THE MONGE-AMPÈRE SYSTEM IN DIMENSION TWO AND CODIMENSION THREE

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Abstract. We revisit the convex integration constructions for the Monge-Ampère system and prove its flexibility in dimension $d = 2$ and codimension $k = 3$, up to $\mathcal{C}^{1,1-1/\sqrt{5}}$. To our knowledge, it is the first result in which the obtained Hölder exponent $1 - \frac{1}{\sqrt{5}}$ is larger than 1/2 but it is not contained in the full flexibility up to $\mathcal{C}^{1,1}$ result. Previous various approaches, based on Kuiper's corrugations, always led to the Hölder regularity not exceeding $\mathcal{C}^{1,1/2}$, while constructions based on the Nash spirals (when applicable) led to the regularity $\mathcal{C}^{1,1}$. Combining the two approaches towards an interpolation between their corresponding exponent ranges has been so far an open problem.

1. Introduction

Consider the following Monge-Ampère system, posed on a 2-dimensional domain ω, to which we seek a 3-dimensional vector field solution v:

$$\begin{aligned} &\mathfrak{Det}\,\nabla^2 v = f \quad \text{ in } \ \omega\subset\mathbb{R}^2,\\ &\text{where } \ \mathfrak{Det}\,\nabla^2 v = \langle \partial_{11}v, \partial_{22}v\rangle - \big|\partial_{12}v\big|^2 \quad \text{ for } \ v:\omega\to\mathbb{R}^3. \end{aligned} \tag{1.1}$$

System (**??**) together with its weak formulation (**??**) below, was introduced in [**?**], in the full generality of arbitrary dimension d (now equal 2) and codimension k (now equal 3). Flexibility of both systems, in the sense of Theorem **??** and Corollary **??** below, was proved in there up to the regularity $\mathcal{C}^{1,\frac{1}{1+d(d+1)/k}}$, and also up to $\mathcal{C}^{1,1}$ when $k \geq d(d+1)$. For $d = 2$, the former assertion means flexibility up to $\mathcal{C}^{1,\frac{1}{1+6/k}}$, and when $k = 1$ this result agrees with flexibility up to $\mathcal{C}^{1,\frac{1}{7}}$ obtained in [**?**], which was subsequently improved to $\mathcal{C}^{1,\frac{1}{5}}$ in [**?**] (and to $\mathcal{C}^{1,\frac{1}{1+4/k}}$ for k arbitrary in [**?**]), and further to $\mathcal{C}^{1,\frac{1}{3}}$ in [**?**]. The findings of [**?**] allowed to obtain [**?**] flexibility up to $\mathcal{C}^{1,1}$ when $k \geq 4$, and up to $\mathcal{C}^{1,\frac{2^k-1}{2^{k+1}-1}}$ for arbitrary $k \geq 1$. Note that this last exponent is less than 1/2 for any k and, in particular, at $k = 3$ it equals 7/15.

The purpose of this paper is to revisit the previous constructions and, by adding a new ingredient, show flexibility of (**??**) in dimension $d = 2$ and codimension $k = 3$ up to:

$$\mathcal{C}^{1,1-1/\sqrt{5}}.$$

To our knowledge, ours is the first result in which the obtained Hölder exponent is larger than 1/2 but it is not covered by the full flexibility up to $\mathcal{C}^{1,1}$. Indeed, the large gap between the exponents' ranges corresponding to codimensions 3 and 4 at the dimension 2 (or codimensions $k = d(d+1) - 1$ and $k = d(d+1)$ at arbitrary d) was due to the two different techniques in the Nash-Kuiper iteration scheme, based, respectively, on Kuiper's corrugations and on Nash's spirals. Combining the two approaches towards an interpolation between their resulting exponents has been so far an open problem.

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Recall that the closely related problem of isometric immersions of a given Riemann metric g:

$$\begin{aligned} &(\nabla u)^T\nabla u = g \quad \text{ in } \ \omega,\\ &\text{for } \ u:\omega\to\mathbb{R}^5, \end{aligned} \tag{1.2}$$

reduces to (**??**) upon taking a family of metrics $\{g_\epsilon = \mathrm{Id}_2 + 2\epsilon^2 A\}_{\epsilon\to 0}$ each a small perturbation of Id_2 with $A:\omega\to\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ satisfying $-\mathrm{curl}\,\mathrm{curl}\, A = f$. Making an ansatz $u_\epsilon = \mathrm{id}_2 + \epsilon v + \epsilon^2 w$ and gathering the lowest order terms in the ϵ-expansions, leads to the following system:

$$\begin{aligned} &\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w = A \quad \text{ in } \ \omega,\\ &\text{for } \ v:\omega\to\mathbb{R}^3, \quad w:\omega\to\mathbb{R}^2. \end{aligned} \tag{1.3}$$

On a simply connected ω, the system (**??**) is further equivalent to: $\mathrm{curl}\,\mathrm{curl}\big(\frac{1}{2}(\nabla v)^T\nabla v\big) = \mathrm{curl}\,\mathrm{curl}\, A$, which is $\mathfrak{Det}\,\nabla^2 v = -\mathrm{curl}\,\mathrm{curl}\, A$. This brings us back to (**??**), reflecting the agreement of the Gaussian curvatures κ of g_ϵ and of surfaces $u_\epsilon(\omega)$, at their lowest order terms:

$$\begin{aligned} &\kappa(g_\epsilon) = -\epsilon^2\mathrm{curl}\,\mathrm{curl}\, A + o(\epsilon^2),\\ &\kappa((\nabla u_\epsilon)^T\nabla u_\epsilon) = -\frac{\epsilon^2}{2}\mathrm{curl}\,\mathrm{curl}\big((\nabla v)^T\nabla v + 2\,\mathrm{sym}\nabla w\big) + o(\epsilon^2) = \epsilon^2\mathfrak{Det}\,\nabla^2 v + o(\epsilon^2). \end{aligned}$$

Our main result pertaining to (**??**) states that a $\mathcal{C}^1$-regular pair (v,w) which is a subsolution, can be uniformly approximated by exact solutions $\{(v_n,w_n)\}_{n=1}^\infty$, as follows:

Theorem 1.1. *Let $\omega\subset\mathbb{R}^2$ be an open, bounded domain. Given the fields $v\in\mathcal{C}^1(\bar\omega,\mathbb{R}^3)$, $w\in\mathcal{C}^1(\bar\omega,\mathbb{R}^2)$ and $A\in\mathcal{C}^{0,\beta}(\bar\omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})$, assume that:*

$$A > \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big) \qquad \text{on } \ \bar\omega, \tag{1.4}$$

in the sense of matrix inequalities. Then, for every exponent α with:

$$\alpha < \min\Big\{\frac{\beta}{2}, 1-\frac{1}{\sqrt{5}}\Big\} \tag{1.5}$$

and for every $\epsilon>0$, there exists $\tilde v\in\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^3)$, $\tilde w\in\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^2)$ such that the following holds:

$$\begin{aligned} &\|\tilde v - v\|_0\leq\epsilon, \qquad \|\tilde w - w\|_0\leq\epsilon,\\ &A - \big(\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v + \mathrm{sym}\nabla\tilde w\big) = 0 \quad \text{ in } \ \bar\omega. \end{aligned}$$

As a byproduct, we obtain the density of solutions to (**??**) in the space of continuous functions:

Corollary 1.2. *For any $f\in L^\infty(\omega,\mathbb{R})$ on an open, bounded, simply connected domain $\omega\subset\mathbb{R}^2$, the following holds. Fix an exponent α in the range (**??**). Then, the set of $\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^3)$ weak solutions to (**??**) is dense in $\mathcal{C}^0(\bar\omega,\mathbb{R}^3)$. Namely, every $v\in\mathcal{C}^0(\bar\omega,\mathbb{R}^3)$ is the uniform limit of some sequence $\{v_n\in\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^3)\}_{n=1}^\infty$, such that:*

$$\mathfrak{Det}\,\nabla^2 v_n = f \qquad \text{on } \ \omega \ \text{ for all } \ n\geq 1.$$

We now give an overview of the techniques and results prior to the present paper.

1.1. **An introduction of the method and the proofs in** [?]**.** Seeking a solution to (**??**) starts with specifying a subsolution, namely a pair (v, w) that satisfies (**??**). The goal is now to modify the fields v, w within their ϵ-neighbourhoods in $\mathcal{C}^0(\bar{\omega})$, with the goal of canceling the positive definite defect field $\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big)$. This is achieved by an inductive procedure of subsequent modifications, referred to as the Nash-Kuiper iteration scheme. Therein, each modification, called a "step", is performed by adding an oscillatory perturbation with values along a chosen codimension direction $E \in \mathbb{R}^3$, having the typical form:

$$x \mapsto \frac{1}{\lambda}a(x)\sin(\lambda\langle x, \eta\rangle)E. \tag{1.6}$$

Above, λ is a (large) amplitude, *sin* is an appropriate periodic function, while the effective amplitude a and the oscillation direction η correspond to one of the three modes in the decomposition of $\mathcal{D}$ into the rank-one "primitive" defects:

$$\mathcal{D}(x) = \sum_{i=1}^{3} a_i(x)^2\eta_i \otimes \eta_i. \tag{1.7}$$

The number three of these modes is due to the space $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ having dimension 3, and (**??**) is in fact only valid in the vicinity of a fixed positive definite matrix, the limitation circumvented by scaling the original $\mathcal{D}$ and adding Id_2 to it. In any case, the aforementioned modification removes the single mode $a^2\eta \otimes \eta$ of $\mathcal{D}$ at a cost of introducing a higher order error, consisting of the following three types of terms:

$$\frac{a\nabla^2\langle v, E\rangle}{\lambda}\sin(\lambda\langle x, \eta\rangle), \qquad \frac{a\nabla^2 a}{\lambda^2}\sin(\lambda\langle x, \eta\rangle), \qquad \frac{\nabla a \otimes \nabla a}{\lambda^2}\sin(\lambda\langle x, \eta\rangle). \tag{1.8}$$

Consider the first term above, which is of the leading order. In magnitude, it is the quotient of the effective amplitude $\|a\|_0$ in (**??**) times the Hessian $\|\nabla^2\langle v, E\rangle\|_0$ of the current field v's appropriate component, and the frequency λ. Since $\|a\|_0$ has the order of $\|\mathcal{D}\|_0^{1/2}$ as seen from (**??**), the single mode $a^2\eta\otimes\eta$ of $\mathcal{D}$ in there is replaced by the error of the order $\|\mathcal{D}\|_0^{1/2}\|\nabla^2\langle v, E\rangle\|_0/\lambda$; while at the same time the second derivatives of v increase by the factor λ due to the form of the perturbation (**??**). The same is true for the second and third modifications, provided that we choose their codimension directions E linearly independent. Concluding, after three steps (three modifications), together constituting what is called a "stage", $\|\mathcal{D}\|_0$ decreases by the factor of λ, and $\|\nabla^2 v\|_0$ increases by λ, while the updated field v differs from the original one in its $\mathcal{C}^1$ norm by the order of $\|a\|_0$ or equivalently of $\|\mathcal{D}\|_0^{1/2}$. Continuing in this manner by iterating on stages, we observe blow-up of the $\mathcal{C}^2$ norm $\|v\|_2$ at the rate λ^n and the decay of the defect $\|\mathcal{D}\|_0$ also at the rate λ^n, which translates to the control of $\|v\|_1$ at the rate $\lambda^{-n/2}$. By interpolation, this iterative sequence is Cauchy in $\mathcal{C}^{1,\alpha}$ and yields a solution to (**??**) for any exponent α such that $\alpha - (1-\alpha)/2 < 0$, namely for $\alpha < 1/3$. This is precisely the result in [**?**] for $d = 2$, $k = 3$.

1.2. **The improved method in** [**?**, **?**]**.** The idea employed in [**?**] and following [**?**, **?**] was to replace (**??**) by another decomposition:

$$\mathcal{D}(x) = a(x)^2\mathrm{Id}_2 + \mathrm{sym}\nabla\Phi(x), \tag{1.9}$$

and transfer its second term in the right hand side into $\mathrm{sym}\nabla w$. Then, it is only necessary to cancel two rank-one modes $a^2 e_1 \otimes e_1$ and $a^2 e_2 \otimes e_2$ in $a^2\mathrm{Id}_2$ rather than the three modes in (**??**), and it is done using two oscillatory perturbation (**??**) achieving values in the two orthogonal codimension directions E_1 and E_2. The remaining direction E_3 may be further used to cancel one of the modes in the decomposition (**??**) of the resulting decreased defect. The fourth modification whose frequency we denote by μ, removes then the second primitive defect in there, but introduces a further error, now of the order $\|\mathcal{D}\|_0^{1/2}\|\nabla^2 v\|_0\lambda/\mu$. To match this term with the previous one, we are bound to choose $\mu = \lambda^2$, which in turn implies the new increase of $\|\nabla^2 v\|_0$ by the factor of λ^2. After completing six steps of this kind, together constituting a "stage" that removes the defect up to the third order, we have decreased $\|\mathcal{D}\|_0$ by the factor of λ^3, while increasing $\|\nabla^2 v\|_0$ by the factor of λ^2. Iterating on stages, the blow-up rate of $\|v\|_2$ is thus λ^{2n} and the decay rate of the defect $\|\mathcal{D}\|_0$ is λ^{3n}, which translates to the control of $\|v\|_1$ at the rate $\lambda^{-3n/2}$. Invoking an interpolation argument as before, the sequence of consecutively perturbed fields v (and likewise w) is Cauchy in $\mathcal{C}^{1,\alpha}$ for any exponent α such that $2\alpha - 3(1-\alpha)/2 < 0$, namely for $\alpha < 3/7$. This was the result obtained in [?] for $k = 3$.

1.3. **A further improvement and proofs in** [?, ?]**.** A subsequent improvement of the Hölder regularity exponent as in Theorem **??**, was motivated by the approach of [?] where, before assigning the effective amplitude a in (**??**) and before cancelling the first mode $a^2 e_1 \otimes e_1$, one iterates the decomposition (**??**) on consecutive errors. This way, an entire block of errors, say up to order N, may be absorbed at once. If the single error consisted only of the first type terms in (**??**), the resulting defect would have the advantageously high order:

$$\|\mathcal{D}\|_0 \left(\frac{\nu}{\lambda}\right)^N, \tag{1.10}$$

where $\nu = \|\nabla^2 v\|_0/\|\mathcal{D}\|_0^{1/2}$ is the frequency at which v oscillates. However, the presence of the second term in (**??**) precludes this construction. This is because each $\|\nabla^2 a\|_0$ has the order of the second derivative of the previously incorporated error which includes terms of the same second type, thus it is of order 1 in λ. Consequently, the order of the defect does not improve upon iteration. An ingenious observation put forward in [?] allows to circumvent this problem, relying on a parallel decomposition construction of the form (**??**) which, if only applied to $\mathcal{D}$ with its $\mathcal{D}_{22}$ component removed, trades one ∂_1 derivative for one ∂_2 derivative in estimating the derivatives of a. Hence, the worst $\partial_{11}a$ component of $\nabla^2 a$, computed from the previous error, is now only of the order 1 in ν instead of λ. After N such iterations, the eventual error quantities are all proportional to (**??**). One then proceeds to canceling the second mode $a^2 e_2 \otimes e_2$ augmented by the so far neglected $\mathcal{D}_{22}$ components, through the second modification (a "step") of the form (**??**), using the same codimension direction, say E_1, as before, a frequency μ and the oscillation direction $\eta = e_2$. This construction only affects the component v_1 of the field v, increasing its second derivative $\|\nabla^2 v_1\|_0$ by a factor μ. At the same time, the new defect consists of the previously generated errors bounded by (**??**), plus terms of the new order $\|\mathcal{D}\|_0\|\nabla^2 v_1\|_0/(\mu/\lambda)$ due to the neglected components oscillating on the length scale λ. This suggests taking $\mu = \lambda^{N+1}$. Consequently we obtain, to the leading order and when N is large, the increase of $\|\nabla^2 v_1\|_0$ by the factor μ and the decrease of $\|\mathcal{D}\|_0$ also by the factor μ. If $k = 1$, as in [?], this procedure constitutes a "stage", which if iterated upon yields the convergence in $\mathcal{C}^{1,\alpha}$, for $\alpha < 1/3$.

If $k = 3$, as in the present case, the procedure in [?] (which covers arbitrary $k \geq 1$) continues from the procedure indicated above with the next pair of modifications (steps), towards cancellation of the present defect. The modifications as in (**??**) achieve values along the second codimension direction E_2 and oscillate with frequency denoted by μ_2. Only the component v_2 is affected and its second derivatives $\|\nabla^2 v_2\|_0$ are of the order $\big(\|\mathcal{D}\|_0/\mu\big)^{1/2}\mu_2$, hence we are bound to take $\mu_2 = \mu^{3/2}$ to achieve matching the order of $\|\nabla^2 v_1\|_0$, namely μ. The current defect decreases by the factor μ_2/μ, implying the original defect $\|\mathcal{D}\|_0$ decrease by the factor μ_2. Finally, the remaining third codimension direction E_3 is employed and we add the two new modifications, now with a frequency μ_3. The order of the obtained $\|\nabla^2 v_3\|_0$ is $\big(\|\mathcal{D}\|_0/\mu_2\big)^{1/2}\mu_3$, hence we take $\mu_3 = \mu\mu_2^{1/2} = \mu^{7/4}$ to match with the order μ_1 of $\|\nabla^2 v_1\|_0$. The final, third order defect displays therefore the decrease by the factor μ_3/μ_2, which in terms of $\|\mathcal{D}\|_0$ yields the decrease factor μ_3. The described construction constitutes a single "stage". Iterating on stages, the blow-up rate of $\|v\|_2$ becomes μ^n, with the decrease rate of $\|\mathcal{D}\|_0$ being $\mu^{7n/4}$ and implying the decay of $\|v\|_1$ at the rate $\mu^{-7n/8}$. By interpolation, the resulting sequence is Cauchy in $\mathcal{C}^{1,\alpha}$ provided that $\alpha - 7(1-\alpha)/8 < 0$, i.e. $\alpha < 7/15$. This was precisely the result in [**?**] for $k = 3$.

1.4. **Proofs in the present paper.** The main new idea behind the proof of Theorem **??** is to iteratively use the decomposition (**??**) as described in subsection 1.3., but instead of cancelling the two principal defects in $a^2\mathrm{Id}_2$ within a single codimension direction, use now two distinct codimensions. These are assigned to the defects of consecutive orders in a circular fashion: E_1, E_2 are assigned to the cancellation of the initial defect $\mathcal{D}$ (with μ as the frequency of the corresponding second modification), then E_3, E_1 to the cancellation of the first order defect (with frequency μ_2), then E_2, E_3 to the second order defect, then E_1, E_2 to the third order, etc. Upon reaching some prescribed order K, we declare this to be a complete "stage". The viability of such construction relies on the set of algebraic conditions satisfied by the progression of frequencies $\{\mu_i\}_{i=1}^K$. One can check that these indeed hold for a specific sequence derived from the Fibonacci sequence, which justifies the presence of the golden ratio in the resulting Hölder exponent range (**??**). More precisely, the main ingredient allowing for the flexibility in Theorem (**??**), is the following "stage" construction:

Theorem 1.3. *Let $\omega \subset \mathbb{R}^2$ be an open, bounded, smooth planar domain. Fix two integers $N, K \geq 4$ and an exponent $\gamma \in (0,1)$. Then, there exists $l_0 \in (0,1)$ depending only on ω, and there exists $\sigma_0 \geq 1$ depending on ω, γ, N, K, such that the following holds. Given the fields $v \in \mathcal{C}^2(\bar\omega + \bar B_{2l}(0), \mathbb{R}^3)$, $w \in \mathcal{C}^2(\bar\omega + \bar B_{2l}(0), \mathbb{R}^2)$, $A \in \mathcal{C}^{0,\beta}(\bar\omega + \bar B_{2l}(0), \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ defined on the closed $2l$-neighbourhood of ω, and given the positive constants $l, \lambda, \mathcal{M}$ with the properties:*

$$l \leq l_0, \qquad \lambda^{1-\gamma} l \geq \sigma_0, \qquad \mathcal{M} \geq \max\{\|v\|_2, \|w\|_2, 1\}, \tag{1.11}$$

there exist $\tilde v \in \mathcal{C}^2(\bar\omega + \bar B_l(0), \mathbb{R}^k)$, $\tilde w \in \mathcal{C}^2(\bar\omega + \bar B_l(0), \mathbb{R}^2)$ such that, denoting the defects:

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big), \qquad \tilde{\mathcal{D}} = A - \big(\frac{1}{2}(\nabla \tilde v)^T\nabla \tilde v + \mathrm{sym}\nabla \tilde w\big), \tag{1.12}$$

the following bounds are valid:

$$\begin{aligned} &\|\tilde v - v\|_1 \leq C\lambda^{\gamma/2}\big(\|\mathcal{D}\|_0^{1/2} + l\mathcal{M}\big), \\ &\|\tilde w - w\|_1 \leq C\lambda^{\gamma}\big(\|\mathcal{D}\|_0^{1/2} + l\mathcal{M}\big)\big(1 + \|\mathcal{D}\|_0^{1/2} + l\mathcal{M} + \|\nabla v\|_0\big), \end{aligned} \tag*{$(1.13)_1$}$$

$$\begin{aligned} &\|\nabla^2\tilde v\|_0 \leq C\,\frac{(\lambda l)^{(F_{K+1}-2)+(F_{K+1}-1)N/2}\lambda^{\gamma/2}}{l}\big(\|\mathcal{D}\|_0^{1/2} + l\mathcal{M}\big), \\ &\|\nabla^2\tilde w\|_0 \leq C\,\frac{(\lambda l)^{(F_{K+1}-2)+(F_{K+1}-1)N/2}\lambda^{\gamma}}{l}\big(\|\mathcal{D}\|_0^{1/2} + l\mathcal{M}\big)\times \\ &\qquad\qquad\qquad\qquad\qquad\qquad \times\big(1 + \|\mathcal{D}\|_0^{1/2} + l\mathcal{M} + \|\nabla v\|_0\big), \end{aligned} \tag*{$(1.13)_2$}$$

$$\|\tilde{\mathcal{D}}\|_0 \leq C\Big(l^\beta\|A\|_{0,\beta} + \frac{\lambda^\gamma}{(\lambda l)^{2(F_K-1)N}}\big(\|\mathcal{D}\|_0 + (l\mathcal{M})^2\big)\Big). \tag*{$(1.13)_3$}$$

Above, $\{F_k\}_{k=0}^\infty$ is the Fibonacci sequence, given by the recursion:

$$F_0 = F_1 = 1, \qquad F_{k+2} = F_k + F_{k+1} \quad \text{for all } \ k \geq 0.$$

The norms of the maps $v, w, A, \mathcal{D}$ and $\tilde{v}, \tilde{w}, \tilde{\mathcal{D}}$ in **??** *-* **??** *are taken on the respective domains of the maps' definiteness. The constants C depend only on ω, γ, N, K.*

By assigning N sufficiently large, we see that the quotient $r_{K,N}$ of the blow-up rate of $\|\nabla^2\tilde{v}\|_0$ with respect to the rate of decay of $\|\tilde{\mathcal{D}}\|_0$, can be taken arbitrarily close to $\frac{F_{K+1}-1}{4(F_K-1)}$, whereas this last quotient approaches the quarter of the golden ratio $\varphi = \frac{1}{2}(1+\sqrt{5})$ for large K:

$$\begin{aligned} \lim_{K\to\infty}\lim_{N\to\infty} r_{K,N} &= \lim_{K\to\infty}\lim_{N\to\infty} \frac{(F_{K+1}-2) + (F_{K+1}-1)N/2}{2N(F_K-1)} \\ &= \lim_{N\to\infty} \frac{F_{K+1}-1}{4(F_K-1)} = \frac{1}{4}\lim_{N\to\infty}\frac{F_{K+1}}{F_K} = \frac{\varphi}{4}. \end{aligned}$$

Since the Hölder regularity exponent deduced from iterating the "stage" as in Theorem **??** depends only on the aforementioned quotient of the blow-up rates, and in fact it equals $\frac{1}{1+2r_{K,N}}$ (see section **??** and Theorem **??**), this implies the range claimed in (**??**).

1.5. **The organization of the paper.** In section **??** we gather the preparatory results: the convolution and commutator estimates; the precise formulation of the decomposition (**??**) and its properties; and the "step" construction for the convex integration algorithm, in which the oscillatory modifications of the form (**??**) are used to cancel a single rank-one primitive defect of the form $a^2(x)e_i \otimes e_i$. In section **??** we carry out the Källen-like iterations, shifting the $\mathcal{D}_{11}$ component of the defect $\mathcal{D}$ onto its $\mathcal{D}_{22}$ component and absorbing a large number N of higher order defects obtained by the consecutive application of the decomposition (**??**). Section **??** determines the sufficient conditions allowing for the iteration: first the conditions on the two frequencies λ, μ of any pair of the modifications that cancel the current defect; then the conditions on the progression of the couples of frequencies $\{\lambda_i, \mu_i\}_{i=1}^K$ in the K pairs of "steps" within a single "stage". In section **??** we declare specific frequencies derived from the Fibonacci sequence, and show that they satisfy the aforementioned conditions, leading to a proof of Theorem **??**. Finally, section **??** gathers our previous results on the iteration on stages, which convert the estimates in Theorem **??** into the final result of Theorem **??**.

1.6. **Notation.** By $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ we denote the space of symmetric 2×2 matrices. The space of Hölder continuous vector fields $\mathcal{C}^{m,\alpha}(\bar{\omega}, \mathbb{R}^k)$ consists of restrictions of all $f \in \mathcal{C}^{m,\alpha}(\mathbb{R}^2, \mathbb{R}^k)$ to the closure of an open, bounded domain $\omega \subset \mathbb{R}^2$. The $\mathcal{C}^m(\bar{\omega}, \mathbb{R}^k)$ norm of such restriction is denoted by $\|f\|_m$, while its Hölder norm in $\mathcal{C}^{m,\gamma}(\bar{\omega}, \mathbb{R}^k)$ is $\|f\|_{m,\gamma}$. By C we denote a universal constant which may change from line to line, but it depends only on the specified parameters. For a matrix $D \in \mathbb{R}^{2\times 2}$ we denote by $D^\wedge$ the matrix with its D_{22} component removed.

2. Preparatory statements

In this section, we gather the regularization, decomposition and perturbation statements that will be used in the course of the convex integration constructions. The first lemma below consists of the basic convolution estimates and the commutator estimate from [**?**]:

Lemma 2.1. *Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d, \mathbb{R})$ be a standard mollifier that is nonnegative, radially symmetric, supported on the unit ball $B(0,1) \subset \mathbb{R}^d$ and such that $\int_{\mathbb{R}^d} \phi \ \mathrm{d}x = 1$. Denote:*

$$\phi_l(x) = \frac{1}{l^d}\phi(\frac{x}{l}) \quad \text{for all } \ l \in (0,1], \ x \in \mathbb{R}^d.$$

Then, for every $f, g \in \mathcal{C}^0(\mathbb{R}^d, \mathbb{R})$ *and every* $m, n \geq 0$ *and* $\beta \in (0, 1]$ *there holds:*

$$\|\nabla^{(m)}(f * \phi_l)\|_0 \leq \frac{C}{l^m}\|f\|_0, \tag{2.1$_1$}$$

$$\|f - f * \phi_l\|_0 \leq C \min\big\{l^2\|\nabla^2 f\|_0, l\|\nabla f\|_0, l^\beta\|f\|_{0,\beta}\big\}, \tag{2.1$_2$}$$

$$\|\nabla^{(m)}\big((fg) * \phi_l - (f * \phi_l)(g * \phi_l)\big)\|_0 \leq Cl^{2-m}\|\nabla f\|_0\|\nabla g\|_0, \tag{2.1$_3$}$$

with a constant $C > 0$ *depending only on the differentiability exponent* m.

The next auxiliary result, put forward in [**?**] (for a self-contained proof, see also [**?**, Lemma 2.3]) is specific to dimension $d = 2$. It allows for the decomposition of the given defect into a multiple of Id_2 (thus two primitive defects of rank 1) and a symmetric gradient, in agreement with the local conformal invariance of any Riemann 2d metric:

Lemma 2.2. *Given a radius* $R > 0$ *and an exponent* $\gamma \in (0, 1)$, *define the linear space* E *consisting of* $\mathcal{C}^{0,\gamma}$*-regular,* $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$*-valued matrix fields* D *on the ball* $\bar{B}_R \subset \mathbb{R}^2$, *whose traceless part* $\dot{D} = D - \frac{1}{2}(\mathit{trace}\ D)\mathrm{Id}_2$ *is compactly supported in* B_R. *There exist linear maps* $\bar{\Psi}$, $\bar{a}$ *in:*

$$\bar{\Psi} : E \to \mathcal{C}^{1,\gamma}(\bar{B}_R, \mathbb{R}^2), \qquad \bar{a} : E \to \mathcal{C}^{0,\gamma}(\bar{B}_R),$$
$$E = \big\{D \in \mathcal{C}^{0,\gamma}(\bar{B}_R, \mathbb{R}^{2\times 2}_{\mathrm{sym}});\ \dot{D} \in \mathcal{C}^{0,\gamma}_c(B_R, \mathbb{R}^{2\times 2}_{\mathrm{sym}})\big\},$$

with the following properties:

(i) *for all* $D \in E$ *there holds:* $D = \bar{a}(D)\mathrm{Id}_2 + \mathrm{sym}\nabla\big(\bar{\Psi}(D)\big)$,

(ii) $\bar{\Psi}(\mathrm{Id}_2) \equiv 0$ *and* $\bar{a}(\mathrm{Id}_2) \equiv 1$ *in* B_R,

(iii) $\|\bar{\Psi}(D)\|_{1,\gamma} \leq C\|\dot{D}\|_{0,\gamma}$ *and* $\|\bar{a}(D)\|_{0,\gamma} \leq C\|D\|_{0,\gamma}$ *with constants* C *depending on* R, γ,

(iv) *for all* $m \geq 1$, *if* $D \in E \cap \mathcal{C}^{m,\gamma}(\bar{B}_R, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ *then* $\bar{\Psi}(D) \in \mathcal{C}^{m+1,\gamma}(\bar{B}_R, \mathbb{R}^2)$ *and* $\bar{a}(D) \in \mathcal{C}^{m,\gamma}(\bar{B}_R)$, *and we have:*

$$\partial_I\bar{\Psi}(D) = \bar{\Psi}(\partial_I D), \quad \partial_I\bar{a}(D) = \bar{a}(\partial_I D) \qquad \text{for all } |I| \leq m.$$

(v) *for all* $m \geq 1$, *if* $D \in E \cap \mathcal{C}^{m,\gamma}(\bar{B}_R, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ *and if additionally* $D_{22} = 0$ *in* $\bar{B}_R$, *then:*

$$\|\partial_2^{(s)}\bar{a}(D)\|_{0,\gamma} \leq C\|\partial_2^{(s)}D\|_{0,\gamma}, \quad \|\partial_1^{(t+1)}\partial_2^{(s)}\bar{a}(D)\|_{0,\gamma} \leq C\|\partial_1^{(t)}\partial_2^{(s+1)}D\|_{0,\gamma},$$

for all $s, t \geq 0$ *such that* $s \leq m$, $t + s + 1 \leq m$, *and with* C *depending only on* R, γ.

As the final preparatory result, we recall the "step" construction from [**?**, Lemma 2.1], in which a single codimension is used to cancel one rank-one defect of the form $a(x)^2 e_i \otimes e_i$:

Lemma 2.3. *Let* $v \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^k)$, $w \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$, $\lambda > 0$ *and* $a \in \mathcal{C}^2(\mathbb{R}^2)$ *be given. Denote:*

$$\Gamma(t) = 2\sin t, \quad \bar{\Gamma}(t) = \frac{1}{2}\cos(2t), \quad \bar{\bar{\Gamma}}(t) = -\frac{1}{2}\sin(2t), \quad \bar{\bar{\bar{\Gamma}}}(t) = 1 - \frac{1}{2}\cos(2t),$$

and for a fixed $i = 1, 2$ *and* $j = 1 \ldots k$ *define:*

$$\tilde{v} = v + \frac{a(x)}{\lambda}\Gamma(\lambda x_i)e_j, \quad \tilde{w} = w - \frac{a(x)}{\lambda}\Gamma(\lambda x_i)\nabla v^j + \frac{a(x)}{\lambda^2}\bar{\Gamma}(\lambda x_i)\nabla a(x) + \frac{a(x)^2}{\lambda}\bar{\bar{\Gamma}}(\lambda x_i)e_i. \tag{2.2}$$

Then, the following identity is valid on $\mathbb{R}^2$*:*

$$\begin{aligned}&\Big(\frac{1}{2}(\nabla\tilde{v})^T\nabla\tilde{v} + \mathrm{sym}\nabla\tilde{w}\Big) - \Big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\Big) - a(x)^2 e_i \otimes e_i \\ &= -\frac{a}{\lambda}\Gamma(\lambda x_i)\nabla^2 v^j + \frac{a}{\lambda^2}\bar{\Gamma}(\lambda x_i)\nabla^2 a + \frac{1}{\lambda^2}\bar{\bar{\bar{\Gamma}}}(\lambda x_i)\nabla a \otimes \nabla a.\end{aligned} \tag{2.3}$$

3. The Källen iteration procedure

Recall that our Nash-Kuiper iteration scheme will proceed as induction on "stages" specified in Theorem **??**. Each such stage is built as an iteration of a given number K of "steps", consecutively decreasing the current defect $\mathcal{D}$ towards the resulting defect $\tilde{\mathcal{D}}$, given as in (**??**). In turn, a single step consists of a double application of Lemma **??**. The first application (corrugation) shifts the $\mathcal{D}_{11}$ component of the current defect $\mathcal{D}$, after its off-diagonal components have been removed via an application of Lemma **??**, onto its $\mathcal{D}_{22}$ component, which is then removed using the second application of Lemma **??**. This defect shift, put forward in [**?**], is achieved by N iterations of the Källen procedure, presented in this section.

Towards its eventual application in section **??**, the field H in Proposition **??** below should be thought of as the (scaled) defect field $\mathcal{D}$, whereas Q is the (scaled) second derivative $\nabla^2 v$ of the current displacement. For a matrix $D \in \mathbb{R}^{2\times 2}$ we denote by $D^\wedge$ the matrix with its D_{22} component removed, namely:

$$D^\wedge = D - D_{22} e_2 \otimes e_2.$$

Proposition 3.1. *Let $\omega \subset \mathbb{R}^2$ be open, bounded, smooth and let $N \geq 1$ and $\gamma \in (0,1)$. Then, there exists $l_0 \in (0,1)$ depending only on ω and $\sigma_0 \geq 1$ depending on ω, γ, N such that the following holds. Given the positive constants $\delta, \eta, \mu, \lambda$ and an integer M, satisfying:*

$$\delta, \eta \leq 2l_0, \qquad \mu \geq \frac{1}{\eta}, \qquad \lambda^{1-\gamma} \geq \mu\sigma_0, \qquad M \geq 0, \tag{3.1}$$

and given the fields $H, Q \in \mathcal{C}^\infty(\bar{\omega} + \bar{B}_{\delta+\eta}(0), \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ with the properties:

$$\begin{aligned} &\|\nabla^{(m)} H\|_0 \leq \mu^m \qquad \text{for all } \ m = 0\ldots M+3N, \\ &\|\nabla^{(m)} Q\|_0 \leq \mu^{m+1} \quad \text{for all } \ m = 0\ldots M+3(N-1), \end{aligned} \tag{3.2}$$

there exist $a \in \mathcal{C}^\infty(\bar{\omega} + \bar{B}_\delta(0), \mathbb{R})$ and $\Psi \in \mathcal{C}^\infty(\bar{\omega} + \bar{B}_\delta(0), \mathbb{R}^2)$ such that, denoting:

$$\mathcal{F} = a^2 \mathrm{Id}_2 + \mathrm{sym}\nabla\Psi - H + \Big(-\frac{a}{\lambda}\Gamma(\lambda x_1) Q + \frac{a}{\lambda^2}\bar{\Gamma}(\lambda x_1)\nabla^2 a + \frac{1}{\lambda^2}\bar{\bar{\bar{\Gamma}}}(\lambda x_1)\nabla a \otimes \nabla a\Big)^\wedge,$$

there hold the estimates:

$$\frac{\tilde{C}}{2}\mu^\gamma \leq a^2 \leq \frac{3\tilde{C}}{2}\mu^\gamma \quad \text{ and } \quad \frac{\tilde{C}^{1/2}}{2}\mu^{\gamma/2} \leq a \leq \frac{3\tilde{C}^{1/2}}{2}\mu^{\gamma/2}, \tag{3.3$_1$}$$

$$\|\nabla^{(m)} a\|_0 \leq C\mu^{\gamma/2}\frac{\lambda^m}{\lambda/\mu} \qquad \text{for all } \ m = 1\ldots M, \tag{3.3$_2$}$$

$$\|\Psi\|_1 \leq C\mu^\gamma \quad \text{ and } \quad \|\nabla^2\Psi\|_0 \leq C\mu^\gamma\lambda, \tag{3.3$_3$}$$

$$\|\nabla^{(m)}\mathcal{F}\|_0 \leq C\mu^\gamma\lambda^{\gamma N}\frac{\lambda^m}{(\lambda/\mu)^N} \qquad \text{for all } \ m = 0\ldots M. \tag{3.3$_4$}$$

The constant $\tilde{C}$ in **??** *depends only on ω, γ. Other constants C depend: in* **??** *on ω, γ, N, M and only $M + 3(N-1) + 1$ derivatives of H and $M + 3(N-2) + 1$ derivatives of Q; in* **??** *on ω, γ, N and only $2 + 3(N-1)$derivatives of H and $2 + 3(N-2)$ derivatives of Q; in* **??** *on ω, γ, N, M, necessitating the full condition (***??***).*

Proposition **??** follows from more detailed estimates, that we gather in:

Theorem 3.2. *Let $\omega, \gamma, N, l_0, \sigma_0$ be as in Proposition **??**. Given the positive parameters $\delta, \eta, \mu, \lambda, M$ satisfying (**??**), and given the fields $H, Q \in \mathcal{C}^\infty(\bar\omega + \bar B_{\delta+\eta}(0), \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ such that:*

$$\begin{aligned} &\|\nabla^{(m)} H\|_0 \leq \mu^m \qquad \textit{for all } m = 0 \dots M+3N, \\ &\|\nabla^{(m)} Q^\wedge\|_0 \leq \mu^{m+1} \quad \textit{for all } m = 0 \dots M + 3(N-1), \end{aligned} \tag{3.4}$$

there exists a family of fields:

$$\big\{a_n \in \mathcal{C}^\infty(\bar\omega + \bar B_\delta(0), \mathbb{R}), \quad \Psi_n \in \mathcal{C}^\infty(\bar\omega + \bar B_\delta(0), \mathbb{R}^2)\big\}_{n=1}^N$$

such that, denoting $\mathcal{E}_0 = 0$ and $\mathcal{E}_n = -\frac{a_n}{\lambda}\Gamma(\lambda x_1) Q + \frac{a_n}{\lambda^2}\bar\Gamma(\lambda x_1)\nabla^2 a_n + \frac{1}{\lambda^2}\bar{\bar{\bar\Gamma}}(\lambda x_1)\nabla a_n \otimes \nabla a_n$ for $n = 1 \dots N$, there holds:

$$a_n^2 \mathrm{Id}_2 + \mathrm{sym}\nabla\Psi_n = H - \mathcal{E}_{n-1}^\wedge \qquad \textit{for all } n = 1 \dots N, \tag{3.5}$$

together with the following bounds, likewise valid for all $n = 1 \dots N$:

$$\frac{\tilde C}{2}\mu^\gamma \leq a_n^2 \leq \frac{3\tilde C}{2}\mu^\gamma \quad \textit{and} \quad \frac{\tilde C^{1/2}}{2}\mu^{\gamma/2} \leq a_n \leq \frac{3\tilde C^{1/2}}{2}\mu^{\gamma/2}, \tag{3.6$_1$}$$

$$\|\partial_2^{(s)} a_n^2\|_0 \leq C\mu^\gamma\mu^s \quad \textit{and} \quad \|\partial_2^s a_n\|_0 \leq C\mu^{\gamma/2}\mu^s \qquad \textit{for all } s = 1\dots M, \tag{3.6$_2$}$$

$$\left.\begin{aligned} &\|\partial_1^{(t+1)}\partial_2^{(s)} a_n^2\|_0 \leq C\mu^\gamma \frac{\lambda^{t+1}\mu^s}{\lambda/\mu} \\ &\qquad \textit{and} \quad \|\partial_1^{(t+1)}\partial_2^{(s)} a_n\|_0 \leq C\mu^{\gamma/2}\frac{\lambda^{t+1}\mu^s}{\lambda/\mu} \end{aligned}\right\} \quad \textit{for all } s+t+1 = 1\dots M, \tag{3.6$_3$}$$

$$\|\Psi_n\|_1 \leq C\mu^\gamma \quad \textit{and} \quad \|\nabla^2\Psi_n\|_0 \leq C\mu^\gamma\lambda, \tag{3.6$_4$}$$

$$\|\partial_1^{(t)}\partial_2^{(s)}\big(\mathcal{E}_n^\wedge - \mathcal{E}_{n-1}^\wedge\big)\|_0 \leq C\mu^\gamma\lambda^{\gamma(n-1)}\frac{\lambda^t\mu^s}{(\lambda/\mu)^n} \qquad \textit{for all } s+t = 0\dots M. \tag{3.6$_5$}$$

*The constant $\tilde C$ in **??** depends only on ω, γ. Other constants C depend: in **??**, **??** on ω, γ, n, M and only $M + 3(n-1)+1$ derivatives of H and $M+3(n-2)+1$ derivatives of $Q^\wedge$; in **??** on ω, γ, n and only $2+3(n-1)$derivatives of H and $2+3(n-2)$ derivatives of $Q^\wedge$; in **??** on ω, γ, N, M, necessitating the full condition (**??**).*

Proof. **1. (Preparatory observations)** We fix a radius $R > 0$ depending only on ω, such that $\bar\omega + \bar B_4(0) \subset B_R(0)$. Also, given $0 < \delta, \eta \leq 2l_0$ we set a cut-off function $\chi \in \mathcal{C}_c^\infty(\omega + B_{\delta+\eta}(0), [0,1])$ with $\chi \equiv 1$ on $\bar\omega + \bar B_\delta(0)$. When $l_0 \ll 1$, it is possible to request that for any function $f \in \mathcal{C}^m(\bar\omega + \bar B_{\delta+\eta}(0), \mathbb{R})$ and any multiindex I with $|I| \leq m$, there holds:

$$\|\partial_I(\chi f)\|_0 \leq C\sum_{I_1+I_2=I}\|\partial_{I_1}\chi\|_0\|\partial_{I_2} f\|_0 \leq C\sum_{I_1+I_2=I}\mu^{|I_1|}\|\partial_{I_2} f\|_0, \tag{3.7}$$

with constants C depending only on ω and m. Further, we directly observe that the second bound in **??** is implied by the first one, because:

$$\Big|\frac{a(x)}{\tilde C^{1/2}\mu^{\gamma/2}} - 1\Big| \leq \Big|\frac{a(x)^2}{\tilde C\mu^\gamma} - 1\Big| \leq \frac{1}{2}.$$

Next, in both **??**, **??** the second bounds follow from the first ones, in view of **??**, and they necessitate the same number of derivatives bounds in condition (**??**). In **??**, we use the one-dimensional Faá di Bruno formula:

$$\begin{aligned}\|\partial_2^{(s)} a_n\|_0 &\leq C\Big\| \sum_{p_1+2p_2+\ldots sp_s=s} a_n^{2(1/2-p_1-\ldots-p_s)} \prod_{z=1}^s \big|\partial_2^{(z)} a_n^2\big|^{p_z}\Big\|_0 \\ &\leq C\|a_n\|_0 \sum_{p_1+2p_2+\ldots sp_s=s} \prod_{z=1}^s \Big(\frac{\|\partial_2^{(z)} a_n^2\|_0}{\tilde C\mu^\gamma}\Big)^{p_z} \leq C\mu^{\gamma/2}\mu^s.\end{aligned}$$

For **??**, we apply the multivariate version of the Faá di Bruno formula. Let Π be the set of all partitions π of the initial multiindex $\{1\}^{t+1}+\{2\}^s$ into multiindices I of lengths $|I|\in[1,t+s+1]$. Denoting by $|\pi|$ the number of multiindices in a partition π, we have:

$$\begin{aligned}\|\partial_1^{(t+1)}\partial_2^{(s)} a_n\|_0 &\leq C\Big\|\sum_{\pi\in\Pi} a_n^{2(1/2-|\pi|)}\prod_{I\in\pi}\partial_I a_n^2\Big\|_0 \leq C\|a_n\|_0 \sum_{\pi\in\Pi}\prod_{I\in\pi}\frac{\|\partial_I a_n^2\|_0}{\tilde C\mu^\gamma} \\ &\leq C\mu^{\gamma/2}\lambda^{t+1}\mu^s\sum_{\pi\in\Pi}\Big(\prod_{I\in\pi,\, 1\in I}\frac{1}{\lambda/\mu}\Big) \leq C\mu^{\gamma/2}\frac{\lambda^{t+1}\mu^s}{\lambda/\mu}.\end{aligned}$$

Finally, applying Faá di Bruno's formula to inverse rather than square root, **??**, **??** yield:

$$\begin{aligned}\|\partial_2^{(s)}\Big(\frac{1}{a_{n+1}+a_n}\Big)\|_0 &\leq \frac{C}{\mu^{\gamma/2}}\sum_{p_1+2p_2+\ldots sp_s=s}\prod_{z=1}^s\Big(\frac{\|\partial_2^{(z)}(a_{n+1}+a_n)\|_0}{\tilde C^{1/2}\mu^{\gamma/2}}\Big)^{p_z} \leq \frac{C}{\mu^{\gamma/2}}\mu^s, \\ \|\partial_1^{(t+1)}\partial_2^{(s)}\Big(\frac{1}{a_{n+1}+a_n}\Big)\|_0 &\leq C\Big\|\sum_{\pi\in\Pi}\big(a_{n+1}+a_n\big)^{-1-|\pi|}\prod_{I\in\pi}\partial_I(a_{n+1}+a_n)\Big\|_0 \\ &\leq \frac{C}{\mu^{\gamma/2}}\sum_{\pi\in\Pi}\prod_{I\in\pi}\frac{\|\partial_I a_{n+1}\|_0+\|\partial_I a_n\|_0}{\tilde C^{1/2}\mu^{\gamma/2}} \\ &\leq \frac{C}{\mu^{\gamma/2}}\lambda^{t+1}\mu^s\sum_{\pi\in\Pi}\Big(\prod_{I\in\pi,\,1\in I}\frac{1}{\lambda/\mu}\Big) \leq \frac{C}{\mu^{\gamma/2}}\frac{\lambda^{t+1}\mu^s}{\lambda/\mu}.\end{aligned} \tag{3.8}$$

with the the same dependence of constants and using the same number of derivatives of H and $Q^\wedge$ as in the corresponding bounds on a_n and a_{n+1}.

2. (Induction base $n=1$ and definition of $\tilde C$) Let the linear maps $\bar a,\bar\Psi$ be as in Lemma **??**, applied with the specified R, γ. From (**??**) and using the bound on $\|H\|_1$ in (**??**), we get:

$$\|\bar a(\chi H)\|_0 \leq C\|\chi H\|_{0,\gamma} \leq C\big(\|H\|_0+\|H\|_0^{1-\gamma}\|\nabla(\chi H)\|_0^\gamma\big) \leq C\mu^\gamma$$

where C depends on ω, γ. We declare $\tilde C$ to be four times the final constant above, leading to:

$$\|\bar a(\chi H)\|_0 \leq \frac{\tilde C}{4}\mu^\gamma. \tag{3.9}$$

This results in the validity of the first bound in **??** in view of (**??**), where we set:

$$a_1^2 = \tilde C\mu^\gamma + \bar a(\chi H), \qquad \Psi_1 = \bar\Psi(\chi H) - \tilde C\mu^\gamma id_2, \tag{3.10}$$

while the identity (**??**) holds (on $\bar\omega+\bar B_\delta(0)$ where $\chi\equiv 1$), because:

$$\begin{aligned}\chi H &= \bar a(\chi H)\mathrm{Id}_2 + \mathrm{sym}\nabla\big(\bar\Psi(\chi H)\big) \\ &= (a_1^2\mathrm{Id}_2 - \tilde C\mu^\gamma\mathrm{Id}_2) + (\mathrm{sym}\nabla\Psi_1 + \tilde C\mu^\gamma\mathrm{Id}_2) = a_1^2\mathrm{Id}_2 + \mathrm{sym}\nabla\Psi_1.\end{aligned}$$

Further, using (**??**) and the bound on $\|H\|_{M+1}$ in (**??**), we obtain for all $m = 1 \dots M$:

$$\begin{aligned}\|\nabla^{(m)} a_1^2\|_0 &= \|\bar{a}(\nabla^{(m)}(\chi H))\|_0 \leq C\|\nabla^{(m)}(\chi H)\|_{0,\gamma} \\ &\leq C\big(\|\nabla^{(m)}(\chi H)\|_0 + \|\nabla^{(m)}(\chi H)\|_0^{1-\gamma}\|\nabla^{(m+1)}(\chi H)\|_0^{\gamma}\big) \\ &\leq C\big(\mu^m + \mu^{m(1-\gamma)}\mu^{(m+1)\gamma}\big) \leq C\mu^\gamma\mu^m,\end{aligned} \tag{3.11}$$

where C depends on ω, γ, M. The above implies both bounds **??** and **??** for a_1^2. Towards **??**, we apply (**??**) and use (**??**) to estimate $\|H\|_{M+3}$ and $\|Q^\wedge\|_M$ in:

$$\begin{aligned}\|\partial_1^{(t)}\partial_2^{(s)}\mathcal{E}_1^\wedge\|_0 \leq C &\sum_{\substack{p_1+q_1+z_1=t \\ q_2+z_2=s}} \Big(\lambda^{p_1-1}\|\nabla^{(q_1+q_2)}a_1\|_0\|\nabla^{(z_1+z_2)}Q^\wedge\|_0 \\ &+ \lambda^{p_1-2}\|\nabla^{(q_1+q_2+1)}a_1\|_0\|\nabla^{(z_1+z_2+1)}a_1\|_0 + \lambda^{p_1-2}\|\nabla^{(q_1+q_2)}a_1\|_0\|\nabla^{(z_1+z_2+2)}a_1\|_0\Big) \\ \leq C &\sum_{\substack{p_1+q_1+z_1=t \\ q_2+z_2=s}} \big(\lambda^{p_1-1}\mu^{\gamma/2}\mu^{q_1+q_2}\mu^{z_1+z_2+1} + \lambda^{p_1-2}\mu^\gamma\mu^{q_1+q_2+z_1+z_2+2}\big) \\ \leq C&\mu^\gamma\Big(\frac{\lambda^t\mu^s}{\lambda/\mu} + \frac{\lambda^t\mu^s}{(\lambda/\mu)^2}\Big) \leq C\mu^\gamma\frac{\lambda^t\mu^s}{\lambda/\mu},\end{aligned}$$

valid for $t + s \leq M$, with C depending on ω, γ, M. The above is precisely **??** since $\mathcal{E}_0 = 0$.

3. (Induction step: bounds ?? – ??) Assume that **??** – **??** and **??** hold up to some $1 \leq n \leq N-1$, necessitating (**??**) to estimate derivatives of H only up to $M + 3n$ and of $Q^\wedge$ up to $M + 3(n-1)$. We will prove the validity of **??** – **??** and **??** at $n+1$, necessitating (**??**) to estimate $\|H\|_{M+3n+1}$ and $\|Q^\wedge\|_{M+3(n-1)+1}$. We start by noting that, as a consequence of **??** in view of (**??**), for all $j = 1 \dots n$ and $s + t = 0 \dots M$ we have:

$$\|\partial_1^{(t)}\partial_2^{(s)}\big(\chi(\mathcal{E}_j - \mathcal{E}_{j-1})^\wedge\big)\|_0 \leq C\frac{\mu^\gamma}{\lambda^\gamma}\lambda^t\mu^s\Big(\frac{\lambda^\gamma}{\lambda/\mu}\Big)^j,$$

with C depending on ω, γ, n, M. This yields:

$$\|\partial_1^{(t)}\partial_2^{(s)}\big(\chi(\mathcal{E}_j - \mathcal{E}_{j-1})^\wedge\big)\|_{0,\gamma} \leq C\mu^\gamma\lambda^t\mu^s\Big(\frac{\lambda^\gamma}{\lambda/\mu}\Big)^j, \tag{3.12}$$

necessitating the bounds on $\|H\|_{M+3n+1}$ and $\|Q^\wedge\|_{M+1+3(n-1)+1}$. Since $\mathcal{E}_0 = 0$, it follows that:

$$\|\partial_1^{(t)}\partial_2^{(s)}(\chi\mathcal{E}_n)\|_{0,\gamma} \leq C\mu^\gamma\lambda^t\mu^s\sum_{j=1}^n\Big(\frac{\lambda^\gamma}{\lambda/\mu}\Big)^j \leq C\mu^\gamma\lambda^t\mu^s\frac{\lambda^\gamma/(\lambda/\mu)}{1-\lambda^\gamma/(\lambda/\mu)} \leq C\mu^\gamma\lambda^\gamma\frac{\lambda^t\mu^s}{\lambda/\mu} \tag{3.13}$$

as $\lambda^\gamma/(\lambda/\mu) \leq 1/\sigma_0 \leq 1/2$ from the third assumption in (**??**). In particular, we get:

$$\|\bar{a}(\chi\mathcal{E}_n^\wedge)\|_0 \leq C\|\chi\mathcal{E}_n^\wedge\|_{0,\gamma} \leq C\mu^\gamma\frac{\lambda^\gamma}{\lambda/\mu} \leq \frac{\tilde{C}}{4}\mu^\gamma$$

provided that σ_0 is large enough, in function of ω, γ, n. Recalling (**??**), the above yields the well definiteness of a_{n+1}^2 together with the first bound in **??**, upon defining:

$$a_{n+1}^2 = \tilde{C}\mu^\gamma + \bar{a}(\chi H - \chi\mathcal{E}_n^\wedge), \qquad \Psi_{n+1} = \bar{\Psi}(\chi H - \chi\mathcal{E}_n^\wedge) - \tilde{C}\mu^\gamma id_2. \tag{3.14}$$

The identity (**??**) clearly holds from Lemma **??** (i). Towards proving **??**, we apply (**??**) and the bound on $\|\nabla^{(m)}(\chi H)\|_{0,\gamma}$ in (**??**), to get:

$$\|\partial_2^{(s)}a_{n+1}^2\|_0 \leq C\|\partial_2^{(s)}(\chi H - \chi\mathcal{E}_n^\wedge)\|_{0,\gamma} \leq C\mu^\gamma\mu^s\Big(1 + \frac{\lambda^\gamma}{\lambda/\mu}\Big) \leq C\mu^\gamma\mu^s$$

for all $s = 1 \dots M$. The estimate in **??** follows by additionally recalling Lemma **??** (v) in:

$$\begin{aligned}\|\partial_1^{(t+1)}\partial_2^{(s)} a_{n+1}^2\|_0 &\leq C\|\partial_1^{(t+1)}\partial_2^{(s)}(\chi H)\|_{0,\gamma} + C\|\partial_1^{(t)}\partial_2^{(s+1)}(\chi \mathcal{E}_n^\wedge)\|_{0,\gamma} \\ &\leq C\mu^\gamma\Big(\mu^{s+t+1} + \lambda^t\mu^{s+1}\frac{\lambda^\gamma}{\lambda/\mu}\Big) \leq C\mu^\gamma\lambda^t\mu^{s+1} \leq C\mu^\gamma\frac{\lambda^{t+1}\mu^s}{\lambda/\mu},\end{aligned}$$

for all $s + t + 1 = 1 \dots M$. In both bounds above the constant C depends on ω, γ, n, M and condition (**??**) has been used only up to $M + 3n+1$ in derivatives of H and up to $M + 3(n-1)+1$ in $Q^\wedge$.

4. (Induction step: the bound ??) In this step we continue the inductive step argument and show **??** at $n+1$, necessitating (**??**) to estimate derivatives of H up to $M + 3(n+1)$ and of $Q^\wedge$ up to $M + 3n$. From the definitions (**??**) and (**??**), it follows that:

$$a_{n+1}^2 - a_n^2 = \bar{a}(\chi(\mathcal{E}_n - \mathcal{E}_{n-1})^\wedge).$$

Consequently, recalling (**??**) we get:

$$\begin{aligned}&\|\partial_2^{(s)}(a_{n+1}^2 - a_n^2)\|_0 \leq C\|\partial_2^{(s)}(\chi(\mathcal{E}_n - \mathcal{E}_{n-1})^\wedge)\|_{0,\gamma} \leq C\mu^\gamma\mu^s\big(\frac{\lambda^\gamma}{\lambda/\mu}\big)^n \\ &\|\partial_1^{(t+1)}\partial_2^{(s)}(a_{n+1}^2 - a_n^2)\|_0 \leq C\|\partial_1^{(t)}\partial_2^{(s+1)}(\chi(\mathcal{E}_n - \mathcal{E}_{n-1})^\wedge)\|_{0,\gamma} \leq C\mu^\gamma\lambda^t\mu^{s+1}\big(\frac{\lambda^\gamma}{\lambda/\mu}\big)^n,\end{aligned}$$

which together with (**??**) implies:

$$\begin{aligned}\|\partial_2^{(s)}(a_{n+1} - a_n)\|_0 &\leq C\sum_{p+q=s}\|\partial_2^{(p)}(a_{n+1}^2 - a_n^2)\|_0\big\|\partial_2^{(q)}\big(\frac{1}{a_{n+1}+a_n}\big)\big\|_0 \\ &\leq C\mu^{\gamma/2}\mu^s\big(\frac{\lambda^\gamma}{\lambda/\mu}\big)^n \\ \|\partial_1^{(t+1)}\partial_2^{(s)}&(a_{n+1} - a_n)\|_0 \\ &\leq C\sum_{\substack{p_1+q_1=t+1\\ p_2+q_2=s}}\|\partial_1^{(p_1)}\partial_2^{(p_2)}(a_{n+1}^2 - a_n^2)\|_0\big\|\partial_1^{(q_1)}\partial_2^{(q_2)}\big(\frac{1}{a_{n+1}+a_n}\big)\big\|_0 \\ &\leq C\mu^{\gamma/2}\frac{\lambda^{t+1}\mu^s}{\lambda/\mu}\big(\frac{\lambda^\gamma}{\lambda/\mu}\big)^n\end{aligned} \tag{3.15}$$

for all $s = 0 \dots M$ and $s + t + 1 = 1 \dots M$, with C depending on ω, γ, n, M and where we used bounds in (**??**) on the derivatives of H up to $M + 3n + 1$ and of $Q^\wedge$ up to $M + 3(n-1) + 1$. Towards proving **??**, we first write:

$$\begin{aligned}\mathcal{E}_{n+1}^\wedge - \mathcal{E}_n^\wedge = &-\frac{a_{n+1} - a_n}{\lambda}\Gamma(\lambda x_1)Q^\wedge + \frac{1}{\lambda^2}\bar{\Gamma}(\lambda x_1)\big(a_{n+1}\nabla^2 a_{n+1} - a_n\nabla^2 a_n\big)^\wedge \\ &+ \frac{1}{\lambda^2}\bar{\bar{\bar{\Gamma}}}(\lambda x_1)\big(\nabla a_{n+1}\otimes\nabla a_{n+1} - \nabla a_n\otimes\nabla a_n\big)^\wedge\end{aligned}$$

with the goal of estimating, for all $s+t=0\dots M$, contribution of the three terms above in the quantity $\|\partial_1^{(t)}\partial_2^{(s)}\big(\mathcal{E}_{n+1}-\mathcal{E}_n\big)^\wedge\|_0$. The first term is:

$$\Big\|\partial_1^{(t)}\partial_2^{(s)}\Big(\frac{a_{n+1}-a_n}{\lambda}\Gamma(\lambda x_1)Q^\wedge\Big)\Big\|_0 \leq C\sum_{\substack{p_1+q_1+z_1=t\\ q_2+z_2=s}}\lambda^{p_1-1}\|\partial_1^{(q_1)}\partial_2^{(q_2)}(a_{n+1}-a_n)\|_0\big\|\nabla^{(z_1+z_2)}Q^\wedge\big\|_0$$

$$\leq C\mu^{\gamma/2}\big(\frac{\lambda^\gamma}{\lambda/\mu}\big)^n\sum_{\substack{p_1+q_1+z_1=t\\ q_2+z_2=s}}\lambda^{p_1-1}\lambda^{q_1}\mu^{q_2}\mu^{z_1+z_2+1}\leq C\mu^{\gamma/2}\big(\frac{\lambda^\gamma}{\lambda/\mu}\big)^n\frac{\lambda^t\mu^s}{\lambda/\mu}=C\mu^{\gamma/2}\lambda^{\gamma n}\frac{\lambda^t\mu^s}{(\lambda/\mu)^{n+1}},$$

where we used (**??**) and the bounds in (**??**) on the derivatives of H up to $M+3n+1$ and of $Q^\wedge$ up to $M+3(n-1)+1$. For the second term, we employ the identity: $a_{n+1}\nabla^2 a_{n+1}-a_n\nabla^2 a_n = (a_{n+1}-a_n)\nabla^2 a_n + a_{n+1}\nabla^2(a_{n+1}-a_n)$ and thus obtain:

$$\begin{aligned}
&\Big\|\partial_1^{(t)}\partial_2^{(s)}\Big(\frac{1}{\lambda^2}\bar\Gamma(\lambda x_1)\big(a_{n+1}\nabla^2 a_{n+1}-a_n\nabla^2 a_n\big)^\wedge\Big)\Big\|_0\\
&\leq C\sum_{\substack{p_1+q_1+z_1=t\\ q_2+z_2=s}}\lambda^{p_1-2}\Big(\|\partial_1^{(q_1)}\partial_2^{(q_2)}(a_{n+1}-a_n)\|_0\big(\|\partial_1^{(z_1+2)}\partial_2^{(z_2)}a_n\|_0+\|\partial_1^{(z_1+1)}\partial_2^{(z_2+1)}a_n\|_0\big)\\
&\qquad\qquad+\big(\|\partial_1^{(q_1+2)}\partial_2^{(q_2)}(a_{n+1}-a_n)\|_0+\|\partial_1^{(q_1+1)}\partial_2^{(q_2+1)}(a_{n+1}-a_n)\|_0\big)\|\partial_1^{(z_1)}\partial_2^{(z_2)}a_{n+1}\|_0\Big)\\
&\leq C\mu^\gamma\big(\frac{\lambda^\gamma}{\lambda/\mu}\big)^n\sum_{\substack{p_1+q_1+z_1=t\\ q_2+z_2=s}}\lambda^{p_1-2}\Big(\lambda^{q_1}\mu^{q_2}\frac{\lambda^{z_1+2}\mu^{z_2}}{\lambda/\mu}+\frac{\lambda^{q_1+2}\mu^{q_2}}{\lambda/\mu}\lambda^{z_1}\mu^{z_2}\Big)\\
&\leq C\mu^\gamma\big(\frac{\lambda^\gamma}{\lambda/\mu}\big)^n\frac{\lambda^t\mu^s}{\lambda/\mu}=C\mu^\gamma\lambda^{\gamma n}\frac{\lambda^t\mu^s}{(\lambda/\mu)^{n+1}},
\end{aligned}$$

where we used (**??**), the already established bounds **??**, **??** up to counter $n+1$, and the bounds in (**??**) on the derivatives of H up to counter $M+3n+3$ and of $Q^\wedge$ up to $M+3(n-1)+3$. For the third term, we use the identity: $\nabla a_{n+1}\otimes\nabla a_{n+1}-\nabla a_n\otimes\nabla a_n = \nabla(a_{n+1}-a_n)\otimes\nabla(a_{n+1}-a_n)+2\,\mathrm{sym}\big(\nabla(a_{n+1}-a_n)\otimes\nabla a_n\big)$ and estimate:

$$\begin{aligned}
&\Big\|\partial_1^{(t)}\partial_2^{(s)}\Big(\frac{1}{\lambda^2}\bar{\bar{\bar\Gamma}}(\lambda x_1)\big(\nabla a_{n+1}\otimes\nabla a_{n+1}-\nabla a_n\otimes\nabla a_n\big)^\wedge\Big)\Big\|_0\\
&\leq C\sum_{\substack{p_1+q_1+z_1=t\\ q_2+z_2=s}}\lambda^{p_1-2}\Big(\|\partial_1^{(q_1+1)}\partial_2^{(q_2)}(a_{n+1}-a_n)\|_0\big(\|\partial_1^{(z_1)}\partial_2^{(z_2)}\nabla(a_{n+1}-a_n)\|_0+\|\partial_1^{(z_1)}\partial_2^{(z_2)}\nabla a_n\|_0\big)\\
&\qquad\qquad+\|\partial_1^{(q_1)}\partial_2^{(q_2+1)}(a_{n+1}-a_n)\|_0\|\partial_1^{(z_1+1)}\partial_2^{(z_2)}a_n\|_0\Big)\\
&\leq C\mu^\gamma\big(\frac{\lambda^\gamma}{\lambda/\mu}\big)^n\sum_{\substack{p_1+q_1+z_1=t\\ q_2+z_2=s}}\lambda^{p_1-2}\Big(\frac{\lambda^{q_1+1}\mu^{q_2}}{\lambda/\mu}\frac{\lambda^{z_1+1}\mu^{z_2}}{\lambda/\mu}+\lambda^{q_1}\mu^{q_2+1}\frac{\lambda^{z_1+1}\mu^{z_2}}{\lambda/\mu}\Big)\\
&\leq C\mu^\gamma\big(\frac{\lambda^\gamma}{\lambda/\mu}\big)^n\frac{\lambda^t\mu^s}{(\lambda/\mu)^2}=C\mu^\gamma\lambda^{\gamma n}\frac{\lambda^t\mu^s}{(\lambda/\mu)^{n+2}},
\end{aligned}$$

where we used (**??**), the already established bounds **??**, **??**, and the bounds in (**??**) on the derivatives of H up to $M+3n+2$ and of $Q^\wedge$ up to $M+3(n-1)+2$. In conclusion:

$$\|\partial_1^{(t)}\partial_2^{(s)}\big(\mathcal{E}_{n+1}-\mathcal{E}_n\big)^\wedge\|_0\leq C\mu^\gamma\lambda^{\gamma n}\frac{\lambda^t\mu^s}{(\lambda/\mu)^{n+1}},$$

which is exactly **??** at $n+1$ and with the right dependence of constants and order of used derivatives, as claimed.

5. (The bound ??) From the definitions (**??**), (**??**) we obtain, for all $n = 1 \dots N$ in virtue of Lemma **??** (iii), (**??**) and the third assumption in (**??**):

$$\|\Psi_n\|_1 \leq C\big(\mu^\gamma + \|\chi H\|_{0,\gamma} + \|\chi \mathcal{E}^\wedge_{n-1}\|_{0,\gamma}\big) \leq C\mu^\gamma\big(1 + \frac{\lambda^\gamma}{\lambda/\mu}\big) \leq C\mu^\gamma$$

$$\|\nabla^2\Psi_n\|_0 \leq C\big(\|\nabla(\chi H)\|_{0,\gamma} + \|\nabla(\chi\mathcal{E}^\wedge_{n-1})\|_{0,\gamma}\big) \leq C\mu^\gamma\big(\mu + \lambda\frac{\lambda^\gamma}{\lambda/\mu}\big) \leq C\mu^\gamma\lambda,$$

with C depending on ω, γ, n and where we used the bounds in (**??**) on the derivatives of H up to $2+3(n-1)$ and of $Q^\wedge$ up to $2+3(n-2)$. This ends the proof of Theorem **??**. ■

We observe that Proposition **??** easily follows from Theorem **??**:

Proof of Proposition ??

Declare $a = a_N$, $\Psi = \Psi_N$. Then, **??** becomes **??**, while **??** and **??** imply **??** since $\mu \leq \lambda$. Further, **??** becomes **??**, and **??** follows from **??** and (**??**), because:

$$\mathcal{F} = a_N^2\mathrm{Id}_2 + \mathrm{sym}\nabla\Psi_N - H + \mathcal{E}^\wedge_N = \mathcal{E}^\wedge_N - \mathcal{E}^\wedge_{N-1}.$$

The proof is done. ■

4. The double step construction and its iteration

The proof of Theorem **??** relies on iterating the corrugation construction which utilizes Proposition **??** in one codimension direction e_α and augments it by cancelling the bulk of the defect $\mathcal{D}_{22}$, now accumulated in its $\mathcal{D}_{22}$ component, via the application of Lemma **??** in another codimension direction e_β. In preparation for this recursion, carried out in Proposition **??**, we first present its building block, relative to a chosen pair $\alpha \neq \beta$, and whose bounds we index using the eventual recursion counter k. The given quantities are referred to through the subscript k while the derived quantities carry the consecutive subscript $k+1$. Namely, we have:

Proposition 4.1. *Let $\omega \subset \mathbb{R}^2$ be open, bounded, smooth and let $N \geq 1$ and $\gamma \in (0,1)$. Then, there exists $l_0 \in (0,1)$ depending only on ω, and $\sigma_0 \geq 1$ depending on ω, γ, N such that the following holds. Given the positive constants δ, η, the positive frequencies:*

$$\mu_{k-1} \leq \lambda_k \leq \mu_k \leq \lambda_{k+1} \leq \mu_{k+1},$$

the positive auxiliary constants $\tilde{C}_k, A_k, B_k$, and an integer M, satisfying:

$$\begin{aligned} &\delta, \eta \leq 2l_0, \qquad \mu_k \geq \frac{1}{\eta}, \qquad \lambda_{k+1}^{1-\gamma} \geq \mu_k\sigma_0, \qquad M \geq 0, \\ &\frac{\mu_k}{\mu_{k-1}} \geq \big(\frac{A_k}{\tilde{C}_k}\big)^{1/2}, \qquad \frac{\mu_{k+1}}{\lambda_{k+1}} \geq \max\Big\{\big(\frac{\lambda_{k+1}}{\mu_k}\big)^{(N-1)/2}, \big(\frac{\lambda_{k+1}}{\mu_k}\big)^{N-1}\frac{(B_k/\tilde{C}_k)^{1/2}}{\mu_k/\lambda_k}\Big\}, \end{aligned} \tag{4.1}$$

and given $v_k \in \mathcal{C}^\infty(\bar\omega + \bar{B}_{\delta+\eta}(0), \mathbb{R}^3)$, $w_k \in \mathcal{C}^\infty(\bar\omega + \bar{B}_{\delta+\eta}(0), \mathbb{R}^2)$, $A_0 \in \mathcal{C}^\infty(\bar\omega + \bar{B}_{\delta+\eta}(0), \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, such that together with the derived field $\mathcal{D}_k = A_0 - \big(\frac{1}{2}(\nabla v_k)^T\nabla v_k + \mathrm{sym}\nabla w_k\big)$, the following bounds hold relative to the two chosen components $\alpha \neq \beta \in \{1,2,3\}$:

$$\begin{aligned} &\|\nabla^{(m)}\mathcal{D}_k\|_0 \leq \tilde{C}_k\mu_k^m && \textit{for all } m = 0\dots M+3(N+1), \\ &\|\nabla^{(m+2)}v_k^\alpha\|_0 \leq A_k^{1/2}\mu_{k-1}^{m+1} && \textit{for all } m = 0\dots M+3N, \\ &\|\nabla^{(m+2)}v_k^\beta\|_0 \leq B_k^{1/2}\lambda_k^{m+1} && \textit{for all } m = 0\dots M+1, \end{aligned} \tag{4.2}$$

there exist $v_{k+1} \in \mathcal{C}^\infty(\bar\omega + \bar B_\delta(0), \mathbb{R}^3)$ *and* $w_{k+1} \in \mathcal{C}^\infty(\bar\omega + \bar B_\delta(0), \mathbb{R}^2)$*, such that denoting the new derived field* $\mathcal{D}_{k+1} = A_0 - \big(\frac{1}{2}(\nabla v_{k+1})^T\nabla v_{k+1} + \mathrm{sym}\nabla w_{k+1}\big)$*, there hold the estimates:*

$$\|(v_{k+1}-v_k)^\alpha\|_1 + \|(v_{k+1}-v_k)^\beta\|_1 \le C\tilde C_k^{1/2}\mu_k^{\gamma/2}, \tag{4.3$_1$}$$

$$\left.\begin{aligned}&\|\nabla^{(m+2)}v^\alpha_{k+1}\|_0 \le C\tilde C_k^{1/2}\mu_k^{\gamma/2}\lambda_{k+1}^{m+1}\\ &\|\nabla^{(m+2)}v^\beta_{k+1}\|_0 \le C\tilde C_k^{1/2}\mu_k^{\gamma/2}\mu_{k+1}^{m+1}\end{aligned}\right\} \quad \textit{for all } \; m = 0\dots M, \tag{4.3$_3$}$$

$$\begin{aligned}&\|w_{k+1}-w_k\|_1 \le C\tilde C_k^{1/2}\mu_k^{\gamma/2}\big(\|\nabla v_k^\alpha\|_0 + \|\nabla v_k^\beta\|_0 + \tilde C_k^{1/2}\mu_k^{\gamma/2}\big),\\ &\|\nabla^2(w_{k+1}-w_k)\|_0 \le C\tilde C_k^{1/2}\mu_k^{\gamma/2}\big(\|\nabla v_k^\alpha\|_0 + \|\nabla v_k^\beta\|_0 + \tilde C_k^{1/2}\mu_k^{\gamma/2}\big)\mu_{k+1},\end{aligned} \tag{4.3$_2$}$$

$$\|\nabla^{(m)}\mathcal{D}_{k+1}\|_0 \le C\tilde C_k\mu_k^\gamma\lambda_{k+1}^{\gamma N}\frac{\mu_{k+1}^m}{(\lambda_{k+1}/\mu_k)^N} \qquad \textit{for all } \; m = 0\dots M. \tag{4.3$_4$}$$

Above, constants C depend: in **??**, **??** *on* ω, γ, N*; in* **??**, **??** *on* ω, γ, N, M.

Proof. **1. (Applying Proposition ??)** We apply Proposition **??** to the same ω, γ, N, l_0, σ_0, δ, η, M, N as in there, and with:

$$\mu = \mu_k, \quad \lambda = \lambda_{k+1}, \quad H = \frac{1}{\tilde C_k}\mathcal{D}_k, \quad Q = \frac{1}{\tilde C_k^{1/2}}\nabla^2 v_k^\alpha,$$

upon validating conditions (**??**) in view of (**??**) and (**??**):

$$\begin{aligned}&\|\nabla^{(m)}H\|_0 \le \mu_k^m && \text{for all } \; m = 0\dots M+3(N+1),\\ &\|\nabla^{(m)}Q\|_0 \le \big(\frac{A_k}{\tilde C_k}\big)^{1/2}\frac{\mu_{k-1}}{\mu_k}\cdot\mu_k^{m+1} \le \mu_k^{m+1} && \text{for all } \; m = 0\dots M+3N.\end{aligned}$$

Having thus obtained the fields $a, \Psi, \mathcal{F}$ with properties **??** – **??**, we define $a_{k+1} \in \mathcal{C}^\infty(\bar\omega + \bar B_\delta(0), \mathbb{R})$, $\Psi_{k+1} \in \mathcal{C}^\infty(\bar\omega + \bar B_\delta(0), \mathbb{R}^2)$, $\mathcal{F}_{k+1} \in \mathcal{C}^\infty(\bar\omega + \bar B_\delta(0), \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ by:

$$a_{k+1} = \tilde C_k^{1/2} a, \qquad \Psi_{k+1} = \tilde C_k\Psi, \qquad \mathcal{F}_{k+1} = \tilde C_k\mathcal{F},$$

so that:

$$\begin{aligned}\mathcal{F}_{k+1} &= a_{k+1}^2\mathrm{Id}_2 + \mathrm{sym}\nabla\Psi_{k+1} - \mathcal{D}_k\\ &+ \Big(-\frac{a_{k+1}}{\lambda_{k+1}}\Gamma(\lambda_{k+1}x_1)\nabla^2 v_k^\alpha + \frac{a_{k+1}}{\lambda_{k+1}^2}\bar\Gamma(\lambda_{k+1}x_1)\nabla^2 a_{k+1} + \frac{1}{\lambda_{k+1}^2}\bar{\bar{\bar\Gamma}}(\lambda_{k+1}x_1)(\nabla a_{k+1})^{\otimes 2}\Big)^\wedge.\end{aligned} \tag{4.4}$$

and the following bounds hold:

$$\frac{\tilde C\tilde C_k}{2}\mu_k^\gamma \le a_{k+1}^2 \le \frac{3\tilde C\tilde C_k}{2}\mu_k^\gamma \quad \text{and} \quad \frac{\tilde C^{1/2}\tilde C_k^{1/2}}{2}\mu_k^{\gamma/2} \le a_{k+1} \le \frac{3\tilde C^{1/2}\tilde C_k^{1/2}}{2}\mu_k^{\gamma/2}, \tag{4.5$_1$}$$

$$\|\nabla^{(m)}a_{k+1}\|_0 \le C\tilde C_k^{1/2}\mu_k^{\gamma/2}\frac{\lambda_{k+1}^m}{\lambda_{k+1}/\mu_k} \quad \text{for all } \; m = 1\dots M+5, \tag{4.5$_2$}$$

$$\|\Psi_{k+1}\|_1 \le C\tilde C_k\mu_k^\gamma \quad \text{and} \quad \|\nabla^2\Psi_{k+1}\|_0 \le C\tilde C_k\mu_k^\gamma\lambda_{k+1}, \tag{4.5$_3$}$$

$$\|\nabla^{(m)}\mathcal{F}_{k+1}\|_0 \le C\tilde C_k\mu_k^\gamma\lambda_{k+1}^{\gamma N}\frac{\lambda_{k+1}^m}{(\lambda_{k+1}/\mu_k)^N} \quad \text{for all } \; m = 0\dots M. \tag{4.5$_4$}$$

where the constant $\tilde C$ depends only on ω, γ, while the constants C depend on: in **??** on ω, γ, N; in **??**, **??** on ω, γ, N, M.

2. (Adding the first corrugation) We define the intermediate fields $\tilde v \in \mathcal{C}^\infty(\bar\omega + \bar B_\delta(0), \mathbb{R}^3)$, $\tilde w \in \mathcal{C}^\infty(\bar\omega + \bar B_\delta(0), \mathbb{R}^2)$ by setting, in accordance with Lemma **??** for $i = 1, j = \alpha$:

$$\begin{aligned}
&\tilde v = v_k + \frac{a_{k+1}}{\lambda_{k+1}}\Gamma(\lambda_{k+1}x_1)e_\alpha,\\
&\tilde w = w_k - \frac{a_{k+1}}{\lambda_{k+1}}\Gamma(\lambda_{k+1}x_1)\nabla v_k^\alpha + \frac{a_{k+1}}{\lambda_{k+1}^2}\bar\Gamma(\lambda_{k+1}x_1)\nabla a_{k+1} + \frac{a_{k+1}^2}{\lambda_{k+1}}\bar{\bar\Gamma}(\lambda_{k+1}x_1)e_1 + \Psi_{k+1}.
\end{aligned}$$

By **??**, **??**, we directly get for all $m = 0\dots M$:

$$\begin{aligned}
&\|(\tilde v - v_k)^\alpha\|_1 \le C\big(\|a_{k+1}\|_0 + \frac{\|\nabla a_{k+1}\|_0}{\lambda_{k+1}}\big) \le C\tilde C_k^{1/2}\mu_k^{\gamma/2}\\
&\|\nabla^{(m+2)}(\tilde v - v_k)^\alpha\|_0 \le C\sum_{p+q=m+2}\lambda_{k+1}^{p-1}\|\nabla^{(q)}a_{k+1}\|_0 \le C\tilde C_k^{1/2}\mu_k^{\gamma/2}\lambda_{k+1}^{m+1},
\end{aligned}\tag{4.6}$$

where C in the first bound depends on ω, γ, N, while in the second bound on ω, γ, N, M. Combining that last bound with (**??**) yields:

$$\|\nabla^{(m+2)}\tilde v^\alpha\|_0 \le C\tilde C_k^{1/2}\mu_k^{\gamma/2}\lambda_{k+1}^{m+1} + A_k^{1/2}\mu_{k-1}^{m+1} \le C\tilde C_k^{1/2}\mu_k^{\gamma/2}\lambda_{k+1}^{m+1}.\tag{4.7}$$

Similarly, **??** – **??** and (**??**) result in the estimates:

$$\begin{aligned}
\|\tilde w - w_k\|_1 \le C\Big(&\tilde C_k\mu_k^\gamma + \|a_{k+1}\|_0\|\nabla v_k^\alpha\|_0 + \|a_{k+1}\|_0^2\\
&+ \frac{\|\nabla a_{k+1}\|_0\|\nabla v_k^\alpha\|_0 + \|a_{k+1}\|_0\|\nabla^2 v_k^\alpha\|_0 + \|a_{k+1}\|_0\|\nabla a_{k+1}\|_0}{\lambda_{k+1}}\\
&+ \frac{\|\nabla a_{k+1}\|_0\|\nabla^2 a_{k+1}\|_0 + \|\nabla a_{k+1}\|_0^2}{\lambda_{k+1}^2}\Big)\\
\le\ & C\tilde C_k^{1/2}\mu_k^{\gamma/2}\big(\|\nabla v_k^\alpha\|_0 + \tilde C_k^{1/2}\mu_k^{\gamma/2}\big),\\
\|\nabla^2(\tilde w - \bar w_k)\|_0 \le C\Big(&\tilde C_k\mu_k^\gamma\lambda_{k+1} + \lambda_{k+1}\big(\|a_{k+1}\|_0\|\nabla v_k^\alpha\|_0 + \|a_{k+1}\|_0^2\big)\\
&+ \big(\|\nabla a_{k+1}\|_0\|\nabla v_k^\alpha\|_0 + \|a_{k+1}\|_0\|\nabla^2 v_k^\alpha\|_0 + \|a_{k+1}\|_0\|\nabla a_{k+1}\|_0\big)\\
&+ \frac{\|\nabla^2 a_{k+1}\|_0\|\nabla v_k^\alpha\|_0 + \|\nabla a_{k+1}\|_0\|\nabla^2 v_k^\alpha\|_0 + \|a_{k+1}\|_0\|\nabla^3 v_k^\alpha\|_0}{\lambda_{k+1}}\\
&+ \frac{\|a_{k+1}\|_0\|\nabla^2 a_{k+1}\|_0 + \|\nabla a_{k+1}\|_0^2}{\lambda_{k+1}}\\
&+ \frac{\|a_{k+1}\|_0\|\nabla^3 a_{k+1}\|_0 + \|\nabla a_{k+1}\|_0\|\nabla^2 a_{k+1}\|_0}{\lambda_{k+1}^2}\Big)\\
\le\ & C\tilde C_k^{1/2}\mu_k^{\gamma/2}\big(\|\nabla v_k^\alpha\|_0 + \tilde C_k^{1/2}\mu_k^{\gamma/2}\big)\lambda_{k+1},
\end{aligned}\tag{4.8}$$

with constants C depending on ω, γ, N. Finally, we note that (??) and (??) yield:

$$
\begin{aligned}
\tilde{\mathcal{D}} &\doteq A_0 - \big(\frac{1}{2}(\nabla\tilde{v})^T\nabla\tilde{v} + \mathrm{sym}\nabla\tilde{w}\big)\\
&= \mathcal{D}_k - a_{k+1}^2 e_1\otimes e_1 - \mathrm{sym}\nabla\Psi_{k+1}\\
&\quad - \Big(- \frac{a_{k+1}}{\lambda_{k+1}}\Gamma(\lambda_{k+1}x_1)\nabla^2 v_k^\alpha + \frac{a_{k+1}}{\lambda_{k+1}^2}\bar{\Gamma}(\lambda_{k+1}x_1)\nabla^2 a_{k+1} + \frac{1}{\lambda_{k+1}^2}\bar{\bar{\bar{\Gamma}}}(\lambda_{k+1}x_1)(\nabla a_{k+1})^{\otimes 2}\Big)\\
&= -\mathcal{F}_{k+1} + \Big(a_{k+1}^2 + \frac{a_{k+1}}{\lambda_{k+1}}\Gamma(\lambda_{k+1}x_1)\partial_{22}v_k^\alpha\\
&\qquad - \frac{a_{k+1}}{\lambda_{k+1}^2}\bar{\Gamma}(\lambda_{k+1}x_1)\partial_{22}a_{k+1} - \frac{1}{\lambda_{k+1}^2}\bar{\bar{\bar{\Gamma}}}(\lambda_{k+1}x_1)(\partial_2 a_{k+1})^2\Big)e_2\otimes e_2.
\end{aligned}
\tag{4.9}
$$

3. (Adding the second corrugation) We define the final fields $v_{k+1}\in\mathcal{C}^\infty(\bar{\omega}+\bar{B}_\delta(0),\mathbb{R}^3)$, $w_{k+1}\in\mathcal{C}^\infty(\bar{\omega}+\bar{B}_\delta(0),\mathbb{R}^2)$ by using Lemma **??** (with $i=2, j=\beta$) and the perturbation amplitude dictated by (??), namely:

$$
\begin{aligned}
& v_{k+1} = \tilde{v} + \frac{b_{k+1}}{\mu_{k+1}}\Gamma(\mu_{k+1}x_2)e_\beta,\\
& w_{k+1} = \tilde{w} - \frac{b_{k+1}}{\mu_{k+1}}\Gamma(\mu_{k+1}x_2)\nabla v_k^\beta + \frac{b_{k+1}}{\mu_{k+1}}\bar{\Gamma}(\mu_{k+1}x_2)\nabla b_{k+1} + \frac{b_{k+1}^2}{\mu_{k+1}}\bar{\bar{\bar{\Gamma}}}(\mu_{k+1}x_2)e_2,\\
& \text{where }\ b_{k+1}^2 = a_{k+1}^2 + \frac{a_{k+1}}{\lambda_{k+1}}\Gamma(\lambda_{k+1}x_1)\partial_{22}v_k^\alpha\\
&\qquad\qquad - \frac{a_{k+1}}{\lambda_{k+1}^2}\bar{\Gamma}(\lambda_{k+1}x_1)\partial_{22}a_{k+1} - \frac{1}{\lambda_{k+1}^2}\bar{\bar{\bar{\Gamma}}}(\lambda_{k+1}x_1)(\partial_2 a_{k+1})^2.
\end{aligned}
\tag{4.10}
$$

Firstly, we argue that $b_{k+1}\in\mathcal{C}^\infty(\bar{\omega}+\bar{B}_\delta(0))$ is well defined and it satisfies:

$$
\frac{\tilde{C}\tilde{C}_k}{4}\mu_k^\gamma \le b_{k+1}^2 \le 2\tilde{C}\tilde{C}_k\mu_k^\gamma \quad \text{and so} \quad \|b_{k+1}\| \le C\tilde{C}_k^{1/2}\mu_k^{\gamma/2}, \tag{4.11$_1$}
$$

$$
\|\nabla^{(m)}b_{k+1}\|_0 \le C\tilde{C}_k^{1/2}\mu_k^{\gamma/2}\frac{\lambda_{k+1}^m}{\lambda_{k+1}/\mu_k} \quad \text{for all }\ m=1\dots M+3, \tag{4.11$_2$}
$$

where C in **??** depends on ω,γ,N, and in **??** on ω,γ,N,M. Indeed by **??**, **??**, (??), and (??) we get, with C depending on ω,γ,N:

$$
\|b_{k+1}^2 - a_{k+1}^2\|_0 \le C\big(\tilde{C}_k^{1/2}A_k^{1/2}\mu_k^{\gamma/2}\frac{\mu_{k-1}}{\lambda_{k+1}} + \tilde{C}_k\mu_k^\gamma\frac{\mu_k}{\lambda_{k+1}}\big) \le C\frac{\tilde{C}_k\mu_k^\gamma}{\lambda_{k+1}/\mu_k} \le \frac{\tilde{C}\tilde{C}_k}{4}\mu_k^\gamma
$$

where the last estimate follows by taking λ_{k+1}/μ_k sufficiently large in function of ω,γ,N, which in turn follows from the third assumption in (??) if σ_0 has been assigned large enough. Combined with **??**, the above yields **??**. Towards **??**, we similarly get, for all $m=1\dots M+3$ and C depending on ω,γ,N,M:

$$
\begin{aligned}
\|\nabla^{(m)}(b_{k+1}^2 - a_{k+1}^2)\|_0 &\le C\Big(\sum_{p+q+z=m}\lambda_{k+1}^{p-1}\|\nabla^{(q)}a_{k+1}\|_0\|\nabla^{(z+2)}v_k^\alpha\|_0\\
&\qquad + \sum_{p+q+z=m}\lambda_{k+1}^{p-2}\big(\|\nabla^{(q)}a_{k+1}\|_0\|\nabla^{(z+2)}a_{k+1}\|_0 + \|\nabla^{(q+1)}a_{k+1}\|_0\|\nabla^{(z+1)}a_{k+1}\|_0\big)\Big)\\
&\le C\lambda_{k+1}^m\Big(\mu_k^{\gamma/2}\frac{\tilde{C}_k^{1/2}A_k^{1/2}}{\lambda_{k+1}/\mu_{k-1}} + \tilde{C}_k\frac{\mu_k^\gamma}{\lambda_{k+1}/\mu_k}\Big) \le C\tilde{C}_k\mu_k^\gamma\frac{\lambda_{k+1}^m}{\lambda_{k+1}/\mu_k}.
\end{aligned}
$$

Since the same bound is enjoyed by $\nabla^{(m)}a_{k+1}^2$, it follows that:

$$\|\nabla^{(m)}b_{k+1}^2\|_0 \leq C\tilde{C}_k\mu_k^\gamma\frac{\lambda_{k+1}^m}{\lambda_{k+1}/\mu_k}.$$

In view of **??** and via the application of Faa di Bruno's formula we conclude **??**:

$$\begin{aligned}\|\nabla^{(m)}b_{k+1}\|_0 &\leq C\Big\|\sum_{p_1+2p_2+\ldots mp_m=m} b_{k+1}^{2(1/2-p_1-\ldots-p_m)}\prod_{z=1}^m\big|\nabla^{(z)}b_{k+1}^2\big|^{p_z}\Big\|_0\\ &\leq C\|b_{k+1}\|_0\sum_{p_1+2p_2+\ldots mp_s=m}\prod_{z=1}^m\Big(\frac{\|\nabla^{(z)}b_{k+1}^2\|_0}{\tilde{C}\tilde{C}_k\mu^\gamma}\Big)^{p_z}\leq C\tilde{C}_k^{1/2}\mu_k^{\gamma/2}\frac{\lambda_{k+1}^m}{\lambda_{k+1}/\mu_k}.\end{aligned}$$

We are now ready to carry out the bounds similar to those done with the first corrugation in step 2. By **??**, **??** we have for all $m=0\ldots M$:

$$\begin{aligned}&\|(\tilde{v}-v_{k+1})^\beta\|_1\leq C\big(\|b_{k+1}\|_0+\frac{\|\nabla b_{k+1}\|_0}{\mu_{k+1}}\big)\leq C\tilde{C}_k^{1/2}\mu_k^{\gamma/2}\\ &\|\nabla^{(m+2)}(\tilde{v}-v_{k+1})^\beta\|_0\leq C\sum_{p+q=m+2}\mu_{k+1}^{p-1}\|\nabla^{(q)}b_{k+1}\|_0\leq C\tilde{C}_k^{1/2}\mu_k^{\gamma/2}\mu_{k+1}^{m+1},\end{aligned}\tag{4.12}$$

where C in the first bound depends on ω,γ,N, while in the second bound on ω,γ,N,M. At this point, recalling (**??**) we conclude **??**, because $v_{k+1}^\alpha=\tilde{v}^\alpha$ and $v_k^\beta=\tilde{v}^\beta$. We likewise get **??**, recalling (**??**) and deducing from (**??**) and (**??**) that:

$$\|\nabla^{(m+2)}v_{k+1}^\beta\|_0\leq C\tilde{C}_k^{1/2}\mu_k^{\gamma/2}\mu_{k+1}^{m+1}+B_k^{1/2}\lambda_k^{m+1}\leq C\tilde{C}_k^{1/2}\mu_k^{\gamma/2}\mu_{k+1}^{m+1},$$

where in the last bound we used (**??**) to observe that:

$$B_k^{1/2}\lambda_k\leq\tilde{C}_k^{1/2}\mu_k\frac{\mu_{k+1}}{\lambda_{k+1}}\leq\tilde{C}_k^{1/2}\mu_{k+1}.$$

Finally, we observe that the formula for $w_{k+1}-\tilde{w}$ is exactly the same as that for $w_k-\tilde{w}$, where a_{k+1} is exchanged for b_{k+1}, λ_{k+1} for μ_{k+1} and ∇v_k^α to ∇v_k^β, and minus the term Ψ_{k+1}. Hence, the same estimate as in (**??**) now yields, in view of **??**, **??**, (**??**):

$$\begin{aligned}&\|\tilde{w}-w_{k+1}\|_1\leq C\tilde{C}_k^{1/2}\mu_k^{\gamma/2}\big(\|\nabla v_k^\beta\|_0+\tilde{C}_k^{1/2}\mu_k^{\gamma/2}\big),\\ &\|\nabla^2(\tilde{w}-\bar{w}_{k+1})\|_0\leq C\tilde{C}_k^{1/2}\mu_k^{\gamma/2}\big(\|\nabla v_k^\beta\|_0+\tilde{C}_k^{1/2}\mu_k^{\gamma/2}\big)\mu_{k+1},\end{aligned}$$

with constants C depending on ω,γ,N. Recalling (**??**), the above yields **??**, where we note that we necessitated bounds on $\|v_k^\beta\|_3$ and $\|b_{k+1}\|_3$.

4. (The bound on the derived deficit) It remains to show **??**. To this end, we use Lemma **??** and recall that $\tilde{\mathcal{D}}=-\mathcal{F}_{k+1}+b_{k+1}^2e_2\otimes e_2$ from (**??**), to write:

$$\begin{aligned}&\mathcal{D}_{k+1}=\tilde{\mathcal{D}}-b_{k+1}^2e_2\otimes e_2-\mathcal{S}_{k+1}=-\mathcal{F}_{k+1}-\mathcal{S}_{k+1},\\ &\text{where }\ \mathcal{S}_{k+1}=-\frac{b_{k+1}}{\mu_{k+1}}\Gamma(\mu_{k+1}x_2)\nabla^2v_k^\beta+\frac{b_{k+1}}{\mu_{k+1}^2}\bar{\Gamma}(\mu_{k+1}x_2)\nabla^2b_{k+1}+\frac{1}{\mu_{k+1}^2}\bar{\bar{\Gamma}}(\mu_{k+1}x_2)(\nabla b_{k+1})^{\otimes 2}.\end{aligned}$$

We now use (**??**) and **??**, **??** to get, for all $m = 0 \dots M$:

$$\begin{aligned} \|\nabla^{(m)}\mathcal{S}_{k+1}\|_0 \leq C\Big(& \sum_{p+q+z=m} \mu_{k+1}^{p-1}\|\nabla^{(q)}b_{k+1}\|_0\|\nabla^{(z+2)}v_k^\beta\|_0 \\ & + \sum_{p+q+z=m} \mu_{k+1}^{p-2}\big(\|\nabla^{(q)}b_{k+1}\|_0\|\nabla^{(z+2)}b_{k+1}\|_0 + \|\nabla^{(q+1)}b_{k+1}\|_0\|\nabla^{(z+1)}b_{k+1}\|_0\big)\Big) \\ \leq C \sum_{p+q+z=m} & \Big(\mu_{k+1}^{p-1}\tilde{C}_k^{1/2}\mu_k^{\gamma/2}\lambda_{k+1}^q B_k^{1/2}\lambda_k^{z+1} + \mu_{k+1}^{p-2}\tilde{C}_k\mu_k^\gamma \frac{\lambda_{k+1}^{q+z+2}}{\lambda_{k+1}/\mu_k}\Big) \\ \leq C\mu_{k+1}^m & \Big(\mu_k^{\gamma/2}\frac{\tilde{C}_k^{1/2}B_k^{1/2}}{\mu_{k+1}/\lambda_k} + \tilde{C}_k\mu_k^\gamma \frac{1}{(\mu_{k+1}/\lambda_{k+1})^2(\lambda_{k+1}/\mu_k)}\Big) \leq C\tilde{C}_k\mu_k^\gamma \frac{\mu_{k+1}^m}{(\lambda_{k+1}/\mu_k)^N}, \end{aligned} \tag{4.13}$$

where the last bound follows from the last assumption in (**??**) because:

$$\begin{aligned} & \frac{B_k^{1/2}}{\mu_{k+1}/\lambda_k} \leq \tilde{C}_k^{1/2}\frac{\mu_k}{\lambda_k}\frac{\mu_{k+1}}{\lambda_{k+1}}\frac{1}{(\lambda_{k+1}/\mu_k)^{N-1}} \leq \frac{1}{(\lambda_{k+1}/\mu_k)^N} \\ \text{and} \quad & \big(\frac{\mu_{k+1}}{\lambda_{k+1}}\big)^2\frac{\lambda_{k+1}}{\mu_k} \geq \big(\frac{\lambda_{k+1}}{\mu_k}\big)^{N-1}\frac{\lambda_{k+1}}{\mu_k} = \big(\frac{\lambda_{k+1}}{\mu_k}\big)^N. \end{aligned}$$

Recalling **??**, the bound in (**??**) yields:

$$\|\nabla^{(m)}\mathcal{D}_{k+1}\|_0 \leq C\tilde{C}_k\mu_k^\gamma\lambda_{k+1}^{\gamma N}\frac{\mu_{k+1}^m}{(\lambda_{k+1}/\mu_k)^N}.$$

for all $m = 0 \dots M$ and with C depending on ω, γ, N, M. We note that for carrying out the whole argument, we necessitated bounds on a_{k+1} and v_k^α up to $\max\{M+4, 5\}$ derivatives, on v_k^β up to $\max\{M+2, 3\}$ derivatives, and on $\mathcal{F}_{k+1}$ up to M derivatives. This justifies the derivative count in (**??**). The proof is done. ∎

We are now ready to present the main result of this section, in which Proposition **??** is iterated with the consecutive choice of α_k, β_k in the mod 3 arithmetic. This construction necessitates specific assumptions on the ratios of the employed frequencies. The viability of these assumptions and existence of frequencies satisfying them will be shown in the next section.

Proposition 4.2. *Let $\omega \subset \mathbb{R}^2$ be open, bounded, smooth and let $N, K \geq 1$ and $\gamma \in (0,1)$. Then, there exists $l_0 \in (0,1)$ depending only on ω, and $\sigma_0 \geq 1$ depending on ω, γ, N, K such that the following holds. Given the positive constants l, η, μ_0 such that:*

$$l + K\eta \leq 2l_0, \qquad \mu_0 \geq \frac{1}{\eta}, \tag{4.14}$$

and given $v_0 \in \mathcal{C}^\infty(\bar{\omega}+\bar{B}_{l+K\eta}(0), \mathbb{R}^3)$, $w_0 \in \mathcal{C}^\infty(\bar{\omega}+\bar{B}_{l+K\eta}(0), \mathbb{R}^2)$, $A_0 \in \mathcal{C}^\infty(\bar{\omega}+\bar{B}_{l+K\eta}(0), \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, such that together with the derived field $\mathcal{D}_0 = A_0 - \big(\frac{1}{2}(\nabla v_0)^T\nabla v_0 + \mathrm{sym}\nabla w_0\big)$, the following bounds hold with some auxiliary constant $\tilde{C}_0 > 0$:

$$\left.\begin{aligned} & \|\nabla^{(m)}\mathcal{D}_0\|_0 \leq \tilde{C}_0\mu_0^m \\ & \|\nabla^{(m+2)}v_0\|_0 \leq \tilde{C}_0^{1/2}\mu_0^{m+1} \end{aligned}\right. \qquad \textit{for all } \ m = 0 \dots 3K(N+1), \tag{4.15}$$

*one can apply Proposition **??** consecutively for $k = 0 \dots K-1$ with:*

$$\alpha_k = (2k) \bmod 3 + 1, \qquad \beta_k = (2k+1) \bmod 3 + 1 \tag{4.16}$$

and with the frequencies $\{\lambda_k, \mu_k\}_{k=1}^K$ *as long as the following conditions are satisfied:*

$$\begin{aligned}
&\lambda_{k+1}^{1-\gamma} \geq \mu_k \sigma_0 \quad \textit{for all } \ k = 0 \dots K-1, \qquad \frac{\mu_1}{\lambda_1} \geq \big(\frac{\lambda_1}{\mu_0}\big)^N, \\
&\frac{\mu_k}{\lambda_k} \geq \max\Big\{ \big(\frac{\lambda_k}{\mu_{k-1}} \frac{\lambda_{k-1}}{\mu_{k-2}}\big)^{N/2}, \frac{\big(\lambda_k/\mu_{k-1}\big)^N (\lambda_{k-1}/\mu_{k-2})^{N/2}}{\mu_{k-1}/\lambda_{k-1}} \Big\} \quad \textit{for all } \ k = 2 \dots K-1, \qquad (4.17)\\
&\frac{\mu_K}{\lambda_K} \geq \max\Big\{ \big(\frac{\lambda_K}{\mu_{K-1}}\big)^{N/2}, \frac{\big(\lambda_K/\mu_{K-1}\big)^N (\lambda_{K-1}/\mu_{K-2})^{N/2}}{\mu_{K-1}/\lambda_{K-1}} \Big\},
\end{aligned}$$

to obtain $\tilde v \in \mathcal{C}^\infty(\bar\omega + \bar B_l(0), \mathbb{R}^3)$ *and* $\tilde w \in \mathcal{C}^\infty(\bar\omega + \bar B_l(0), \mathbb{R}^2)$*, such that denoting the new derived field* $\tilde{\mathcal{D}} = A_0 - \big(\frac{1}{2}(\nabla \tilde v)^T \nabla \tilde v + \mathrm{sym} \nabla \tilde w\big)$*, the following bounds hold:*

$$\|\tilde v - v_0\|_1 \leq C \tilde C_0^{1/2} \Lambda^{\gamma/2}, \qquad (4.18)_1$$

$$\|\nabla^2 \tilde v\|_0 \leq C \tilde C_0^{1/2} \Lambda^{\gamma/2} \mu_K \Big(\prod_{k=0}^{K-2} \frac{1}{(\lambda_{k+1}/\mu_k)^{N/2}} \Big) \Big(1 + \frac{(\lambda_{K-1}/\mu_{K-2})^{N/2}}{\mu_K/\mu_{K-1}} \Big), \qquad (4.18)_2$$

$$\begin{aligned}
&\|\tilde w - w_0\|_1 \leq C \tilde C_0^{1/2} \Lambda^\gamma \big(\|\nabla v_0\|_0 + \tilde C_0^{1/2} \big) \\
&\|\nabla^2 (\tilde w - w_0)\|_0 \leq C \tilde C_0^{1/2} \Lambda^\gamma \mu_K \Big(\prod_{k=0}^{K-2} \frac{1}{(\lambda_{k+1}/\mu_k)^{N/2}} \Big) \times \\
&\qquad\qquad \times \Big(1 + \frac{(\lambda_{K-1}/\mu_{K-2})^{N/2}}{\mu_K/\mu_{K-1}} \Big) \big(\|\nabla v_0\|_0 + \tilde C_0^{1/2} \big),
\end{aligned} \qquad (4.18)_3$$

$$\|\tilde{\mathcal{D}}\|_0 \leq C \tilde C_0 \Lambda^\gamma \prod_{k=0}^{K-1} \frac{1}{(\lambda_{k+1}/\mu_k)^N}, \qquad (4.18)_4$$

Above, constants C *depend on:* ω, γ, N, K*, and we denoted:* $\Lambda = \prod_{k=0}^K (\mu_k \lambda_k^N)$.

Proof. **1. (Setting the inductive quantities)** For $k = 0 \dots K-1$ we will define the fields:

$$\begin{aligned}
&v_{k+1} \in \mathcal{C}^\infty(\bar\omega + \bar B_{l+(K-(k+1))\eta}, \mathbb{R}^3), \quad w_{k+1} \in \mathcal{C}^\infty(\bar\omega + \bar B_{l+(K-(k+1))\eta}, \mathbb{R}^2), \\
&\mathcal{D}_{k+1} = A_0 - \big(\frac{1}{2} (\nabla v_{k+1})^T \nabla v_{k+1} + \mathrm{sym} \nabla w_{k+1} \big),
\end{aligned}$$

by applying Proposition **??** to the previous fields $v_k \in \mathcal{C}^\infty(\bar\omega + \bar B_{l+(K-k)\eta}, \mathbb{R}^3)$, $w_k \in \mathcal{C}^\infty(\bar\omega + \bar B_{l+(K-k)\eta}, \mathbb{R}^2)$ with the fixed parameters N, γ, the given frequencies $\mu_{k-1} \leq \lambda_k \leq \mu_k \leq \lambda_{k+1} \leq \mu_{k+1}$ the inductive parameters δ_k, η_k satisfying $\delta_k + \eta_k \leq 2 l_0$ and $M_k \geq 0$ where:

$$\mu_{-1} = \lambda_0 = \mu_0, \qquad \delta_k = l + (K - (k+1))\eta, \quad \eta_k = \eta, \qquad M_k = 3(K - (k+1))(N+1),$$

and the auxiliary constants $\tilde C_k, A_k, B_k$ defined as follows:

$$\begin{aligned}
&A_1 = A_0 = B_0 = \tilde C_0, \\
&\tilde C_{k+1} = C \tilde C_k \frac{\mu_k^\gamma \lambda_{k+1}^{\gamma N}}{(\lambda_{k+1}/\mu_k)^N} \qquad \text{for all } \ k = 0 \dots K-1, \\
&A_k = C \tilde C_{k-2} \mu_{k-2}^\gamma \qquad \text{for all } \ k = 2 \dots K+1, \\
&B_k = C \tilde C_{k-1} \mu_{k-1}^\gamma \qquad \text{for all } \ k = 1 \dots K,
\end{aligned} \qquad (4.19)$$

where constants C depend on ω, γ, N, K, k. The codimension components α_k, β_k are as in (**??**) and we call $\gamma_k \in \{1,2,3\} \setminus \{\alpha_k, \beta_k\}$ the remaining component:

$$\gamma_k = \alpha_{k+1} = (2k+2) \bmod 3 + 1.$$

The applicability of Proposition **??** relies on the following bounds, that will be validated throughout the proof for all $k = 0 \dots K$:

$$\begin{aligned} &\|\nabla^{(m)}\mathcal{D}_k\|_0 \leq \tilde{C}_k \mu_k^m, && \|\nabla^{(m+2)} v_k^{\alpha_k}\|_0 \leq A_k^{1/2} \mu_{k-1}^{m+1}, \\ &\|\nabla^{(m+2)} v_k^{\beta_k}\|_0 \leq B_k^{1/2} \lambda_k^{m+1}, && \|\nabla^{(m+2)} v_k^{\gamma_k}\|_0 \leq A_{k+1}^{1/2} \mu_k^{m+1}. \end{aligned} \tag{4.20}$$

Eventually, we will set $\tilde{v} = v_K$ and $\tilde{w} = w_K$ and deduce the bounds **??** – **??**.

2. (Induction base $k = 0$ and $k = 1$) We have $\alpha_0 = 1$, $\beta_0 = 2$, $\gamma_0 = 3$ and so (**??**) holds with $k = 0$. We now check conditions in (**??**). The first five conditions are valid by assumption, while the last one becomes: $\mu_1/\lambda_1 \geq (\lambda_1/\mu_0)^{N-1}$ and it is implied by the third assumption in (**??**). Consequently, Proposition **??** yields:

$$\|\nabla^{(m)}\mathcal{D}_1\|_0 \leq C\tilde{C}_0 \mu_0^{\gamma} \lambda_1^{\gamma N} \frac{\mu_1^m}{(\lambda_1/\mu_0)^N} \doteq \tilde{C}_1 \mu_1^m, \quad \|\nabla^{(m+2)} v_1^1\|_0 \leq C\tilde{C}_0^{1/2} \mu_0^{\gamma/2} \lambda_1^{m+1} \doteq B_1^{1/2} \lambda_1^{m+1},$$

$$\|\nabla^{(m+2)} v_1^2\|_0 \leq C\tilde{C}_0^{1/2} \mu_0^{\gamma/2} \mu_1^{m+1} \doteq A_2^{1/2} \mu_1^{m+1}, \quad \|\nabla^{(m+2)} v_1^3\|_0 \leq \tilde{C}_0^{1/2} \mu_0^{m+1} \leq A_1^{1/2} \mu_0^{m+1}.$$

where constants C depend on ω, γ, N, K and where we defined the new quantities $\tilde{C}_1, B_1, A_2$ according to (**??**). The above bounds are exactly (**??**) at $k = 1$.

Continuing, for $k = 1$ we have $\alpha_1 = 3$, $\beta_1 = 1$, $\gamma_1 = 2$. We need to the check the last two conditions in (**??**). This is validated as follows, in virtue of (**??**):

$$\begin{aligned} &\big(\frac{A_1}{\tilde{C}_1}\big)^{1/2} = \frac{(\lambda_1/\mu_0)^{N/2}}{C\mu_0^{\gamma/2}\lambda_1^{\gamma N/2}} \leq \big(\frac{\lambda_1}{\mu_0}\big)^{N/2} \leq \big(\frac{\mu_1}{\lambda_1}\big)^{1/2} \leq \frac{\mu_1}{\mu_0}, \\ &\big(\frac{\lambda_2}{\mu_1}\big)^{(N-1)/2} \leq \big(\frac{\lambda_2}{\mu_1}\big)^{N/2} \leq \big(\frac{\lambda_2}{\mu_1}\frac{\lambda_1}{\mu_0}\big)^{N/2} \leq \frac{\mu_2}{\lambda_2}, \\ &\big(\frac{\lambda_2}{\mu_1}\big)^{N-1} \frac{(B_1/\tilde{C}_1)^{1/2}}{\mu_1/\lambda_1} = \frac{(\lambda_2/\mu_1)^{N-1}}{\mu_1/\lambda_1} \big(\frac{C(\lambda_1/\mu_0)^N}{\lambda_1^{\gamma N}}\big)^{1/2} \leq \frac{(\lambda_2/\mu_1)^N (\lambda_1/\mu_0)^{N/2}}{\mu_1/\lambda_1} \frac{C}{\lambda_2/\mu_1} \leq \frac{\mu_2}{\lambda_2}, \end{aligned}$$

provided that σ_0 has been chosen large enough (in function of ω, γ, N, K) to ensure that the last quotient above is less than 1 in virtue of the first assumption in (**??**) at $k = 1$. Consequently, Proposition **??** implies:

$$\|\nabla^{(m)}\mathcal{D}_2\|_0 \leq C\tilde{C}_1 \mu_1^{\gamma} \lambda_2^{\gamma N} \frac{\mu_2^m}{(\lambda_2/\mu_1)^N} \doteq \tilde{C}_2 \mu_2^m, \quad \|\nabla^{(m+2)} v_2^3\|_0 \leq C\tilde{C}_1^{1/2} \mu_1^{\gamma/2} \lambda_2^{m+1} \doteq B_2^{1/2} \lambda_2^{m+1},$$

$$\|\nabla^{(m+2)} v_2^1\|_0 \leq C\tilde{C}_0^{1/2} \mu_1^{\gamma/2} \mu_2^{m+1} \doteq A_3^{1/2} \mu_2^{m+1}, \quad \|\nabla^{(m+2)} v_2^2\|_0 = \|\nabla^{(m+2)} v_1^2\|_0 \leq A_2^{1/2} \mu_1^{m+1},$$

where constants C depend on ω, γ, N, K and where we relied on the definition (**??**). The above bounds are exactly (**??**) at $k = 2$.

3. (Induction step) Let now $k = 2 \dots K-1$ and assume (**??**). To verify the fifth assumption in (**??**), we use (**??**) and (**??**) in:

$$\begin{aligned} \big(\frac{A_k}{\tilde{C}_k}\big)^{1/2} &= \Big(\frac{C\tilde{C}_{k-2}\mu_{k-2}^{\gamma}}{\tilde{C}_k}\Big)^{1/2} = \Big(\frac{C\mu_{k-2}^{\gamma}}{(\tilde{C}_k/\tilde{C}_{k-1})(\tilde{C}_{k-1}/\tilde{C}_{k-2})}\Big)^{1/2} \\ &= \Big(\frac{C(\lambda_k/\mu_{k-1})^N(\lambda_{k-1}/\mu_{k-2})^N}{\mu_{k-1}^{\gamma}\lambda_k^{\gamma N}\lambda_{k-1}^{\gamma N}}\Big)^{1/2} \leq C\big(\frac{\lambda_k}{\mu_{k-1}}\frac{\lambda_{k-1}}{\mu_{k-2}}\big)^{N/2} \leq C\frac{\mu_k}{\lambda_k} \leq \frac{\mu_k}{\mu_{k-1}}, \end{aligned}$$

because λ_k/μ_{k-1} is larger than the constant C depending on ω, γ, N, K provided that σ_0 in the first assumption in (**??**) is sufficiently large. For the last assumption in (**??**), we first observe:

$$\Big(\frac{\lambda_{k+1}}{\mu_k}\Big)^{N-1}\frac{(B_k/\tilde C_k)^{1/2}}{\mu_k/\lambda_k} = \frac{(\lambda_{k+1}/\mu_k)^{N-1}}{\mu_k/\lambda_k}\Big(\frac{C\mu_{k-1}^\gamma}{\tilde C_k/\tilde C_{k-1}}\Big)^{1/2} = \frac{(\lambda_{k+1}/\mu_k)^{N-1}}{\mu_k/\lambda_k}\Big(\frac{C(\lambda_k/\mu_{k-1})^N}{\lambda_k^{\gamma N}}\Big)^{1/2}$$
$$\leq \frac{(\lambda_{k+1}/\mu_k)^N(\lambda_k/\mu_{k-1})^{N/2}}{\mu_k/\lambda_k}\frac{C}{\lambda_{k+1}/\mu_k} \leq \frac{\mu_{k+1}}{\lambda_{k+1}},$$

where in the last bound above we used (**??**) and, again the fact that $\lambda_{k+1}/\mu_k \geq C$ is σ_0 if sufficiently large. It remains to estimate $(\lambda_{k+1}/\mu_k)^{(N-1)/2}$. In case of $k \leq K-2$, we get:

$$\Big(\frac{\lambda_{k+1}}{\mu_k}\Big)^{(N-1)/2} \leq \Big(\frac{\lambda_{k+1}}{\mu_k}\Big)^{N/2} \leq \Big(\frac{\lambda_{k+1}}{\mu_k}\frac{\lambda_k}{\mu_{k-1}}\Big)^{N/2} \leq \frac{\mu_{k+1}}{\lambda_{k+1}}$$

by (**??**). Likewise, for $k = K-1$:

$$\Big(\frac{\lambda_{k+1}}{\mu_k}\Big)^{(N-1)/2} \leq \Big(\frac{\lambda_K}{\mu_{K-1}}\Big)^{N/2} \leq \frac{\mu_K}{\lambda_K} = \frac{\mu_{k+1}}{\lambda_{k+1}}.$$

This ends the verification of (**??**). We may now apply Proposition **??** to get:

$$\begin{aligned}
&\|\nabla^{(m)}\mathcal{D}_{k+1}\|_0 \leq C\tilde C_k\mu_k^\gamma\lambda_{k+1}^{\gamma N}\frac{\mu_{k+1}^m}{(\lambda_{k+1}/\mu_k)^N} \doteq \tilde C_{k+1}\mu_{k+1}^m,\\
&\|\nabla^{(m+2)}v_{k+1}^{\beta_{k+1}}\|_0 = \|\nabla^{(m+2)}v_{k+1}^{\alpha_k}\|_0 \leq C\tilde C_k^{1/2}\mu_k^{\gamma/2}\lambda_{k+1}^{m+1} \doteq B_{k+1}^{1/2}\lambda_{k+1}^{m+1},\\
&\|\nabla^{(m+2)}v_{k+1}^{\gamma_{k+1}}\|_0 = \|\nabla^{(m+2)}v_{k+1}^{\beta_k}\|_0 \leq C\tilde C_k^{1/2}\mu_k^{\gamma/2}\mu_{k+1}^{m+1} \doteq A_{k+2}^{1/2}\mu_{k+1}^{m+1},\\
&\|\nabla^{(m+2)}v_{k+1}^{\alpha_{k+1}}\|_0 = \|\nabla^{(m+2)}v_{k+1}^{\gamma_k}\|_0 = \|\nabla^{(m+2)}v_k^{\gamma_k}\|_0 \leq A_{k+1}^{1/2}\mu_k^{m+1}.
\end{aligned}$$

where we used (**??**), to obtain (**??**) at $k+1$.

4. (Gathering bounds on v_K, w_K) Recall that we have set $\tilde v = v_K$, $\tilde w = w_K$ and hence $\tilde{\mathcal{D}} = \mathcal{D}_K$. Applying (**??**) at the final counter value K and recalling the progression in (**??**), we obtain the following bounds, in which C depend on ω, γ, N, K:

$$\|\mathcal{D}_K\|_0 \leq \tilde C_K \leq C\tilde C_0\prod_{k=0}^{K-1}\frac{\mu_k^\gamma\lambda_{k+1}^{\gamma N}}{(\lambda_{k+1}/\mu_k)^N} \leq C\tilde C_0\Lambda^\gamma\prod_{k=0}^{K-1}\frac{1}{(\lambda_{k+1}/\mu_k)^N},$$

which is exactly **??**. Further, by (**??**) we get:

$$\begin{aligned}
\|\nabla^2 v_K\|_0 &\leq A_K^{1/2}\mu_{K-1} + B_K^{1/2}\lambda_K + A_{K+1}^{1/2}\mu_K \leq C\big(\tilde C_{K-2}^{1/2}\mu_{K-2}^{\gamma/2}\mu_{K-1} + \tilde C_{K-1}^{1/2}\mu_{K-1}^{\gamma/2}\mu_K\big)\\
&\leq C\tilde C_0^{1/2}\Big(\Big(\prod_{k=0}^{K-3}\frac{\mu_k^{\gamma/2}\lambda_{k+1}^{\gamma N/2}}{(\lambda_{k+1}/\mu_k)^{N/2}}\Big)\mu_{K-2}^{\gamma/2}\mu_{K-1} + \Big(\prod_{k=0}^{K-2}\frac{\mu_k^{\gamma/2}\lambda_{k+1}^{\gamma N/2}}{(\lambda_{k+1}/\mu_k)^{N/2}}\Big)\mu_{K-1}^{\gamma/2}\mu_K\Big)\\
&\leq C\tilde C_0^{1/2}\Lambda^{\gamma/2}\Big(\prod_{k=0}^{K-2}\frac{1}{(\lambda_{k+1}/\mu_k)^{N/2}}\Big)\Big(1 + \frac{(\lambda_{K-1}/\mu_{K-2})^{N/2}}{\mu_K/\mu_{K-1}}\Big)\mu_K,
\end{aligned} \tag{4.21}$$

which is **??**. To obtain **??**, we sum the estimates **??** in:

$$\|v_K - v_0\|_1 \leq \sum_{k=0}^{K-1}\|v_{k+1} - v_k\|_1 \leq C\sum_{k=0}^{K-1}\tilde C_k^{1/2}\mu_k^{\gamma/2} \leq C\tilde C_0^{1/2}\prod_{k=0}^{K-1}\mu_k^{\gamma/2} \leq C\tilde C_0^{1/2}\Lambda^{\gamma/2},$$

because for all $k = 1 \dots K-1$ there holds: $\tilde C_{k+1} \leq C\tilde C_k \mu_k^\gamma/\sigma_0^N$ by (**??**). Similarly, there follows the first bound in **??**, in view of **??** and the previous estimate:

$$\begin{aligned}\|w_K - w_0\|_1 &\leq \sum_{k=0}^{K-1} \|w_{k+1} - w_k\|_1 \leq C \sum_{k=0}^{K-1} \tilde C_k^{1/2}\mu_k^{\gamma/2}\big(\|\nabla v_k\|_0 + \tilde C_k^{1/2}\mu_k^{\gamma/2}\big) \\ &\leq C\tilde C_0^{1/2}\Big(\prod_{k=0}^{K-1}\mu_k^{\gamma/2}\Big)\Big(\|\nabla v_0\|_0 + \tilde C_0^{1/2}\prod_{k=0}^{K-1}\mu_k^{\gamma/2}\Big) \leq C\tilde C_0^{1/2}\Lambda^\gamma\big(\|\nabla v_0\| + \tilde C_0^{1/2}\big).\end{aligned}$$

To complete the claim in **??**, we use the second bound in **??** to get:

$$\begin{aligned}\|\nabla^2(w_K - w_0)\|_0 &\leq \sum_{k=0}^{K-1} \|\nabla^2(w_{k+1} - w_k)\|_0 \\ &\leq C \sum_{k=0}^{K-1} \tilde C_k^{1/2}\mu_k^{\gamma/2}\mu_{k+1}\big(\|\nabla v_k\|_0 + \tilde C_k^{1/2}\mu_k^{\gamma/2}\big) \\ &\leq C \sum_{k=0}^{K-1} \tilde C_k^{1/2}\mu_k^{\gamma/2}\mu_{k+1}\Big(\|\nabla v_0\|_0 + \tilde C_0^{1/2}\prod_{k=0}^{K-1}\mu_k^{\gamma/2}\Big) \\ &\leq C\big(\|\nabla v_0\|_0 + \tilde C_0^{1/2}\big)\Big(\prod_{k=0}^{K-1}\mu_k^{\gamma/2}\Big)\sum_{k=0}^{K-1}\tilde C_k^{1/2}\mu_k^{\gamma/2}\mu_{k+1}.\end{aligned} \tag{4.22}$$

Observe now that $\{\tilde C_k^{1/2}\mu_{k+1}\mu_k^{\gamma/2}\}_{k=0}^{K-2}$ is a nonincreasing sequence, because from (**??**):

$$\frac{\tilde C_{k-1}^{1/2}\mu_k\mu_{k-1}^{\gamma/2}}{\tilde C_k^{1/2}\mu_{k+1}\mu_k^{\gamma/2}} = C\frac{(\lambda_k/\mu_{k-1})^{N/2}}{\mu_k^{\gamma/2}\lambda_k^{\gamma N/2}(\mu_{k+1}/\mu_k)} \leq \frac{C}{\lambda_{k+1}/\mu_k}\frac{(\lambda_k/\mu_{k-1})^{N/2}}{\mu_{k+1}/\lambda_{k+1}} \leq 1 \quad \text{for all } \ k = 1\dots K-2.$$

Indeed, the first term in the right hand side above is less than 1 for σ_0 large enough by the first condition in (**??**), while the second term is less than 1 by the second and third conditions in (**??**). Consequently, using the calculation in (**??**), estimates in (**??**) yield:

$$\begin{aligned}\|\nabla^2(w_K - w_0)\|_0 &\leq C\big(\|\nabla v_0\|_0 + \tilde C_0^{1/2}\big)\Big(\prod_{k=0}^{K-1}\mu_k^{\gamma/2}\Big)\big(\tilde C_{K-2}^{1/2}\mu_{K-2}^{\gamma/2}\mu_{K-1} + \tilde C_{K-1}^{1/2}\mu_{K-1}^{\gamma/2}\mu_K\big) \\ &\leq C\tilde C_0^{1/2}\big(\|\nabla v_0\|_0 + \tilde C_0^{1/2}\big)\Lambda^\gamma\Big(\prod_{k=0}^{K-2}\frac{1}{(\lambda_{k+1}/\mu_k)^{N/2}}\Big)\Big(1 + \frac{(\lambda_{K-1}/\mu_{K-2})^{N/2}}{\mu_K/\mu_{K-1}}\Big)\mu_K.\end{aligned}$$

This is exactly the second bound in **??**. The proof is done. ■

5. The Fibonacci frequencies and a proof of Theorem ??

In this section we complete the inductive procedure put forward in Proposition **??**, by specifying a progression of frequencies which satisfy all the conditions in (**??**). We will then prove the single "stage" estimates in Theorem **??**, which will yield Theorem **??**, as described in the next section. We use the Fibonacci sequence $\{F_k\}_{k=0}^\infty$, where:

$$F_0 = F_1 = 1, \qquad F_{k+2} = F_k + F_{k+1} \quad \text{for all } \ k \geq 0.$$

Proposition 5.1. *Let $N \geq 1$, $K \geq 4$ and let $\gamma \in (0,1)$ satisfy:*

$$\gamma \leq \frac{1}{(F_{K+2} - 3)(1 + N/2)}. \tag{5.1}$$

Then, for every parameters $\mu_0, \sigma_0, \sigma \geq 1$ such that:

$$\frac{\sigma}{(\mu_0\sigma)^\gamma} \geq \sigma_0, \tag{5.2}$$

conditions (??) are satisfied for the sequence $\{\lambda_k, \mu_k\}_{k=1}^K$ defined in:

$$\left.\begin{array}{l} \lambda_k = \mu_0\sigma^{(F_{k+2}-2)+(F_{k+2}-3)N/2} \\ \mu_k = \mu_0\sigma^{(F_{k+2}-2)+(F_{k+3}-3)N/2} \end{array}\right\} \quad \textit{for all } \ k = 1\dots K-1, \\ \lambda_K = \mu_0\sigma^{(2F_K-2)+(F_{K+2}-3)N/2}, \qquad \mu_K = \mu_0\sigma^{(2F_K-2)+(3F_K-3)N/2}. \tag{5.3}$$

Proof. **1. (Conditions not involving γ)** We first directly derive the formulas on the quotients of the frequencies $\{\lambda_i, \mu_i\}$, which will be used below:

$$\frac{\lambda_k}{\mu_{k-1}} = \sigma^{F_k}, \quad \frac{\mu_k}{\lambda_k} = \sigma^{F_{k+1}N/2} \qquad \text{for all } \ k = 1\dots K-1, \\ \frac{\lambda_K}{\mu_{K-1}} = \sigma^{2F_K - F_{K+1}} = \sigma^{F_{K-2}}, \qquad \frac{\mu_K}{\lambda_K} = \sigma^{3F_K - F_{k+2}N/2} = \sigma^{F_{K-2}N/2}. \tag{5.4}$$

We now validate all conditions (??), apart from the first one, as equalities. All formulas follow by (??). Regarding the second condition, we have:

$$\frac{\mu_1}{\lambda_1} = \sigma^{F_2N/2} = \sigma^N = \big(\frac{\lambda_1}{\mu_0}\big)^N.$$

The third condition holds, because for $k = 2\dots K-1$:

$$\big(\frac{\lambda_k}{\mu_{k-1}}\frac{\lambda_{k-1}}{\mu_{k-2}}\big)^{N/2} = \sigma^{(F_k+F_{k-1})N/2} = \sigma^{F_{k+1}N/2} = \frac{\mu_k}{\lambda_k}, \\ \frac{(\lambda_k/\mu_{k-1})^N(\lambda_{k-1}/\mu_{k-2})^{N/2}}{\mu_{k-1}/\lambda_{k-1}} = \sigma^{F_kN+F_{k-1}N/2-F_kN/2} = \sigma^{F_{k+1}N/2} = \frac{\mu_k}{\lambda_k}.$$

Likewise, the last condition in (??) follows from:

$$\big(\frac{\lambda_K}{\mu_{K-1}}\big)^{N/2} = \sigma^{F_{K-2}N/2} = \frac{\mu_K}{\lambda_K}, \\ \frac{(\lambda_K/\mu_{K-1})^N(\lambda_{K-1}/\mu_{K-2})^{N/2}}{\mu_{K-1}/\lambda_{K-1}} = \sigma^{F_{K-2}N+F_{K-1}N/2-F_KN/2} = \sigma^{F_{K-2}N/2} = \frac{\mu_K}{\lambda_K}.$$

2. (Condition involving γ) To prove the first condition in (??), we use (??) to check that for all $k = 1\dots K-1$ there holds:

$$\frac{\lambda_k^{1-\gamma}}{\mu_{k-1}} = \frac{1}{\lambda_k^\gamma}\frac{\lambda_k}{\mu_{k-1}} = \frac{\sigma^{F_k}}{\lambda_k^\gamma} = \frac{\sigma}{(\mu_0\sigma)^\gamma}\sigma^{(F_k-1)-(F_{k+2}-3)(1+N/2)\gamma} \geq \frac{\sigma}{(\mu_0\sigma)^\gamma} \geq \sigma_0,$$

by (??) and because $(F_k-1)-(F_{k+2}-3)(1+N/2)\gamma$ is always nonnegative. Indeed, at $k=1$ this expression is null, while for $k>1$ it is at least $1-(F_{k+2}-3)(1+N/2)\gamma$. The assumption (??) guarantees that this last expression is nonnegative. Similarly:

$$\frac{\lambda_K^{1-\gamma}}{\mu_{K-1}} = \frac{1}{\lambda_K^\gamma}\frac{\lambda_K}{\mu_{K-1}} = \frac{\sigma^{F_{K-2}}}{\lambda_K^\gamma} \geq \frac{\sigma}{(\mu_0\sigma)^\gamma}\sigma^{(F_{K-2}-1)-(F_{K+2}-3)(1+N/2)\gamma} \geq \frac{\sigma}{(\mu_0\sigma)^\gamma} \geq \sigma_0,$$

since $(F_{K-2}-1)-(F_{K+2}-3)(1+N/2)\gamma \geq 0$ when $K \geq 4$ and under (??). ∎

It is useful to separately derive the quantities that appear in the estimates **??** – **??**, under the definition of frequencies in (??). These are:

Lemma 5.2. *Let $N, K \geq 4$, $\mu_0 \geq 1$ and let $\{\lambda_k, \mu_k\}_{k=1}^K$ be given by (**??**). Then there hold:*

$$\prod_{k=0}^{K-1} \frac{1}{(\lambda_{k+1}/\mu_k)^N} = \frac{1}{\sigma^{2(F_K-1)N}}, \tag{5.5$_1$}$$

$$\mu_K\Big(\prod_{k=0}^{K-2} \frac{1}{(\lambda_{k+1}/\mu_k)^{N/2}}\Big)\Big(1 + \frac{(\lambda_{K-1}/\mu_{K-2})^{N/2}}{\mu_K/\mu_{K-1}}\Big) \leq 2\mu_0\sigma^{(F_{K+1}-2)+(F_{K+1}-1)N/2}. \tag{5.6$_2$}$$

Proof. Using (**??**) and the formula $\displaystyle\sum_{i=1}^{j} F_i = F_{j+2} - 2$, we obtain **??**:

$$\prod_{k=0}^{K-1} \frac{1}{(\lambda_{k+1}/\mu_k)^N} = \sigma^{-(\sum_{k=1}^{K-1} F_k)N}\sigma^{-F_{K-2}N} = \sigma^{-(2F_K-2)N}.$$

Towards **??**, note that:

$$3F_K - 3 + F_{K-1} - (F_{K+1} + F_{K-2} - 2) = 2F_K - F_{K-2} - 1 = F_{K+1} - 1,$$

which implies, for $N \geq 4$:

$$\begin{aligned}
&\mu_K\Big(\prod_{k=0}^{K-2} \frac{1}{(\lambda_{k+1}/\mu_k)^{N/2}}\Big)\Big(1 + \frac{(\lambda_{K-1}/\mu_{K-2})^{N/2}}{\mu_K/\mu_{K-1}}\Big) \\
&\qquad = \frac{\mu_0\sigma^{(2F_K-2)+(3F_K-3)N/2}}{\sigma^{(\sum_{k=1}^{K-1} F_k)N/2}}\Big(1 + \frac{\sigma^{F_{K-1}N/2}}{\sigma^{F_{K-2}(1+N/2)}}\Big) \\
&\qquad \leq 2\,\frac{\mu_0\sigma^{(2F_K-2)+(3F_K-3)N/2}}{\sigma^{(F_{K+1}-2)N/2}} \cdot \frac{\sigma^{F_{K-1}N/2}}{\sigma^{F_{K-2}(1+N/2)}} = 2\mu_0\sigma^{(F_{K+1}-2)+(F_{K+1}-1)N/2},
\end{aligned}$$

since then $F_{K-2}(1+N/2) \leq 2F_{K-3} + F_{K-2}N/2 \leq (F_{K-3}+F_{K-2})N/2$. The proof is done. ■

We are now ready to complete the "stage" construction in our convex integration algorithm:

Proof of Theorem ??.

1. (Setting the initial quantities) For given $N, K \geq 4$ and γ that satisfies (**??**), we take l_0 as in Proposition **??** and σ_0 increased $(K+1)$ times. Let v, w be as in the statement of the theorem, together with the positive constants $l, \lambda, \mathcal{M}$ satisfying (**??**). Denote:

$$\eta = \frac{l}{K+1}, \quad \mu_0 = \frac{1}{\eta}.$$

We first construct the fields $v_0 \in \mathcal{C}^\infty(\bar\omega + \bar B_{l+K\eta}(0), \mathbb{R}^3)$, $w_0 \in \mathcal{C}^\infty(\bar\omega + \bar B_{l+K\eta}(0), \mathbb{R}^2)$, $A_0 \in \mathcal{C}^\infty(\bar\omega + \bar B_{l+K\eta}(0), \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ by using the mollification kernel as in Lemma **??**:

$$v_0 = v * \phi_{\eta/2}, \quad w_0 = w * \phi_{\eta/2}, \quad A_0 = A * \phi_{\eta/2}, \quad \mathcal{D}_0 = A_0 - \big(\frac{1}{2}(\nabla v_0)^T\nabla v_0 + \mathrm{sym}\nabla w_0\big).$$

From Lemma **??**, we deduce the initial bounds, where constants C depend only on ω, K, m:

$$\|v_0 - v\|_1 + \|w_0 - w\|_1 \leq Cl\mathcal{M}, \tag{5.7$_1$}$$

$$\|A_0 - A\|_0 \leq Cl^\beta \|A\|_{0,\beta}, \tag{5.7$_2$}$$

$$\|\nabla^{(m+1)} v_0\|_0 + \|\nabla^{(m+1)} w_0\|_0 \leq \frac{C}{l^m} l\mathcal{M} \quad \text{ for all } \ m \geq 1, \tag{5.7$_3$}$$

$$\|\nabla^{(m)}\mathcal{D}_0\|_0 \leq \frac{C}{l^m}\big(\|\mathcal{D}\|_0 + (l\mathcal{M})^2\big) \qquad \text{ for all } \ m \geq 0. \tag{5.7$_4$}$$

Indeed, **??**, **??** follow from **??** and in view of the lower bound on $\mathcal{M}$. Similarly, **??** follows by applying **??** to $\nabla^2 v$ and $\nabla^2 w$ with the differentiability exponent $m-1$. Since:

$$\mathcal{D}_0 = \mathcal{D} * \phi_{\eta/2} - \frac{1}{2}\big((\nabla v_0)^T \nabla v_0 - ((\nabla v)^T \nabla v) * \phi_{\eta/2}\big),$$

we get **??** by applying **??** to $\mathcal{D}$, and **??** to ∇v.

2. (Applying Proposition ??) Since $l + K\eta \leq 2l \leq 2l_0$, we may apply Proposition **??** to $v_0, w_0, \mathcal{D}_0$ with the parameters:

$$\tilde{C}_0 = C\big(\|\mathcal{D}\|_0 + (l\mathcal{M})^2\big), \quad \lambda_0 = \mu_0 = \frac{K+1}{l},$$

consistent with **??**, and with frequencies $\{\lambda_k, \mu_k\}_{k=1}^K$ given in Proposition **??** for $\sigma > 1$ in:

$$\sigma = \lambda l \qquad \text{so that:} \ \ \frac{\sigma}{(\mu_0\sigma)^\gamma} = \frac{\lambda l}{((K+1)\lambda)^\gamma} \geq \frac{\lambda^{1-\gamma} l}{K+1} \geq \sigma_0$$

In conclusion, Proposition **??** yields $\tilde{v} \in \mathcal{C}^\infty(\bar{\omega} + \bar{B}_l(0), \mathbb{R}^3)$, $\tilde{w} \in \mathcal{C}^\infty(\bar{\omega} + \bar{B}_l(0), \mathbb{R}^2)$, with **??** – **??**. These imply, in virtue of Lemma **??** and the initial bounds **??** – **??**:

$$\begin{aligned}
&\|\tilde{v} - v\|_1 \leq C\Lambda^{\gamma/2}\big(\|\mathcal{D}\|_0^{1/2} + l\mathcal{M}\big),\\
&\|\tilde{w} - w\|_1 \leq C\Lambda^{\gamma}\big(\|\mathcal{D}\|_0^{1/2} + l\mathcal{M}\big)\big(1 + \|\mathcal{D}\|_0^{1/2} + l\mathcal{M} + \|\nabla v\|_0\big),\\
&\|\nabla^2\tilde{v}\|_0 \leq C\Lambda^{\gamma/2}\frac{(\lambda l)^{(F_{K+1}-2)+(F_{K+1}-1)N/2}}{l}\big(\|\mathcal{D}\|_0^{1/2} + l\mathcal{M}\big),\\
&\|\nabla^2\tilde{w}\|_0 \leq C\Lambda^{\gamma}\frac{(\lambda l)^{(F_{K+1}-2)+(F_{K+1}-1)N/2}}{l}\big(\|\mathcal{D}\|_0^{1/2} + l\mathcal{M}\big)\big(1 + \|\mathcal{D}\|_0^{1/2} + l\mathcal{M} + \|\nabla v\|_0\big),\\
&\|\tilde{\mathcal{D}}\|_0 \leq Cl^\beta\|A\|_{0,\beta} + C\Lambda^{\gamma/2}\frac{1}{(\lambda l)^{2(F_K-1)N}}\big(\|\mathcal{D}\|_0 + (l\mathcal{M})^2\big),
\end{aligned} \tag{5.8}$$

with C depending on ω, γ, N, K. The quantity Λ, may be estimated by:

$$\Lambda = \prod_{k=1}^K (\mu_k\lambda_k^N) \leq \mu_0^{K(N+1)}\sigma^{P(N,K)} \leq C\frac{(\lambda l)^{P(N,K)}}{l^{K(N+1)}},$$

where $P(N,K)$ is a second order polynomial in N, with coefficients depending on K.

3. (Reparametrizing γ) Since $P(N,K)$ is (much) larger than $(2F_K - 2)N$, we get:

$$\begin{aligned}
P(N,K) - K(N+1) &\geq 2(F_K-2)N - K(N+1)\\
&= (F_K - K)(N+1) + (F_K-2)(N-3) + 2(F_K-3) \geq 0,
\end{aligned}$$

because $N, K \geq 4$. Consequently:

$$\Lambda \leq C\lambda^{P(N,K)}$$

so that in (**??**) one can replace each occurrence of Λ^γ by $\lambda^{\bar{\gamma}}$ with $\bar{\gamma} = \gamma P(N,K) \geq \gamma$. However, condition $\lambda^{1-\bar{\gamma}} l \geq \sigma_0$ implies $\lambda^{1-\gamma} l \geq \sigma_0$, so the bounds (**??**) imply those in **??** – **??**, albeit within a smaller range of γ than that indicated in (**??**), still depending only on N, K. Similarly, we finally observe that if the statement of Theorem **??** holds for all sufficiently small γ, then it is valid for all $\gamma \in (0,1)$, as stated. The proof is done. ■

6. The Nash-Kuiper scheme and a proof of Theorem ??

The proof of Theorem **??** relies on iterating Theorem **??** according to the Nash-Kuiper scheme. We quote the main recursion result given in [**?**, **?**]:, similar to [**?**, section 6], but now involving the Hölder norms, as is necessary in view of the decomposition Lemma **??**.

Theorem 6.1. [**?**, Theorem 1.4] [**?**, Lemma 5.2] *Let $\omega \subset \mathbb{R}^d$ be an open, bounded and smooth domain, and let $k, J, S \geq 1$. Assume that there exists $l_0 \in (0,1)$ such that the following holds for every $l \in (0, l_0]$. Given $v \in \mathcal{C}^2(\bar\omega + \bar B_{2l}(0), \mathbb{R}^k)$, $w \in \mathcal{C}^2(\bar\omega + \bar B_{2l}(0), \mathbb{R}^d)$, $A \in \mathcal{C}^{0,\beta}(\bar\omega + \bar B_{2l}(0), \mathbb{R}^{d\times d}_{\mathrm{sym}})$, and $\gamma, \lambda, \mathcal{M}$ which satisfy, together with $\sigma_0 \geq 1$ that depends on ω, k, S, J, γ:*

$$\gamma \in (0,1), \qquad \lambda^{1-\gamma} l > \sigma_0, \qquad \mathcal{M} \geq \max\{\|v\|_2, \|w\|_2, 1\}, \tag{6.1}$$

there exist $\tilde v \in \mathcal{C}^2(\bar\omega + \bar B_l(0), \mathbb{R}^k)$, $\tilde w \in \mathcal{C}^2(\bar\omega + \bar B_l(0), \mathbb{R}^d)$ satisfying:

$$\begin{aligned}
&\|\tilde v - v\|_1 \leq C\lambda^{\gamma/2}\big(\|\mathcal{D}\|_0^{1/2} + l\mathcal{M}\big),\\
&\|\tilde w - w\|_1 \leq C\lambda^{\gamma}\big(\|\mathcal{D}\|_0^{1/2} + l\mathcal{M}\big)\big(1 + \|\mathcal{D}\|_0^{1/2} + l\mathcal{M} + \|\nabla v\|_0\big),\\
&\|\nabla^2 \tilde v\|_0 \leq C\frac{(\lambda l)^J}{l}\lambda^{\gamma/2}\big(\|\mathcal{D}\|_0^{1/2} + l\mathcal{M}\big),\\
&\|\nabla^2 \tilde w\|_0 \leq C\frac{(\lambda l)^J}{l}\lambda^{\gamma}\big(\|\mathcal{D}\|_0^{1/2} + l\mathcal{M}\big)\big(1 + \|\mathcal{D}\|_0^{1/2} + l\mathcal{M} + \|\nabla v\|_0\big),\\
&\|\tilde{\mathcal{D}}\|_0 \leq C\Big(l^\beta \|A\|_{0,\beta} + \frac{\lambda^\gamma}{(\lambda l)^S}\big(\|\mathcal{D}\|_0 + (l\mathcal{M})^2\big)\Big).
\end{aligned}$$

with constants C depending only on ω, k, J, S, γ, and with the defects, as usual, denoted by:

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big), \qquad \tilde{\mathcal{D}} = A - \big(\frac{1}{2}(\nabla \tilde v)^T\nabla \tilde v + \mathrm{sym}\nabla \tilde w\big).$$

Then, for every triple of fields v, w, A as above, which additionally satisfy the defect smallness condition $0 < \|\mathcal{D}\|_0 \leq 1$, and for every exponent α in the range:

$$0 < \alpha < \min\Big\{\frac{\beta}{2}, \frac{S}{S+2J}\Big\}, \tag{6.2}$$

there exist $\bar v \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^k)$ and $\bar w \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^d)$ with the following properties:

$$\begin{aligned}
&\|\bar v - v\|_1 \leq C\big(1 + \|\nabla v\|_0\big)^2\|\mathcal{D}_0\|_0^{1/4}, \quad \|\bar w - w\|_1 \leq C(1 + \|\nabla v\|_0)^3\|\mathcal{D}\|_0^{1/4},\\
&A - \big(\frac{1}{2}(\nabla \bar v)^T\nabla \bar v + \mathrm{sym}\nabla \bar w\big) = 0 \quad \text{in } \bar\omega.
\end{aligned}$$

The constants C above depend only on ω, k, A and α.

Clearly, Theorem **??** and Theorem **??** yield together the following result below, where we compute $\frac{S}{S+2J} = \frac{1}{1+2J/S}$, with φ denoting the golden ratio in:

$$\begin{aligned}
&\frac{J}{S} = \frac{(F_{K+1} - 2) + (F_{K+1} - 1)N/2}{2N(F_K - 1)} \to \frac{F_{K-1} - 1}{4(F_K - 1)} \quad \text{as } K \to \infty\\
&\text{and:} \quad \frac{F_{K-1} - 1}{4(F_K - 1)} \to \frac{\varphi}{4} = \frac{1 + \sqrt{5}}{8} \quad \text{as } N \to \infty.
\end{aligned}$$

Corollary 6.2. *Let $\omega \subset \mathbb{R}^2$ be an open, bounded and smooth domain. Fix any α as in (**??**). Then, there exists $l_0 \in (0,1)$ such that, for every $l \in (0, l_0]$, and for every $v \in \mathcal{C}^2(\bar\omega + \bar B_{2l}(0), \mathbb{R}^3)$, $w \in \mathcal{C}^2(\bar\omega + \bar B_{2l}(0), \mathbb{R}^2)$, $A \in \mathcal{C}^{0,\beta}(\bar\omega + \bar B_{2l}(0), \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ such that:*

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big) \quad \textit{satisfies} \quad 0 < \|\mathcal{D}\|_0 \leq 1,$$

there exist $\tilde v \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^3)$, $\tilde w \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^2)$ with the following properties:

$$\begin{aligned} &\|\tilde v - v\|_1 \leq C(1 + \|\nabla v\|_0)^2 \|\mathcal{D}\|_0^{1/4}, \quad \|\tilde w - w\|_1 \leq C(1 + \|\nabla v\|_0)^3 \|\mathcal{D}\|_0^{1/4}, \\ &A - \big(\frac{1}{2}(\nabla \tilde v)^T\nabla \tilde v + \mathrm{sym}\nabla \tilde w\big) = 0 \quad \textit{in } \ \bar\omega. \end{aligned}$$

The norms in the left hand side above are taken on $\bar\omega$, and in the right hand side on $\bar\omega + \bar B_{2l}(0)$. The constants C depend only on ω, A and α.

The proof of Theorem **??** is consequently the same as the proof of Theorem 1.1 in [**?**], in section 5 in there. We replace ω by its smooth superset, and apply the basic stage construction in order to first decrease $\|\mathcal{D}\|_0$ below 1. Then, Corollary **??** yields the theorem. ■

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