THE MONGE-AMPÈRE SYSTEM IN DIMENSION TWO: A FURTHER REGULARITY IMPROVEMENT

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Abstract. We prove a convex integration result for the Monge-Ampère system introduced in [11], in case of dimension $d = 2$ and arbitrary codimension $k \geq 1$. Our prior result [12] stated flexibility up to the Hölder regularity $\mathcal{C}^{1,\frac{1}{1+4/k}}$, whereas presently we achieve flexibility up to $\mathcal{C}^{1,1}$ when $k \geq 4$ and up to $\mathcal{C}^{1,\frac{2^k-1}{2^{k+1}-1}}$ for any k. This first result uses the approach closest to that of Källen [9] in the context of the isometric immersion problem, while the second result uses the double iteration procedure from [11] combined with the approach of Cao-Hirsch-Inauen [1], agreeing with it for $k = 1$ at the Hölder regularity up to $\mathcal{C}^{1,1/3}$.

1. Introduction

In this paper we present a new bound for the admissible Hölder continuity exponent for weak solutions of the Monge-Ampère system in dimension $d = 2$ and arbitrary codimension $k \geq 1$:

$$\begin{aligned} &\mathfrak{Det}\,\nabla^2 v = f \quad \text{ in } \ \omega \subset \mathbb{R}^2, \\ &\text{where } \ \mathfrak{Det}\,\nabla^2 v = \langle \partial_{11} v, \partial_{22} v\rangle - \big|\partial_{12} v\big|^2 \quad \text{ for } \ v : \omega \to \mathbb{R}^k. \end{aligned} \tag{1.1}$$

The closely related problem of isometric immersions of a given Riemannian metric g:

$$\begin{aligned} &(\nabla u)^T \nabla u = g \quad \text{ in } \ \omega, \\ &\text{for } \ u : \omega \to \mathbb{R}^{2+k}, \end{aligned} \tag{1.2}$$

reduces to (1.1) upon taking the family of metrics $\{g_\epsilon = \mathrm{Id}_2 + \epsilon A\}_{\epsilon \to 0}$, each a small perturbation of Id_2, making an ansatz $u_\epsilon = \mathrm{id}_2 + \epsilon v + \epsilon^2 w$, and gathering the lowest order terms in the ϵ-expansions. This leads to the following system:

$$\begin{aligned} &\frac{1}{2}(\nabla v)^T \nabla v + \mathrm{sym}\nabla w = A \quad \text{ in } \ \omega, \\ &\text{for } \ v : \omega \to \mathbb{R}^k, \quad w : \omega \to \mathbb{R}^2. \end{aligned} \tag{1.3}$$

On a simply connected ω, the system (1.3) is equivalent to: $\mathrm{curl}\,\mathrm{curl}\,\big(\frac{1}{2}(\nabla v)^T \nabla v\big) = \mathrm{curl}\,\mathrm{curl}\, A$, and further to: $\mathfrak{Det}\,\nabla^2 v = -\mathrm{curl}\,\mathrm{curl}\, A$, reflecting the agreement of the Gaussian curvatures of g_ϵ and the surfaces $u_\epsilon(\omega)$ at their lowest order terms in ϵ, and bringing us back to (1.1).

Systems (1.1) and (1.3) were introduced and studied in [11] for arbitrary dimensions d and k, where their flexibility, in the sense of Theorem 1.1 below, was proved up to the regularity $\mathcal{C}^{1,\frac{1}{1+d(d+1)/k}}$. For $d = 2$, this means flexibility up to $\mathcal{C}^{1,\frac{1}{1+6/k}}$, and when $k = 1$ this result agrees with flexibility up to $\mathcal{C}^{1,\frac{1}{7}}$ obtained in [13], which was subsequently improved to $\mathcal{C}^{1,\frac{1}{5}}$ in [2] (and to $\mathcal{C}^{1,\frac{1}{1+4/k}}$ for k arbitrary in [12]), and further to $\mathcal{C}^{1,\frac{1}{3}}$ in [1]. The main purpose of the present paper is to increase the aforementioned Hölder exponents in the case of codimension

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$k>1$, obtaining flexibility up to $\mathcal{C}^{1,1}$ when $k\geq 4$ and up to $\mathcal{C}^{1,\frac{2^k-1}{2^{k+1}-1}}$ for arbitrary $k\geq 1$. Consequently, this sets the state of the art in establishing the density of $\mathcal{C}^{1,\alpha}$ solutions to (1.1) in $\mathcal{C}^0$, as follows:

$$k=1 \rightsquigarrow \alpha<1/3, \quad k=2 \rightsquigarrow \alpha<3/7, \quad k=3 \rightsquigarrow \alpha<7/15, \quad k\geq 4 \rightsquigarrow \alpha<1.$$

The large gap between exponents at $k=3$ and $k=4$ is due to the two different techniques in the Nash-Kuiper iteration scheme: for $k\geq 4$ our approach is closest to that used by Källen [9] (in the context of isometric immersions, and without specifying the resulting Hölder regularity), while for arbitrary k we utilize the double induction procedure from [11] combined with the decomposition lemma of Cao-Hirsch-Inauen leading in [1] to the aforementioned regularity at $k=1$. This approach seemingly yields only the critical exponent $1/2$ as $k\to\infty$. Combining the two approaches towards a better interpolation between their corresponding exponents is an open problem.

We also point out that according to a result due to Poznak (see [7, Chapter 2.3]), any smooth 2-dimensional metric has a smooth local embedding in $\mathbb{R}^4$, namely a solution of (1.2) with $k=2$. Our type of density results, albait only addressing the Hölder continuous solutions, are stronger in the following sense: rather than yielding existence of a single solution, they imply that an arbitrary subsolution to (1.3) or (1.2) can be approximated by a $\mathcal{C}^{1,\alpha}$ solution.

Indeed, our main result pertaining to (1.3) states that a $\mathcal{C}^1$-regular pair (v,w) which is a subsolution, can be uniformly approximated by exact solutions $\{(v_n,w_n)\}_{n=1}^\infty$, as follows:

Theorem 1.1. *Let $\omega\subset\mathbb{R}^2$ be an open, bounded domain and let $k\geq 1$ be the given codimension. Given the fields $v\in\mathcal{C}^1(\bar\omega,\mathbb{R}^k)$, $w\in\mathcal{C}^1(\bar\omega,\mathbb{R}^2)$ and $A\in\mathcal{C}^{0,\beta}(\bar\omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})$, assume that:*

$$\mathcal{D}=A-\big(\frac{1}{2}(\nabla v)^T\nabla v+\mathrm{sym}\nabla w\big) \quad \textit{satisfies} \quad \mathcal{D}>c\,\mathrm{Id}_d \ \ \textit{on} \ \ \bar\omega,$$

for some $c>0$, in the sense of matrix inequalities. Then, for every exponent α with:

$$\begin{aligned} &\alpha<\min\Big\{\frac{\beta}{2},1\Big\} && \textit{for } k\geq 4\\ &\alpha<\min\Big\{\frac{\beta}{2},\frac{2^k-1}{2^{k+1}-1}\Big\} && \textit{for any } k, \end{aligned} \tag{1.4}$$

and for every $\epsilon>0$, there exists $\tilde v\in\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^k)$, $\tilde w\in\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^d)$ such that the following holds:

$$\|\tilde v-v\|_0\leq\epsilon, \quad \|\tilde w-w\|_0\leq\epsilon, \tag*{(1.5)$_1$}$$

$$A-\big(\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v+\mathrm{sym}\nabla\tilde w\big)=0 \quad \textit{in} \ \ \bar\omega. \tag*{(1.5)$_2$}$$

The above result implies the aforementioned density of solutions to (1.1), as in [11]:

Corollary 1.2. *For any $f\in L^\infty(\omega,\mathbb{R})$ on an open, bounded, simply connected domain $\omega\subset\mathbb{R}^2$, the following holds. Fix $k\geq 1$ and fix an exponent α in the range (1.4). Then the set of $\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^k)$ weak solutions to (1.1) is dense in $\mathcal{C}^0(\bar\omega,\mathbb{R}^k)$. Namely, every $v\in\mathcal{C}^0(\bar\omega,\mathbb{R}^k)$ is the uniform limit of some sequence $\{v_n\in\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^k)\}_{n=1}^\infty$, such that:*

$$\mathfrak{Det}\,\nabla^2 v_n=f \quad \textit{on} \ \ \omega \ \ \textit{for all} \ \ n=1\ldots\infty.$$

The main new technical ingredient allowing for the flexibility stated in Theorem 1.1, is the "stage"-type constructions in the following two results:

Theorem 1.3. *Let $\omega\subset\mathbb{R}^2$ be an open, bounded, smooth planar domain and let $k\geq 4$. Fix an exponent $\gamma\in(0,1)$ and an integer $N\geq 1$. Then, there exists $l_0\in(0,1)$ depending only on ω, and there exists $\sigma_0\geq 1$ depending on ω,γ,N, such that the following holds. Given $v\in\mathcal{C}^2(\bar\omega+\bar B_{2l}(0),\mathbb{R}^k)$, $w\in\mathcal{C}^2(\bar\omega+\bar B_{2l}(0),\mathbb{R}^2)$, $A\in\mathcal{C}^{0,\beta}(\bar\omega+\bar B_{2l}(0),\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ defined on the closed $2l$-neighbourhood of ω, and given constants $l,\lambda,M>0$ with the properties:*

$$l\leq l_0,\qquad \lambda^{1-\gamma}l\geq\sigma_0,\qquad M\geq\max\{\|v\|_2,\|w\|_2,1\},\tag{1.6}$$

there exist $\tilde v\in\mathcal{C}^2(\bar\omega+\bar B_l(0),\mathbb{R}^k)$, $\tilde w\in\mathcal{C}^2(\bar\omega+\bar B_l(0),\mathbb{R}^2)$ such that, denoting the defects:

$$\mathcal{D}=A-\big(\frac{1}{2}(\nabla v)^T\nabla v+\mathrm{sym}\nabla w\big),\qquad \tilde{\mathcal{D}}=A-\big(\frac{1}{2}(\nabla \tilde v)^T\nabla \tilde v+\mathrm{sym}\nabla \tilde w\big),$$

the following bounds are valid:

$$\begin{aligned}&\|\tilde v-v\|_1\leq C\lambda^{\gamma/2}\big(\|\mathcal{D}\|_0^{1/2}+lM\big),\\ &\|\tilde w-w\|_1\leq C\lambda^{\gamma}\big(\|\mathcal{D}\|_0^{1/2}+lM\big)\big(1+\|\mathcal{D}\|_0^{1/2}+lM+\|\nabla v\|_0\big),\end{aligned}\tag{1.7$_1$}$$

$$\begin{aligned}&\|\nabla^2\tilde v\|_0\leq C\lambda\lambda^{\gamma/2}\big(\|\mathcal{D}\|_0^{1/2}+lM\big),\\ &\|\nabla^2\tilde w\|_0\leq C\lambda\lambda^{\gamma}\big(\|\mathcal{D}\|_0^{1/2}+lM\big)\big(1+\|\mathcal{D}\|_0^{1/2}+lM+\|\nabla v\|_0\big),\end{aligned}\tag{1.7$_2$}$$

$$\|\tilde{\mathcal{D}}\|_0\leq C\big(l^\beta\|A\|_{0,\beta}+\frac{\lambda^\gamma}{(\lambda l)^N}\big(\|\mathcal{D}\|_0+(lM)^2\big).\tag{1.7$_3$}$$

The norms of the maps $v,w,A,\mathcal{D}$ and $\tilde v,\tilde w,\tilde{\mathcal{D}}$ are taken on the respective domains of the maps' definiteness. The constants C depend only on ω,γ,N.

Theorem 1.4. *Let $\omega\subset\mathbb{R}^2$ be an open, bounded, smooth planar domain. Given any codimension $k\geq 1$, the result in Theorem 1.3 holds true, for each sufficiently small γ (in function of N, k), with (1.7)$_1$ – (1.7)$_3$ replaced with:*

$$\begin{aligned}&\|\tilde v-v\|_1\leq C\lambda^{\gamma/2}\big(\|\mathcal{D}\|_0^{1/2}+lM\big),\\ &\|\tilde w-w\|_1\leq C\lambda^{\gamma}\big(\|\mathcal{D}\|_0^{1/2}+lM\big)\big(1+\|\mathcal{D}\|_0^{1/2}+lM+\|\nabla v\|_0\big),\end{aligned}\tag{1.8$_1$}$$

$$\begin{aligned}&\|\nabla^2\tilde v\|_0\leq C\frac{(\lambda l)^{N+1}\lambda^{\gamma/2}}{l}\big(\|\mathcal{D}\|_0^{1/2}+lM\big),\\ &\|\nabla^2\tilde w\|_0\leq C\frac{(\lambda l)^{N+1}\lambda^{\gamma}}{l}\big(\|\mathcal{D}\|_0^{1/2}+lM\big)\big(1+\|\mathcal{D}\|_0^{1/2}+lM+\|\nabla v\|_0\big),\end{aligned}\tag{1.8$_2$}$$

$$\|\tilde{\mathcal{D}}\|_0\leq C\Big(l^\beta\|A\|_{0,\beta}+\frac{\lambda^\gamma}{(\lambda l)^{\frac{2(N^2-1)}{N+3}\big(1-\big(\frac{N-1}{2(N+1)}\big)^k\big)}}\big(\|\mathcal{D}\|_0+(lM)^2\big)\Big).\tag{1.8$_3$}$$

By assigning N sufficiently large, we see that the quotient r of the blow-up rate of $\|\nabla^2\tilde v\|_0$ with respect to the rate of decay of $\|\tilde{\mathcal{D}}\|_0$ can be taken arbitrarily close to 0 in Theorem 1.3 and arbitrarily close to $\frac{1}{2\left(1-\frac{1}{2^k}\right)}$ in Theorem 1.4. Since the Hölder regularity exponent equals $\frac{1}{1+2r}$ (see section 5 and Theorem 5.1), this implies the respective ranges in (1.4).

The layout of the paper is as follows. In section 2 we gather the preparatory results: the mollification and the commutator estimates, followed by two decompositions of the $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$-valued matrix fields on $\omega\subset\mathbb{R}^2$, into a symmetric gradient and a multiple of Id_2. The first decomposition, used towards Theorem 1.3, passes through solving the Poisson equation with Dirichlet boundary data and thus it does not commute with differentiation. The second decomposition, on which Theorem 1.4 relies, uses convolution with the Poisson kernel and thus it has the desired properties including commuting with differentiation. We then present two convex integration "step" constructions: the first one with Nash's spirals as the oscillatory perturbations added to the fields (v, w), and the second one with Kuiper's corrugations.

Section 3 contains a proof of Theorem 1.3, where Källen's iteration procedure is used in codimension $k \geq 4$ allowing for an almost complete absorption of the diagonalized deficit field $\mathcal{D}$ in a single step. Section 4 contains the proof of Theorem 1.4, which combines iterating on the codimension with Källen's iterations within each step, that instead of absorbing the full error as explained before, essentially transfers one of its (diagonal) modes onto the other existing mode. In [1] this observation allowed to argue that a single convex integration step, necessitating only one codimension $k = 1$, suffices to absorb the first order deficit, thus yielding the regularity exponent $1/3$. Presently, we iterate this construction over k codimensions, superposing perturbation with appropriately increasing frequencies. Each iteration then reduces the deficit by a factor commensurate with the ratio of the current and the previous perturbation frequencies, while the second derivative of v increases by the factor of the next frequency times the square root of the deficit, see the estimates $(4.4)_1$ - $(4.4)_3$. This leads to the relative estimates in $(1.8)_2$ and $(1.8)_3$ whereas the C^1 norm of the accumulated perturbation remains controlled as in $(1.8)_1$. An interpolation to $\mathcal{C}^{1,\alpha}$ and the Nash-Kuiper iteration on stages is the content of section 5, where we complete the proof of Theorem 1.1, frequently referring to [12].

We point out that our presentation is modular, namely we separate the "stage" estimates in Theorems 1.3 and 1.4, from the Nash-Kuiper iteration in Theorem 5.1. This latter construction automatically yields a density result with the maximal Hölder regularity exponent given in function of the respective rates of second derivative blow-up and of the deficit decrease in a stage. This way, a future improvement of the stage estimates will directly yield the regularity improvement as well. This important point was more convoluted in the presentation of [1].

1.1. **Notation.** By $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ we denote the space of symmetric 2×2 matrices. The space of Hölder continuous vector fields $\mathcal{C}^{m,\alpha}(\bar{\omega}, \mathbb{R}^k)$ consists of restrictions of all $f \in \mathcal{C}^{m,\alpha}(\mathbb{R}^2, \mathbb{R}^k)$ to the closure of an open, bounded domain $\omega \subset \mathbb{R}^2$. The $\mathcal{C}^m(\bar{\omega}, \mathbb{R}^k)$ norm of such restriction is denoted by $\|f\|_m$, while its Hölder norm in $\mathcal{C}^{m,\gamma}(\bar{\omega}, \mathbb{R}^k)$ is $\|f\|_{m,\gamma}$. By C we denote a universal constant which may change from line to line, but it depends only on the specified parameters.

2. Preparatory statements

In this section, we gather the regularization, decomposition and perturbation statements that will be used in the course of the convex integration constructions. The first lemma below consists of the basic convolution estimates and the commutator estimate from [3]:

Lemma 2.1. *Let $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^d, \mathbb{R})$ be a standard mollifier that is nonnegative, radially symmetric, supported on the unit ball $B(0,1) \subset \mathbb{R}^d$ and such that $\int_{\mathbb{R}^d} \phi \,\mathrm{d}x = 1$. Denote:*

$$\phi_l(x) = \frac{1}{l^d}\phi(\frac{x}{l}) \quad \textit{for all } \; l \in (0,1], \; x \in \mathbb{R}^d.$$

Then, for every $f, g \in \mathcal{C}^0(\mathbb{R}^d, \mathbb{R})$ and every $m, n \geq 0$ and $\beta \in (0,1]$ there holds:

$$\|\nabla^{(m)}(f * \phi_l)\|_0 \leq \frac{C}{l^m}\|f\|_0, \tag{2.1$_1$}$$

$$\|f - f * \phi_l\|_0 \leq C \min\big\{l^2\|\nabla^2 f\|_0, l\|\nabla f\|_0, l^\beta\|f\|_{0,\beta}\big\}, \tag{2.1$_2$}$$

$$\|\nabla^{(m)}\big((fg) * \phi_l - (f * \phi_l)(g * \phi_l)\big)\|_0 \leq Cl^{2-m}\|\nabla f\|_0\|\nabla g\|_0, \tag{2.1$_3$}$$

with a constant $C > 0$ depending only on the differentiability exponent m.

The next two auxiliary results are specific to dimension $d = 2$. They allow for the decomposition of the given defect into a multiple of Id_2 (thus two primitive defects of rank 1) and a symmetric gradient, in agreement with the local conformal invariance of any Riemann metric:

Lemma 2.2. *Let $\omega \subset \mathbb{R}^2$ be an open, bounded and Lipschitz set. There exist maps:*

$$\bar{\Psi} : L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \to H^1(\omega, \mathbb{R}^2), \qquad \bar{a} : L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \to L^2(\omega, \mathbb{R}),$$

which are linear, continuous, and such that:

(i) *for all $D \in L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ there holds: $D = \bar{a}(D)\mathrm{Id}_2 + \mathrm{sym}\nabla\big(\bar{\Psi}(D)\big)$,*

(ii) *$\bar{\Psi}(\mathrm{Id}_2) \equiv 0$ and $\bar{a}(\mathrm{Id}_2) \equiv 1$ in ω,*

(iii) *for all $m \geq 0$ and $\gamma \in (0,1)$, if ω is $\mathcal{C}^{m+2,\gamma}$ regular then the maps $\bar{\Psi}$ and $\bar{a}$ are continuous from $\mathcal{C}^{m,\gamma}(\bar{\omega}, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ to $\mathcal{C}^{m+1,\gamma}(\bar{\omega}, \mathbb{R}^2)$ and to $\mathcal{C}^{m,\gamma}(\bar{\omega}, \mathbb{R})$, respectively, so that:*

$$\|\bar{\Psi}(D)\|_{m+1,\gamma} \leq C\|D\|_{m,\gamma} \;\; \textit{and} \;\; \|\bar{a}(D)\|_{m,\gamma} \leq C\|D\|_{m,\gamma} \qquad \textit{for all} \;\; D \in L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}). \tag{2.2}$$

The constants C above depend on ω, m, γ but not on D. Also, there exists $l_0 > 0$ depending only on ω, such that (2.2) are uniform on the closed l-neighbourhoods $\{\bar{\omega} + \bar{B}_l(0)\}_{l\in(0,l_0)}$ of ω.

The proof of the above [2, Proposition 3.1] is direct, namely one may assign: $\bar{a}(D) = D_{11} - \partial_{11}\Delta^{-1}(D_{11} - D_{12}) - \partial_{12}\Delta^{-1}(2D_{12})$, where Δ^{-1} corresponds to solving the Poisson problem on ω with the Dirichlet boundary condition. Consequently, $\bar{a}$ does not commute with taking partial derivatives of the input matrix field D. An improvement of this construction, which is crucial for the proof in [1], is given in:

Lemma 2.3. *Given a radius $R > 0$ and an exponent $\gamma \in (0,1)$, define the linear space E consisting of $\mathcal{C}^{0,\gamma}$-regular, $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$-valued matrix fields D on the ball $\bar{B}_R \subset \mathbb{R}^2$, whose traceless part $\dot{D} = D - \frac{1}{2}(\mathrm{trace}\, D)\mathrm{Id}_2$ is compactly supported in B_R. There exist linear maps $\bar{\Psi}$, $\bar{a}$ in:*

$$\begin{aligned} &\bar{\Psi} : E \to \mathcal{C}^{1,\gamma}(\bar{B}_R, \mathbb{R}^2), \qquad \bar{a} : E \to \mathcal{C}^{0,\gamma}(\bar{B}_R), \\ &E = \big\{D \in \mathcal{C}^{0,\gamma}(\bar{B}_R, \mathbb{R}^{2\times 2}_{\mathrm{sym}});\ \dot{D} \in \mathcal{C}^{0,\gamma}_c(B_R, \mathbb{R}^{2\times 2}_{\mathrm{sym}})\big\}, \end{aligned}$$

with the following properties:

(i) *for all $D \in E$ there holds: $D = \bar{a}(D)\mathrm{Id}_2 + \mathrm{sym}\nabla\big(\bar{\Psi}(D)\big)$,*

(ii) *$\bar{\Psi}(\mathrm{Id}_2) \equiv 0$ and $\bar{a}(\mathrm{Id}_2) \equiv 1$ in B_R,*

(iii) *$\|\bar{\Psi}(D)\|_{1,\gamma} \leq C\|\dot{D}\|_{0,\gamma}$ and $\|\bar{a}(D)\|_{0,\gamma} \leq C\|D\|_{0,\gamma}$ with constants C depending on R, γ,*

(iv) *for all $m \geq 1$, if $D \in E \cap \mathcal{C}^{m,\gamma}(\bar{B}_R, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ then $\bar{\Psi}(D) \in \mathcal{C}^{m+1,\gamma}(\bar{B}_R, \mathbb{R}^2)$ and $\bar{a}(D) \in \mathcal{C}^{m,\gamma}(\bar{B}_R)$, and we have:*

$$\partial_I \bar{\Psi}(D) = \bar{\Psi}(\partial_I D), \quad \partial_I \bar{a}(D) = \bar{a}(\partial_I D) \qquad \textit{for all} \;\; |I| \leq m.$$

(v) *for all $m \geq 1$, if $D \in E \cap \mathcal{C}^{m,\gamma}(\bar{B}_R, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ and if additionally $D_{22} = 0$ in $\bar{B}_R$, then:*

$$\|\partial_2^{(s)}\bar{a}(D)\|_{0,\gamma} \leq C\|\partial_2^{(s)}D\|_{0,\gamma}, \quad \|\partial_1^{(t+1)}\partial_2^{(s)}\bar{a}(D)\|_{0,\gamma} \leq C\|\partial_1^{(t)}\partial_2^{(s+1)}D\|_{0,\gamma},$$

for all $s, t \geq 0$ such that $s \leq m$, $t + s + 1 \leq m$, and with C depending only on R, γ.

Proof. For $f \in \mathcal{C}_c(\mathbb{R}^2)$ we set $\psi[f] = \Gamma * f \in \mathcal{C}^1(\mathbb{R}^2)$ with $\Gamma(x) = \frac{1}{2\pi}\log|x|$, namely:

$$\psi[f](x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x - y| f(y)\,\mathrm{d}y.$$

Recall [6] that the operator $\psi[\cdot]$ is well defined, linear and satisfies for any $\gamma\in(0,1)$ and $m\geq 1$:

$$\text{if } f\in\mathcal{C}^{0,\gamma}_c(\mathbb{R}^2) \text{ then } \psi[f]\in\mathcal{C}^{2,\gamma}(\mathbb{R}^2) \text{ and } \Delta\psi[f]=f \text{ in } \mathbb{R}^2, \tag{2.3$_1$}$$

$$\begin{split}&\text{if } f\in\mathcal{C}^{m,\gamma}_c(\mathbb{R}^2) \text{ then } \psi[f]\in\mathcal{C}^{m+2,\gamma}(\mathbb{R}^2)\\ &\qquad\qquad \text{and } \partial_I\psi[f]=\psi[\partial_I f] \text{ for any multiindex } I \text{ with } |I|\leq m.\end{split} \tag{2.3$_2$}$$

Moreover, if $\operatorname{supp} f\subset B_R$ then the following holds with a constant C depending only of R,γ:

$$\|\nabla\psi[f]\|_{\mathcal{C}^{1,\gamma}(\bar B_R)}\leq C\|f\|_{0,\gamma} \tag{2.3$_3$}$$

Given $D: B_R\to\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ such that $\dot D\in\mathcal{C}_c(B_R,\mathbb{R}^{2\times2}_{\mathrm{sym}})$, define the continuous vector field on $\mathbb{R}^2$:

$$\bar\Psi(D)=\bar\Psi(\dot D)=\Big(\partial_1\psi[D_{11}-D_{22}]+\partial_2\psi[2D_{12}],\ \partial_1\psi[2D_{12}]-\partial_2\psi[D_{11}-D_{22}]\Big).$$

If $\dot D\in\mathcal{C}^{0,\gamma}_c(B_R,\mathbb{R}^{2\times2}_{\mathrm{sym}})$ then $(2.3)_1$ implies that $\bar\Psi(D)\in\mathcal{C}^{1,\gamma}(\mathbb{R}^2,\mathbb{R}^2)$, together with $\partial_1\bar\Psi^1-\partial_2\bar\Psi^2=\Delta\psi[D_{11}-D_{22}]=D_{11}-D_{22}$ and $\partial_1\bar\Psi^2+\partial_2\bar\Psi^1=\Delta\psi[2D_{12}]=2D_{12}$, so that:

$$D-\operatorname{sym}\nabla(\bar\Psi(D))=\big(D_{11}-\partial_1\bar\Psi^1(D)\big)\mathrm{Id}_2=\big(D_{22}-\partial_2\bar\Psi^2(D)\big)\mathrm{Id}_2.$$

Consequently, there follows (i) and (ii) if we set, for all $D\in E$:

$$\bar a(D)=D_{11}-\partial_1\bar\Psi^1(D)=D_{22}-\partial_2\bar\Psi^2(D).$$

Clearly, (iii) follows from $(2.3)_3$ while (iv) is a consequence of $(2.3)_2$. For (v), the above definition of $\bar a$ and the assumption $D_{22}=0$, yield:

$$\begin{split}\partial_2^{(s)}\bar a(D)&=-\partial_2^{(s+1)}\bar\Psi^2(D)=-\partial_{12}\psi[2\partial_2^{(s)}D_{12}]+\partial_{22}\psi[\partial_2^{(s)}(D_{11}-D_{22})],\\ \partial_1^{(t+1)}\partial_2^{(s)}\bar a(D)&=-\partial_1^{(t+1)}\partial_2^{(s+1)}\bar\Psi^2(D)\\ &=-\partial_{11}\psi[2\partial_1^{(t)}\partial_2^{(s+1)}D_{12}]+\partial_{12}\psi[\partial_1^{(t)}\partial_2^{(s+1)}(D_{11}-D_{22})].\end{split}$$

This ends the proof in virtue of $(2.3)_3$. ■

As the final preparatory result, we recall two different single "step" constructions from [11]. The first one uses Nash's spirals, necessitating two codimension directions in order to cancel each of the non-zero entries of the given nonnegative defect in the diagonal form:

Lemma 2.4. *Assume that $k\geq 4$. Let $v\in\mathcal{C}^2(\mathbb{R}^2,\mathbb{R}^k)$, $w\in\mathcal{C}^1(\mathbb{R}^2,\mathbb{R}^2)$, $\lambda>0$ and $a\in\mathcal{C}^2(\mathbb{R}^2)$. Denote $\Gamma(t)=\sin t$, $\bar\Gamma(t)=\cos t$ and define:*

$$\begin{split}\tilde v&=v+\frac{a(x)}{\lambda}\Big(\Gamma(\lambda x_1)e_1+\bar\Gamma(\lambda x_1)e_2+\Gamma(\lambda x_2)e_3+\bar\Gamma(\lambda x_3)e_4\Big),\\ \tilde w&=w-\frac{a(x)}{\lambda}\Big(\Gamma(\lambda x_1)\nabla v^1+\bar\Gamma(\lambda x_1)\nabla v^2+\Gamma(\lambda x_2)\nabla v^3+\bar\Gamma(\lambda x_2)\nabla v^4\Big).\end{split} \tag{2.4}$$

Then, the following identity is valid on $\mathbb{R}^2$:

$$\begin{split}&\big(\frac12(\nabla\tilde v)^T\nabla\tilde v+\operatorname{sym}\nabla\tilde w\big)-\big(\frac12(\nabla v)^T\nabla v+\operatorname{sym}\nabla w\big)-\frac{a(x)^2}{2}\mathrm{Id}_2\\ &=-\frac{a}{\lambda}\Big(\Gamma(\lambda x_1)\nabla^2v^1+\bar\Gamma(\lambda x_1)\nabla^2v^2+\Gamma(\lambda x_2)\nabla^2v^3+\bar\Gamma(\lambda x_2)\nabla^2v^4\Big)+\frac{1}{\lambda^2}\nabla a\otimes\nabla a.\end{split} \tag{2.5}$$

The second "step" construction uses Kuiper's corrugations, in which a single codimension is used to cancel one rank-one defect of the form $a(x)^2e_i\otimes e_i$:

Lemma 2.5. *Let $v \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^k)$, $w \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$, $\lambda > 0$ and $a \in \mathcal{C}^2(\mathbb{R}^2)$ be given. Denote: $\Gamma(t) = 2\sin t$, $\bar\Gamma(t) = -\frac{1}{2}\sin(2t)$, and for a fixed $i = 1 \dots 2$ and $j = 1 \dots k$ define:*

$$\tilde v = v + \frac{a(x)}{\lambda}\Gamma(\lambda x_i) e_j, \qquad \tilde w = w - \frac{a(x)}{\lambda}\Gamma(\lambda x_i)\nabla v^j + \frac{a(x)^2}{\lambda}\bar\Gamma(\lambda x_i) e_i. \tag{2.6}$$

Then, the following identity is valid on $\mathbb{R}^2$:

$$\begin{aligned} &\big(\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v + \mathrm{sym}\nabla\tilde w\big) - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big) - a(x)^2 e_i\otimes e_i \\ &= -\frac{a}{\lambda}\Gamma(\lambda x_i)\nabla^2 v^j + \frac{1}{2\lambda^2}\Gamma(\lambda x_i)^2\nabla a\otimes\nabla a - \frac{1}{\lambda}\bar\Gamma(\lambda x_i)\,\mathrm{sym}\big(\nabla(a^2)\otimes e_i\big). \end{aligned} \tag{2.7}$$

3. A proof of Theorem 1.3

In the proof below, the constants $C > 1$ may change from line to line, but they depend only on ω, γ, m and r (and thus, ultimately, on N), unless specified otherwise.

Proof of Theorem 1.3

1. (Preparing the data) Let l_0 be as in Lemma 2.2, and ϕ_l as in Lemma 2.1. For $l \in (0, l_0]$ we define the following smoothed data functions on the l-thickened set $\bar\omega + \bar B_l(0)$:

$$v_0 = v * \phi_l, \quad w_0 = w * \phi_l, \quad A_0 = A * \phi_l, \quad \mathcal{D}_0 = A_0 - \big(\frac{1}{2}(\nabla v_0)^T\nabla v_0 + \mathrm{sym}\nabla w_0\big).$$

From Lemma 2.1, we deduce the initial bounds, where constants C depend only on m and ω:

$$\|v_0 - v\|_1 + \|w_0 - w\|_1 \le ClM, \tag{3.1$_1$}$$

$$\|A_0 - A\|_0 \le Cl^\beta\|A\|_{0,\beta}, \tag{3.1$_2$}$$

$$\|\nabla^{(m+1)}v_0\|_0 + \|\nabla^{(m+1)}w_0\|_0 \le \frac{C}{l^m}lM \quad \text{for all } m \ge 1, \tag{3.1$_3$}$$

$$\|\nabla^{(m)}\mathcal{D}_0\|_0 \le \frac{C}{l^m}\big(\|\mathcal{D}\|_0 + (lM)^2\big) \quad \text{for all } m \ge 0. \tag{3.1$_4$}$$

Indeed, $(3.1)_1$, $(3.1)_2$ follow from $(2.1)_2$ and in view of the lower bound on M. Similarly, $(3.1)_3$ follows by applying $(2.1)_1$ to $\nabla^2 v$ and $\nabla^2 w$ with the differentiability exponent $m-1$. Since:

$$\mathcal{D}_0 = \mathcal{D} * \phi_l - \frac{1}{2}\big((\nabla v_0)^T\nabla v_0 - ((\nabla v)^T\nabla v) * \phi_l\big),$$

we get $(3.1)_4$ by applying $(2.1)_1$ to $\mathcal{D}$, and $(2.1)_3$ to ∇v.

2. (Induction definition: iterative decomposition of deficits) Let the linear maps $\bar a$, $\bar\Psi$ be as in Lemma 2.2. Also, let $r_0 > 0$ be a constant depending on ω, γ, such that $\|\bar a(D) - 1\|_0 \le \frac{1}{2}$ whenever $\|D - \mathrm{Id}_2\|_{0,\gamma} \le r_0$. For $r = 0 \dots N$ we iteratively define the perturbation amplitudes $a_r \in \mathcal{C}^\infty(\bar\omega + \bar B_l(0), \mathbb{R})$ and the correction fields $\Psi_r \in \mathcal{C}^\infty(\bar\omega + \bar B_l(0), \mathbb{R}^2)$ by setting:

$$\begin{aligned} &a_0 = 0, \qquad a_r = \Big(2\bar a\big(\tilde C\mathrm{Id}_2 + \mathcal{D}_0 - \mathcal{E}_{r-1}\big)\Big)^{1/2}, \quad \Psi_r = \bar\Psi\big(\tilde C\mathrm{Id}_2 + \mathcal{D}_0 - \mathcal{E}_{r-1}\big), \\ &\text{with } \ \tilde C = \frac{2}{r_0}\Big(\|\mathcal{D}_0\|_{0,\gamma} + \frac{1}{l^\gamma}\big(\|\mathcal{D}\|_0 + (lM)^2\big)\Big), \end{aligned}$$

and with the error fields $\mathcal{E}_r \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ given by the right hand side of (2.5), namely:

$$\mathcal{E}_r = -\frac{a_r}{\lambda}\Big(\Gamma(\lambda x_1)\nabla^2 v^1 + \bar\Gamma(\lambda x_1)\nabla^2 v^2 + \Gamma(\lambda x_2)\nabla^2 v^3 + \bar\Gamma(\lambda x_2)\nabla^2 v^4\Big) + \frac{1}{\lambda^2}\nabla a_r\otimes\nabla a_r.$$

Our definition of a_r is correctly posed if only $\|\mathcal{D}_0 - \mathcal{E}_{r-1}\|_{0,\gamma} \leq r_0\tilde{C}$. To this end, we right away observe that $\|\mathcal{D}_0\|_{0,\gamma} \leq r_0\tilde{C}/2$, while we will prove that the second condition in (1.6) implies:

$$\|\mathcal{E}_r\|_{0,\gamma} \leq \frac{r_0\tilde{C}}{2} \quad \text{for all } \ r = 0\dots N-1. \tag{3.2}$$

Note that then automatically there holds for all $r = 1\dots N$:

$$\begin{aligned} &\tilde{C}\mathrm{Id}_2 + \mathcal{D}_0 - \mathcal{E}_{r-1} = \frac{1}{2}(a_r)^2\mathrm{Id}_2 + \mathrm{sym}\nabla\Psi_r \\ &\text{and} \qquad (a_r)^2 \in [\tilde{C}, 3\tilde{C}] \text{ in } \bar{\omega} + \bar{B}_l(0). \end{aligned} \tag{3.3}$$

For the future estimate of derivatives of a_r of order $m \geq 1$, we use Faá di Bruno's formula, $(3.1)_4$ and the bound (2.2) in Lemma 2.2 in:

$$\begin{aligned} \|\nabla^{(m)} a_r\|_0 & \leq C\Big\| \sum_{p_1+2p_2+\dots mp_m=m} a_r^{2(1/2-p_1-\dots-p_m)} \prod_{t=1}^m \big|\nabla^{(t)}(a_r)^2\big|^{p_t}\Big\|_0 \\ & \leq C \sum_{p_1+2p_2+\dots mp_m=m} \frac{1}{\tilde{C}^{(p_1+\dots+p_m)-1/2}} \prod_{t=1}^m \big(\tilde{C} + \|\mathcal{D}_0\|_{t,\gamma} + \|\mathcal{E}_{r-1}\|_{t,\gamma}\big)^{p_t} \\ & \leq C\tilde{C}^{1/2} \sum_{p_1+2p_2+\dots mp_m=m} \prod_{t=1}^m \Big(\frac{1}{l^t} + \frac{\|\mathcal{E}_{r-1}\|_{t,\gamma}}{\tilde{C}}\Big)^{p_t}, \end{aligned} \tag{3.4}$$

Additionally, applying Faá di Bruno's formula to the inverse rather than square root, we get:

$$\|\nabla^{(m)}\Big(\frac{1}{a_r + a_{r-1}}\Big)\|_0 \leq \frac{C}{\tilde{C}^{1/2}} \sum_{p_1+2p_2+\dots mp_m=m} \prod_{t=1}^m \Big(\frac{\|\nabla^{(t)}(a^r + a^{r-1})\|_0}{\tilde{C}^{1/2}}\Big)^{p_t} \tag{3.5}$$

The formulas (3.4), (4.10) hold for all $r = 1\dots N$ with constants C depending on ω, γ, m.

3. (Inductive estimates) In steps 4-5 we will prove the following estimates, valid for all $r = 1\dots N$, with constants C depending, in line with our convention, only on ω, γ, m, r:

$$\|a_r\|_0 \leq C\tilde{C}^{1/2} \tag*{$(3.6)_1$}$$

$$\|\nabla^{(m)} a_r\|_0 \leq C\frac{\lambda^m\lambda^\gamma}{\lambda l}\tilde{C}^{1/2} \quad \text{for all } \ m \geq 1, \tag*{$(3.6)_2$}$$

$$\|\nabla^{(m)}\mathcal{E}_r\|_0 \leq C\frac{\lambda^m}{\lambda l}\tilde{C} \quad \text{for all } \ m \geq 0, \tag*{$(3.6)_3$}$$

$$\|\nabla^{(m)}\big(\mathcal{E}_r - \mathcal{E}_{r-1}\big)\|_0 \leq C\frac{\lambda^m\lambda^{2(r-1)\gamma}}{(\lambda l)^r}\tilde{C} \quad \text{for all } \ m \geq 0. \tag*{$(3.6)_4$}$$

In general, $C \to \infty$ as $m \to \infty$ or $r \to \infty$, so it is crucial that eventually only finitely many of bounds above are used. We now check that $(3.6)_1$, $(3.6)_2$ are already valid at $r = 1$. Indeed, $(3.6)_1$ is a consequence of (3.3) since $\mathcal{E}_0 = 0$, whereas $(3.6)_2$ follows from (3.4):

$$\|\nabla^{(m)} a_1\|_0 \leq C\frac{\tilde{C}^{1/2}}{l^m} \leq C\frac{\lambda^m}{\lambda l}\tilde{C}^{1/2}.$$

We now observe that $(3.6)_3$ always follows from $(3.6)_1$ and $(3.6)_2$, because:

$$\begin{aligned}\|\nabla^{(m)}\mathcal{E}_r\|_0 &\leq C\sum_{p+q+t=m}\lambda^{p-1}\|\nabla^{(q)}a_r\|_0\|\nabla^{(t+2)}v_0\|_0 + C\sum_{q+t=m}\lambda^{-2}\|\nabla^{(q+1)}a_r\|_0\|\nabla^{(t+1)}a_1\|_0\\ &\leq C\Big(\sum_{p+t=m}\frac{\lambda^{p-1}}{l^{t+1}}\tilde{C}^{1/2}(lM) + \sum_{p+q+t=m,q\neq 0}\frac{\lambda^{p+q-1}\lambda^{\gamma}}{(\lambda l)l^{t+1}}\tilde{C}^{1/2}(lM) + \sum_{q+t=m}\frac{\lambda^{q+t+2}\lambda^{2\gamma}}{\lambda^2(\lambda l)^2}\tilde{C}\Big)\\ &\leq C\lambda^m\Big(\sum_{p+t=m}\frac{1}{(\lambda l)^{t+1}} + \sum_{p+q+t=m,q\neq 0}\frac{\lambda^{\gamma}}{(\lambda l)^{t+2}} + \sum_{q+t=m}\frac{\lambda^{2\gamma}}{(\lambda l)^2}\Big)\tilde{C} \leq C\frac{\lambda^m}{\lambda l}\tilde{C},\end{aligned}$$

if only $\lambda^{2\gamma}\leq\lambda l$. Strengthening this working assumption to:

$$\frac{\lambda l}{\lambda^{2\gamma}} \geq \frac{2C}{r_0}, \tag{3.7}$$

we additionally arrive at (3.2). Concluding and since $(3.6)_3$, $(3.6)_4$ are equivalent at $r=1$, we note that we have proven $(3.6)_1$ - $(3.6)_4$ and (3.2) at their lowest counter r value.

4. (Proof of the inductive estimates) Assume that the bounds $(3.6)_1$–$(3.6)_4$ hold up to some $1\leq r\leq N-1$. We will prove their validity at $r+1$. By (3.2) and (3.3) we directly get $(3.6)_1$, whereas (3.4) and $(3.6)_3$ yield $(3.6)_2$, since for all $m\geq 1$:

$$\begin{aligned}\|\nabla^{(m)}a_{r+1}\|_0 &\leq C\tilde{C}^{1/2}\sum_{p_1+2p_2+\ldots mp_m=m}\prod_{t=1}^m\Big(\frac{1}{l^t}+\frac{\lambda^t\lambda^{\gamma}}{\lambda l}\Big)^{p_t}\\ &\leq C\tilde{C}^{1/2}\sum_{p_1+2p_2+\ldots mp_m=m}\prod_{t=1}^m\Big(\frac{\lambda^t\lambda^{\gamma}}{\lambda l}\Big)^{p_t}\\ &= C\tilde{C}^{1/2}\lambda^m\sum_{p_1+2p_2+\ldots mp_m=m}\Big(\frac{\lambda^{\gamma}}{\lambda l}\Big)^{p_1+p_2+\ldots p_m} \leq C\frac{\lambda^m\lambda^{\gamma}}{\lambda l}\tilde{C}^{1/2}.\end{aligned}$$

We have already justified $(3.6)_3$ and (3.2) in the previous step, so it remains to show $(3.6)_4$. Towards this end, note first the rough bound below, in view of (4.10) and $(3.6)_1$–$(3.6)_4$:

$$\|\nabla^{(m)}\Big(\frac{1}{a_{r+1}+a_r}\Big)\|_0 \leq \frac{C}{\tilde{C}^{1/2}}\sum_{p_1+2p_2+\ldots mp_m=m}\prod_{t=1}^m\Big(\frac{\lambda^t\lambda^{\gamma}}{\lambda l}\Big)^{p_t} \leq \frac{C}{\tilde{C}^{1/2}}\lambda^m.$$

Further, since $(a_{r+1})^2-(a_r)^2 = -2\bar{a}(\mathcal{E}_r-\mathcal{E}_{r-1})$, it follows that:

$$\begin{aligned}\|\nabla^{(m)}\big(a_{r+1}-a_r\big)\|_0 &\leq C\sum_{q+t=m}\|\nabla^{(q)}\big((a_{r+1})^2-(a_r)^2\big)\|_0\|\nabla^{(t)}\Big(\frac{1}{a^{r+1}+a^r}\Big)\|_0\\ &\leq \frac{C}{\tilde{C}^{1/2}}\sum_{q+t=m}\|\mathcal{E}_r-\mathcal{E}_{r-1}\|_{q,\gamma}\lambda^t \leq C\tilde{C}^{1/2}\lambda^m\frac{\lambda^{(2r-1)\gamma}}{(\lambda l)^r},\end{aligned} \tag{3.8}$$

for all $m\geq 0$. Finally, writing:

$$\begin{aligned}\mathcal{E}_{r+1}-\mathcal{E}_r = &-\frac{a_{r+1}-a_r}{\lambda}\Big(\Gamma(\lambda x_1)\nabla^2 v_0^1 + \bar{\Gamma}(\lambda x_1)\nabla^2 v_0^2 + \Gamma(\lambda x_2)\nabla^2 v_0^3 + \bar{\Gamma}(\lambda x_2)\nabla^2 v_0^4\Big)\\ &+\frac{1}{\lambda^2}\Big(\big(\nabla a_{r+1}-\nabla a_r\big)\otimes\nabla a_{r+1} + \nabla a_r\otimes\big(\nabla a_{r+1}-\nabla a_r\big)\Big),\end{aligned}$$

we conclude, in virtue of (4.14), $(3.6)_2$ and $(3.1)_2$:

$$\begin{aligned}\|\nabla^{(m)}(\mathcal{E}_{r+1}-\mathcal{E}_r)\|_0 \leq C &\sum_{p+q+t=m} \lambda^{p-1}\|\nabla^{(q)}\big(a_{r+1}-a_r\big)\|_0\|\nabla^{(t+2)}v_0\|_0 \\ &+ C\sum_{q+t=m}\lambda^{-2}\|\nabla^{(q+1)}(a_{r+1}-a_r)\|_0\big(\|\nabla^{(t+1)}a_{r+1}\|_0+\|\nabla^{(t+1)}a_r\|_0\big) \\ \leq C\tilde C^{1/2}&\sum_{p+q+t=m}\lambda^{p+q-1}\frac{\lambda^{(2r-1)\gamma}}{(\lambda l)^r}\frac{lM}{l^{t+1}} + C\tilde C\sum_{q+t=m}\lambda^{-2}\frac{\lambda^{q+1}\lambda^{(2r-1)\gamma}}{(\lambda l)^r}\frac{\lambda^{t+1}\lambda^\gamma}{\lambda l} \\ \leq C\tilde C\lambda^m &\sum_{p+q+t=m}\frac{\lambda^{(2r-1)\gamma}}{(\lambda l)^r(\lambda l)^{t+1}} + C\tilde C\lambda^m\frac{\lambda^{2r\gamma}}{(\lambda l)^{r+1}} \leq C\frac{\lambda^m}{(\lambda l)^{r+1}}\tilde C.\end{aligned}$$

This ends the proof of $(3.6)_4$ and thus of all the inductive estimates, under (3.7).

5. (End of proof) Define $\tilde v\in\mathcal{C}^\infty(\bar\omega+\bar B_l(0),\mathbb{R}^k)$ and $\tilde w\in\mathcal{C}^\infty(\bar\omega+\bar B_l(0),\mathbb{R}^2)$ according to the "step" construction in Lemma 2.4, involving the periodic functions $\Gamma,\bar\Gamma$:

$$\begin{aligned}&\tilde v = v_0+\frac{a_N}{\lambda}\Big(\Gamma(\lambda x_1)e_1+\bar\Gamma(\lambda x_1)e_2+\Gamma(\lambda x_2)e_3+\bar\Gamma(\lambda x_2)e_4\Big),\\ &\tilde w = w_0-\frac{a_N}{\lambda}\Big(\Gamma(\lambda x_1)\nabla v_0^1+\bar\Gamma(\lambda x_1)\nabla v_0^2+\Gamma(\lambda x_2)\nabla^2 v_0^3+\bar\Gamma(\lambda x_2)\nabla v_0^4\Big)+\Psi_N-\tilde C id_2.\end{aligned}$$

We now show that $(3.6)_1$ - $(3.6)_4$ imply the type of bounds claimed in the Theorem. Observe first that, in virtue of (3.7):

$$\begin{aligned}&\|\Psi_N\|_1\leq C\|\tilde C\mathrm{Id}_2+\mathcal{D}_0-\mathcal{E}_{N-1}\|_{0,\gamma}\leq C\tilde C,\\ &\|\nabla^2\Psi_N\|_0\leq C\|\tilde C\mathrm{Id}_2+\mathcal{D}_0-\mathcal{E}_{N-1}\|_{1,\gamma}\leq C\Big(\tilde C+\frac{\tilde C}{l}+\frac{\lambda\lambda^\gamma}{\lambda l}\tilde C\Big)\leq C\lambda\tilde C.\end{aligned}$$

To prove $(1.7)_1$, we use the above, $(3.6)_1$, $(3.6)_2$ and $(3.1)_1$, $(3.1)_3$:

$$\begin{aligned}\|\tilde v-v\|_1 &\leq \|v_0-v\|_1+C\Big(\|a_N\|_0+\frac{\|\nabla a_N\|_0}{\lambda}\Big)\leq C\Big(lM+\tilde C^{1/2}+\frac{\lambda^\gamma}{\lambda l}\tilde C^{1/2}\Big)\leq C\tilde C^{1/2},\\ \|\tilde w-w\|_1 &\leq \|w_0-w\|_1+C\Big(\tilde C+\|a_N\|_0\|\nabla v_0\|_0+\frac{\|\nabla a_N\|_0\|\nabla v_0\|_0+\|a_N\|_0\|\nabla^2 v_0\|_0}{\lambda}\Big)+\|\Psi_N\|_1\\ &\leq C\Big(lM+\tilde C^{1/2}\|\nabla v_0\|_0+\frac{lM}{\lambda l}\tilde C^{1/2}+\frac{\lambda^\gamma}{\lambda l}\tilde C^{1/2}\|\nabla v_0\|_0+\tilde C\Big)\\ &\leq C\tilde C^{1/2}\big(1+\|\nabla v_0\|_0+lM+\tilde C^{1/2}\big)\leq C\tilde C^{1/2}\big(1+\tilde C^{1/2}+\|\nabla v\|_0\big).\end{aligned}$$

Similarly, there follows $(1.7)_2$:

$$\begin{aligned}\|\nabla^2\tilde v\|_0 &\leq \|\nabla^2 v_0\|_0+C\Big(\lambda\|a_N\|_0+\|\nabla a_N\|_0+\frac{\|\nabla^2 a_N\|_0}{\lambda}\Big)\\ &\leq C\Big(M+(\frac{\lambda^\gamma}{l}+\lambda)\tilde C^{1/2}\Big)\leq C\lambda\tilde C^{1/2},\\ \|\nabla^2\tilde w\|_0 &\leq \|\nabla^2 w_0\|_0+C\Big(\lambda\|a_N\|_0\|\nabla v_0\|_0+\big(\|\nabla a_N\|_0\|\nabla v_0\|_0+\|a_N\|_0\|\nabla^2 v_0\|_0\big)\\ &\qquad+\frac{\|\nabla^2 a_N\|_0\|\nabla v_0\|_0+\|\nabla a_N\|_0\|\nabla^2 v_0\|_0+\|a_N\|_0\|\nabla^3 v_0\|_0}{\lambda}\Big)+\|\nabla^2\Psi_N\|_0\\ &\leq C\Big(M+\big(1+\lambda+\frac{1}{l}\big)\tilde C^{1/2}(\|\nabla v\|_0+lM)+\tilde C^{1/2}M+\lambda\tilde C\Big)\\ &\leq C\lambda\tilde C^{1/2}\big(1+\tilde C^{1/2}+\|\nabla v\|_0\big).\end{aligned}$$

Finally, (2.5) and (3.3) yield $(1.7)_3$, in virtue of the decomposition:

$$\tilde{\mathcal{D}} = (A - A_0) + \mathcal{D}_0 - \Big(\frac{(a_N)^2}{2}\mathrm{Id}_2 + \mathcal{E}_N + \mathrm{sym}\nabla\Psi_N - \tilde{C}\mathrm{Id}_2\Big) = (A - A_0) - (\mathcal{E}_N - \mathcal{E}_{N-1}),$$

and further, in view of $(3.1)_2$, $(3.6)_4$:

$$\|\tilde{\mathcal{D}}\|_0 \leq \|A - A_0\|_0 + \|\mathcal{E}_N - \mathcal{E}_{N-1}\|_0 \leq C\Big(l^\beta\|A\|_{0,\beta} + C\frac{\lambda^{2(N-1)\gamma}}{(\lambda l)^N}\tilde{C}\Big).$$

We now summarize the obtained bounds, under the assumption $\frac{\lambda l}{\lambda^{2\gamma}} \geq C$, in the following form:

$$\begin{aligned}
&\|\tilde{v} - v\|_1 \leq C\lambda^{\gamma/2}\big(\|\mathcal{D}\|_0^{1/2} + lM\big),\\
&\|\tilde{w} - w\|_1 \leq C\lambda^{\gamma}\big(\|\mathcal{D}\|_0^{1/2} + lM\big)\big(1 + \|\mathcal{D}\|_0^{1/2} + lM + \|\nabla v\|_0\big),\\
&\|\nabla^2\tilde{v}\|_0 \leq C\lambda\lambda^{\gamma/2}\big(\|\mathcal{D}\|_0^{1/2} + lM\big),\\
&\|\nabla^2\tilde{w}\|_0 \leq C\lambda\lambda^{\gamma}\big(\|\mathcal{D}\|_0^{1/2} + lM\big)\big(1 + \|\mathcal{D}\|_0^{1/2} + lM + \|\nabla v\|_0\big),\\
&\|\tilde{\mathcal{D}}\|_0 \leq C\big(l^\beta\|A\|_{0,\beta} + \frac{\lambda^{2N\gamma}}{(\lambda l)^N}\big(\|\mathcal{D}\|_0 + (lM)^2\big).
\end{aligned}$$

The claimed $(1.7)_1$ - $(1.7)_3$ are obtained by rescaling $2N\gamma$ to γ. The proof is done. ■

4. A proof of Theorem 1.4

In the proof below, the constants $C > 1$ may change from line to line, but they depend only on ω, γ, k, m, s, t and r (and thus, ultimately, on N), unless specified otherwise.

Proof of Theorem 1.4

1. (Preparing the data) Fix $R, l_0 > 0$ so that $\bar{\omega} \subset B_R$ and $2l_0 < 1 < \mathrm{dist}(\bar{\omega}, \partial B_R)$. Given l, v, w, A as in the statement of the Theorem, we define the smoothed data on $\bar{\omega} + \bar{B}_{2l-l/(2k)}$:

$$v_0 = v * \phi_{l/(2k)}, \quad w_0 = w * \phi_{l/(2k)}, \quad A_0 = A * \phi_{l/(2k)}, \quad \mathcal{D}_0 = A_0 - \big(\frac{1}{2}(\nabla v_0)^T\nabla v_0 + \mathrm{sym}\nabla w_0\big),$$

where we used the mollifier $\phi_{l/(2k)}$ as in Lemma 2.1. Similarly to Step 1 of the proof of Theorem 1.3, there follow the bounds $(3.1)_1$ - $(3.1)_4$ with constants C depending only on ω, k and m.

We also set a cut-off function $\chi_0 \in \mathcal{C}_c^\infty(\omega + B_{2l-l/(2k)}, [0,1])$ with $\chi_0 = 1$ on $\bar{\omega} + \bar{B}_{2l-l/k}$, and for all $i = 1\ldots(k-1)$ we set the intermediate cut-off functions $\chi_i \in \mathcal{C}_c^\infty(\omega + B_{2l-il/k}, [0,1])$ with $\chi_i = 1$ on $\bar{\omega} + \bar{B}_{2l-(i+1)l/k}$. When $l_0 \ll 1$, it is possible to request that for any $f \in \mathcal{C}^m(\bar{\omega} + \bar{B}_{2l-il/k})$ and any multiindex I with $|I| \leq m$, there holds:

$$\|\partial_I(\chi_i f)\|_0 \leq C\sum_{I_1+I_2=I}\|\partial_{I_1}\chi_i\|_0\|\partial_{I_2}f\|_0 \leq C\sum_{I_1+I_2=I}\frac{1}{l^{|I_1|}}\|\partial_{I_2}f\|_0, \tag{4.1}$$

with constants C depending only on ω, k and m.

In the course of the proof below, we will inductively construct the intermediate data:

$$\begin{aligned}
&v_i \in \mathcal{C}^\infty(\bar{\omega} + \bar{B}_{2l-il/k}, \mathbb{R}^k), \quad w_i \in \mathcal{C}^\infty(\bar{\omega} + \bar{B}_{2l-il/k}, \mathbb{R}^2),\\
&\mathcal{D}_i = A_0 - \big(\frac{1}{2}(\nabla v_i)^T\nabla v_i + \mathrm{sym}\nabla w_i\big) \in \mathcal{C}^\infty(\bar{\omega} + \bar{B}_{2l-il/k}, \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \qquad \text{for all } \ i = 1\ldots k,
\end{aligned}$$

and, eventually, set $(\tilde v, \tilde w) = (v_k, w_k)$. All definitions will rely on the family of pairs of frequencies $\{\lambda_i, \bar\lambda_i\}_{i=0}^k$ that will be specified later, assumed to satisfy the monotonicity property:

$$\lambda_0 = \bar\lambda_0 = \frac{1}{l}, \quad \lambda = \lambda_1 \leq \bar\lambda_1 \ldots \leq \lambda_i \leq \bar\lambda_i \ldots \leq \lambda_k \leq \bar\lambda_k, \tag{4.2}$$

and on the positive constants (where we set $\tilde C_{-1} = 0$):

$$\tilde C_i = \frac{2}{r_0}\|\chi_i \mathcal{D}_i\|_{0,\gamma} + \bar\lambda_i^\gamma \frac{\big(\|\mathcal{D}\|_0 + (lM)^2\big)}{(\bar\lambda_i l)^2} + \tilde C_{i-1}\bar\lambda_i^\gamma\Big(\frac{\lambda_i^{(N-1)\gamma}}{(\lambda_i/\bar\lambda_{i-1})^N} + \frac{\lambda_i}{\bar\lambda_i}\Big) \quad \text{for } \ i = 0\ldots k. \tag{4.3}$$

The first term above is crucial, while the other two terms follow from the technical considerations and are of at most the same order. We will show that for all $i = 1\ldots k$ there holds:

$$\|v_i - v\|_1 \leq C\tilde C_0^{1/2}, \qquad \|w_i - w\|_1 \leq C\tilde C_0^{1/2}\big(1 + \tilde C_0^{1/2} + \|\nabla v\|_0\big), \tag{4.4$_1$}$$

$$\|\nabla^2 v_i\|_0 \leq C\sum_{j=0}^{i-1}\tilde C_j^{1/2}\bar\lambda_{j+1}, \quad \|\nabla^2 w_i\|_0 \leq C\sum_{j=0}^{i-1}\tilde C_j^{1/2}\bar\lambda_{j+1}\big(1 + \tilde C_0^{1/2} + \|\nabla v\|_0\big), \tag{4.4$_2$}$$

$$\|\partial_1^{(t)}\partial_2^{(s)}\mathcal{D}_i\|_0 \leq C\frac{\tilde C_i}{\bar\lambda_i^\gamma}\lambda_i^t\bar\lambda_i^s \qquad \text{for all } \ s,t \geq 0. \tag{4.4$_3$}$$

Observe that (4.1) and $(4.4)_3$ result in:

$$\|\partial_1^{(t)}\partial_2^{(s)}(\chi_i\mathcal{D}_i)\|_{0,\gamma} \leq C\tilde C_i\lambda_i^t\bar\lambda_i^s \qquad \text{for all } \ s,t \geq 0. \tag{4.5}$$

The bounds $(4.4)_3$, (4.5) are already satisfied at $i = 0$ because of $(3.1)_1$ - $(3.1)_4$, and we have:

$$\tilde C_0 \leq C\frac{1}{l^\gamma}\big(\|\mathcal{D}\|_0 + (lM)^2\big). \tag{4.6}$$

2. (Induction definition: iterative decomposition of deficits) We now define the main quantities in the construction of (v_i, w_i) from (v_{i-1}, w_{i-1}) for $i = 1\ldots k$.

Let the linear maps $\bar a, \bar\Psi$ be as in Lemma 2.3. Also, let $r_0 > 0$ be a constant depending on ω, γ, such that $\|\bar a(D) - 1\|_0 \leq \frac{1}{2}$ whenever $\|D - \mathrm{Id}_2\|_{0,\gamma} \leq r_0$. For all $i = 1\ldots k$ and $r = 0\ldots N$, we define the perturbation amplitudes $a_r^i \in \mathcal{C}^\infty(\bar B_R)$ and the corrections $\Psi_r^i \in \mathcal{C}^\infty(\bar B_R, \mathbb{R}^2)$ by:

$$\begin{aligned} a_0^{i+1} = 0, \qquad (a_r^{i+1})^2 &= \bar a\Big(\tilde C_i \mathrm{Id}_2 + \chi_i\mathcal{D}_i - \chi_i\mathcal{E}_{r-1}^{i+1}\Big), \\ \Psi_r^{i+1} &= \bar\Psi\Big(\tilde C_i \mathrm{Id}_2 + \chi_i\mathcal{D}_i - \chi_i\mathcal{E}_{r-1}^{i+1}\Big), \qquad \text{for } \ i = 0\ldots k-1, \ r = 1\ldots N. \end{aligned}$$

The error fields $\mathcal{E}_r^i \in \mathcal{C}^\infty(\bar\omega + \bar B_{2l-l/(2k)}, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ above are given by the right hand side of (2.7) (taken with $i, j = 1$) after removing their $[\cdot]_{22}$ entries, namely:

$$\begin{aligned} \mathcal{E}_r^i = &-\frac{a_r^i}{\lambda_i}\Gamma(\lambda_i x_1)\big(\nabla^2 v_0^i - (\partial_{22}v_0^i)e_2^{\otimes 2}\big) + \frac{1}{2\lambda_i^2}\Gamma(\lambda_i x_1)^2\big(\nabla a_r^i \otimes \nabla a_r^i - (\partial_2 a_r^i)^2 e_2^{\otimes 2}\big) \\ &- \frac{1}{\lambda_i}\bar\Gamma(\lambda_i x_1)\,\mathrm{sym}\big(\nabla(a_r^i)^2 \otimes e_1\big) \quad \text{for } \ i = 1\ldots k. \end{aligned}$$

Note that $\tilde C_i \mathrm{Id}_2 + \chi_i\mathcal{D}_i - \chi_i\mathcal{E}_{r-1}^{i+1}$ belongs to the space E as in Lemma 2.3, and so our definition of a_r^{i+1} is correctly posed if only $\|\chi_i\mathcal{D}_i - \chi_i\mathcal{E}_{r-1}^{i+1}\|_{0,\gamma} \leq r_0\tilde C_i$. To this end, we right away observe that $\|\chi_i\mathcal{D}_i\|_{0,\gamma} \leq r_0\tilde C_i/2$, while we will show that the second condition in (1.6) implies:

$$\|\chi_i\mathcal{E}_r^{i+1}\|_{0,\gamma} \leq \frac{r_0\tilde C_i}{2} \qquad \text{for all } \ 0 = 1\ldots k-1, \ r = 0\ldots N-1. \tag{4.7}$$

Also, then automatically there holds, for all $i = 0 \dots k-1$, $r = 1 \dots N$:

$$\begin{aligned} &\tilde{C}_i \mathrm{Id}_2 + \chi_i \mathcal{D}_i - \chi_i \mathcal{E}_{r-1}^{i+1} = (a_r^{i+1})^2 \, \mathrm{Id}_2 + \mathrm{sym}\nabla\Psi_r^{i+1} \\ &\text{and} \quad (a_r^{i+1})^2 \in \Big[\frac{\tilde{C}_i}{2}, \frac{3\tilde{C}_i}{2}\Big] \ \text{ in } \ \bar{B}_R. \end{aligned} \tag{4.8}$$

3. (Inductive estimates) Fix $i = 0 \dots k-1$ and assume $(4.4)_3$. We right away note that at $i = 0$, we indeed have $(4.4)_3$ because of $(3.1)_4$. In steps 4-6 below we will prove the following estimates, valid for all $r = 1 \dots N$, with constants C depending, in line with our convention, only on ω, k, γ, t, s, r:

$$\|\partial_2^{(s)} (a_r^{i+1})^2\|_0 \leq C\tilde{C}_i \bar{\lambda}_i^s \quad \text{and} \quad \|\partial_2^{(s)} a_r^{i+1}\|_0 \leq C\tilde{C}_i^{1/2} \bar{\lambda}_i^s \qquad \text{for all } \ s \geq 0, \tag{4.9_1}$$

$$\begin{aligned} &\|\partial_1^{(t+1)}\partial_2^{(s)} (a_r^{i+1})^2\|_0 \leq C\tilde{C}_i \frac{\lambda_{i+1}^{t+1}\bar{\lambda}_i^s}{(\lambda_{i+1}/\bar{\lambda}_i)} \\ &\text{and} \quad \|\partial_1^{(t+1)}\partial_2^{(s)} a_r^{i+1}\|_0 \leq C\tilde{C}_i^{1/2} \frac{\lambda_{i+1}^{t+1}\bar{\lambda}_i^s}{(\lambda_{i+1}/\bar{\lambda}_i)} \end{aligned} \qquad \text{for all } \ s,t \geq 0, \tag{4.9_2}$$

$$\|\partial_1^{(t)}\partial_2^{(s)} \big(\mathcal{E}_r^{i+1} - \mathcal{E}_{r-1}^{i+1}\big)\|_0 \leq C\tilde{C}_i \lambda_{i+1}^t \bar{\lambda}_i^s \frac{\lambda_{i+1}^{(r-1)\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^r} \qquad \text{for all } \ s,t \geq 0. \tag{4.9_3}$$

We observe that, at any counter value $r \geq 1$, the latter estimates in both $(4.9)_1$ and $(4.9)_2$ follow from the former ones, in view of (4.8). Indeed, this implication in $(4.9)_1$ is trivial at $s = 0$, whereas for $s \geq 1$ we use the Faá di Bruno formula:

$$\begin{aligned} \|\partial_2^{(s)} a_r^{i+1}\|_0 &\leq C\Big\| \sum_{p_1+2p_2+\dots sp_s=s} (a_r^{i+1})^{2(1/2-p_1-\dots-p_s)} \prod_{z=1}^{s} \big|\partial_2^{(z)}(a_r^{i+1})^2\big|^{p_z} \Big\|_0 \\ &\leq C\|a_r^{i+1}\|_0 \sum_{p_1+2p_2+\dots sp_s=s} \prod_{z=1}^{s} \Big(\frac{\|\partial_2^{(z)}(a_r^{i+1})^2\|_0}{\tilde{C}_i}\Big)^{p_z} \leq C\tilde{C}_i^{1/2}\bar{\lambda}_i^s. \end{aligned}$$

For $(4.9)_2$, we apply the multivariate version of the Faá di Bruno formula. We note that the above estimate is just a particular case of the more general formula below, but we first separated the more familiar one-dimensional version for clarity. Let Π be the set of all partitions π of the initial multiindex $\{1\}^{t+1} + \{2\}^s$ into multiindices I of lengths $|I| \in [0, t+s+1]$ (some of them possibly empty). Denoting by $|\pi|$ the number of multiindices in the given partition π, we have:

$$\begin{aligned} \|\partial_1^{(t+1)}\partial_2^{(s)} a_r^{i+1}\|_0 &\leq C\Big\| \sum_{\pi\in\Pi} (a_r^{i+1})^{2(1/2-|\pi|)} \prod_{I\in\pi} \partial_I (a_r^{i+1})^2 \Big\|_0 \leq C\|a_r^{i+1}\|_0 \sum_{\pi\in\Pi}\prod_{I\in\pi} \frac{\|\partial_I(a_r^{i+1})^2\|_0}{\tilde{C}_i} \\ &\leq C\tilde{C}_i^{1/2}\lambda_{i+1}^{t+1}\bar{\lambda}_i^s \sum_{\pi\in\Pi} \Big(\prod_{I\in\pi,\ 1\in I} \frac{\bar{\lambda}_i}{\lambda_{i+1}} \Big) \leq C\tilde{C}_i^{1/2}\lambda_{i+1}^{t+1}\bar{\lambda}_i^s \frac{\bar{\lambda}_i}{\lambda_{i+1}}. \end{aligned}$$

Additionally, applying Faá di Bruno's formula to the inverse rather than square root, we get:

$$
\begin{aligned}
&\|\partial_2^{(s)}\Big(\frac{1}{a_r^{i+1}+a_{r+1}^{i+1}}\Big)\|_0 \leq \frac{C}{\tilde C_i^{1/2}}\sum_{p_1+2p_2+\ldots sp_s=s}\prod_{z=1}^{s}\Big(\frac{\|\partial_2^{(z)}(a_r^{i+1}+a_{r+1}^{i+1})\|_0}{\tilde C_i^{1/2}}\Big)^{p_z} \leq \frac{C}{\tilde C_i^{1/2}}\bar\lambda_i^s,\\
&\|\partial_1^{(t+1)}\partial_2^{(s)}\Big(\frac{1}{a_r^{i+1}+a_{r+1}^{i+1}}\Big)\|_0 \leq C\Big\|\sum_{\pi\in\Pi}\big(a_r^{i+1}+a_{r+1}^{i+1}\big)^{-1-|\pi|}\prod_{I\in\pi}\partial_I(a_r^{i+1}+a_{r+1}^{i+1})\Big\|_0\\
&\qquad\qquad\leq \frac{C}{\tilde C_i^{1/2}}\sum_{\pi\in\Pi}\prod_{I\in\pi}\frac{\|\partial_I a_r^{i+1}\|_0+\|\partial_I a_{r+1}^{i+1}\|_0}{\tilde C_i^{1/2}}\\
&\qquad\qquad\leq \frac{C}{\tilde C_i^{1/2}}\lambda_{i+1}^{t+1}\bar\lambda_i^s\sum_{\pi\in\Pi}\Big(\prod_{I\in\pi,\ 1\in I}\frac{\bar\lambda_i}{\lambda_{i+1}}\Big) \leq \frac{C}{C_i^{1/2}}\lambda_{i+1}^{t+1}\bar\lambda_i^s\frac{\bar\lambda_i}{\lambda_{i+1}}.
\end{aligned}
\tag{4.10}
$$

4. (Induction base: $r=1$**)** In this step, we check that $(4.9)_1$ – $(4.9)_3$ are valid at the lowest counter value $r=1$. Indeed, for all $m\geq 0$ there holds, by Lemma 2.3 (iii) and (iv):

$$
\begin{aligned}
\|\nabla^{(m)}(a_1^{i+1})^2\|_0 &= \sum_{|I|=m}\Big\|\bar a\Big(\partial_I\big(\tilde C_i\mathrm{Id}_2+\chi_i\mathcal{D}_i\big)\Big)\Big\|_0\\
&\leq C\|\nabla^{(m)}\big(\tilde C_i\mathrm{Id}_2+\chi_i\mathcal{D}_i\big)\|_{0,\gamma} \leq C\Big(\tilde C_i+\|\nabla^{(m)}(\chi_i\mathcal{D}_i)\|_{0,\gamma}\Big) \leq C\tilde C_i\bar\lambda_i^m,
\end{aligned}
\tag{4.11}
$$

in view of (4.5) and since $\mathcal{E}_0^{i+1}=0$. This yields, as explained in step 3, that $\|\nabla^{(m)}a_i^{i+1}\|_0 \leq \tilde C_i^{1/2}\bar\lambda_i^m$, implying $(4.9)_1$ and $(4.9)_2$, as $\lambda_{i+1}\geq\bar\lambda_i$. Further:

$$
\begin{aligned}
\|\partial_1^{(t)}\partial_2^{(s)}\mathcal{E}_1^{i+1}\|_0 &\leq C\sum_{\substack{p_1+q_1+z_1=t\\ q_2+z_2=s}}\lambda_{i+1}^{p_1-1}\|\nabla^{(q_1+q_2)}a_1^{i+1}\|_0\|\nabla^{(z_1+z_2+2)}v_0\|_0\\
&+C\sum_{\substack{p_1+q_1+z_1=t\\ q_2+z_2=s}}\lambda_{i+1}^{p_1-2}\|\nabla^{(q_1+q_2+1)}a_1^{i+1}\|_0\|\nabla^{(z_1+z_2+1)}a_1^{i+1}\|_0 + C\sum_{p+q=t}\lambda_{i+1}^{p-1}\|\nabla^{(q+s+1)}(a_1^{i+1})^2\|_0\\
&\leq C\sum_{\substack{p_1+q_1+z_1=t\\ q_2+z_2=s}}\Big(\frac{\lambda_{i+1}^{p_1-1}\bar\lambda_i^{q_1+q_2}}{l^{z_1+z_2+1}}\tilde C_i^{1/2}(lM)+\lambda_{i+1}^{p_1-2}\bar\lambda_i^{q_1+q_2+z_1+z_2+2}\tilde C_i\Big)+C\sum_{p+q=t}\lambda_{i+1}^{p-1}\bar\lambda_i^{q+s+1}\tilde C_i\\
&\leq C\tilde C_i\lambda_{i+1}^t\bar\lambda_i^s\sum_{p_1+q_1+z_1=t}\Big(\big(\frac{\bar\lambda_i}{\lambda_{i+1}}\big)^{q_1+z_1+1}\frac{lM}{\tilde C_i(\bar\lambda_i l)^{z_1+z_2+1}}+\big(\frac{\bar\lambda_i}{\lambda_{i+1}}\big)^{q_1+z_1+2}\Big)\\
&\quad + C\tilde C_i\lambda_{i+1}^t\bar\lambda_i^s\sum_{p+q=t}\big(\frac{\bar\lambda_i}{\lambda_{i+1}}\big)^{q+1} \leq C\tilde C_i\frac{\lambda_{i+1}^t\bar\lambda_i^s}{(\lambda_{i+1}/\bar\lambda_i)},
\end{aligned}
$$

for all $s,t\geq 0$, because $(lM)/(\bar\lambda_i l)\leq\tilde C_i^{1/2}$ by (4.3). The above is precisely $(4.9)_3$ since $\mathcal{E}_0=0$.

5. (Proof of the inductive estimates (4.7), $(4.9)_1$ and $(4.9)_2$) Assume that the bounds $(4.9)_1$ – $(4.9)_3$ hold up to some $1\leq r\leq N-1$. We will prove their validity at $r+1$. We start by noting a direct consequence of $(4.9)_3$ in view of (4.1) and the interpolation inequality:

$$
\|\partial_1^{(t)}\partial_2^{(s)}\big(\chi_i(\mathcal{E}_r^{i+1}-\mathcal{E}_{r-1}^{i+1})\big)\|_{0,\gamma} \leq C\tilde C_i\lambda_{i+1}^t\bar\lambda_i^s\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar\lambda_i)^r} \qquad \text{for all } s,t\geq 0. \tag{4.12}
$$

Further, since $\mathcal{E}_0 = 0$, it follows that:

$$\|\partial_1^{(t)}\partial_2^{(s)}(\chi_i\mathcal{E}_r^{i+1})\|_{0,\gamma} \leq C\tilde{C}_i\lambda_{i+1}^t\bar{\lambda}_i^s\sum_{j=1}^r\Big(\frac{\lambda_{i+1}^\gamma}{\lambda_{i+1}/\bar{\lambda}_i}\Big)^j \leq C\tilde{C}_i\lambda_{i+1}^t\bar{\lambda}_i^s\frac{\lambda_{i+1}^\gamma}{(\lambda_{i+1}/\bar{\lambda}_i)}. \tag{4.13}$$

In particular, the requirement (4.7) is automatically justified, because:

$$\|\chi_i\mathcal{E}_r^{i+1}\|_{0,\gamma} \leq C\tilde{C}_i\frac{\lambda_{i+1}^\gamma}{(\lambda_{i+1}/\bar{\lambda}_i)} \leq \frac{r_0\tilde{C}_i}{2},$$

provided that the following working assumption holds:

$$\frac{(\lambda_{i+1}/\bar{\lambda}_i)}{\lambda_{i+1}^\gamma} \geq \frac{2C}{r_0} \quad \text{ for all } \ i = 0\dots k-1. \tag{4.14}$$

To prove $(4.9)_1$, we use Lemma 2.3 (iii), (iv) and argue as in (4.11) in view of (4.13):

$$\begin{aligned}
&\|\partial_2^{(s)}(a_{r+1}^{i+1})^2\|_0 = \Big\|\bar{a}\Big(\partial_2^{(s)}\big(\tilde{C}_i\mathrm{Id}_2 + \chi_i\mathcal{D}_i - \chi_i\mathcal{E}_r^{i+1}\big)\Big)\Big\|_0 \\
&\leq C\Big(\tilde{C}_i + \|\partial_2^{(s)}(\chi_i\mathcal{D}_i)\|_{0,\gamma} + \|\partial_2^{(s)}(\chi_i\mathcal{E}_r^{i+1})\|_{0,\gamma}\Big) \leq C\tilde{C}_i\Big(1 + \bar{\lambda}_i^s + \frac{\lambda_{i+1}^\gamma\bar{\lambda}_i^s}{(\lambda_{i+1}/\bar{\lambda}_i)}\Big) \leq C\tilde{C}_i\bar{\lambda}_i^s.
\end{aligned}$$

Finally, $(4.9)_2$ follows by additionally invoking Lemma 2.3 (v) in:

$$\begin{aligned}
\|\partial_1^{(t+1)}\partial_2^{(s)}(a_{r+1}^{i+1})^2\|_0 &\leq \Big\|\bar{a}\Big(\partial_1^{(t+1)}\partial_2^{(s)}\big(\tilde{C}_i\mathrm{Id}_2 + \chi_i\mathcal{D}_i\big)\Big)\Big\|_0 + \Big\|\partial_1^{(t+1)}\partial_2^{(s)}\Big(\bar{a}\big(\chi_i\mathcal{E}_r^{i+1}\big)\Big)\Big\|_0 \\
&\leq C\Big(\|\partial_1^{(t+1)}\partial_2^{(s)}\big(\chi_i\mathcal{D}_i\big)\|_{0,\gamma} + \|\partial_1^{(t)}\partial_2^{(s+1)}\big(\chi_i\mathcal{E}_r^{i+1}\big)\|_{0,\gamma}\Big) \\
&\leq C\tilde{C}_i\Big(\lambda_i^{t+1}\bar{\lambda}_i^s + \lambda_{i+1}^t\bar{\lambda}_i^{s+1}\frac{\lambda_{i+1}^\gamma}{(\lambda_{i+1}/\bar{\lambda}_i)}\Big) \leq C\tilde{C}_i\lambda_{i+1}^t\bar{\lambda}_i^{s+1}.
\end{aligned}$$

6. (Proof of the inductive estimate $(4.9)_3$) Noting that:

$$(a_{r+1}^{i+1})^2 - (a_r^{i+1})^2 = -\bar{a}\big(\chi_i(\mathcal{E}_r^{i+1} - \mathcal{E}_{r-1}^{i+1})\big)$$

and recalling the inductive assumption $(4.9)_3$, we get from Lemma 2.3 (v) and (4.12):

$$\begin{aligned}
&\|\partial_2^{(s)}\big((a_{r+1}^{i+1})^2 - (a_r^{i+1})^2\big)\|_0 \leq C\|\partial_2^{(s)}\big(\chi_i(\mathcal{E}_r^{i+1} - \mathcal{E}_{r-1}^{i+1})\big)\|_{0,\gamma} \leq C\tilde{C}_i\bar{\lambda}_i^s\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^r}, \\
&\|\partial_1^{(t+1)}\partial_2^{(s)}\big((a_{r+1}^{i+1})^2 - (a_r^{i+1})^2\big)\|_0 \leq C\|\partial_1^{(t)}\partial_2^{(s+1)}\big(\chi_i(\mathcal{E}_r^{i+1} - \mathcal{E}_{r-1}^{i+1})\big)\|_{0,\gamma} \\
&\qquad\qquad\leq C\tilde{C}_i\lambda_{i+1}^{t+1}\bar{\lambda}_i^s\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^{r+1}},
\end{aligned} \tag{4.15}$$

valid for all $s,t\geq 0$. We now combine the above with (4.10) to get:

$$\begin{aligned}
&\|\partial_2^{(s)}\big(a_{r+1}^{i+1}-a_r^{i+1}\big)\|_0 \leq C\sum_{p+q=s}\|\partial_2^{(p)}\big((a_{r+1}^{i+1})^2-(a_r^{i+1})^2\big)\|_0\Big\|\partial_2^{(q)}\Big(\frac{1}{a_{r+1}^{i+1}+a_r^{i+1}}\Big)\Big\|_0\\
&\qquad\qquad \leq C\tilde C_i^{1/2}\bar\lambda_i^s\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar\lambda_i)^r},\\
&\|\partial_1^{(t+1)}\partial_2^{(s)}\big(a_{r+1}^{i+1}-a_r^{i+1}\big)\|_0\\
&\quad\leq C\sum_{\substack{p_1+q_1=t+1\\ p_2+q_2=s}}\|\partial_1^{(p_1)}\partial_2^{(p_2)}\big((a_{r+1}^{i+1})^2-(a_r^{i+1})^2\big)\|_0\Big\|\partial_1^{(q_1)}\partial_2^{(q_2)}\Big(\frac{1}{a_{r+1}^{i+1}+a_r^{i+1}}\Big)\Big\|_0\\
&\quad\leq C\tilde C_i^{1/2}\lambda_{i+1}^{t+1}\bar\lambda_i^s\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar\lambda_i)^{r+1}}+C\tilde C_i^{1/2}\sum_{\substack{p_1+q_1=t+1\\ p_2+q_2=s,\,q_1\geq 1}}\lambda_{i+1}^{p_1+q_1}\bar\lambda_i^{p_2+q_2}\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar\lambda_i)^{r+1}}\\
&\quad\leq C\tilde C_i^{1/2}\lambda_{i+1}^{t+1}\bar\lambda_i^s\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar\lambda_i)^{r+1}}.
\end{aligned}\tag{4.16}$$

Towards proving $(4.9)_3$, we first write:

$$\begin{aligned}
\mathcal{E}_{r+1}^{i+1}-\mathcal{E}_r^{i+1}=&-\frac{a_{r+1}^{i+1}-a_r^{i+1}}{\lambda_{i+1}}\Gamma(\lambda_{i+1}x_1)\Big(\nabla^2v_0^i-(\partial_{22}v_0^i)e_2^{\otimes 2}\Big)\\
&+\frac{1}{2\lambda_{i+1}^2}\Gamma(\lambda_{i+1}x_1)^2\Big(\big(\nabla a_{r+1}^{i+1}-\nabla a_r^{i+1}\big)\otimes\nabla a_{r+1}^{i+1}+\nabla a_r^{i+1}\otimes\big(\nabla a_{r+1}^{i+1}-\nabla a_r^{i+1}\big)\\
&\qquad\qquad-\big((\partial_2a_{r+1}^{i+1})^2-(\partial_2a_r^{i+1})^2\big)e_2^{\otimes 2}\Big)\\
&-\frac{1}{\lambda_{i+1}}\bar\Gamma(\lambda_{i+1}x_1)\,\mathrm{sym}\big(\nabla((a_{r+1}^{i+1})^2-(a_r^{i+1})^2)\otimes e_1\big),
\end{aligned}$$

with the goal of estimating, for all $s,t\geq 0$:

$$\begin{aligned}
&\|\partial_1^{(t)}\partial_2^{(s)}\big(\mathcal{E}_{r+1}^{i+1}-\mathcal{E}_r^{i+1}\big)\|_0\\
&\quad\leq C\sum_{\substack{p_1+q_1+z_1=t\\ q_2+z_2=s}}\lambda_{i+1}^{p_1-1}\|\partial_1^{(q_1)}\partial_2^{(q_2)}(a_{r+1}^{i+1}-a_r^{i+1})\|_0\big\|\nabla^{(z_1+z_2+2)}v_0\big\|_0\\
&\qquad+C\sum_{\substack{p_1+q_1+z_1=t\\ q_2+z_2=s}}\lambda_{i+1}^{p_1-2}\|\partial_1^{(q_1)}\partial_2^{(q_2)}\nabla(a_{r+1}^{i+1}-a_r^{i+1})\|_0\times\\
&\qquad\qquad\times\Big(\|\partial_1^{(z_1)}\partial_2^{(z_2)}\nabla a_r^{i+1}\|_0+\|\partial_1^{(z_1)}\partial_2^{(z_2)}\nabla a_{r+1}^{i+1}\|_0\Big)\\
&\qquad+C\sum_{p+q=t}\lambda_{i+1}^{p-1}\|\partial_1^{(q)}\partial_2^{(s)}\nabla((a_{r+1}^{i+1})^2-(a_r^{i+1})^2)\|_0.
\end{aligned}\tag{4.17}$$

The first term above is bounded, in virtue of (4.16) and $(3.1)_2$, by:

$$
\begin{aligned}
&C\tilde{C}_i^{1/2}(lM)\lambda_{i+1}^t\Big(\sum_{\substack{p_1+z_1=t\\ q_2+z_2=s}} \frac{\bar{\lambda}_i^{q_2}}{\lambda_{i+1}^{z_1+1}}\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^r}\frac{1}{l^{z_1+z_2+1}}\\
&\qquad\qquad + \sum_{\substack{p_1+q_1+z_1=t\\ q_1\geq 1\\ q_2+z_2=s}} \frac{\bar{\lambda}_i^{q_2}}{\lambda_{i+1}^{q_1+z_1+1}}\frac{\lambda_{i+1}^{q_1}\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^{r+1}}\frac{1}{l^{z_1+z_2+1}}\Big)\\
&\leq C\tilde{C}_1^{1/2}(lM)\lambda_{i+1}^t\bar{\lambda}_i^s\Big(\sum_{p_1+z_1=t}\frac{1}{(\lambda_{i+1}l)^{z_1+1}}\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^r} + \sum_{p_1+q_1+z_1=t}\frac{1}{(\lambda_{i+1}l)^{z_1+1}}\frac{\lambda_{i+!}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^{r+1}}\Big)\\
&\leq C\tilde{C}_i\lambda_{i+1}^t\bar{\lambda}_i^s\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^{r+1}},
\end{aligned}
$$

in virtue of (4.3). Similarly, the third term in (4.17) is estimated through (4.15), by:

$$
C\tilde{C}_i\lambda_{i+1}^t\Big(\frac{\bar{\lambda}_i^{s+1}}{\lambda_{i+2}}\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^r} + \sum_{p+q=t,\ q\geq 1}\bar{\lambda}_i^s\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^{r+1}} \leq C\tilde{C}_i\lambda_{i+1}^t\bar{\lambda}_i^s\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^{r+i}}.
$$

For the middle term in (4.17), we obtain the bound:

$$
\begin{aligned}
&C\tilde{C}_i\lambda_{i+1}^t\Bigg(\sum_{q_2+z_2=s}\frac{1}{\lambda_{i+1}^2}\frac{\bar{\lambda}_i^{q_2+1}\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^r}\bar{\lambda}_i^{z_2+1} + \sum_{\substack{p_1+z_1=t\\ z_1\geq 1\\ q_2+z_2=s}}\frac{1}{\lambda_{i+1}^{z_1+2}}\frac{\bar{\lambda}_i^{q_2+1}\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^r}\lambda_{i+1}^{z_1}\bar{\lambda}_i^{z_2+1}\\
&\qquad + \sum_{\substack{p_1+q_1=t\\ q_1\geq 1\\ q_2+z_2=s}}\frac{1}{\lambda_{i+1}^{q_1+2}}\frac{\lambda_{i+1}^{q_1+1}\bar{\lambda}_i^{q_2}\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^{r+1}}\bar{\lambda}_i^{z+2+1}\\
&\qquad + \sum_{\substack{p_1+q_1+z_1=t\\ q_1\geq 1,\, z_1\geq 1\\ q_2+z_2=s}}\frac{1}{\lambda_{i+1}^{q_1+z_1+2}}\frac{\lambda_{i+1}^{q_1+1}\bar{\lambda}_i^{q_2}\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^{r+1}}\lambda_{i+1}^{z_1}\bar{\lambda}_i^{z_2+1}\Bigg)\\
&\leq C\tilde{C}_i\lambda_{i+1}^t\bar{\lambda}_i^s\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^{r+2}},
\end{aligned}
$$

where we repeatedly used (4.16) and $(4.9)_1$, $(4.9)_2$. In conclusion, (4.17) yields $(3.6)_4$:

$$
\|\partial_1^{(t)}\partial_2^{(s)}\big(\mathcal{E}_{r+1}^{i+1}-\mathcal{E}_r^{i+1}\big)\|_0 \leq C\tilde{C}_i\lambda_{i+1}^t\bar{\lambda}_i^s\frac{\lambda_{i+1}^{r\gamma}}{(\lambda_{i+1}/\bar{\lambda}_i)^{r+1}}.
$$

This ends the proof of all the inductive estimates, under the assumption (3.7).

7. (Adding the first corrugation) Define the perturbed fields $\bar{v}_i\in\mathcal{C}^\infty(\bar{\omega}+\bar{B}_{2l-il/k},\mathbb{R}^k)$ and $\bar{w}_i\in\mathcal{C}^\infty(\bar{\omega}+\bar{B}_{2l-il/k},\mathbb{R}^2)$ in accordance with Lemma 2.5 (here, $i=1\dots k$):

$$
\bar{v}_i = v_{i-1}+\frac{a_N^i}{\lambda_i}\Gamma(\lambda_i x_1)e_i,\quad \bar{w}_i = w_{i-1}-\frac{a_N^i}{\lambda_i}\Gamma(\lambda_i x_1)\nabla v_0^i + \frac{(a_N^i)^2}{\lambda_i}\bar{\Gamma}(\lambda_i x_1)e_1 + \Psi_N^i - \tilde{C}_{i-1}id_2.
$$

Observe that, recalling $(4.9)_1$, $(4.9)_2$, there holds for all $s,t\geq 0$:

$$\begin{aligned}\|\partial_1^{(t)}\partial_2^{(s)}(\bar v_i^i - v_{i-1}^i)\|_0 &\leq C\sum_{p+q=t}\lambda_i^{p-1}\|\partial_1^{(q)}\partial_2^{(s)}a_N^i\|_0 \\ &\leq C\tilde C_{i-1}^{1/2}\lambda_i^t\bar\lambda_{i-1}^s\Big(\frac{1}{\lambda_i}+\sum_{p+q=t,\ q\geq 1}\frac{\bar\lambda_{i-1}}{\lambda_i^2}\Big)\leq C\tilde C_i\lambda_i^t\bar\lambda_{i-1}^s\frac{1}{\lambda_i}.\end{aligned}\tag{4.18}$$

In particular, by $(3.1)_1$, $(3.1)_3$, the above yields (since $v_{i-1}^i = v_0^i$):

$$\begin{aligned}&\|\bar v_i^i - v^i\|_1 \leq \|v_0 - v\|_1 + \|\bar v_i^i - v_{i-1}^i\|_1 \leq C(lM + C_{i-1}^{1/2}) \leq C\tilde C_0^{1/2},\\ &\|\partial_1^{(t)}\partial_2^{(s)}\nabla^2\bar v_i^i\|_0 \leq \|\nabla^{(t+s+2)}v_0\|_0 + C\tilde C_i^{1/2}\lambda_i^{t+1}\bar\lambda_{i-1}^s \leq C\tilde C_i^{1/2}\lambda_i^{t+1}\bar\lambda_{i-1}^s,\end{aligned}\tag{4.19}$$

by (4.6) so that $(lM)^2\leq C\tilde C_0$, and by (4.3) which yields $lM\leq \tilde C_{i-1}^{1/2}(\lambda_i l)$, and as long as:

$$\tilde C_{i-1}\leq C\tilde C_0.\tag{4.20}$$

Now, by Lemma 2.3, (4.5), (4.13) and (4.14) we obtain:

$$\begin{aligned}&\|\Psi_N^i\|_1 \leq C\big(\|\chi_{i-1}\mathcal{D}_{i-1}\|_{0,\gamma} + \|\chi_{i-1}\mathcal{E}_{N-1}^i\|_{0,\gamma}\big)\leq C\tilde C_{i-1},\\ &\|\nabla^2\Psi_N^i\|_0 \leq C\big(\|\chi_{i-1}\mathcal{D}_{i-1}\|_{1,\gamma} + \|\chi_{i-1}\mathcal{E}_{N-1}^i\|_{1,\gamma}\big)\leq C\Big(\tilde C_{i-1}\bar\lambda_{i-1} + \frac{\lambda_i\lambda^\gamma}{(\lambda_i/\bar\lambda_i)}\tilde C\Big)\leq C\tilde C_{i-1}\lambda_i,\end{aligned}$$

and consequently, $(4.9)_1$, $(4.9)_2$ $(3.1)_1$, $(3.1)_3$ result in the estimate:

$$\begin{aligned}\|\bar w_i - w_{i-1}\|_1 &\leq C\Big(\tilde C_{i-1} + \|a_N^i\|_0\|\nabla v_0\|_0 + \|(a_N^i)^2\|_0 \\ &\qquad + \frac{\|\nabla a_N^i\|_0\|\nabla v_0\|_0 + \|a_N^i\|_0\|\nabla^2 v_0\|_0 + \|\nabla(a_N^i)^2\|_0}{\lambda_i}\Big)\\ &\leq C\Big(\tilde C_{i-1} + \tilde C_{i-1}^{1/2}\|\nabla v_0\|_0 + \frac{\tilde C_{i-1}^{1/2}}{\bar\lambda_{i-1}/\lambda_i}\|\nabla v_0\|_0 + \frac{lM}{\lambda_i l}\tilde C_{i-1}^{1/2} + \tilde C\Big)\\ &\leq C\tilde C_{i-1}^{1/2}\big(\|\nabla v_0\|_0 + \tilde C_{i-1}^{1/2}\big)\leq C\tilde C_{i-1}^{1/2}\big(lM + \tilde C_{i-1}^{1/2} + \|\nabla v\|_0\big)\\ &\leq C\tilde C_0^{1/2}\big(\tilde C_0^{1/2} + \|\nabla v\|_0\big),\\ \|\nabla^2\bar w_i - \nabla^2\bar w_{i-1}\|_0 &\leq \|\nabla^2\Psi_N\|_0 + C\Big(\lambda_i\big(\|a_N^i\|_0\|\nabla v_0\|_0 + \|(a_N^i)^2\|_0\big)\\ &\qquad + \big(\|\nabla a_N^i\|_0\|\nabla v_0\|_0 + \|a_N^i\|_0\|\nabla^2 v_0\|_0 + \|\nabla(a_N^i)^2\|_0\big)\\ &\qquad + \frac{\|\nabla^2 a_N^i\|_0\|\nabla v_0\|_0 + \|\nabla a_N^i\|_0\|\nabla^2 v_0\|_0 + \|a_N^i\|_0\|\nabla^3 v_0\|_0 + \|\nabla^2(a_N^i)^2\|_0}{\lambda_i}\Big)\\ &\leq C\Big(\lambda_i\tilde C_{i-1} + \big(\lambda_i + \bar\lambda_{i-1}\big)\tilde C_{i-1}^{1/2}(\|\nabla v\|_0 + lM) + \tilde C_{i-1}^{1/2}\frac{lM}{l}\Big)\\ &\leq C\lambda_i\tilde C_{i-1}^{1/2}\big(lM + \tilde C_{i-1}^{1/2} + \|\nabla v\|_0\big)\leq C\tilde C_{i-1}\lambda_i\big(\tilde C_0^{1/2} + \|\nabla v\|_0\big).\end{aligned}\tag{4.21}$$

Finally, we note that (2.7) and (4.8) yield the decomposition on $\bar\omega + \bar B_{2l-il/k}$:

$$\begin{aligned}\bar{\mathcal{D}}_i &\doteq A_0 - \big(\frac12 (\nabla\bar v_i)^T\nabla\bar v_i + \mathrm{sym}\nabla\bar w_i\big)\\ &= \mathcal{D}_{i-1} - \Big((a_N^i)^2 e_1\otimes e_1 + \mathcal{E}_N^i + \mathrm{sym}\nabla\Psi_N^i - \tilde C_{i-1}\mathrm{Id}_2\Big)\\ &\quad - \Big(-\frac{a_N^i}{\lambda_i}\Gamma(\lambda_i x_1)\partial_{22}v_0^i + \frac{1}{2\lambda_i^2}\Gamma(\lambda_i x_1)^2(\partial_2 a_N^i)^2\Big)e_2\otimes e_2\\ &= -\big(\mathcal{E}_N^i - \mathcal{E}_{N-1}^i\big) + \Big((a_N^i)^2 + \frac{a_N^i}{\lambda_i}\Gamma(\lambda_i x_1)\partial_{22}v_0^i - \frac{1}{2\lambda_i^2}\Gamma(\lambda_i x_1)^2(\partial_2 a_N^i)^2\Big)e_2\otimes e_2.\end{aligned} \tag{4.22}$$

8. (Adding the second corrugation) We now update of $\bar v_i, \bar w_i$, to new fields $v_i \in \mathcal{C}^\infty(\bar\omega + \bar B_{2l-il/k}, \mathbb{R}^k)$, $w_i \in \mathcal{C}^\infty(\bar\omega + \bar B_{2l-il/k}, \mathbb{R}^2)$ by using Lemma 2.5 (with $i = 2, j = 1$) and the perturbation amplitude dictated by (4.22), namely:

$$\begin{aligned}&v_i = \bar v_i + \frac{b^i(x)}{\bar\lambda_i}\Gamma(\bar\lambda_i x_2)e_i, \quad w_1 = \bar w_1 - \frac{b^i(x)}{\bar\lambda_i}\Gamma(\bar\lambda_i x_2)\nabla\bar v_i^i + \frac{b^i(x)^2}{\bar\lambda_i}\bar\Gamma(\bar\lambda_i x_2)e_2,\\ &\text{where } (b^i)^2 = (a_N^i)^2 + \frac{a_N^i}{\lambda_i}\Gamma(\lambda_i x_1)\partial_{22}v_0^i - \frac{1}{2\lambda_i^2}\Gamma(\lambda_i x_1)^2(\partial_2 a_N^i)^2.\end{aligned} \tag{4.23}$$

Firstly, we argue that $b \in \mathcal{C}^\infty(\bar\omega + \bar B_{2l-il/k})$ is well defined, since by $(3.1)_3$, $(4.9)_1$, $(4.9)_2$:

$$\Big\|\frac{a_N^i}{\lambda_i}\Gamma(\lambda x_1)\partial_{22}v_0^i - \frac{1}{2\lambda_i^2}\Gamma(\lambda_i x_1)^2(\partial_2 a_N^i)^2\Big\|_0 \le C\Big(\tilde C_{i-1}^{1/2}\frac{(lM)}{\lambda_i l} + \tilde C_{i-1}\Big(\frac{\bar\lambda_{i-1}}{\lambda_i}\Big)^2\Big) \le C\tilde C_{i-1}\frac{\bar\lambda_{i-1}}{\lambda_i}.$$

Assigning $\bar\lambda_{i-1}/\lambda_i$ small as guaranteed by (4.14), the bound in (4.8) implies:

$$(b^i)^2 \in \Big[\frac{\tilde C_{i-1}}{3}, 2\tilde C_{i-1}\Big] \quad \text{in } \bar\omega + \bar B_{2l-il/k}. \tag{4.24}$$

Likewise, for every $t, s \ge 0$ we obtain the first set of bounds in:

$$\begin{aligned}&\|\partial_2^{(s)}(b^i)^2\|_0 \le C\tilde C_{i-1}\bar\lambda_{i-1}^s \qquad \|\partial_1^{(t+1)}\partial_2^{(s)}(b^i)^2\|_0 \le C\tilde C_{i-1}\lambda_i^t\bar\lambda_{i-1}^{s+1},\\ &\|\partial_2^{(s)}b^i\|_0 \le C\tilde C_{i-1}^{1/2}\bar\lambda_{i-1}^s, \qquad \|\partial_1^{(t+1)}\partial_2^{(s)}b^i\|_0 \le C\tilde C_{i-1}^{1/2}\lambda_i^t\bar\lambda_{i-1}^{s+1},\end{aligned} \tag{4.25}$$

while the second set of bounds follows by the Faá di Bruno formula as in step 4. Further:

$$\begin{aligned}&\|\partial_2^{(s)}(v_i^i - \bar v_i^i)\|_0 \le C\sum_{p+q=s}\bar\lambda_i^{p-1}\|\partial_2^{(q)}b^i\|_0 \le C\tilde C_{i-1}^{1/2}\frac{\bar\lambda_i^s}{\bar\lambda_i},\\ &\|\partial_1^{(t+1)}\partial_2^{(s)}(v_i^i - \bar v_i^i)\|_0 \le C\sum_{p+q=s}\bar\lambda_i^{p-1}\|\partial_1^{(t+1)}\partial_2^{(q)}b^i\|_0\\ &\qquad\qquad \le C\tilde C_{i-1}^{1/2}\bar\lambda_i^s\sum_{p+q=s}\frac{\lambda_i^t\bar\lambda_{i-i}^{q+1}}{\bar\lambda_i^{q+1}} \le C\tilde C_{i-1}^{1/2}\lambda_i^t\lambda_i^s\frac{\bar\lambda_{i-1}}{\bar\lambda_i},\end{aligned} \tag{4.26}$$

as in (4.18), which combined with (4.19) implies that:

$$\begin{aligned}&\|v_i^i - v^i\|_1 \le \|\bar v_i^i - v^i\|_1 + C\tilde C_{i-1}^{1/2} \le C\tilde C_0^{1/2},\\ &\|\nabla^2 v_i^i\|_0 \le C\tilde C_{i-1}^{1/2}\Big(\lambda_i + \bar\lambda_i + \bar\lambda_{i-1}\Big) \le C\tilde C_{i-1}^{1/2}\bar\lambda_i,\end{aligned} \tag{4.27}$$

and further:

$$\begin{aligned}
&\|w_i - w_{i-1}\|_1 \leq \|\bar w_i - w_{i-1}\|_1 \\
&\qquad\qquad + C\Big(\|b^i\|_0\|\nabla \bar v_i^i\|_0 + \|(b^i)^2\|_0 + \frac{\|\nabla b^i\|_0\|\nabla \bar v_i^i\|_0 + \|b^i\|_0\|\nabla^2\bar v_i^i\|_0 + \|\nabla (b^i)^2\|_0}{\bar\lambda_i}\Big) \\
&\leq C\tilde C_{i-1}^{1/2}\big(\tilde C_0^{1/2} + \tilde C_{i-1}^{1/2} + \|\nabla v\|_0\big) \leq C\tilde C_{i-1}^{1/2}\big(\tilde C_0^{1/2} + \|\nabla v\|_0\big), \\
&\|\nabla^2 w_i - \nabla^2 w_{i-1}\|_0 \leq \|\nabla^2\bar w_i - \nabla^2 w_{i-1}\|_0 + C\Big(\bar\lambda_i\big(\|b^i\|_0\|\nabla\bar v_i^i\|_0 + \|(b^i)^2\|_0\big) \\
&\qquad\qquad + \|\nabla b^i\|_0\|\nabla \bar v_i^i\|_0 + \|b^i\|_0\|\nabla^2\bar v_i^i\|_0 + \|\nabla (b^i)^2\|_0 \\
&\qquad\qquad + \frac{\|\nabla^2 b^i\|_0\|\nabla \bar v_i^i\|_0 + \|\nabla b^i\|_0\|\nabla^2\bar v_i^i\|_0 + \|b^i\|_0\|\nabla^3\bar v_i^i\|_0 + \|\nabla^2 (b^i)^2\|_0}{\bar\lambda_i}\Big) \\
&\leq C\tilde C_{i-1}^{1/2}\bar\lambda_i\big(\tilde C_0^{1/2} + \tilde C_{i-1}^{1/2} + \|\nabla v\|_0\big) \leq C\tilde C_{i-1}^{1/2}\bar\lambda_i\big(\tilde C_0^{1/2} + \|\nabla v\|_0\big),
\end{aligned} \tag{4.28}$$

where we used $\|\nabla^3\bar v_i^i\|_0 \leq C\tilde C_{i-1}^{1/2}\lambda_i^2$.

Finally, Lemma 2.5 implies:

$$\begin{aligned}
&\big(\frac{1}{2}(\nabla v_i)^T\nabla v_i + \mathrm{sym}\nabla w_i\big) - \big(\frac{1}{2}(\nabla \bar v_i)^T\nabla \bar v_i + \mathrm{sym}\nabla \bar w_i\big) = (b^i)^2 e_2\otimes e_2 + \mathcal F^i \\
&\text{with }\ \mathcal F^i = -\frac{b^i}{\bar\lambda_i}\Gamma(\bar\lambda_i x_2)\nabla^2\bar v_i^i + \frac{1}{2\bar\lambda_i^2}\Gamma(\bar\lambda_i x_2)^2\nabla b^i\otimes\nabla b^i - \frac{1}{\bar\lambda_i}\bar\Gamma(\bar\lambda_i x_2)\,\mathrm{sym}\big(\nabla (b^i)^2\otimes e_2\big),
\end{aligned} \tag{4.29}$$

where the bounds (4.25), (4.19) yield for all $s,t\geq 0$:

$$\begin{aligned}
\|\partial_2^{(s)}\mathcal F^i\|_0 \leq\ & C\sum_{p+q+z=s}\bar\lambda_i^{p-1}\|\partial_2^{(q)}b^i\|_0\|\partial_2^{(z)}\nabla^2\bar v_i^i\| \\
& + C\sum_{p+q+z=s}\bar\lambda_i^{p-2}\|\partial_2^{(q)}\nabla b^i\|_0\|\partial_2^{(z)}\nabla b^i\|_0 + C\sum_{p+q=s}\bar\lambda_i^{p-1}\|\partial_2^{(q)}\nabla (b^i)^2\|_0 \\
\leq\ & C\tilde C_{i-1}\bar\lambda_i^s\Big(\sum_{p+q+z=s}\frac{\bar\lambda_{i-1}^q\lambda_i\bar\lambda_{i-1}^z}{\bar\lambda_i^{q+z+1}} + \sum_{p+q+z=s}\frac{\bar\lambda_{i-1}^{q+1}\bar\lambda_{i-1}^{z+1}}{\bar\lambda_i^{q+z+2}} + \sum_{p+q=s}\frac{\bar\lambda_{i-1}^{q+1}}{\bar\lambda_i^{q+1}}\Big) \leq C\tilde C_{i-1}\bar\lambda_i^s\frac{\lambda_i}{\bar\lambda_i},
\end{aligned}$$

$$
\begin{aligned}
\|\partial_i^{(t+1)}\partial_2^{(s)}\mathcal{F}^i\|_0 \leq\, & C \sum_{\substack{p_1+q_1+z_1=s\\ q_2+z_2=t+1}} \bar\lambda_i^{p_i-1}\|\partial_i^{(q_2)}\partial_2^{(q_1)}b^i\|_0\|\partial_1^{(z_2)}\partial_2^{(z_1)}\nabla^2\bar v_i^i\| \\
& + C \sum_{\substack{p_1+q_1+z_1=s\\ q_2+z_2=t+1}} \bar\lambda_i^{p_1-2}\|\partial_1^{(q_2)}\partial_2^{(q_1)}\nabla b^i\|_0\|\partial_1^{(z_2)}\partial_2^{(z_1)}\nabla b^i\|_0 \\
& + C\sum_{p+q=s}\bar\lambda_i^{p-1}\|\partial_1^{(t+1)}\partial_2^{(q)}\nabla(b^i)^2\|_0 \\
\leq C\tilde C_{i-1}\bar\lambda_i^s\Big(& \sum_{p_1+q_1+z_1=s}\frac{\bar\lambda_{i-1}^{q_1}\lambda_i^{t+2}\bar\lambda_{i-1}^{z_1}}{\bar\lambda_i^{q_1+z_1+1}} + \sum_{\substack{p_1+q_1+z_1=s\\ q_2+z_2=t+1,\ q_2\geq 1}}\frac{\bar\lambda_{i-1}^{q_1+1}\lambda_i^{q_2-1}}{\bar\lambda_i^{q_1+z_1+1}}\lambda_i^{z_2+1}\bar\lambda_{i-1}^{z_1} \\
& + \sum_{p_1+q_1+z_1=s}\frac{\bar\lambda_{i-1}^{q_1+1}}{\bar\lambda_i^{q_1+z_1+2}}\lambda_i^{t+1}\bar\lambda_{i-1}^{z_1+1} + \sum_{\substack{p_1+q_1+z_1=s\\ q_2+z_2=t+1,\ q_2,z_2\geq 1}}\frac{\lambda_i^{q_2}\bar\lambda_{i-1}^{z_2}\lambda_{i-1}^{z_1+1}\bar\lambda_{i-1}^{q_1+1}}{\bar\lambda_i^{q_1+z_1+2}} \\
& + \sum_{p+q=s}\frac{\lambda_i^{t+1}\bar\lambda_{i-1}^{q+1}}{\bar\lambda_i^{q+1}}\Big) \leq C\tilde C_{i-1}\bar\lambda_i^{t+1}\lambda_i^s\frac{\lambda_i}{\bar\lambda_i}.
\end{aligned}
$$

Since the new defect can be written in virtue of (4.22) and (4.29) as:

$$
\begin{aligned}
\mathcal{D}_i &= \bar{\mathcal{D}}_i - \Big(\big(\frac12(\nabla v_i)^T\nabla v_i + \mathrm{sym}\nabla w_i\big) - \big(\frac12(\nabla\bar v_i)^T\nabla\bar v_i + \mathrm{sym}\nabla\bar w_1\big)\Big) \\
&= -(\mathcal{E}_N^i - \mathcal{E}_{N-1}^i) - \mathcal{F}^i,
\end{aligned}
$$

then $(4.9)_3$ together with the above developed bounds on the derivatives of $\mathcal{F}$ imply:

$$
\begin{aligned}
\|\partial_1^{(t)}\partial_2^{(s)}\mathcal{D}_i\|_0 &\leq C\tilde C_{i-1}\lambda_i^t\bar\lambda_{i-1}^s\frac{\lambda_i^{(N-1)\gamma}}{(\lambda_i/\bar\lambda_{i-1})^N} + C\tilde C_{i-1}\lambda_i^t\bar\lambda_i^s\frac{\lambda_i}{\bar\lambda_i} \\
&\leq C\tilde C_{i-1}\lambda_i^t\bar\lambda_i^s\Big(\frac{\lambda_i^{(N-1)\gamma}}{(\lambda_i/\bar\lambda_{i-1})^N} + \frac{\lambda_i}{\bar\lambda_i}\Big) \quad \text{for all } \ t,s\geq 0.
\end{aligned}
\tag{4.30}
$$

9. (Conclusion of the proof) From (4.30) we see that $(4.4)_3$ is satisfied, together with:

$$
\tilde C_i \leq C\tilde C_{i-1}\bar\lambda_i^\gamma\Big(\frac{\lambda_i^{(N-1)\gamma}}{(\lambda_i/\bar\lambda_{i-1})^N} + \frac{\lambda_i}{\bar\lambda_i}\Big) + \bar\lambda_i^\gamma\frac{\big(\|\mathcal{D}\|_0 + (lM)^2\big)}{(\bar\lambda_i l)^2} \quad \text{for all } \ i=1\dots k. \tag{4.31}
$$

Also, from (4.27), (4.28) we get $(4.4)_1$, $(4.4)_2$, provided that (4.14), (4.20) hold, namely:

$$
\tilde C_i \leq C\tilde C_0 \quad \text{and} \quad \frac{(\lambda_i/\bar\lambda_{i-1})}{\lambda_i^\gamma} \geq \frac{2C}{r_0} \quad \text{for all } i=1\dots k. \tag{4.32}
$$

We now make a scaling assumption, towards obtaining the final form of the constants $\tilde C_i$. Set:

$$
\frac{\bar\lambda_i}{\lambda_i} = \Big(\frac{\lambda_i}{\bar\lambda_{i-1}}\Big)^N \quad \text{for all } i=1\dots k, \tag{4.33}
$$

which equivalently reads: $\lambda_i^{N+1} = \bar\lambda_i\bar\lambda_{i-1}^N$ and further: $\big(\lambda_i/\bar\lambda_{i-1}\big)^N = \big(\bar\lambda_i/\bar\lambda_{i-1}\big)^{N/(N+1)}$. We also assume the following condition:

$$
\frac{(\bar\lambda_i/\bar\lambda_{i-1})}{\bar\lambda_i^{\gamma N(N+1)}} \geq \Big(\frac{2C}{r_0}\Big)^{N+1} \quad \text{for all } i=1\dots k. \tag{4.34}
$$

Observe that the above implies the second condition in (4.32) because:

$$\frac{\lambda_i/\bar\lambda_{i-1}}{\lambda_i^\gamma} \geq \frac{\lambda_i/\bar\lambda_{i-1}}{\bar\lambda_i^\gamma} = \Big(\frac{\bar\lambda_i/\bar\lambda_{i-1}}{\bar\lambda_i^{(N+1)\gamma}}\Big)^{1/(N+1)} \geq \frac{2C}{r_0},$$

whereas the first condition in (4.32) also then holds, as (4.31) becomes in view of (4.34):

$$\begin{split} \tilde C_i & \leq C\tilde C_{i-1}\frac{\bar\lambda_i^{N\gamma}}{(\lambda_i/\bar\lambda_{i-1})^N} + \bar\lambda_i^\gamma\frac{\big(\|\mathcal{D}\|_0+(lM)^2\big)}{(\bar\lambda_i l)^2} \\ & \leq C\frac{\tilde C_{i-1}}{(\bar\lambda_i/\bar\lambda_{i-1})^{(N-1)/(N+1)}} + \bar\lambda_i^\gamma\frac{\big(\|\mathcal{D}\|_0+(lM)^2\big)}{(\bar\lambda_i l)^2}, \end{split}$$

so that a straightforward induction argument shows that:

$$\tilde C_i \leq C\frac{\tilde C_0}{(\bar\lambda_i/\bar\lambda_0)^{(N-1)/(N+1)}} = C\frac{\tilde C_0}{(\bar\lambda_i l)^{(N-1)/(N+1)}} \quad \text{for all } i=1\dots k. \tag{4.35}$$

Also, we note that the monotonicity of the sequence:

$$\frac{1}{l} = \bar\lambda_0 \leq \frac{(\lambda l)^{N+1}}{l} = \bar\lambda_1 \leq \bar\lambda_2\ldots \leq \bar\lambda_i \ldots \leq \bar\lambda_k, \tag{4.36}$$

implies the monotonicity properties in (4.2), because then $\lambda_i = (\bar\lambda_i\bar\lambda_{i-1}^N)^{1/(N+1)} \geq \bar\lambda_{i-1}$ in view of (4.33) and thus also $\bar\lambda_i/\lambda_i\geq 1$, for all $i=1\dots k$.

Finally, we assign the progression of frequences in (4.36) motivated by $(4.4)_2$, namely we request:

$$\tilde C_{i-1}^{1/2}\bar\lambda_i \leq C\tilde C_0^{1/2}\bar\lambda_1 \quad \text{for all } i=1\dots k, \tag{4.37}$$

which in view of (4.35) is implied by: $\bar\lambda_i l \leq C(\bar\lambda_i l)(\bar\lambda_{i-1}l)^{(N-1)/(2(N+1))}$. We thus set:

$$\bar\lambda_i l = C(\bar\lambda_i l)(\bar\lambda_{i-1}l)^{(N-1)/(2(N+1))}.$$

The above is a straightforward recursion, which in the closed form yields the formula:

$$\bar\lambda_i l = (\bar\lambda_1 l)^{\alpha_i} \quad \text{with } \alpha_i = \frac{1-\Big(\frac{N-1}{2(N+1)}\Big)^i}{1-\frac{N-1}{2(N+1)}} \quad \text{for all } i=1\dots k,$$

that indeed is compatible with (4.36). Towards verifying (4.34), the above implies:

$$\frac{\bar\lambda_i/\bar\lambda_{i-1}}{\bar\lambda_i^{N(N+1)\gamma}} = \frac{(\bar\lambda_1 l)^{\alpha_i-\alpha_{i-1}}}{(\bar\lambda_1 l)^{N(N+1)\gamma\alpha_i}} l^{N(N+1)\gamma} = \frac{(\lambda l)^{(N+1)(\alpha_i-\alpha_{i-1}-N(N+1)\gamma\alpha_i+N\gamma)}}{\lambda^{N(N+1)\gamma}},$$

Since the exponent in the numerator term above, for every $i=1\dots k$ can be estimated by:

$$\alpha_i-\alpha_{i-1}-N(N+1)\gamma\alpha_i+N\gamma \geq \Big(\frac{N-1}{2(N+1)}\Big)^k - \gamma N(2N+3) \geq \frac12\Big(\frac{N-1}{2(N+1)}\Big)^k,$$

if only γ is sufficiently small in function of N and k, we see that (4.34) is implied by:

$$\frac{\lambda l}{\lambda^{2N\left(\frac{2(N+1)}{N-1}\right)^k\gamma}} \geq \Big(\frac{2C}{r_0}\Big)^{2\left(\frac{2(N+1)}{N-1}\right)^k}, \tag{4.38}$$

while $(4.4)_1$-$(4.4)_3$ result in:

$$\|v_k - v\|_1 \le C\tilde{C}_0^{1/k}, \qquad \|w_k - w\|_1 \le C\tilde{C}_0^{1/2}\big(1 + \tilde{C}_0^{1/2} + \|\nabla v\|_0\big),$$
$$\|\nabla^2 v_k\|_0 \le C\tilde{C}_0^{1/2}\bar{\lambda}_1, \qquad \|\nabla^2 w_k\|_0 \le C\tilde{C}_0^{1/2}\bar{\lambda}_1\big(1 + \tilde{C}_0^{1/2} + \|\nabla v\|_0\big),$$
$$\|\mathcal{D}_k\|_0 \le C\frac{\tilde{C}_k}{\bar{\lambda}_k^{\gamma}} \le C\frac{\tilde{C}_0}{(\lambda l)^{(N-1)\alpha_k}}.$$

We now summarize the obtained bounds, under the assumption (4.38), in the following form:

$$\begin{aligned}
&\|v_k - v\|_1 \le C\lambda^{\gamma/2}\big(\|\mathcal{D}\|_0^{1/2} + lM\big),\\
&\|w_k - w\|_1 \le C\lambda^{\gamma}\big(\|\mathcal{D}\|_0^{1/2} + lM\big)\big(1 + \|\mathcal{D}\|_0^{1/2} + lM + \|\nabla v\|_0\big),\\
&\|\nabla^2 v_k\|_0 \le C\frac{(\lambda l)^{N+1}\lambda^{\gamma/2}}{l}\big(\|\mathcal{D}\|_0^{1/2} + lM\big),\\
&\|\nabla^2 w_k\|_0 \le C\frac{(\lambda l)^{N+1}\lambda^{\gamma}}{l}\big(\|\mathcal{D}\|_0^{1/2} + lM\big)\big(1 + \|\mathcal{D}\|_0^{1/2} + lM + \|\nabla v\|_0\big),\\
&\|\mathcal{D}_k\|_0 \le C\frac{\lambda^{\gamma}}{(\lambda l)^{\frac{2(N^2-1)}{N+3}\left(1-\left(\frac{N-1}{2(N+1)}\right)^k\right)}}\big(\|\mathcal{D}\|_0 + (lM)^2\big)
\end{aligned}$$

The claimed $(1.8)_1$ - $(1.8)_3$ follow by rescaling $2N\big(\frac{N-1}{2(N+1)}\big)^k\gamma$ to γ. The proof is done. ■

5. A proof of Theorem 1.1

The proof of Theorem 1.1 relies on iterating Theorems 1.3 and 1.4 according to the Nash-Kuiper scheme, whose proof and estimates involving the Hölder exponent, as in the decomposition Lemma 2.2, were given [12]. We need to further adjust these iteration estimates in view of the new assumption in (1.6). Recall the following:

Theorem 5.1. [12, Theorem 1.4] *Let $\omega \subset \mathbb{R}^d$ be open, bounded and smooth, and let $k, J, S \ge 1$. Assume that there exists $l_0 \in (0,1)$ such that the following holds for every $l \in (0, l_0]$. Given $v \in \mathcal{C}^2(\bar{\omega} + \bar{B}_{2l}(0), \mathbb{R}^k)$, $w \in \mathcal{C}^2(\bar{\omega} + \bar{B}_{2l}(0), \mathbb{R}^d)$, $A \in \mathcal{C}^{0,\beta}(\bar{\omega} + \bar{B}_{2l}(0), \mathbb{R}^{d\times d}_{\mathrm{sym}})$, and γ, λ, M with:*

$$\gamma \in (0,1), \qquad \lambda > \frac{1}{l}, \qquad M \ge \max\{\|v\|_2, \|w\|_2, 1\}, \tag{5.1}$$

there exist $\tilde{v} \in \mathcal{C}^2(\bar{\omega} + \bar{B}_l(0), \mathbb{R}^k)$, $\tilde{w} \in \mathcal{C}^2(\bar{\omega} + \bar{B}_l(0), \mathbb{R}^d)$ satisfying:

$$\begin{aligned}
&\|\tilde{v} - v\|_1 \le C\lambda^{\gamma/2}\big(\|\mathcal{D}\|_0^{1/2} + lM\big),\\
&\|\tilde{w} - w\|_1 \le C\lambda^{\gamma}\big(\|\mathcal{D}\|_0^{1/2} + lM\big)\big(1 + \|\mathcal{D}\|_0^{1/2} + lM + \|\nabla v\|_0\big),\\
&\|\nabla^2 \tilde{v}\|_0 \le C\frac{(\lambda l)^J}{l}\lambda^{\gamma/2}\big(\|\mathcal{D}\|_0^{1/2} + lM\big),\\
&\|\nabla^2 \tilde{w}\|_0 \le C\frac{(\lambda l)^J}{l}\lambda^{\gamma}\big(\|\mathcal{D}\|_0^{1/2} + lM\big)\big(1 + \|\mathcal{D}\|_0^{1/2} + lM + \|\nabla v\|_0\big),\\
&\|\tilde{\mathcal{D}}\|_0 \le C\Big(l^{\beta}\|A\|_{0,\beta} + \frac{\lambda^{\gamma}}{(\lambda l)^S}\big(\|\mathcal{D}\|_0 + (lM)^2\big)\Big).
\end{aligned}$$

with constants C depending only on ω, k, J, S, γ, and with the defects, as usual, denoted by: $\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big)$ and $\tilde{\mathcal{D}} = A - \big(\frac{1}{2}(\nabla\tilde{v})^T\nabla\tilde{v} + \mathrm{sym}\nabla\tilde{w}\big)$.

Then, for every v, w, A as above which additionally satisfy $0 < \|\mathcal{D}\|_0 \leq 1$, and for every α in:

$$0 < \alpha < \min\Big\{\frac{\beta}{2}, \frac{S}{S+2J}\Big\}, \tag{5.2}$$

there exist $\bar v \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^k)$ and $\bar w \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^d)$ with the following properties:

$$\begin{aligned} &\|\bar v - v\|_1 \leq C\big(1 + \|\nabla v\|_0\big)^2 \|\mathcal{D}_0\|_0^{1/4}, \quad \|\bar w - w\|_1 \leq C(1 + \|\nabla v\|_0)^3 \|\mathcal{D}\|_0^{1/4}, \\ &A - \big(\frac{1}{2}(\nabla \bar v)^T \nabla \bar v + \mathrm{sym}\nabla \bar w\big) = 0 \quad \text{in } \bar\omega. \end{aligned}$$

The constants C above depend only on ω, k, A and α.

The above formulation does not allow iterating on the construction in Theorems 1.3 and 1.4, because of the assumption $\lambda^{1-\gamma} l \geq \sigma_0$. We however observe:

Lemma 5.2. *Theorem 5.1 remains valid if (5.1) is replaced by a more restrictive assumption:*

$$\gamma \in (0,1), \qquad \lambda^{1-\gamma} l > \sigma_0, \qquad M \geq \max\{\|v\|_2, \|w\|_2, 1\}, \tag{5.3}$$

where $\sigma_0 \geq 1$ is a given constant depending on ω, k, S, J, γ.

Proof. We only indicate changes in the proof of Theorem 1.3 in [12], referring to the formulas numbering in there. In step 1, the general requirement (4.3) is now replaced by:

$$\begin{aligned} &l_{i+1} \leq \frac{l_i}{2}, \quad l_i \lambda_i^{1-\gamma} > \sigma_0, \quad M_i \geq \max\{\|v_i\|_2, \|w_i\|_2, 1\}, \quad M_i \nearrow \infty, \\ &\|\mathcal{D}_i\|_0 \leq (l_i M_i)^2, \quad l_i M_i \to 0, \end{aligned}$$

taking into account the middle assumption in (5.3). In step 2, formulas (4.6), (4.8) and (4.11) are augmented by the additional requirement on b in the definition $\lambda_i = b/l_i^a$:

$$b^{1/2} \geq \sigma_0.$$

Steps 3 - 5 remain unchanged. In steps 6 and 7, we augment (4.22) and (4.24) by the same bound above, and proof of their viability is the same. Step 8 remains unaltered. ■

We note that Lemma 5.2 automatically yields the following result below, where we compute $\frac{S}{S+2J} = \frac{1}{1+2J/S}$, with: $\frac{J}{S} = \frac{1}{N} \to 0$ as $N \to \infty$ when $k \geq 4$, while in the general case:

$$\frac{J}{S} = \frac{N+3}{2(N-1)\big(1 - \big(\frac{N-1}{2(N+1)}\big)^k\big)} \to \frac{1}{2\big(1 - \frac{1}{2^k}\big)} = \frac{2^{k-1}}{2^k - 1} \quad \text{as } N \to \infty$$

Corollary 5.3. *Let $\omega \subset \mathbb{R}^2$ be an open, bounded and smooth domain, and let $k \geq 1$. Fix any α as in (1.4). Then, there exists $l_0 \in (0,1)$ such that, for every $l \leq l_0$, given $v \in \mathcal{C}^2(\bar\omega + \bar B_{2l}(0), \mathbb{R}^k)$, $w \in \mathcal{C}^2(\bar\omega + \bar B_{2l}(0), \mathbb{R}^2)$, $A \in \mathcal{C}^{0,\beta}(\bar\omega + \bar B_{2l}(0), \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, such that:*

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T \nabla v + \mathrm{sym}\nabla w\big) \quad \text{satisfies} \quad 0 < \|\mathcal{D}\|_0 \leq 1,$$

there exist $\tilde v \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^k)$, $\tilde w \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^2)$ with the following properties:

$$\begin{aligned} &\|\tilde v - v\|_1 \leq C(1 + \|\nabla v\|_0)^2 \|\mathcal{D}\|_0^{1/4}, \quad \|\tilde w - w\|_1 \leq C(1 + \|\nabla v\|_0)^3 \|\mathcal{D}\|_0^{1/4}, \\ &A - \big(\frac{1}{2}(\nabla \tilde v)^T \nabla \tilde v + \mathrm{sym}\nabla \tilde w\big) = 0 \quad \text{in } \bar\omega. \end{aligned}$$

The norms in the left hand side above are taken on $\bar\omega$, and in the right hand side on $\bar\omega + \bar B_{2l}(0)$. The constants C depend only on ω, k, A and α.

The proof of Theorem 1.1 is consequently the same as the proof of Theorem 1.1 in [12], in section 5 in there. We replace ω by its smooth superset, and apply the basic stage construction in order to first decrease $\|\mathcal{D}\|_0$ below 1. Then, Corollary 5.3 yields the result. ■

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