}(y) = T_p^{0,\infty}(y)\setminus B_\epsilon(0)$, each oriented along its indicated unit direction vector y and having the aperture angle α as in (29). The integral averages ⨍ are taken with respect to the singular measure $|z|^{-n-2s}\,\mathrm{d}z$. Note that $\mathcal{A}_\epsilon u$ is well defined for any bounded, Borel function u, and in particular it does not necessitate the existence or the knowledge of $\nabla u(x)$, which was essential in (29). The following asymptotic expansion is then valid for functions u that are $\mathcal{C}^2$ in the vicinity of a given $x\in\mathbb{R}^n$ with $\nabla u(x)\neq 0$, and uniformly continuous away from x:

$$\mathcal{A}_\epsilon u(x) = u(x) + \frac{s}{(2-2s)(n+p-2)}\epsilon^{2s}\cdot\Delta_p^s u(x) + o(\epsilon^{2s}) \qquad \text{as } \ \epsilon\to 0+. \tag{30}$$

The error quantity $o(\epsilon^{2s})$ blows up to ∞ as $s\to 1-$. In [Lew15] we also propose another nonlinear average of a combined local - non-local nature, which is superior to (30), because the said error term is then be made uniform in the whole considered range $s\in(\frac12,1)$.

The second set of results concerns the operator $\mathcal{A}_\epsilon$ and the truncated version of the expansion (30), aiming at an approximation scheme for solutions to the problem:

$$\Delta_p^s u = 0 \ \text{ in } \ \Omega, \qquad u = F \ \text{ in } \ \mathbb{R}^n\setminus\Omega, \tag{31}$$

posed on a given an open bounded domain $\Omega\subset\mathbb{R}^n$ and with a bounded Borel data function $F:\mathbb{R}^n\setminus\Omega\to\mathbb{R}$.

Consider the following family of non-local averaging problems:

$$u_\epsilon(x) = \begin{cases} \mathcal{A}_\epsilon u_\epsilon(x) & \text{for } x \in \Omega \\ F(x) & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{32}$$

Then, for every $\epsilon > 0$ there exists exactly one solution u_ϵ which is bounded Borel on $\mathbb{R}^n$ (and continuous in Ω). For Ω satisfying the exterior cone condition and for uniformly continuous F, any sequence $\{u_\epsilon\}_{\epsilon\to 0}$ has a further subsequence converging uniformly in $\mathbb{R}^n$ to a continuous limit u that is a viscosity solution to (31). To this end, each $u_\epsilon(x)$ is shown to be the value of the following zero-sum two-players game, which is a non-local version of the Tug-of-War with noise introduced in [87].

In this game, each Player chooses a unit direction vector according to their own strategy, based on the knowledge of all prior moves and random outcomes. With equal probabilities, direction from Player I or Player II is picked; this resulting direction is called y. The current game position x is then updated to a next position within the shifted and truncated cone $x + T_p^{\epsilon,\infty}(y)$, randomly according to the probability-normalisation of the measure $|z|^{-n-2s}\mathrm{d}z$ on $T_p^{\epsilon,\infty}(y)$. Such process, started at $x_0 \in \mathbb{R}^n$ is stopped the first time τ when $x_\tau \notin \Omega$, whereas Player I collects from their opponent the payoff given by the value $F(x_\tau)$. The expected value of the payoff, under condition that both Players play optimally has the min-max property, yielding the solution u_ϵ to (32).

Convergence as $\epsilon \to 0$ is obtained by showing the approximate equicontinuity of $\{u_\epsilon\}_{\epsilon\to 0}$, for which the sufficient condition is expressed via "game-regularity" of the boundary points. This general condition, implied in particular by the exterior cone condition on $\partial\Omega$, is similar in spirit to the celebrated Doob's boundary regularity criterion for Brownian motion. We also prove uniqueness of viscosity solutions to (31) under a more restrictive assumption which necessitates extending Δ_p^s to include the case $\nabla u(x) = 0$. It is not clear if solutions to (31) as posed originally, are unique.

3.7. Lipschitz regularity of graph Laplacians on random data clouds. In [CGLew] we study Lipschitz regularity of elliptic PDEs on geometric graphs, constructed from random data points. The data points are sampled from a distribution supported on a smooth manifold. The family of equations that we study arises in data analysis in the context of graph-based learning and contains, as important examples, the equations satisfied by graph Laplacian eigenvectors. In particular, we prove high probability interior and global Lipschitz estimates for solutions of graph Poisson equations. Our results can be used to show that graph Laplacian eigenvectors are, with high probability, essentially Lipschitz regular with constants depending explicitly on their corresponding eigenvalues. Our analysis relies on a probabilistic coupling argument of suitable random walks at the continuum level, and an interpolation method for extending functions on random point clouds to the continuum manifold. As a byproduct of our general regularity results, we obtain high probability L^∞ and approximate $\mathcal{C}^{0,1}$ convergence rates for the convergence of graph Laplacian eigenvectors towards eigenfunctions of the corresponding weighted Laplace-Beltrami operators.

Of a possible independent interest is the following continuum Lipschitz estimate for functions $f : \mathcal{M} \to \mathbb{R}$ defined on a m-dimensional manifold $\mathcal{M}$, equipped with the geodesic distance $d_\mathcal{M}$ and the volume form $\mathrm{dVol}_\mathcal{M}$:

$$|f(x) - f(y)| \leq C\big(\|\Delta_\epsilon f\|_{L^\infty(\mathcal{M})} + \|f\|_{L^\infty(\mathcal{M})}\big) \cdot (d_\mathcal{M}(x,y) + \epsilon) \tag{33}$$

The continuum *non-local* Laplacian Δ_ϵ operator is given by with respect to the probability density ρ of the point cloud $\mathcal{X}_n \subset \mathcal{M}$ and a nonnegative, nonincreasing kernel $\eta : [0,1] \to \mathbb{R}$ satisfying $\int_{B(0,1)\subset\mathbb{R}^m} \eta(|w|)\,\mathrm{d}w = 1$:

$$\Delta_\epsilon f(x) \doteq \frac{1}{\epsilon^{m+2}} \int_\mathcal{M} \eta\left(\frac{d_\mathcal{M}(x,y)}{\epsilon}\right) \big(f(x) - f(y)\big)\rho(y)\,\mathrm{dVol}_\mathcal{M}(y) \quad \text{ for all } \ x \in \mathcal{M}. \tag{34}$$

We remark that the non-local Laplacian Δ_ϵ can be thought of intuitively as the $n \to \infty$, $\epsilon > 0$ counterpart of the graph Laplacian on $\mathcal{X}_n$. The estimate parallel to (33) in the graph setting is proved by using an interpolation map that extends functions on $\mathcal{X}_n$ to functions on $\mathcal{M}$ in such a way that the non-local Laplacian Δ_ϵ is controlled by the graph Laplacian, and then applying (33) to the interpolated function.

The Lipschitz estimate (33) is proved with a probabilistic argument, not related to the randomness of the data points, but using techniques in [LewPe2]. For an arbitrary pair of points $x, y \in \mathcal{M}$ we consider discrete time random walks $\{X_k\}_{k=1}^\infty$ and $\{Y_k\}_{k=1}^\infty$ with state space $\mathcal{M}$ starting at x and y respectively, both of which have as generator an operator closely related to Δ_ϵ. These walks are coupled to encourage coalescence; we consider a stopping time τ, defined as the first time at which either the walks have gotten sufficiently close to each other or have drifted apart a certain order-one distance. For the appropriately coupled walks, we provide basic quantitative estimates for τ, show that τ is not expected to be too large, and also that the probability of the walks being close to each other at τ is close to one (i.e. the walks do coalesce). We then use martingale techniques to bound the difference $|f(x) - f(y)|$ in terms of $|f(X_\tau) - f(Y_\tau)|$, the point being that while x, y may be of order-one apart, the points X_τ and Y_τ will be closer together (with high probability), thus allowing to estimate $|f(x) - f(y)|$ in terms of $|f(\tilde{x}) - f(\tilde{y})|$ for $\tilde{x}, \tilde{y}$ that are closer together than the original x, y. From there, we follow an iteration argument to eventually obtain the desired regularity bound (33).

4 Calculus of variations on thin elastic shells.

Elastic thin objects (such as rods, plates, shells) of various geometries are ubiquitous in the physical world and the understanding of laws governing their equilibria has many applications. Until recently, in the main focus of mathematical elasticity, there has been the linear theory which deals with relatively small scale deformations. The situation becomes more complicated once the deformations are large, and different theories have been proposed based on empiric observations. The strength of the variational approach lies in the fact that it can predict the appropriate model together with the response of the elastic body for the given scaling of forces or kinematic boundary conditions without any a priori assumptions other than the general principles.

4.1. The analytical set-up. Let S be a compact, connected, oriented 2d surface in $\mathbb{R}^3$, whose unit normal vector is denoted by $\vec{n}(x)$. Consider a family $\{S^h\}$ of shells of small thickness h around S:

$$S^h = \{z = x + t\vec{n}(x);\ x \in S,\ \ -h/2 < t < h/2\}, \qquad 0 < h < h_0 << 1, \tag{35}$$

The elastic energy (scaled per unit thickness) of a deformation $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ is then given by:

$$E^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h), \tag{36}$$

with the stored-energy density W obeying (2). The objective is now to describe the limiting behavior, as $h \to 0$, of critical points (or directly, of the minimizers) u^h of the following total energy functionals, subject to applied external forces $f^h \in L^2(S^h, \mathbb{R}^3)$:

$$J^h(u^h) = E^h(u^h) - \frac{1}{h}\int_{S^h} f^h u^h, \tag{37}$$

The classical approach is to propose a formal asymptotic expansion for the solutions (in other words an Ansatz) and derive the corresponding limiting theory by taking the leading order terms of the 3d Euler-Lagrange equations of (37). The more rigorous variational approach of Γ-convergence was more recently applied in this context (see section 3.3, also the review paper [Lew10]). Among other features, such approach provides a rigorous justification of convergence of minimizers of (37) to minimizers of suitable lower dimensional limit energies, under the sole assumption that f^h obey a prescribed scaling law.

It can be shown [40, LewMP1] that if $f^h \approx h^\alpha$, then the minimizers u^h of (37) automatically satisfy:

$$E^h(u^h) \approx h^\beta \tag{38}$$

with $\beta = \alpha$ if $0 \le \alpha \le 2$ and $\beta = 2\alpha - 2$ if $\alpha > 2$. The main part of the analysis consists therefore of identifying the Γ-limit $\mathcal{I}_\beta$ of the energies $h^{-\beta}E^h$ as $h \to 0$, for a given scaling $\beta \ge 0$, but without making any a priori assumptions on the form of the minimizing deformations u^h.

4.2. Conjecture on the infinite hierarchy of shell models. If the deformations u^h as above are compatible with the Kirchhoff-Love Ansatz $u^h(x+t\vec{n}) = u^h(x)+t\vec{N}^h(x)$ (here, $\vec{N}^h$ denotes the unit normal to the deformed surface $u^h(S)$), then formal calculations show that:

$$E^h(u^h) \approx \int_S |\delta g_S|^2 + h^2\int_S |\delta\Pi_S|^2 \qquad \text{as } h \to 0. \tag{39}$$

Above, δg_S and $\delta\Pi_S$ stand for, respectively, the change in metric (first fundamental form) and the shape operator (second fundamental form), between the image surface $u^h(S)$ and the reference mid-surface S. The two terms in (39) correspond, in order of appearance, to the stretching and bending energies, while the factor h^2 in the second term points to the fact that a thin body undergoes bending more easily than stretching. For a plate (i.e. $S \subset \mathbb{R}^2$) the energy (39) is known in the material science literature as the Föppl-von Kármán functional [21], and in all instances when the 2d theory has been rigorously derived, the validity of both the Kirchhoff-Love Ansatz and of the asymptotic formula (39) have been always confirmed.

Writing the expansions of u^h, δg_S, $\delta \Pi_S$ and equating terms of same orders, we arrived [LewP1] at formulating the following conjecture, consistent with all the so far established results; for plates in [40, 21] and for shells in [38, LewMP1, LewMP2, LewMP3, 21]. Namely, the limiting functional $\mathcal{I}_\beta$ corresponding to the scaling (38) with $\beta > 2$, is defined on the space $\mathcal{V}_N$ of N-th order infinitesimal isometries, where:

$$\beta \in [\beta_{N+1}, \beta_N), \qquad \text{with } \beta_i = 2 + \frac{2}{i-1} \qquad \forall i \geq 2.$$

The space $\mathcal{V}_N$ consists of N-tuples $(V_1, \ldots, V_N)$ of displacements $V_i : S \longrightarrow \mathbb{R}^3$ such that the resulting deformations $u^\epsilon = \mathrm{id} + \sum_{i=1}^N \epsilon^i V_i$ of S preserve its metric up to order ϵ^N. Further:

(i) When $\beta = \beta_{N+1}$ then $\mathcal{I}_\beta = \int_S \mathcal{Q}_2\left(x, \delta_{N+1} g_S\right) + \int_S \mathcal{Q}_2\left(x, \delta_1 \Pi_S\right)$ where $\delta_{N+1} g_S$ is the change of metric on S of the order ϵ^{N+1}, generated by the family of deformations u^ϵ and $\delta_1 \Pi_S$ is the first order change in the second fundamental form. The quadratic forms $\mathcal{Q}_2(x, \cdot)$ are nondegenerate, positive definite, derived from $D^2 W(\mathrm{Id})$.

(ii) When $\beta \in (\beta_{N+1}, \beta_N)$ then $I_\beta = \int_S \mathcal{Q}_2\left(x, \delta_1 \Pi_S\right)$.

(iii) The constraint of N-th order infinitesimal isometry $\mathcal{V}_N$ may be relaxed to that of $\mathcal{V}_M$, $M < N$, if S has the following *matching property.* For every $(V_1, \ldots V_M) \in \mathcal{V}_M$ there exist sequences of corrections $V_{M+1}^\epsilon, \ldots V_N^\epsilon$, uniformly bounded in ϵ, such that $\tilde{u}^\epsilon$ below preserve the metric on S up to order ϵ^N:

$$\tilde{u}^\epsilon = \mathrm{id} + \sum_{i=1}^{M} \epsilon^i V_i + \sum_{i=M+1}^{N} \epsilon^i V_i^\epsilon \tag{40}$$

4.3. The generalized von Kármán model (N = 1). In [LewMP1, LewMP2], the desired limiting model has been identified in the above framework for $\beta \geq 4$ and for an arbitrary surface S. Confirming the conjecture, the limiting admissible deformations u of S are only those whose first order term (modulo a rigid motion) in the expansion of $u - \mathrm{id}$ with respect to h, is an element V of the class $\mathcal{V}_1$ of *infinitesimal isometries* of S. The space $\mathcal{V}_1$ consists of vector fields $V \in W^{2,2}(S, \mathbb{R}^3)$ with skew-symmetric covariant gradient (denoted by A). Equivalently, the change of metric on S induced by $\mathrm{id} + hV$ is at most of order h^2 for each $V \in \mathcal{V}_1$.

When $\beta > 4$ (so that $N = 1$) the Γ-limit of $h^{-\beta} J^h$ in (37) is given by $J(V, \bar{Q}) = \mathcal{I}_\beta(V) - \int_S f \cdot \bar{Q} V$ defined for $V \in \mathcal{V}_1$ and $\bar{Q} \in SO(3)$, where:

$$\mathcal{I}_\beta(V) = \frac{1}{24} \int_S \mathcal{Q}_2\Big(x, (\nabla(A\vec{n}) - A\Pi)_{tan}\Big)\, \mathrm{d}x, \tag{41}$$

measuring the first order change, produced by V, in the second fundamental form Π of S.

For $\beta = 4$ the Γ-limit (which is the generalization of the von Kármán functional [40] to shells), contains also a stretching term, measuring the total second order change in the metric of S:

$$\mathcal{I}_4(V, B_{tan}) = \frac{1}{2} \int_S \mathcal{Q}_2\left(x, B_{tan} - \frac{1}{2}(A^2)_{tan}\right) + \frac{1}{24} \int_S \mathcal{Q}_2\Big(x, (\nabla(A\vec{n}) - A\Pi)_{tan}\Big). \tag{42}$$

It involves a symmetric matrix field B_{tan} belonging to the finite strain space: $\mathcal{B} = \mathrm{cl}_{L^2(S)}\Big\{\mathrm{sym}\nabla w^h; \quad w^h \in W^{1,2}(S, \mathbb{R}^3)\Big\}$. The two terms in (42) are stretching and bending energies of a sequence of deformations $v^h = \mathrm{id} + hV + h^2 w^h$ of S which is induced by a first order displacement $V \in \mathcal{V}_1$ and second order displacements w^h satisfying $\lim_{h\to 0} \mathrm{sym}\nabla w^h = B_{tan}$. The crucial property of (42) or (41) is the one-to-one correspondence between the minimizing sequences u^h of the total energies J^h, and their approximations (modulo rigid motions $\bar{Q}x + c$) given by v^h as above with $(V, B_{tan}, \bar{Q})$ minimizing J and $f = \lim_{h\to 0} 1/h^3 f^h$.

The functional (42) (natural from the energy minimization point of view) was so far absent from the literature. We stress the ansatz-free nature of our results. Indeed, we prove (through a compactness argument) that any deformation satisfying the corresponding energy bound must be of the form v^h above.

4.4. The matching property and density of Sobolev infinitesimal isometries. In [LewMP1] we introduced the class of "approximately robust" surfaces, defined by the property that any $V_1 \in \mathcal{V}_1$ can be matched, through a lower order correction as in (40), with an element of $\mathcal{V}_2$. Hence the stretching term (42) can be dropped and $\mathcal{I}_4$ reduces to the linear functional (41) for all $\beta \geq 4$. The class of approximately robust surfaces includes surfaces of revolutions, convex surfaces and developable surfaces, but excludes any surface with a flat part. Instead, for plates, any member of a dense subset of $\mathcal{V}_2$ can be matched with an exact isometry [40]. As a consequence, the plate theory for any $\beta \in (2, 4)$ reduces to minimizing bending energy constrained to $\mathcal{V}_2$.

Towards analyzing more general surfaces S and the scaling exponent $\beta < 4$, in [LewMP3] we derived a matching property for elliptic S (when Π is strictly definite up to the boundary). Let S and ∂S be of class $\mathcal{C}^{3,\alpha}$ for some $\alpha \in (0, 1)$. Then, given $V \in \mathcal{V}_1 \cap \mathcal{C}^{2,\alpha}(\bar{S})$, there exists a sequence w^h equibounded in $\mathcal{C}^{2,\alpha}(\bar{S}, \mathbb{R}^3)$, such that for all small $h > 0$ the map $v^h = \mathrm{id} + hV + h^2 w^h$ is an (exact) isometry. Clearly, this result fulfills only partially the requirement in (iii) of the conjecture 4.2., as the elements of $\mathcal{V}_1$ are only $W^{2,2}$ regular. Indeed, in most Γ-convergence analyses, a key step is to prove density of suitable more regular mappings in the space of admissible mappings for the limiting problem. Results in this direction, for Sobolev spaces of isometries and infinitesimal isometries of flat regions, have been established [85, 80, 45].

In the general setting of S with nontrivial geometry, even though $\mathcal{V}_1$ is a linear space and assuming S to be $\mathcal{C}^\infty$, the mollification techniques do not guarantee that elements of $\mathcal{V}_1$ can be approximated by smooth infinitesimal isometries. An interesting example, discovered by Cohn-Vossen [102], gives a closed smooth surface of non-negative curvature for which $\mathcal{C}^\infty \cap \mathcal{V}_1$ consists only of trivial fields with constant gradient, whereas $\mathcal{C}^2 \cap \mathcal{V}_1$ contains non-trivial elements. We however proved [LewMP3] that on elliptic S of class $\mathcal{C}^{m+2,\alpha}$ with $\mathcal{C}^{m+1,\alpha}$ boundary ($\alpha \in (0, 1)$ and $m > 0$), for every $V \in \mathcal{V}_1$ there exists a sequence $V_n \in \mathcal{V}_1 \cap \mathcal{C}^{m,\alpha}(\bar{S}, \mathbb{R}^3)$ such that $\lim_{n\to\infty} \|V_n - V\|_{W^{2,2}(S)} = 0$. Here we adapted some techniques of Nirenberg [82], used previously in the context of the Weyl problem (on the immersability of all positive curvature metrics on a 2d sphere).

In a similar spirit, in [HLewP] we performed a detailed analysis of first order $W^{2,2}$ Sobolev-regular infinitesimal isometries on developable surfaces without affine regions; we addressed their compensated regularity, rigidity, density and matching properties. Our results depend on the regularity of the surface and a convexity property: we proved that any $\mathcal{C}^{2N-1,1}$ regular first order infinitesimal isometry on a developable $\mathcal{C}^{2N,1}$ surface with a positive lower bound on the mean curvature, can be matched to an Nth-order infinitesimal isometry.

4.5. Intermediate theories for $2 < \beta < 4$ (N $\geq$ 2) and elliptic/developable shells. Ultimately, the main result of [LewMP3] states that for elliptic surfaces of sufficient regularity, the Γ-limit of (36) for the scaling regime $2 < \beta < 4$ is still given by the energy functional (41) over the linear space $\mathcal{V}_1$.

Likewise, combining the results of [HLewP] with a density result for $W^{2,2}$ first order isometries on developable surfaces, we proved that the limit theories for the energy scalings of the order lower than $h^{2+2/N}$ collapse all into the linear theory. Our method is to inductively solve the linearized metric equation $\mathrm{sym}\nabla w = B$ on the surface with suitably chosen right hand sides, a process during which we lose regularity: consequently, if the surface is $\mathcal{C}^\infty$ we can establish the total collapse of all small slope theories, as in the elliptic mid-surface scenario.

4.6. Convergence of equilibria. When $\beta \geq 4$, also the equilibria of (37) converge to solutions of the Euler-Lagrange equations of the functional (42) or (41), as the thickness $h \to 0$ [Lew9]. Notice that the same statement for minimizers follows directly from the earlier Γ-convergence result, while here the novelty is that the same convergence holds for possibly non-minimizing equilibria as well. The definition of "an equilibrium of the 3d energy" may be understood in two apparently different manners, corresponding to passing with the scaling of the variation to 0 outside or inside the integral sign in (36). The main convergence result of [Lew9] (which covers also the plate case, discussed earlier in [80]) follows with either of these definitions of equilibria.

In the same vein, in [LewL] we prove convergence of critical points to the nonlinear elastic energies J^h of 3d thin incompressible plates, to critical points of the 2d energy obtained as the Γ-limit of J^h in the von Kármán scaling regime. The presence of incompressibility constraint requires to restrict the class of admissible test functions to bounded divergence-free variations on the 3d deformations. This poses new technical obstacles, which we resolve by means of introducing 3d extensions and truncations of the 2d limiting deformations.

5 The Korn inequality. Dimension reduction in fluid dynamics.

Thin domains are also encountered in the study of many problems in fluid mechanics, with examples coming from lubrication, meteorology, blood circulation or ocean dynamics. The study of global existence and asymptotic properties of solutions to the Navier-Stokes system in thin 3d domains began with Raugel and Sell [91, 90]. In particular, they proved global existence of strong solutions for large initial data and in presence of large forcing, for the sufficiently thin 3d product domains $\Omega^h = \mathcal{T}^2 \times (0,h)$, where $\mathcal{T}^2$ is the 2d torus. Further generalizations for various boundary conditions followed (see the references in [47]).

In order to study the dynamics of the Navier-Stokes system and the long time existence of its solutions, under the Navier boundary conditions and in thin shells around a given mid-surface S:

$$S^h = \Big\{x + t\vec{n}(x);\ x \in S,\ \ -h/2 < t < h/2\Big\}, \qquad 0 < h \ll 1,$$

one necessitates the rely on the Korn-Poincaré inequality [61, 60]:

$$\|u\|_{W^{1,2}(S^h)} \leq C_h \|\text{sym } \nabla u\|_{L^2(S^h)}. \tag{43}$$

Indeed, in order to define the relevant Stokes operator one uses the symmetric bilinear form $B(u,v) = \int \text{sym}\nabla u : \text{sym}\nabla v$ rather than the usual $\int \nabla u : \nabla v$. The energy methods give then bounds for the quantity: $\|\text{sym}\nabla u^h\|_{L^2(S^h)}$ of a solution flow u^h in S^h, while, in order to establish compactness in the limit as $h \to 0$, one needs bounds for the $W^{1,2}$ norm of u^h, with constants independent of h. Hence (43) with uniform C_h, provides a necessary uniform equivalence of the norms $\|u^h\|_{W^{1,2}}$ and $\|\text{sym}\nabla u^h\|_{L^2}$ on S^h. Starting with the original papers of Korn [61], Korn's inequality has also been widely used as a basic tool for the existence of solutions of the linearized displacement-traction equations in elasticity [44, 21].

5.1. The uniform Korn-Poincaré inequality. In the paper [LewM1] we studied (43) under the tangential boundary conditions for u. It is a classical result that on each open domain S^h, the inequality (43) is valid under the condition of perpendicularity to the appropriate kernel, given in this case by the linear maps with skew gradient that are themselves tangential at the boundary of S^h. We proved sharp results about the blow-up of Korn's constant C_h in this setting, as h goes to 0. Namely, the constants C_h remain uniformly bounded for vector fields u in any family of cones (with angle $< \pi/2$, uniform in h) around the orthogonal complement of extensions of Killing fields on S. We also showed that this condition is optimal, as every Killing field admits a family of extensions u^h, for which the ratio $C_h = \|u^h\|_{W^{1,2}(S^h)}/\|\text{sym } \nabla u^h\|_{L^2(S^h)}$ blows up as $h \to 0$.

5.2. The optimal constants in Korn's and the geometric rigidity estimates. In paper [LewM2] we were concerned with the optimal constants: in the Korn inequality under tangential boundary conditions on bounded sets $\Omega \subset \mathbb{R}^n$, and in the geometric rigidity estimate on the whole $\mathbb{R}^2$. We proved that the latter constant equals $\sqrt{2}$, and we discussed the relation of the former constants with the optimal Korn's constants under Dirichlet boundary conditions and in the whole $\mathbb{R}^n$, which are well known to equal $\sqrt{2}$. We also discussed the attainability of these constants and the structure of deformations/displacement fields in the optimal sets.

5.3. A rigorous justification of the Euler and Navier-Stokes equations with geometric effects. In paper [BFLewN] we derive the 1d isentropic Euler and Navier-Stokes equations describing the motion of a gas through a nozzle of variable cross-section as the asymptotic limit of the 3d isentropic Navier-Stokes system in a cylinder, the diameter of which tends to zero. The method is based on the relative energy inequality satisfied by any weak solution of the 3d Navier-Stokes system and a further variant of the Korn-Poincaré inequality on thin thin channels (with crossections of arbitrary geometry).

6 Topics in viscoelasticity.

The evolutionary equations of isothermal viscoelasticity are given by the balance of linear momentum:

$$u_{tt} - \text{div}\Big(DW(\nabla u) + \mathcal{Z}(\nabla u, \nabla u_t)\Big) = 0. \tag{44}$$

Indeed, the Euler-Lagrange equations of (36) yield precisely the inviscid static version of (44). Here $u : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^3$ denotes the deformation of a reference configuration $\Omega \subset \mathbb{R}^3$ which models a viscoelastic body with constant temperature and density. The flux $DW : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}^{3\times 3}$ is the Piola-Kirchhoff stress tensor equal, in agreement with 2nd law of thermodynamics, to the derivative of elastic energy density W with properties as in section 1. The viscous stress tensor $\mathcal{Z} : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ is compatible with the principles of continuum mechanics: balance of angular momentum, frame invariance, and Claussius-Duhem inequality. Namely: skew $\big(F^{-1}\mathcal{Z}(F,Q)\big) = 0$ for every $F, Q \in \mathbb{R}^{3\times 3}$ with $\det F > 0$, $\mathcal{Z}(RF, R_tF + RQ) = R\mathcal{Z}(F,Q)$ for every path of rotations $R : \mathbb{R}_+ \to SO(3)$, and $\mathcal{Z}(F,Q) : Q \geq 0$.

6.1. Existence and stability of viscoelastic shock profiles. In [BLewZ] we carried out the analytical and numerical study of the existence and stability of viscous shock profiles to (45) below. Following [3] in the incompressible shear flow case, we restrict our attention to the subclass of planar solutions, which are solutions depending only on a single coordinate direction: $u(x) = x + v(x_3)$. Denoting $V = (v^1_{x_3}, v^2_{x_3}, 1 + v^3_{x_3}, v^1_t, v^2_t, v^3_t)$, the system (44) can be equivalently written in the canonical first order hyperbolic-parabolic form:

$$V_t + G(V)_x = (B(V)V_x)_x, \tag{45}$$

where we now write $x := x_3$ and where B is a symmetric, semi-positive definite tensor. We proved that the resulting equations fall into the class of symmetrizable hyperbolic–parabolic systems studied in [73, 74, 75, 89, 112], hence spectral stability implies linearized and nonlinear stability with sharp rates of decay. This important point was previously left undecided, due to a lack of the necessary abstract stability framework.

We further considered a simple prototypical elastic energy density and viscous tensors:

$$W_0(F) = |F^TF - \mathrm{Id}|^2, \qquad \mathcal{Z}_1(F,Q) = 2F\mathrm{sym}(F^TQ), \qquad \mathcal{Z}_2(F,Q) = 2(\det F)\mathrm{sym}(QF^{-1})F^{-1,T}.$$

The rationale for $\mathcal{Z}_2$ is that the Cauchy stress tensor $T_2 = 2(\det F)^{-1}\mathcal{Z}_2F^T = 2\mathrm{sym}(QF^{-1})$ is the Lagrangian version of $2\mathrm{sym}\nabla\nu$ written in terms of the velocity $\nu = u_t$ in Eulerian coordinates. For incompressible fluids we have: $2\mathrm{div}(\mathrm{sym}\nabla\nu) = \Delta\nu$, giving the usual parabolic viscous regularization.

The new contributions of [BLewZ] beyond [3] were: treatment of the compressible case, consideration of large-amplitude waves, formulation of a rigorous nonlinear stability theory including verification of stability of small-amplitude Lax waves, and the systematic incorporation of numerical Evans function computations determining stability of large-amplitude or nonclassical type shock profiles. In the numerical study, we sampled from a broad range of parameters and checked stability of the Lax and over-compressive profiles, whenever their endstates fell into the hyperbolic region of (45). All the over 8,000 Evans function calculations, were consistent with stability.

6.2. A local existence result for a system of viscoelasticity withphysical viscosity. In [LewMu3] we proved the local in time existence of regular solutions to the system of equations of isothermal viscoelasticity with clamped boundary conditions. We deal with a general form of viscous stress tensor $\mathcal{Z}(F, \dot F)$, assuming a Korn-type condition on its derivative $D_{\dot F}\mathcal{Z}(F, \dot F)$. This condition is compatible with the balance of angular momentum, frame invariance and the Claussius-Duhem inequality. We give examples of linear and nonlinear (in $\dot F$) tensors $\mathcal{Z}$ satisfying these required conditions.

6.3. The stress-assisted diffusion systems. In paper [LewMu4] we are concerned with two systems of coupled PDEs in the description of stress-assisted diffusion. The main system below:

$$\begin{cases} u_{tt} - \mathrm{div}\Big(D_FW(\phi, \nabla u)\Big) = 0 \\ \phi_t = \Delta\Big(D_\phi W(\phi, \nabla u)\Big). \end{cases} \tag{46}$$

consists of a balance of linear momentum in the deformation field $u : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$, and the diffusion law of the scalar field $\phi : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ representing the inhomogeneity factor in the elastic energy density W. The field ϕ may be interpreted as the local swelling/shrinkage rate in morphogenesis at polymerization, or the localized conformation in liquid crystal elastomers. We proved the local and global in time existence of the classical solutions to (46) and its quasistatic counterpart. Our results are also applicable in the context of the non-Euclidean elasticity.

7 Topics in combustion. Traveling fronts in Boussinesq equations.

The Boussinesq-type system of reactive flows is a physical model in the description of flame propagation in a gravitationally stratified medium [111]. The system is given as the reaction-advection-diffusion equation for the reaction progress T (interpreted as temperature), coupled to the fluid motion through the advection velocity, and the Navier-Stokes equations for the incompressible flow u driven by the temperature-dependent force term. In non-dimensional variables [7, 105], the system takes the form:

$$\begin{aligned} T_t + u\cdot\nabla T - \Delta T &= f(T) \\ u_t + u\cdot\nabla u - \nu\Delta u + \nabla p &= T\vec{\rho} \\ \mathrm{div}\, u &= 0. \end{aligned} \tag{47}$$

Here $\nu > 0$ corresponds to the Prandtl number and the vector $\vec{\rho} = \rho\vec{g}$ is the non-dimensional gravity $\vec{g}$ scaled by the Rayleigh number $\rho > 0$. The reaction rate is given by a nonnegative ignition type Lipschitz function f. The recent numerical results, motivated by the astrophysical context [105, 106], suggest that the initial perturbation in T in a channel with heat-impermeable boundary, either quenches or develops a curved front, which eventually stabilizes and propagates as a traveling wave. We hence consider the system (47) in an infinite cylinder $D \subset \mathbb{R}^3$ with a smooth, connected crossection $\Omega \subset \mathbb{R}^2$, and look for the traveling waves in (u, T) connecting $(0, 1)$ to $(0, 0)$ and satisfying the Neumann boundary conditions in T.

7.1. Some related results. For the single temperature equation in (47), when u is an imposed flow of shear type (uni-directional and incompressible), the existence and uniqueness of a multidimensional traveling wave, stable in both linear and nonlinear sense, has been proved in [6, 96] (see also [110]). For the coupled system (47) in a 2d infinite vertical strip, it has been showed in [23] that non-planar traveling fronts cannot exist if the aspect ratio (the ratio of the width of the domain and the thickness of the planar front) is sufficiently small. In the same regime, the planar wave ($u \equiv 0$) which corresponds to a traveling solution of the reaction-diffusion equation in T, is nonlinearly stable: it attracts all solutions of the Cauchy problem, asymptotically in time. For large aspect ratios, the planar fronts are linearly unstable and there as a bifurcation at a critical Rayleigh number $\rho_c > 0$; for any $\rho > \rho_c$ there exist non-planar fronts whose Rayleigh number belong to (ρ_c, ρ) [104].

7.2. Existence of traveling waves in non-vertical domains. The situation is different when the channel D is not aligned with $\vec{g}$. As shown in [7], a (necessarily non-planar) traveling front exists for all aspect rations, in $n = 2$ dimensional channels, under the no-stress boundary conditions in u. In papers [CLewR, Lew8, LewMu2] we extended this result for the Dirichlet (no-slip) conditions, and in the following situations. In [CLewR] existence was established for all $n = 2$ dimensional strips; in [Lew8] for channels of arbitrary dimension n, any crossection Ω and any Rayleigh number, but for a simplified system corresponding to the infinite Prandtl number $\nu = \infty$, when the Navier-Stokes part of (47) is replaced by the Stokes system. In [LewMu2] we treated $n = 3$ dimensional channels with any crossection Ω, Prandtl and Rayleigh numbers, but again for a simplified system, with the advection term $u \cdot \nabla u$ neglected. In [LewMu2] we also proved the same result for the full 3d system, under an explicit thinness condition involving ν, $\vec{\rho}$ and $|\Omega|$ (essentially, Ω is thin in the direction of $\vec{g}$).

7.3. A weak Xie's estimate. A method for showing the existence of a traveling wave is to apply Leray-Schauder degree on compactified domains $R_a = [-a, a] \times \Omega$, where one solves the reaction equation, while the flow equations are solved in the full channel D. The main task is then to obtain uniform bounds, which are independent of a, in order to recover the traveling wave in the limit as $a \to \infty$. The crucial estimate one needs to achieve in this setting is for the supremum of the solution u to Stokes system in D.

The known proofs of the inequality $\|u\|_{L^\infty} \leq C_\Omega \|\nabla u\|_{L^2}^{1/2} \|\mathcal{P}\Delta u\|_{L^2}^{1/2}$ ($\mathcal{P}$ being the Helmholtz projection), are based on the a-priori estimates in [2] which hold for smooth domains. Therefore the constant C_Ω depends strongly on the boundary curvature, and becomes unbounded as Ω tends to any domain with a reentrant corner. This is not enough for closing the bounds, as one does not know whether a complicated relation involving C_Ω and various other parameters can actually be realized, and if it can then for which class of channels. It has been conjectured by Xie [109] that C_Ω is actually an independent constant, equal to $1/\sqrt{3\pi}$. This is still an open question (a related estimate has been established in [108] for the Laplacian). However, using a recent commutator estimate in [67] we noticed [LewMu2] that one can have: $\|u\|_{L^\infty(D)} \leq \frac{2}{\sqrt{2\pi\nu}} \|\nabla u\|_{L^2(D)}^{1/2} \|\mathcal{P}\Delta u\|_{L^2(D)}^{1/2} + C_\Omega \|\nabla u\|_{L^2(D)}$. Despite involvement of the lower order terms, the constant at the highest order is uniform, as needed.

7.4. Stability of the Stokes-Boussinesq system. In [LewR] we considered, as above, the Stokes-Boussinesq (and the stationary Navier-Stokes-Boussinesq) equations in a slanted, i.e. not aligned with the gravity's direction, 3d channel and with an arbitrary Rayleigh number. For the front-like initial data, under the no-slip boundary condition for the flow and no-flux boundary condition for the reactant temperature, we derived uniform estimates on the burning rate and the flow velocity, interpreted as stability results for the laminar front.

7.5. Temporal asymptotics for the p'th power viscous gas. In my two early papers [LewW, LewMu1] we studied the Navier-Stokes equations of a compressible, viscous and heat-conducting gas, written in Lagrangian coordinates, and with the pressure law $\mathcal{P} = e\xi^p/c_v$ (here e is the internal energy, $c_v > 0$ the specific heat, ξ the specific volume and $p \geq 1$):

$$\begin{aligned} \xi_t &= v_x, \qquad v_t = (-\mathcal{P} + \mu v_x/\xi)\,, \\ c_v\theta_t &= (-\mathcal{P} + \mu v_x/\xi)\, v_x + (\kappa\theta_x/\xi)_x + \delta f(\xi,\theta,z), \\ z_t &= (\sigma z_x/\xi^2)_x - f(\xi,\theta,z). \end{aligned}$$

In [LewW] the reaction rate δ was 0, while in [LewMu1] the dynamic combustion was allowed, through the intensity function of the form $f(\xi,\theta,z) = z^m \tilde{f}(\xi,\theta,z)$, where $m \geq 1$ is an integer and $\tilde{f}$ is positive and bounded (locally in $1/\xi$, globally in other variables). Under the Dirichlet boundary condition in v, and Dirichlet or Neumann homogeneous boundary conditions in θ, we proved that the global solution tends to the equilibrium at an exponential rate when $\delta = 0$ or $m = 1$ [LewW], or at an algebraic rate when we admit the nonlinearity in the combustion term [LewMu1].

8 Well posedness of systems of conservation laws.

Following my Ph.D. thesis, I have studied the Cauchy problem for $n \times n$ hyperbolic systems of conservation laws in one space dimension. These are the first order nonlinear PDEs of the form:

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x). \tag{48}$$

Here $u = u(x,t) \in \mathbb{R}^n$, $x \in \mathbb{R}$, and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a smooth flux, satisfying the usual assumptions of strict hyperbolicity and genuine nonlinearity/linear degeneracy of the characteristic fields [101, 28, 12]. Several fundamental laws of continuum mechanics take form of (48) [28]. In the setting of one space dimension, it has been established, thanks to Bressan et all., that (48) is well-posed in the class of initial data $\bar{u} \in L^1 \cap BV$ with sufficiently small total variation [14, 8, 12]. The entropy solutions of (48) constitute then a semigroup $S(t,\bar{u})$ which is Lipschitz continuous with respect to both time and initial data. The semigroup S with these properties is unique and its trajectories are the limits of piecewise constant approximate solutions, obtained e.g. by the method of wave front tracking or Glimm's scheme. As proved in [8], they are also the vanishing viscosity limits, that is limits as $\epsilon \to 0$ of unique smooth solutions to $u_t + f(u)_x = \epsilon u_{xx}$ satisfying the initial condition in (48).

8.1. Uniqueness of solutions to the Cauchy problem. Within the framework of small total variation, in a work with Bressan [BrLew2], we proposed a sufficient condition under which any BV solution to (48) automatically coincides with a trajectory of the unique semigroup S. Namely, we proved that every Lax admissible weak solution of (48) has this property if and only if it has locally bounded variation along the family of space-like curves, whose Lipschitz constant is smaller than a fixed positive number. By uniqueness of the semigroup, there followed a uniqueness result for (48), within the class of solutions having the mentioned property. It remains an open question if the validity of this condition is automatically implied by other regularity properties of solutions, for example by the entropy or Lax admissibility itself.

8.2. Well-posedness of systems with $\mathbf{L^\infty}$ data. In [Lew1], a system of a balance and transport laws:

$$u_t + f(u)_x = g(u), \qquad \theta_t + h(u)\theta_x = 0, \tag{49}$$

was studied, using the theory of generalized characteristics and ODEs with discontinuous right hand side. Under the condition of strict hyperbolicity, which implies the transversality condition on the related ODE:

$$x' = h(u(x,t)), \tag{50}$$

the Hölder well-posedness of the system (49) was proved, with initial data $\bar{u} \in L^1 \cap L^\infty$, $\bar{\theta} \in C^0$. Nonetheless, the problem (50) may be ill posed, due to the unboundedness of the total variation of u.

8.3. The Riemann problem with large data. In this project, the main concern was the existence and stability of solutions to (48) in the vicinity of a self-similar entropy solution $u_0(t,x) = u_0(x/t)$ to a given Riemann problem (u_l, u_r), without any restriction on the strength of the discontinuity $\|u_r - u_l\|$. Because of the finite propagation speed, these issues are related to the local in time well posedness of the Cauchy problem (48) with initial data having bounded (but possibly large) total variation. As noted in [50], there are examples of systems and initial data, for which solutions blow up in finite time, due to the interaction of a number of large waves. In our analysis, all large waves are traveling apart from each other and never interact. Still, however, it is possible that the control on the time dependent amount of perturbation (measured in the BV or the L^1 norms) is lost. Well-posedness can hence be achieved only under additional assumptions on the waves in u_0. Generalizing the works [15, 98, 13], which analyzed several particular patterns of waves, we introduced such conditions which are (in order of strength): Finiteness Condition, BV Stability Condition and L^1 Stability Condition [LewT, Lew6, Lew7, Lew2, Lew4, Lew5, LewZ].

Roughly speaking, these conditions require that in some norm (provided by a set of weights), the total amount of the scattered waves $v(t, x)$, evolving according to the linear hyperbolic system:

$$v_t + \big[\mathrm{D}f(u_0) \cdot v\big]_x = 0, \tag{51}$$

supplemented by appropriate boundary conditions across the jumps in u_0, is smaller that the total weight of the scattered incoming waves. A review paper [Lew3] explains our results in the case of multiple large shocks.

8.4. Stability results. In papers [LewT, Lew2, Lew3, Lew6, Lew5] we proved that if the Finiteness Condition for the wave pattern u_0 holds, then any Riemann problem (u^-, u^+) in the vicinity of the original one (u_l, u_r), has a unique self-similar solution, attaining $n+1$ states, consecutively connected by $(n-M)$ weak waves and M strong waves. This essentially follows by the implicit function theorem.

Further, if the BV Stability Condition holds, then there exists $\delta > 0$ such that for every $\bar{u}$ in the set:

$$\mathrm{cl}_{L^1_{loc}} \big\{ w : \mathbb{R} \longrightarrow \mathbb{R}^n; \qquad ||w \circ \phi - u_0(1, \cdot)||_{L^\infty} + TV(w \circ \phi - u_0(1, \cdot)) < \delta \tag{52}$$

$$\text{for some increasing diffeomorphism } \phi : \mathbb{R} \longrightarrow \mathbb{R} \big\}, \tag{53}$$

the Cauchy problem (48) has a global entropy weak solution $u(t, x)$.

In case the L^1 Stability Condition is satisfied, there exists a semigroup $S : \mathcal{D} \times [0, \infty) \longrightarrow \mathcal{D}$, defined on a closed domain $\mathcal{D} \subset L^1_{loc}(\mathbb{R}, \mathbb{R}^n)$, containing the set in (52) (for some $\delta > 0$), such that the following holds. (i) $||S(\bar{u}, t) - S(\bar{v}, s)||_{L^1} \leq L(|t - s| + ||\bar{u} - \bar{v}||_{L^1})$ for all $\bar{u}, \bar{v} \in \mathcal{D}$, all $t, s \geq 0$ and a uniform constant L, (ii) for all $\bar{u} \in \mathcal{D}$, the trajectory $t \mapsto S(\bar{u}, t)$ is the entropy admissible solution to (48). As a corollary, we obtained the local existence and stability for arbitrarily large BV initial data [Lew5]. The uniqueness is also achieved within a class of functions having locally bounded total variation, as in [BrLew2].

8.5. Stability conditions. In [Lew4, Lew7, Lew5, LewZ], we further discussed the three conditions and found their equivalent forms, requiring that, roughly speaking, the eigenvalues of suitable matrices related to wave transmissions - reflections are smaller than 1 in absolute value, or that they evolve in a prescribed way along a continuous rarefaction wave in u_0. We also validated the conditions for particular systems (notably, the Euler system of γ-gas-law) and compared with other stability conditions.

In [LewZ] we compared our inviscid conditions for large-amplitude shock wave patterns with the "slow eigenvalue", or low-frequency, stability conditions obtained by Lin and Schecter [66] through a vanishing viscosity analysis of the Dafermos regularization. Under the structural assumption that scattering coefficients for each component wave are positive, we showed that BV and L^1 inviscid stability is equivalent to respective versions of low-frequency Dafermos-regularized stability. We gave examples demonstrating the role of cancellation (in linearized behavior) in the presence of negative scattering coefficients.

9 Multiplicity results for forced oscillations on manifolds.

In my early works [LewS1, LewS2, LewS3], we considered the system of second order ODEs:

$$\ddot{x}_\pi = h(x, \dot{x}) + \lambda f(t, x, \dot{x}), \quad \lambda \geq 0 \tag{54}$$

on a manifold M, where $(x, \dot{x}) \in TM$ and $\ddot{x}_\pi(t)$ is the orthogonal projection of $\ddot{x}(t)$ on $T_{x(t)}M$. The vector fields f and h are tangent to M, with f of period $T > 0$ in t. In [LewS1] we discussed the case of M compact, and extended known results on the structure of the set of T-periodic solutions to (54). Using degree (of tangent vector fields) argument we proved existence of a global branch of T-periodic solutions, bifurcating from the set of constant solutions of $\ddot{x}_\pi = h(x, \dot{x})$. We showed that for a generic class of vector fields $h = h(x)$ there are at least the Euler-Poincaré $|\chi(M)|$ solutions of period T for $\lambda > 0$ sufficiently small. This result can be refined when h is the gradient of a functional on M; then, for a generic family of functionals, there are at least a number of periodic solutions equal to the sum of the Betti numbers of M.

In [LewS2] we discussed the case of noncompact M, generalized the results of [LewS1] to all h in an open dense (in appropriate topology) subset of C^r tangent vector fields. In [LewS3], M is the product of two differentiable manifolds $M_1 \times M_2$ and the vector fields h and f have the form $h = (h_1, 0)$ and $f = (f_1, f_2)$. In this situation, we proved the existence of a global branch of T-periodic solutions bifurcating from the set of zeros of the vector field $(p, q) \mapsto \big(h_1(p, q), \frac{1}{T}\int_0^T f_2(t, p, q)\ \mathrm{d}t\big)$. This result extends and unifies previous results obtained in [36].

10 Current research interests

In this section, I will briefly indicate my research plans for the near future.

In connection with Tug-of-War related topics (section 3), I would like to solve the conjecture that, regardless of the boundary regularity of Ω, the family $\{u^\epsilon\}_{\epsilon\to 0}$ of game values converges pointwise in Ω to the Perron solution u of the Dirichlet problem: $\Delta_p u = 0$ in Ω and $u = F$ on $\partial\Omega$. This claim is supported by the Wiener resolutivity, valid for every continuous $F : \partial\Omega \to \mathbb{R}$. Further, I would like to show that such pointwise limit obeys $u = F$ precisely at boundary points that are game-regular. In the same vein, there should be an independent proof that game-regularity is equivalent to the nonlinear potential theoretic p-Wiener regularity criterion.

I plan to further study asymptotic mean value formulas with varying dimensionality, in the Heisenberg group and other subriemannian geometries. I would also like to look at the non-local version of the Tug-of-War games with noise, in connection to the fractional p-Laplacian. Another topic of interest is pursuing the interpretation of the Robin boundary conditions via discrete walks (and games in the context of $p \neq 2$) for less regular boundaries. In particular, the Reifenberg flat domains, which are fractal but satisfy both interior and exterior tubular entry criterion, seem to be suitable for this analysis.

The game-theoretical approach developed so far should have its counterparts in the discrete setting of random graphs. I would like to further study the random Tug-of-War on random geometric graphs, where many questions and notions are relevant to problems encountered in the mathematical theory of machine learning. This is a rapidly emerging field: while we have recently seen successful applications of machine learning and data science, the results are largely empirical and we do not understand when algorithms work or fail. There is an opportunity for new mathematical theory to play an important role to better understand existing algorithms, and aid in the development of faster and more effective ones.

In connection with the topic of convex integration (section 2), I would like to consider the rigidity and flexibility questions in presence of boundary conditions. For the Monge-Ampr̀e equation, there are two types of boundary conditions: either constraining only the leading variable v (similarly to the incompressible Euler's equations), or both v and the auxiliary "Lagrange-multiplier" variable w (similar to the compressible case). In the latter formulation, one should first find natural compatibility conditions between the two boundary value functions.

I would like to use techniques developed for the Monge-Ampère equations posed in two space dimensions, to study higher-dimensional problems. In particular, questions of rigidity and flexibility of the two-Hessian equation in $\mathbb{R}^N$ seem to be likely tractable.

In connection with the curvature-driven shape formation (section 1), I would like to fully understand the implications of the flexibility thresholds in both the isometric immersion and the Monge-Ampère problems, on the scalings of the non-Euclidean elasticity energy infima. It should be possible to prove rigorously, that the low regularity solutions to the effective constraint equation, i.e.: $(\nabla y)^T\nabla y = G_{2\times 2}$ for arbitrary prestrain G, or $\det \nabla^2 v = \mathrm{curl}^T\mathrm{curl}\, G_{2\times 2}$ for G that is a perturbation of Id_3, can be lifted to construct the three-dimensional deformations with the energy scaling exponent (less than 2 but bigger than $\frac{5}{3}$) related to the regularity exponent.

Other projects consist of: developing the complete hierarchy of the singular limits in the regimes of small curvatures, corresponding to metrics departing from Id_3 at various orders of parameter thickness; completing the full hierarchy of Γ-limits in the "oscillatory case"; generalizing the recently completed "non-oscillatory case" hierarchy to submanifolds of Riemannian manifolds of arbitrary dimensions and codimensions; obtaining new constrained models via dimension reduction in plasticity; and addressing the time-dependent problems, at least in some partial cases of parameter ranges of interest. I would also like to study the non-Euclidean elastic energies, both in the bulk and via dimension reduction, in the context of numerical analysis and compare the obtained results with the available experimental data.

References

[ALewP] A. Acharya, M. Lewicka and R. Pakzad, *A note on the metric-restricted inverse design problem*, Nonlinearity **29** (2016), 1769–1797.

[BLewZ] B. Barker, M. Lewicka and K. Zumbrun, *Existence and stability of viscoelastic shock profiles*, Arch. Rational Mech. Anal. **200**, Number 2, (2011) 491–532.

[BLewS] K. Bhattacharya, M. Lewicka and M. Schaffner, *Plates with incompatible prestrain*, Arch. Rational Mech. Anal. **221**(1), (2016) 143–181.

[BFLewN] P. Bella, E. Feireisl, M. Lewicka and A. Novotny, *A rigorous justification of the Euler and Navier-Stokes equations with geometric effects*, SIAM J. Math. Anal. **48**(6), 3907–3930 (2017).

[BrLew3] A. Bressan and M. Lewicka, *A model of controlled growth*, Archive for Rational Mechanics and Analysis **227**(3), 1223–1266 (2018).

[BrLew2] A. Bressan and M. Lewicka, *A uniqueness condition for hyperbolic systems of conservation laws*, Discrete and Continuous Dynamical Systems, **6** no. 3 (2000), 673–682.

[BrLew1] A. Bressan and M. Lewicka, *Shift differentials of maps in BV Spaces*, in: 'Nonlinear Theory of Generalized Functions – Proceedings of the Workshop Nonlinear Theory of Nonlinear Functions, Vienna 1997', Chapman&Hall.

[CGLew] J. Calder, N. Garcia Trillos and M. Lewicka, *L*ipschitz regularity of graph Laplacians on random data clouds, submitted (2020).

[CLew] L. Codenotti and M. Lewicka, *Visualization of the convex integration solutions to the Monge-Ampere equation*, AIMS: Evolution Equations and Control Theory **8(2)**, 273–300 (2019).

[CLewM] L. Codenotti, M. Lewicka and J. Manfredi, *The discrete approximations to the double-obstacle problem, and optimal stopping of the tug-of-war games*, Trans. Amer. Math. Soc. **369**, 7387–7403 (2017).

[CLewR] P. Constantin, M. Lewicka and L. Ryzhik, *A note on traveling waves in the 2D Navier-Stokes-Boussinesq system with the no-slip boundary condition*, Nonlinearity **19**, 2605–2615 (2006).

[DELew] F. del Teso, J. Endal and M. Lewicka, *On asymptotic expansions for the fractional infinity Laplacian*, submitted (2020).

[HLewP] P. Hornung, M. Lewicka and M. Pakzad, *Infinitesimal isometries on developable surfaces and asymptotic theories for thin developable shells*, Journal of Elasticity **111(1)**, 1–19 (2013).

[JLew] S. Jimenez-Bolanos and M. Lewicka, *Dimension reduction for thin films prestrained by shallow curvature*, submitted (2020).

[Lew15] M. Lewicka, *Non-local Tug-of-War with noise for the geometric fractional p-Laplacian*, submitted (2020).

[Lew14] M. Lewicka, *A course on Tug of War games. (Introduction and basic constructions. Tug of War with noise.)*, Springer Universitext, 254 pp, (2020).

[Lew13] M. Lewicka, *Quantitative immersability of Riemann metrics and the infinite hierarchy of prestrained shell models*, Archive for Rational Mechanics and Analysis **236**, 1677–1707 (2020).

[Lew12] M. Lewicka, *Random Tug of War games for the p-Laplacian:* $1 < p < \infty$, accepted in: Indiana Univ. Math. J. (2019).

[Lew11] M. Lewicka, *Morphogenesis by growth and non-Euclidean elasticity: scaling laws and thin film models*, Progress in Nonlinear Differential Equations and Their Applications, **60**, 433–445, Springer Basel AG.

[Lew10] M. Lewicka, *Reduced theories in nonlinear elasticity*, in: "Nonlinear Conservation Laws and Applications" IMA Volume 153 in Mathematics and its Applications, Springer (2011) 393–404.

[Lew9] M. Lewicka, *A note on convergence of low energy critical points of nonlinear elasticity functionals, for thin shells of arbitrary geometry*, ESAIM: Control, Optimisation and Calculus of Variations, **17** (2011), 493–505.

[Lew8] M. Lewicka, *Existence of traveling waves in the Stokes-Boussinesq system for reactive flow*, J. Differential Equations, **237** (2007), no. 2, 343–371.

[Lew7] M. Lewicka, *Stability conditions for strong rarefaction waves*, SIAM J. Math. Anal. **36** (2005), no. 4, 1346–1369.

[Lew6] M. Lewicka, *Lyapunov functional for solutions of systems of conservation laws containing a strong rarefaction*, SIAM J. Math. Anal. **36** (2005), no. 5, 1371–1399.

[Lew5] M. Lewicka, *The well posedness for hyperbolic systems of conservation laws with large BV data*, Arch. Rational Mech. Anal. **173** (2004), 415–445.

[Lew4] M. Lewicka, *Stability conditions for patterns of non-interacting large shock waves*, SIAM J. Math. Anal., **32** no. 5 (2001), 1094–1116.

[Lew3] M. Lewicka, *On the L^1 stability of multi-shock solutions to the Riemann problem*, International Series of Numerical Mathematics, **141** (2001), 653–662.

[Lew2] M. Lewicka, *L^1 stability of patterns of non-interacting large shock waves*, Indiana Univ. Math. J., **49** (2000), 1515–1537.

[Lew1] M. Lewicka, *On the well posedness of a system of balance laws with L^∞ data*, Rend. Sem. Mat. Univ. Padova, **102** (1999), 319–340.

[LewL] M. Lewicka and H. Li, *Convergence of equilibria for incompressible elastic plates in the von Karman regime*, Communications on Pure and Applied Analysis **14**, Issue 1 (January 2015), doi: 10.3934/cpaa.2014.14.

[LewLu] M. Lewicka and D. Lucic, *Dimension reduction for thin films with transversally varying prestrian: the oscillatory and the non-oscillatory case*, Communications on Pure and Applied Mathematics **73(9)**, 1880–1932 (2020).

[LewMaP3] M. Lewicka, L. Mahadevan and M. Pakzad, *The Monge-Ampere constraint: matching of isometries, density and regularity, and elastic theories of shallow shells*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis **34**, Issue 1, (2017), 45–67.

[LewMaP2] M. Lewicka, L. Mahadevan and M. Pakzad, *Models for elastic shells with incompatible strains*, Proceedings of the Royal Society A **470** (2014), 21–65.

[LewMaP1] M. Lewicka, L. Mahadevan and M. Pakzad, *The Föppl-von Kármán equations for plates with incompatible strains*, Proceedings of the Royal Society A **467** (2011), 402–426.

[LewM2] M. Lewicka and J.J. Manfredi, *The obstacle problem for the p-Laplacian via Tug-of-War games*, Probability Theory and Related Fields **167**, Issue 12 (2017), 349–378.

[LewM1] M. Lewicka and J.J. Manfredi, *Game theoretical methods in PDEs*, Bollettino dell'Unione Matematica Italiana **7**, Issue 3, (2014), 211–216.

[LewMR] M. Lewicka, J. Manfredi and D. Ricciotti, *Random walks and random Tug of War in the Heisenberg group*, Mathematische Annalen **377**, 797-846 (2020).

[LewMP3] M. Lewicka, M.G. Mora and M. Pakzad, *The matching property of infinitesimal isometries on elliptic surfaces and elasticity of thin shells*, Arch. Rational Mech. Anal. (3) **200** (2011), 1023–1050.

[LewMP2] M. Lewicka, M.G. Mora and M. Pakzad, *A nonlinear theory for shells with slowly varying thickness*, C.R. Acad. Sci. Paris, Ser I **347** (2009), 211–216.

[LewMP1] M. Lewicka, M.G. Mora and M. Pakzad, *Shell theories arising as low energy Γ-limit of 3d nonlinear elasticity*, Annali della Scuola Normale Superiore di Pisa Cl. Sci. (5) Vol. IX (2010), 1–43.

[LewMu4] M. Lewicka and P.B. Mucha, *A local and global well-posedness results for the general stress-assisted diffusion systems*, Journal of Elasticity **123**, Issue 1 (2016) 19–41.

[LewMu3] M. Lewicka and P.B. Mucha, *A local existence result for a system of viscoelasticity with physical viscosity*, AIMS: Evolution Equations and Control Theory **2**, Issue 2, 337 - 353 (2013).

[LewMu2] M. Lewicka and P.B. Mucha, *On the existence of traveling waves in the 3D Boussinesq system*, Commun. Math. Phys. **292** (2009), 417–429.

[LewMu1] M. Lewicka and P.B. Mucha, *On temporal asymptotics for the p'th power viscous reactive gas*, Nonlinear Anal. **57** (2004), no. 7-8, 951–969.

[LewM2] M. Lewicka and S. Müller, *A note on the optimal constants in Korn's and geometric rigidity estimates in bounded and unbounded domains*, Indiana Univ. Math. J. **65** No. 2 (2016), 377-397.

[LewM1] M. Lewicka and S. Müller, *The uniform Korn - Poincaré inequality in thin domains*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis **28**, Issue 3, (May-June 2011) 443–469.

[LewO] M. Lewicka and P. Ochoa, *On the variational limits of lattice energies on prestrained elastic bodies*, in: "Differential Geometry and Continuum Mechanics" Editors Gui-Qiang G. Chen, Michael Grinfeld and R.J. Knops, ISBN: 978-3-319-18572-9 (2015), 281–306.

[LewOP] M. Lewicka, P. Ochoa and R. Pakzad, *Variational models for prestrained plates with Monge-Ampere constraint*, Diff. Int. Equations **28**, no 9-10 (2015), 861–898.

[LewP4] M. Lewicka and R. Pakzad, *Prestrained elasticity: from shape formation to Monge-Ampère anomalies*, an invited paper in the Notices of the AMS, January 2016.

[LewP3] M. Lewicka and M. Pakzad, *Convex integration for the Monge-Ampere equation*, Analysis and PDE **10** (2017).

[LewP2] M. Lewicka and M. Pakzad, *Scaling laws for non-Euclidean plates and the $W^{2,2}$ isometric immersions of Riemannian metrics*, ESAIM: Control, Optimisation and Calculus of Variations **17**, no 4 (2011), 1158–1173.

[LewP1] M. Lewicka and M. Pakzad, *The infinite hierarchy of elastic shell models: some recent results and a conjecture*, Infinite Dimensional Dynamical Systems, Fields Institute Communications **64** (2013), 407–420.

[LewPe3] M. Lewicka and Y. Peres, *The Robin mean value equation II: Asymptotic Hölder regularity*, submitted (2019).

[LewPe2] M. Lewicka and Y. Peres, *The Robin mean value equation I: A random walk approach to the third boundary value problem*, submitted (2019).

[LewPe1] M. Lewicka and Y. Peres, *Which domains have two-sided supporting unit spheres at every boundary point?*, Expositiones Mathematicae **38(4)**, 548–558 (2020).

[LewR] M. Lewicka and M. Raoofi, *A stability result for the Stokes-Boussinesq equations in infinite 3d channels*, Communications on Pure and Applied Analysis **12**, Issue 6, 2615 - 2625 (2013).

[LewRa] M. Lewicka and A. Raoult, *Thin structures with imposed metric*, ESAIM: Proceedings and Surveys **62**, 79–90 (2018).

[LewRaR] M. Lewicka, A. Raoult and D. Ricciotti, *Plates with incompatible prestrain of higher order*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis **34**(7), 1883–1912 (2017).

[LewS3] M. Lewicka and M. Spadini, *Branches of forced oscillations in degenerate systems of second order ODEs*, Nonlinear Analysis **68** (2008), 2623–2628.

[LewS2] M. Lewicka and M. Spadini, *A remark on the genericity of multiplicity results for forced oscillations on manifolds*, Annali di Mat. Pura ed Applicata **181** (2002), 85–94.

[LewS1] M. Lewicka and M. Spadini, *On the genericity of the multiplicity results for forced oscillations on compact manifolds*, Nonlinear Diff. Equ. Appl. **6** (1999), 357–369.

[LewT] M. Lewicka and K. Trivisa, *On the L^1 well posedness of systems of conservation laws near solutions containing two large shocks*, J. Differential Equations **179** (2002), 133–177.

[LewW] M. Lewicka and S.J. Watson, *Temporal asymptotics for the p'th power newtonian fluid in one space dimension*, Z. Angew. Math. Phys. **54** (2003), no. 4, 633–651.

[LewZ] M. Lewicka and K. Zumbrun, *Spectral stability conditions for shock wave patterns*, Journal of Hyperbolic Equations **4** (2007), no. 1, 1–16.

Other cited publications :

[1] E. Acerbi, G. Buttazzo and D. Percivale, *A variational definition for the strain energy of an elastic string*, J. Elasticity **25** (1991), 137–148.

[2] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*, Comm. Pure Appl. Math., **17** (1964), 35–92.

[3] S. Antmann and R. Malek-Madani, *Travelling waves in nonlinearly viscoelastic media and shock structure in elastic media,* Quart. Appl. Math. 46 (1988) 77–93.

[4] B. Audoly and A. Boudaoud, *Self-similar structures near boundaries in strained systems*, Phys. Rev. Lett. **91**, (2004) 086105–086108.

[5] J.M. Ball, *Some open problems in elasticity. Geometry, mechanics, and dynamics*, Springer, New York (2002), 3–59.

[6] H. Berestycki, *The influence of advection on the propagation of fronts in reaction-diffusion equations*, Nonlinear PDEs in Condensed Matter and Reactive Flows, NATO Science Series C, **569**, H. Berestycki and Y. Pomeau eds, Kluwer, Doordrecht, 2003.

[7] H. Berestycki, P. Constantin and L. Ryzhik, *Non-planar fronts in Boussinesq reactive flows*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **23** (2006), no. 4, 407–437.

[8] S. Bianchini and A. Bressan, *Vanishing viscosity solutions of nonlinear hyperbolic systems*, Annals of Math., (2) **161** (2005), no. 1, 223–342.

[9] Bjorland, C., Caffarelli, L. and Figalli, A., *Nonlocal Tug-of-War and the inifnity fractional Laplacian*, Communications on Pure and Applied Mathematics, **65**, 337–380, (2012).

[10] Bjorland, C., Caffarelli, L. and Figalli, A., *Non-local gradient dependent operators*, Advances in Mathematics **230(4-6)**, 1859-1894, (2012).

[11] Yu. F. Borisov, *The parallel translation on a smooth surface. III.*, Vestnik Leningrad. Univ. **14** (1959) no. 1, 34–50.

[12] A. Bressan, *Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem*, Oxford University Press, 2000.

[13] A. Bressan and R.M. Colombo, *Unique solutions of* 2×2 *conservation laws with large data*, Indiana U. Math. J. 44 (1995), 677-725.

[14] A. Bressan, T.P. Liu and T. Yang, *L^1 stability estimates for $n \times n$ conservation laws*, Arch. Rational Mech. Anal. **149** (1999), 1–22.

[15] A. Bressan and A. Marson, *A variational calculus for discontinuous solutions of systems of conservation laws*, Comm. Partial Differential Equations **20** (1995), 1491–1552.

[16] T. Buckmaster, C. De Lellis, P. Isett, and L. Szekelyhidi, Jr., *Anomalous dissipation for* $1/5$*-Hölder Euler flows*, Annals of Mathematics, 2015.

[17] T. Buckmaster, C. De Lellis, and L. Szekelyhidi, Jr., *Transporting microstructures and dissipative Euler flows*, to appear.

[18] T. Buckmaster, C. De Lellis and L. Szekelyhidi Jr., *Dissipative Euler flows with Onsager-critical spatial regularity*, Communications on Pure and Applied Mathematics, 2015.

[19] A. Choffrut and L. Szekelyhidi, Jr., *Weak solutions to the stationary incompressible Euler equations*, to appear in SIAM Journal of Mathematical Analysis.

[20] P.G. Ciarlet, *A justification of the von Kármán equations*, Arch. Rational Mech. Anal. **73** (1980), 349–389.

[21] P.G. Ciarlet, *Mathematical Elasticity*, North-Holland, Amsterdam (2000).

[22] P. Constantin, W. E and E. S. Titi, *Onsager's conjecture on the energy conservation for solutions of Euler's equation*, Comm. Math. Phys. **165** (1994), no. 1, 207–209.

[23] P. Constantin, A. Kiselev and L. Ryzhik, *Fronts in reactive convection: bounds, stability and instability* Comm. Pure Appl. Math., **56** (2003), 1781–1803.

[24] S. Conti, C. De Lellis and L. Székelyhidi Jr., *h-principle and rigidity for* $\mathcal{C}^{1,\alpha}$ *isometric embeddings*, Proceedings of the Abel Symposium 2010.

[25] S. Conti and G. Dolzmann, Γ*-convergence for incompressible elastic plates*, Calc. Var. Partial Differential Equations **34** (2009), no. 4, 531–551.

[26] S. Conti and F. Maggi, *Confining thin sheets and folding paper*, Arch. Ration. Mech. Anal. **187** (2008), no. 1, 1–48.

[27] S. Conti, F. Maggi and S. Müller, *Rigorous derivation of Föppl's theory for clamped elastic membranes leads to relaxation*, SIAM J. Math. Anal. **38** (2006), no. 2, 657–680.

[28] C. Dafermos, *Hyperbolic conservation laws in continuum physics*, Springer-Verlag 1999.

[29] S. Dain, *Generalized Korn's inequality and conformal Killing vectors*, Calc. Var. Partial Differential Equations **25** (2006), no. 4, 535–540.

[30] G. Dal Maso, *An introduction to* Γ*-convergence*, Progress in Nonlinear Differential Equations and their Applications **8**, Birkhäuser, MA, (1993).

[31] C. De Lellis and L. Székelyhidi Jr., *The Euler equations as a differential inclusion*, Ann. of Math. (2) **170** (2009), no. 3, 1417–1436.

[32] C. De Lellis and L. Székelyhidi Jr., *Dissipative continuous Euler flows*, Invent. Math. **193** (2013), no. 2, 377–407.

[33] J. Dervaux and M. Ben Amar, *Morphogenesis of growing soft tissues*, Phys. Rev. Lett. **101**, (2008) 068101–068104.

[34] G.L. Eyink, *Energy dissipation without viscosity in ideal hydrodynamics I. Fourier analysis and local energy transfer*, Physica D: Nonlinear Phenomena **78**, Issues 3-4, (1994), Pages 222–240.

[35] D.D. Fox, A. Raoult and J.C. Simo, *A justification of nonlinear properly invariant plate theories*, Arch. Rational Mech. Anal. **124** (1993), 157–199.

[36] M. Furi and M.P. Pera, *A continuation principle for periodic solutions of forced motion equations on manifolds and applications to bifurcation theory*, Pacific J.Math. **160** (1993), 219–244.

[37] E. Fried and N. Kirby, *Gamma-limit of a model for the elastic energy of an inextensible ribbon*, Journal of Elasticity **119** (2015), Issue 12, 35–47.

[38] G. Friesecke, R. James, M.G. Mora and S. Müller, *Derivation of nonlinear bending theory for shells from three-dimensional nonlinear elasticity by* Γ*-convergence*, C. R. Math. Acad. Sci. Paris, **336** (2003), no. 8, 697–702.

[39] G. Friesecke, R. James and S. Müller, *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity*, Comm. Pure. Appl. Math., **55** (2002), 1461–1506.

[40] G. Friesecke, R. James and S. Müller, *A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence*, Arch. Ration. Mech. Anal., **180** (2006), no. 2, 183–236.

[41] Q. Han and J.-X. Hong, *Isometric embedding of Riemannian manifolds in Euclidean spaces*, Mathematical Surveys and Monographs, **130** American Mathematical Society, Providence, RI (2006).

[42] J.-X. Hong and C. Zuily, *Isometric embedding of the 2-sphere with nonnegative curvature in* $\mathbb{R}^3$, Math. Z., **219** (1995), 323–334.

[43] J. Horak, G.J. Lord and M.A. Peletier, *Cylinder buckling: the mountain pass as an organizing center*, SIAM J. Appl. Math. **66** (2006), no. 5, 1793–1824.

[44] C.O. Horgan, *Korn's inequalities and their applications in continuum mechanics*, SIAM Rev., **37** (1995), no. 4, 491–511.

[45] P. Hornung, *Approximating* $W^{2,2}$ *isometric immersions*, C. R. Math. Acad. Sci. Paris, **346**, no. 3-4, 189–192 (2008).

[46] J. A. Iaia, *Isometric embeddings of surfaces with nonnegative curvature in* $\mathbb{R}^3$, Duke Math. J., **67** (1992), 423–459.

[47] D. Iftimie, G. Raugel and G.R. Sell *Navier-Stokes equations in thin 3D domains with Navier boundary conditions*, Indiana Univ. Math. J. **56** (2007), no. 3, 1083–1156.

[48] P. Isett, *Hölder continuous Euler flows in three dimensions with compact support in time*, Annals of Mathematics Studies **196** (2017)

[49] P. Isett, *A proof of Onsagers conjecture* **188** Issue 3, (2018) 871–963.

[50] H. K. Jenssen, *Blowup for systems of conservation laws*, SIAM J. Math. Anal. **31** no.4 (2000), 894–908.

[51] R.L. Jerrard, *Some remarks on Monge-Ampère functions*, Singularities in PDE and the calculus of variations, CRM Proc. Lecture Notes, **44**, (2008), 89–112.

[52] R.L. Jerrard, *Some rigidity results related to Monge-Ampère functions*, Canad. J. Math. **62**, no. 2, (2010), 320–354.

[53] R.L. Jerrard and M.R. Pakzad, *Sobolev spaces of isometric immersions of arbitrary dimension and co-dimension*, Annali di Matematica Pura ed Applicata **196** Issue 2, (2017), 687–716

[54] J. Kim, J.A. Hanna, M. Byun, C.D. Santangelo, R.C. Hayward, *Designing responsive buckled surfaces by halftone gel lithography*, Science, **335**, (2012), 1201–1205.

[55] Y. Klein, E. Efrati and E. Sharon, *Shaping of elastic sheets by prescription of Non-Euclidean metrics*, Science, **315** (2007), 1116–1120.

[56] Y. Klein, S. C. Venkataramani and E. Sharon, *Experimental Study of Shape Transitions and Energy Scaling in Thin Non-Euclidean Plates* Physical Review Letters, PRL 106, 118303 (2011).

[57] R. Kohn, *Parabolic PDEs and Deterministic Games*, SIAM News, Vol 40, Number 8 (2007).

[58] R. Kohn and S. Serfaty, *A deterministic-control-based approach to motion by curvature*, Comm. Pure Appl. Math. 59 (3) (2006), 344–407.

[59] R. Kohn and S. Serfaty, *A deterministic-control-based approach to fully nonlinear parabolic and elliptic equations*, Comm. Pure Appl. Math. 63, 2010, 1298–1350.

[60] V. Kondratiev and O. Oleinik, *On Korn's inequalities*, C.R. Acad. Sci. Paris, **308** Serie I (1989), 483–487.

[61] A. Korn, *Über einige Ungleichungen, welche in der Theorie der elastischen und elektrischen Schwingungen eine Rolle spielen*, Bull. Int. Cracovie Akademie Umiejet, Classe des Sci. Math. Nat., (1909) 705–724.

[62] N. H. Kuiper, *On C^1 isometric embeddings, I, II*, Indag. Math., **17** (1955), 545–556, 683–689.

[63] H. LeDret and A. Raoult, *The nonlinear membrane model as a variational limit of nonlinear three-dimensional elasticity*, J. Math. Pures Appl., **73** (1995), 549–578.

[64] H. LeDret and A. Raoult, *The membrane shell model in nonlinear elasticity: a variational asymptotic derivation*, J. Nonlinear Sci. **6** (1996), 59–84.

[65] H. Li and M. Chermisi, *The von Karman theory for incompressible elastic shells*, Calculus of Variations and PDE **48**, Issue 1-2, (2013), pp. 185–209.

[66] X.B. Lin and S. Schecter, *Stability of self-similar solutions of the Dafermos regularization of a system of conservation laws*, SIAM J. Math. Anal. **35** (2003), no. 4, 884–92.

[67] J-G. Liu, J. Liu and R.L. Pego, *Stability and convergence of efficient Navier-Stokes solvers via a commutator estimate*, Comm. Pure Appl. Math. **60** (2007), no. 10, 1443–1487.

[68] Z. Liu and M.R. Pakzad, *Rigidity and regularity of co-dimension one Sobolev isometric immersions*, to appear in Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5).

[69] L. Mahadevan and H. Liang, *The shape of a long leaf*, Proc. Nat. Acad. Sci. (2009).

[70] J. Manfredi, M. Parviainen and J. Rossi, *On the definition and properties of p-harmonious functions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. 11(2), 215–241, (2012).

[71] J. Manfredi, M. Parviainen and J. Rossi, *An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games*, SIAM J. Math. Anal. 42 (2010), no. 5, 2058–2081.

[72] M. Marder and N. Papanicolaou, *Geometry and Elasticity of Strips and Flowers*, Journal of Statistical Physics, **125**, no 5-6, 1065–1092.

[73] C. Mascia and K. Zumbrun, *Stability of small-amplitude shock profiles of symmetric hyperbolic-parabolic systems,* Comm. Pure Appl. Math. 57 (2004), no. 7, 841–876.

[74] C. Mascia and K. Zumbrun, *Pointwise Green function bounds for shock profiles of systems with real viscosity.* Arch. Ration. Mech. Anal. 169 (2003), no. 3, 177–263;

[75] C. Mascia and K. Zumbrun, *Stability of large-amplitude viscous shock profiles of hyperbolic-parabolic systems,* Arch. Ration. Mech. Anal. 172 (2004), no. 1, 93–131;

[76] R. Monneau, *Justification of nonlinear Kirchhoff-Love theory of plates as the application of a new singular inverse method*, Arch. Rational Mech. Anal. **169** (2003), 1–34.

[77] M.G. Mora and S. Müller, *Convergence of equilibria of three-dimensional thin elastic beams*, Proc. Roy. Soc. Edinburgh Sect. A **138**, no. 4, (2008) 873–896.

[78] M.G. Mora and L. Scardia, *Convergence of equilibria of thin elastic plates under physical growth conditions for the energy density*, Preprint (2008).

[79] S. Müller and M. Pakzad, *Regularity properties of isometric immersions*, Math. Zeit. **251**, no. 2 (2005), 313–331.

[80] S. Müller and M. Pakzad, *Convergence of equilibria of thin elastic plates – the von Kármán case*, Comm. Partial Differential Equations, **33** , Issue 6 (2008), 1018–1032.

[81] J. Nash, *$\mathcal{C}^1$ isometric imbeddings*, Ann. Math., **60**, (1954), 383–396.

[82] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*, Comm. Pure Appl. Math., **6** (1953), 337–394.

[83] J.A. Nitsche, *On Korn's second inequality*, RAIRO Anal. Numer. **15** (1981) 237–248.

[84] M. Ortiz and G. Gioia, *The morphology and folding patterns of buckling-driven thin-film blisters, J. Mech. Phys. Solids*, **42** (1994), pp. 531–559.

[85] M. Pakzad, *On the Sobolev space of isometric immersions*, Journal of Differential Geometry, **66** (2004), 47–69.

[86] Y. Peres, O. Schramm, S. Sheffield and D. Wilson, *Tug-of-war and the infinity Laplacian*, J. Amer. Math. Soc. **22** (2009), 167–210.

[87] Y. Peres and S. Sheffield, *Tug-of-war with noise: a game theoretic view of the p-Laplacian*, Duke Math. J. **145**(1) (2008), 91–120.

[88] A.V. Pogorelov, *An example of a two-dimensional Riemannian metric admitting no local realisation in* E_3, Dokl. Akad. Nauk SSSR **198** (1971), No.1, 729–730.

[89] M. Raoofi and K. Zumbrun, *Stability of undercompressive viscous shock profiles of hyperbolic-parabolic systems,* J. Differential Equations, (2009) 1539–1567.

[90] G. Raugel, *Dynamics of partial differential equations on thin domains*, in CIME Course, Montecatini Terme, Lecture Notes in Mathematics, **1609** (1995), Springer Verlag, 208–315.

[91] G. Raugel and G.R. Sell, *Navier-Stokes equations on thin 3D domains. I: Global attractors and global regularity of solutions*, J. Amer. Math. Soc. **6** (1993), 503–568.

[92] A. Raoult, *Construction d'un modèle d'évolution de plaques avec terme d'inertie de rotation*, Ann. Mat. Pura Appl. **CXXXIX** (1985), 361–400.

[93] A. Raoult, *Analyse mathématique de quelques modèles de plaques et de poutres élastiques ou élasto-plastiques*, Doctorat d'état, Paris (1988).

[94] Yu.G. Reshetnyak, *Stability theorems in geometry and analysis* Kluwer Academic Publishers Group, Dordrecht, 1994.

[95] E.K. Rodriguez, A. Hoger and A. McCulloch, J.Biomechanics **27**, 455 (1994).

[96] J.M. Roquejoffre, *Eventual monotonicity and convergence to traveling fronts for the solutions of parabolic equations in cylinders*, Ann. Inst. Henri Poincare, **14** (1997), no 4, 499–552.

[97] B. Schmidt, *Plate theory for stressed heterogeneous multilayers of finite bending energy*, J. Math. Pures Appl. **88** (2007), 107–122.

[98] S. Schochet, *Sufficient conditions for local existence via Glimm's scheme for large BV data*, J. Differential Equations **89** (1991), 317–354.

[99] E. Sharon, M. Marder and H.L. Swinney, *Leaves, flowers and garbage bags: Making waves*, American Scientist, **92** (2004) 254–261.

[100] E. Sharon, B. Roman, M Marder, G.S. Shin and H.L. Swinney, *Mechanics: Buckling cascades in free sheets*, Nature, **419** (2002) 579–580.

[101] J. Smoller, *Shock waves and reaction-diffusion equations*, Springer-Verlag, New York, 1994.

[102] M. Spivak, *A Comprehensive Introduction to Differential Geometry, Vol V*, 2nd edition, Publish or Perish Inc. (1979).

[103] V. Šverák, *On regularity for the Monge-Ampère equation without convexity assumptions*, preprint, Heriot-Watt University (1991).

[104] R. Texier-Picard and V. Volpert, *Problemes de reaction-diffusion-convection dans des cylindres non bornes*, C. R. Acad. Sci. Paris Sr. I Math., **333** (2001), 1077–1082.

[105] N. Vladimirova and R. Rosner, *Model flames in the Boussinesq limit: the efects of feedback*, Phys. Rev. E., **67** (2003), 066305.

[106] N. Vladimirova and R. Rosner, *Model flames in the Boussinesq limit: the case of pulsating fronts*, Phys. Rev. E., **71** (2005), 067303.

[107] T.H. Ware, M.E. McConney, J.J. Wie, V.P. Tondiglia, T.J. White, *Voxelated liquid crystal elastomers*, Science, Vol. **347**, no. 6225 (2015) 982–984, `DOI:10.1126/science.1261019`

[108] W. Xie, *A sharp pointwise bound for functions with L^2-Laplacians and zero boundary values of arbitrary three-dimensional domains*, Indiana Univ. Math. J. **40** (1991), no. 4, 1185–1192.

[109] W. Xie, *On a three-norm inequality for the Stokes operator in nonsmooth domains*, The Navier-Stokes equations II—theory and numerical methods, Springer Lecture Notes in Math., **1530** (1992), 310–315.

[110] J. Xin, *Front propagation in heterogeneous media*, SIAM Review **42** (2000), no 2, 161–230.

[111] Ya.B. Zeldovich, G.I. Barenblatt, V.B. Librovich and G.M. Makhviladze, *The Mathematical Theory of Combustion and Explosions*, Consultants Bureau, New York, 1985.

[112] K. Zumbrun, *Stability of large-amplitude shock waves of compressible Navier–Stokes equations,* with an appendix by Helge Kristian Jenssen and Gregory Lyng, in Handbook of mathematical fluid dynamics. Vol. III, 311–533, North-Holland, Amsterdam, (2004).