EXISTENCE OF TRAVELING WAVES IN THE STOKES-BOUSSINESQ SYSTEM FOR REACTIVE FLOWS

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Abstract. We consider the Stokes-Boussinesq equations in a slanted (that is, not aligned with gravity's direction) cylinder of any dimension and with an arbitrary Rayleigh number. We prove the existence of a non-planar traveling wave solution, propagating at a constant speed, and satisfying the Dirichlet boundary condition in the velocity and the Neumann condition in the temperature.

1. Introduction

In this paper we study the existence of traveling wave solutions to the system of reactive Boussinesq equations. This system is given as the advection-reaction-diffusion equation for the temperature T coupled with the Stokes equation for the incompressible flow u driven by T. After passing to non-dimensional variables [BCR], the system takes the form:

$$\begin{aligned} T_t + u \cdot \nabla T - \Delta T &= f(T), \\ -\Delta u + \nabla p &= T\rho\vec{g} \\ \mathrm{div}\ u &= 0. \end{aligned} \tag{1.1}$$

The nonlinearity f is a nonnegative Lipschitz function and throughout we assume it to be of the ignition type. Vector $\vec{g}$ represents the scaled gravity and the constant $\rho > 0$ corresponds to the Rayleigh number. We consider the system (1.1) in an infinite cylinder $D \subset \mathbf{R}^n$, not aligned with $\vec{g}$ and with a smooth crossection Ω. We want to look for the traveling waves in (u, T) connecting $(0, 1)$ to $(0, 0)$ and satisfying the Neumann boundary conditions in T and the Dirichlet (that is, *'no-slip'*) boundary conditions in u.

The system (1.1) can be derived, under the assumption that the Prandtl number σ equals $+\infty$, from a more complete Navier-Stokes-Boussinesq system:

$$\begin{aligned} T_t + u \cdot \nabla T - \Delta T &= f(T) \\ \frac{1}{\sigma}(u_t + u \cdot \nabla u) - \Delta u + \nabla p &= T\rho\vec{g} \\ \mathrm{div}\ u &= 0. \end{aligned} \tag{1.2}$$

For a review of recent results on front propagation we refer to [X] or [B]. In connection with the single temperature equation in (1.1) when u is an imposed flow of shear type, the existence and uniqueness of a multidimensional traveling wave has been proved in [BLL]; this wave is stable both in the linear and the nonlinear sense [R]. As a next step, while studying the coupled system (1.2) or (1.1) in which the flow is affected by the evolution of the temperature of the reactant, one may ask whether there exists a traveling wave in (u, T) whose T- component does not correspond to a traveling solution of the reaction-advection-diffusion equation in T seen from the previous theory, and in particular that is not planar, so that $u \not\equiv 0$. The numerical computations in [VR] suggest that non-planar fronts exist and are stable for large Rayleigh numbers ρ.

In the setting of $n = 2$ and an infinite cylinder aligned with $\vec{g}$, it has been proved in [CKR] that non-planar traveling fronts cannot exist if the aspect ratio (the ratio of the width of the domain and the so-called thickness of the planar front) is sufficiently small. In the same regime, the planar wave (which corresponds to a traveling solution of the reaction-diffusion equation in T), is nonlinearly stable, that is it attracts all solutions of the Cauchy problem asymptotically in time. On the other hand, when the strip is wide and the Rayleigh number is large, the planar fronts are linearly unstable. Moreover, as shown in [TV] there exists a bifurcation at a critical value $\rho_c > 0$; for any sufficiently small interval $[\rho_c, \rho]$ there exist non-planar fronts whose Rayleigh number belongs to this interval.

The situation is quite different when D is not aligned with $\vec{g}$. As shown by Berestycki, Constantin and Ryzhik in [BCR] (still in the setting $n = 2$), a traveling front exists always, independently of the width of the strip and ρ, under the no-stress boundary condition on u. As observed in [BKV], this traveling front cannot be planar.

For a viscous fluid, the no-stress boundary condition is artificial and should be replaced by the no-slip condition. Indeed, in [CLR] Constantin, Lewicka and Ryzhik prove the same existence result for the full system (1.2) and $n = 2$. The goal of the present paper is to extend this result to smooth cylinders of any dimension n and for the system (1.1).

It seems that unlike in the case $n = 2$, for $n \geq 3$ the related analysis should be done separately for systems (1.1) and (1.2). On one hand the estimates on the propagation speed obtained from the reaction-advection-diffusion equation in T are too weak in presence of the quadratic terms in (1.2), while on the other hand, more refined bounds on the propagation speed, coming from the Navier-Stokes equation in u, may not be true in the former case. The case of the full system (1.2) is at the center of our attention and we will address it in a separate paper.

Our main result, whose precise formulation is contained in Theorem 5.1, can be stated as follows:

Theorem 1.1. *Assume that the unbounded direction of D is not aligned with the gravity vector $\vec{g}$. Then there exists a smooth, non-planar traveling wave solution $T(x-ct,\cdot)$, $u(x-ct,\cdot)$ to (1.1) with the ignition type nonlinearity f, which satisfies the boundary conditions (2.4). When moreover the nonlinearity satisfies the smallness condition:*

$$f(T) \le C_\Omega[(T-\theta_0)_+]^n,$$

then this solution satisfies $\theta_- = 1$ in (2.4). Above θ_0 is the ignition temperature and $C_\Omega > 0$ is a constant, depending only on ρ^{n-2} and the geometry of Ω.

The smallness assumption above is made for purely technical reasons and was also present in the two-dimensional setting of [BCR] and [CLR], in which case $C_\Omega = |\Omega|^{-2}$.

Our paper is organized as follows. In section 2 we introduce a version of the problem, posed on compact domains increasing to D. In order to show that each of these approximating problems has a solution, we want to use the Leray-Schauder degree argument and hence we need the a-priori bounds on the solutions. Since we eventually want to pass to the limit with the lengths of the domains $2a$ and recover the traveling wave as the limit of the approximate solutions, we need the a-priori bounds to be independent of a. This is the crucial point; we prove such bounds in section 3 and the major difference with respect to [CLR] is that for $n \ge 3$ the estimates using L^2 norms do not suffice and have to be done instead in L^p, with $p \ge n$. The same estimates remain valid in the setting of the Navier-Stokes-Boussinesq system (1.2); in appendix B we remark that other improvements are necessary to close the bounds for this case. In appendix A we give a proof of the L^p elliptic estimates for the Stokes system on bounded domains. We finally show the existence of the approximate solutions on compact domains, satisfying uniform bounds, in section 4. We prove that the limit of these solutions is a non-planar traveling wave in section 5, where we also discuss the wave's limits at $\pm\infty$.

2. The setting of the problem and its compact approximations

We study the system (1.1) where the unknown functions $T \in \mathbf{R}$, $u \in \mathbf{R}^n$ and $p \in \mathbf{R}$ are defined in an infinite cylinder $D \subset \mathbf{R}^n$ with a smooth, connected crossection $\Omega \subset \mathbf{R}^{n-1}$. The 'gravity vector' $\vec{\rho} = \rho \cdot \vec{g}$ is not parallel to the unbounded direction of D. By an elementary change of variables we can without loss of generality restrict our attention to the horizontal cylinder:

$$D = (-\infty,\infty)\times\Omega = \{(x,\tilde{x});\ x\in\mathbf{R},\ \tilde{x}\in\Omega\} \tag{2.1}$$

and

$$\vec{\rho}\cdot e_n \ne 0. \tag{2.2}$$

The nonlinear Lipschitz continuous function f is assumed to be of 'ignition type':

$$f(T) = 0 \text{ on } (-\infty,\theta_0]\cup[1,\infty), \qquad f(T) > 0 \text{ on } (\theta_0, 1) \tag{2.3}$$

for some 'ignition temperature' $\theta_0 \in (0, 1)$.

We impose the following boundary conditions:

$$
\begin{aligned}
&T \to \theta_- \text{ as } x \to -\infty,\ T \to 0 \text{ as } x \to \infty,\ \frac{\partial T}{\partial \vec{n}} = 0 \text{ on } \partial D,\\
&u = 0 \text{ on } \partial D,\ u \to 0 \text{ as } x \to \pm\infty,
\end{aligned} \tag{2.4}
$$

where by $\vec{n}$ we denote the unit normal to ∂D, and θ_- is some equilibrium of f. Of particular interest is $\theta_- = 1$. We want to study the existence of a traveling wave solution to (1.1) (2.4): $T(x - ct, \tilde{x})$, $u(x - ct, \tilde{x})$, with the speed c to be determined. Such a front satisfies:

$$-cT_x + u \cdot \nabla T - \Delta T = f(T), \tag{2.5}$$

$$-\Delta u + \nabla p = T\vec{\rho}, \tag{2.6}$$

$$\text{div } u = 0, \tag{2.7}$$

together with (2.4).

Towards the proof of Theorem 1.1 we will first replace the problem (2.4) - (2.7) with its approximation on compact domains, which will allow to use the Leray-Schauder degree theory. Let $R_a = [-a, a] \times \Omega$ and let D_a be a smooth domain such that $R_a \subset D_a \subset R_{a+1}$ (see figure 1).

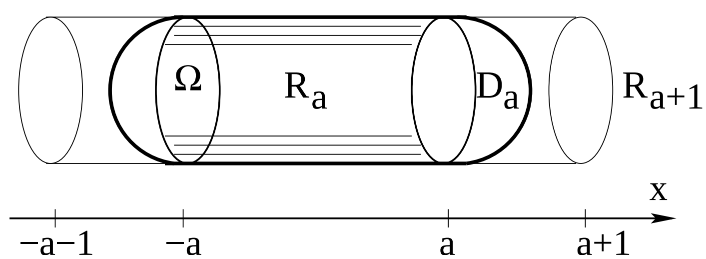


FIGURE 1

We want that (2.5) be satisfied in R_a and (2.6) in D_a, where T is extended on D_a by the odd reflection across the vertical boundary of R_a:

$$T(x, \tilde{x}) = \begin{cases} -T(-2a - x, \tilde{x}) + 2T(-a, \tilde{x}) & \text{for } x < -a \\ -T(2a - x, \tilde{x}) + 2T(a, \tilde{x}) & \text{for } x > a. \end{cases} \tag{2.8}$$

Notice that one clearly has:

$$||T||_{\mathcal{C}^{1,\alpha}(D_a)} \leq 2||T||_{\mathcal{C}^{1,\alpha}(R_a)}, \quad ||\nabla T||_{L^2(D_a)} \leq 2||\nabla T||_{L^2(R_a)}. \tag{2.9}$$

The boundary conditions for the compactified problem are:

$$
\begin{gathered}
u = 0 \text{ on } \partial D_a,\\
T(-a,\tilde{x}) = 1,\ T(a,\tilde{x}) = 0 \text{ for } \tilde{x}\in\Omega,\\
\frac{\partial T}{\partial \vec{n}}(x,\tilde{x}) = \frac{\partial T}{\partial \vec{n}}(x,\tilde{x}) = 0 \text{ for } x\in[-a,a] \text{ and } \tilde{x}\in\partial\Omega,
\end{gathered}
\tag{2.10}
$$

together with the following normalization condition:

$$
\max\{T(x,\tilde{x});\ x\in[0,a],\ \tilde{x}\in\Omega\} = \theta_0. \tag{2.11}
$$

Our first result is:

Theorem 2.1. *For $a > 0$ sufficiently large, there exist $c \in \mathbf{R}$, $T \in \mathcal{C}^{1,\alpha}(R_a)$, $u \in \mathcal{C}^{2,\alpha}(D_a)$, $p \in \mathcal{C}^{1,\alpha}(D_a)$ solving the problem (2.5) - (2.11) and satisfying the following uniform bound:*

$$
|c| + ||T||_{\mathcal{C}^{1,\alpha}(R_a)} + ||u||_{\mathcal{C}^{2,\alpha}(D_a)} + ||\nabla T||_{L^2(R_a)} + ||u||_{W^{3,2}(D_a)} + \int_{R_a} f(T) \leq C. \tag{2.12}
$$

Above C does not depend on a and on none of the estimated quantities.

We postpone the proof of Theorem 2.1 to section 4.

Remark 2.2. The normalization condition (2.11) is the same as in [BNS]. Its role is to single out a correct approximation of the traveling wave in T, in the moving frame which chooses (to fix the ideas) to have $f(T(x,\cdot)) = 0$ for $x \geq 0$.

3. Uniform bounds

The aim of this section is to prove the a-priori bound as in (2.12):

Theorem 3.1. *Let $a > 0$ be sufficiently large and let $c \in \mathbf{R}$, $T \in \mathcal{C}^{1,\alpha}(D_a)$, $u \in \mathcal{C}^{2,\alpha}(D_a)$, $p \in \mathcal{C}^{1,\alpha}(D_a)$ satisfy (2.5) - (2.11). Then the bound (2.12) holds.*

The proof will be achieved through a sequence of lemmas estimating various norms of the quantities c, T, u. Throughout by C we denote the generic constant, the value of which may change from line to line, but does not depend on a, c, T, u. We also use the notation $A \preceq B$ meaning $A \leq C(1 + B)$.

In all the subsequent lemmas we assume that $a > 0$ is sufficiently large and that c, T, u, p having regularity as in Theorem 3.1 satisfy (2.5) - (2.11).

Lemma 3.2. *There holds:*

$$
T(x,\tilde{x}) \in [0,1] \qquad \textit{for all } (x,\tilde{x}) \in R_a \tag{3.1}
$$

$$
T(x,\tilde{x}) \leq \theta_0 \qquad \textit{for all } x > 0,\ \tilde{x}\in\Omega. \tag{3.2}
$$

Proof. Because of the nonnegativity of f, the maximum principle and the Hopf lemma can be applied to (2.5) on R_a, which gives $T \geq 0$ in R_a. Applying the same results to T on $[0, a] \times \Omega$ we obtain (3.2) in view of the normalization (2.11).

Now the function $1 - T$ satisfies the equation:

$$\Delta(1 - T) - u \cdot \nabla(1 - T) + c(1 - T)_x - \frac{f(T)}{1 - T}(1 - T) = 0,$$

where the lowest order coefficient is nonpositive. Again, we conclude that $1 - T \geq 0$ in R_a, proving (3.1). ∎

Lemma 3.3. $|c| \preceq ||u||_{L^\infty}$.

Proof. The proof follows as in [BLL, BCR]. Consider the function $\psi(x, \tilde{x}) = e^{-\alpha(x+a)}$. For every $\alpha > 0$ such that:

$$c \geq \alpha + ||u||_{L^\infty} + \frac{M}{\alpha} \tag{3.3}$$

with $M = ||f'||_{L^\infty}$, ψ is a supersolution of (2.5), so that it satisfies:

$$\begin{aligned} -\Delta\psi + u \cdot \nabla\psi - c\psi_x \geq \frac{f(T)}{T} \times \psi \qquad & \text{in } R_a, \\ \psi(x, \tilde{x}) \geq T(x, \tilde{x}) \qquad & \text{for } \tilde{x} \in \Omega \text{ and } x \in \{-a, a\}. \end{aligned}$$

We now show that for such values of α one has $T \leq \psi$ in R_a. Define:

$$A_0 = \inf\{A \geq 1;\ T \leq A\psi \text{ in } R_a\}.$$

We have that $A_0 \in [1, \infty)$, as $T \leq A\psi$ for A sufficiently large. Clearly, $A_0\psi$ is also a supersolution of (2.5), and the nonnegative function $A_0\psi - T$ achieves its minimum $(= 0)$ in R_a. By the Hopf lemma and the maximum principle [GT] we conclude that this minimum has to be attained in the set $\Omega \times \{-a, a\}$. This implies that $A_0 = 1$.

Now, using the normalization (2.11) we see that $\theta_0 \leq e^{-a\alpha}$. Therefore, if $\alpha > \ln(\theta_0^{-1})/a$, the negation of (3.3) must hold:

$$c < \alpha + ||u||_{L^\infty} + \frac{M}{\alpha}.$$

Therefore:

$$c \leq ||u||_{L^\infty} + \inf_{\alpha \geq \ln(\theta_0^{-1})/a} (\alpha + \frac{M}{\alpha}) \leq ||u||_{L^\infty} + 1 + M.$$

In order to prove the lower bound, we consider the following subsolution of (2.5): $\phi(x, \tilde{x}) = 1 - e^{\alpha(x-a)}$ where $\alpha > 0$ satisfies

$$c \leq -\alpha - ||u||_{L^\infty}.$$

We reason as before and obtain that if $1 - e^{-\alpha a} \leq \theta_0$. Hence:

$$c \geq -||u||_{L^\infty} - \inf_{\alpha \geq \ln(1-\theta_0)^{-1}/a} (\alpha) \geq -||u||_{L^\infty} - 1.$$

This proves the desired inequality. ■

Lemma 3.4. *For every $p \geq 2$ there holds:*

$$||\nabla^2 T||_{L^p(R_a)} \leq C(||\Delta T||_{L^p(R_a)} + ||\nabla T||_{L^p(R_a)}).$$

Proof. Define $S(x, \tilde{x}) = \frac{1}{|\Omega|}\int_\Omega T(x, \cdot)$. Of course, the function S depends only on the variable x. The function $T - S$ satisfies the Dirichlet condition on the vertical part of ∂R_a and the Neumann condition on the horizontal part of ∂R_a. Therefore, by standard elliptic estimates up to the boundary [ADN, GT] we obtain:

$$||\nabla^2(T - S)||_{L^p} \leq C(||\Delta(T - S)||_{L^p} + ||T - S||_{L^p}).$$

Hence:

$$(3.4)\quad ||\nabla^2 T||_{L^p} \leq ||\nabla^2(T - S)||_{L^p} + ||\nabla^2 S||_{L^p} \leq C(||\Delta T||_{L^p} + ||\nabla^2 S||_{L^p} + ||T - S||_{L^p}).$$

Moreover:

$$(3.5)\qquad ||\nabla^2 S||^p_{L^p} = ||S_{xx}||^p_{L^p} = \int_{R_a} \left| \frac{1}{|\Omega|} \int_\Omega \Delta T \right|^p \leq C \int_{R_a} \int_\Omega |\Delta T|^p \leq C||\Delta T||^p_{L^p},$$

because $\int_\Omega T_{xx} = \int_\Omega \Delta T$ in view of the Neumann boundary condition for T. On the other hand by the Poincare' inequality:

$$||T - S||_{L^p} \leq C||\nabla T||_{L^p}.$$

Using (3.4) and (3.5) we conclude the proof. ■

Lemma 3.5. *For every natural $p \geq 2$ we have:*

$$(3.6)\qquad ||\nabla T||_{L^p(R_a)} \preceq (|c| + ||u||_{L^\infty(D_a)})^{\frac{p-1}{p}},$$

$$(3.7)\qquad \int_{R_a} f(T) \preceq |c| + ||u||_{L^\infty(D_a)}.$$

Proof. **1.** We first prove (3.6) for $p = 2$ and (3.7). Multiplying (2.5) by $1 - T$ and using boundary conditions together with the incompressibility of u on R_a we obtain:

$$\int_{R_a} |\nabla T|^2 + \int_\Omega T_x(a, \cdot) = \frac{|\Omega|c}{2} - \frac{1}{2}\int_\Omega u^1(a, \cdot) - \int_{R_a} (1 - T)f(T).$$

Since the last term above is nonpositive, we obtain:

$$(3.8)\qquad \int_{R_a} |\nabla T|^2 \leq \frac{|\Omega|}{2}(|c| + ||u||_{L^\infty}) - \int_\Omega T_x(a, \cdot).$$

We now reproduce the argument from [BCR] to bound the term $-\int_\Omega T_x(a, \cdot)$. Consider the quantity $I(x) = \frac{1}{|\Omega|}\int_\Omega T(x, \cdot)$. Integrating (2.5) on Ω, we notice that I

satisfies:

$$-I_{xx} = \frac{1}{|\Omega|}\int_\Omega f(T) - u\cdot\nabla T + cT_x, \qquad (3.9)$$
$$I(-a) = 1,\ \ I(a) = 0.$$

By an elementary and explicit calculation (see [BCR]) one obtains:

$$-I_x(a) = \frac{1}{2a} + \frac{1}{2a|\Omega|}\int_{-a}^{a}(a+x)\int_\Omega (f(T) - u\cdot\nabla T + cT_x)(x,\tilde{x})\ \mathrm{d}\tilde{x}\mathrm{d}x.$$

Using the boundary conditions and the incompressibility of u, we obtain:

$$-I_x(a) = \frac{1}{2a} + \frac{1}{|R_a|}\int_{R_a}(a+x)f(T) + \frac{1}{|R_a|}\int_{R_a} u^1 T - \frac{c}{|R_a|}\int_{R_a} T.$$

We now notice that $\int_{R_a} xf(T) \leq 0$ by (3.2). Therefore, using (3.1) we obtain:

$$-I_x(a) \leq \frac{1}{2a} + \frac{1}{|R_a|}\int_{R_a} f(T) + |c| + ||u||_{L^\infty}. \qquad (3.10)$$

Now, integrating (3.9) on $[-a,a]$, we obtain:

$$\frac{1}{|\Omega|}\int_{R_a} f(T) = c + I_x(-a) - I_x(a) - \frac{1}{|\Omega|}\int_\Omega u^1(-a,\cdot).$$

Using (3.10) and noting that $I_x(-a) \leq 0$, we continue:

$$\frac{1}{|\Omega|}\int_{R_a} f(T) \leq 2(|c| + ||u||_{L^\infty}) + \frac{1}{2a} + \frac{1}{2|\Omega|}\int_{R_a} f(T)$$

which implies (3.7). Together with (3.10) and (3.8) we also conclude that $\int_{R_a}|\nabla T|^2 \preceq |c| + ||u||_{L^\infty}$.

2. We proceed by induction on $p \geq 3$. Assume that (3.6) holds for $p-1$, that is:

$$\int_{R_a} |\nabla T|^{p-1} \preceq (|c| + ||u||_{L^\infty})^{p-2}. \qquad (3.11)$$

Multiply the equation (2.5) by $T|\nabla T|^{p-2}$ to obtain:

$$\begin{aligned}\left|-\int_{R_a}\Delta T\cdot T|\nabla T|^{p-2}\right| \\ &= \left|c\int T_x T|\nabla T|^{p-2} - \int u\cdot\nabla T\cdot T|\nabla T|^{p-2} + \int f(T)T|\nabla T|^{p-2}\right| \\ &\leq (|c| + ||u||_{L^\infty})\cdot\int_{R_a}|\nabla T|^{p-1} + \int f(T)|\nabla T|^{p-2},\end{aligned}$$

where we used Lemma 3.2 and the boundedness of f. Now recall (3.11) and (3.7) and write:

$$\int f(T)|\nabla T|^{p-2} \leq ||\nabla T||^{p-2}_{L^{p-1}(R_a)} \cdot ||f(T)||_{L^{p-1}(R_a)}$$
$$\preceq (|c| + ||u||_{L^\infty})^{\frac{(p-2)^2+1}{p-1}} \leq (|c| + ||u||_{L^\infty})^{p-1}.$$

Therefore we obtain:

$$\left|\int \Delta T \cdot T|\nabla T|^{p-2}\right| \preceq (|c| + ||u||_{L^\infty})^{p-1}. \tag{3.12}$$

On the other hand, one has:

$$\int_{R_a} |\nabla T|^p = -\int \Delta T \cdot T|\nabla T|^{p-2} - \int_{R_a} T \cdot \nabla T \cdot \nabla|\nabla T|^{p-2} + \int_\Omega |\nabla T|^{p-1}(-a, \cdot). \tag{3.13}$$

By Lemma 3.4, equation (2.5) and (3.7) we obtain:

$$||\nabla^2 T||_{L^{p-1}(R_a)} \preceq ||\Delta T||_{L^{p-1}(R_a)} + ||\nabla T||_{L^{p-1}(R_a)}$$
$$\preceq (|c| + ||u||_{L^\infty}) \cdot ||\nabla T||_{L^{p-1}(R_a)} + (|c| + ||u||_{L^\infty})^{\frac{1}{p-1}} + ||\nabla T||_{L^{p-1}}.$$

Thus, in view of (3.11) we may estimate the second term in the right hand side of (3.13) by:

$$\begin{aligned}\int_{R_a} |\nabla T|^{p-2}|\nabla^2 T| &\leq ||\nabla T||^{p-2}_{L^{p-1}(R_a)} \cdot ||\nabla^2 T||_{L^{p-1}(R_a)} \\ &\preceq (|c| + ||u||_{L^\infty})^{\frac{(p-2)^2}{p-1}+1+\frac{p-2}{p-1}} = (|c| + ||u||_{L^\infty})^{p-1}.\end{aligned} \tag{3.14}$$

To estimate the third term in (3.13), consider the function $|\nabla T|^{p-1}$ on $R'_a = [-a, -a+1] \times \Omega$. By (3.11) and (3.14) we have:

$$\| \, |\nabla T|^{p-1}\|_{W^{1,1}(R'_a)} \preceq (|c| + ||u||_{L^\infty})^{p-1}.$$

Since the trace space of $W^{1,1}(R'_a)$ is embedded in $L^1(\partial R'_a)$, we obtain:

$$\| \, |\nabla T|^{p-1}(-a, \cdot)\|_{L^1(\Omega)} \preceq (|c| + ||u||_{L^\infty})^{p-1}. \tag{3.15}$$

Now, combining (3.13) with (3.12), (3.14) and (3.15), in view of Lemma 3.2 we conclude (3.6). ■

We now turn to estimating norms of u in terms of ∇T. To do this, we need the following:

Lemma 3.6. *For every $a > 1$ and $p \geq 2$ there exists $q \in \mathcal{C}^1(R_a)$ with:*

$$||T\vec{\rho} - \nabla q||_{L^p(R_a)} \leq C||\nabla T||_{L^p(R_a)},$$

where the constant C depends on the exponent p but not on a or T.

Proof. First, extend the function $T\vec{\rho}$ to a $\mathcal{C}^{1,\alpha}$ function $h = (h^1 \dots h^n)$ defined on the domain $R'_a = [-a-1, a+1] \times \Omega'$, where Ω' is a box $[-b, b]^{n-1}$ containing Ω. We may without loss of generality assume that:

$$||\nabla h||_{L^p(R'_a)} \leq 2||\nabla T||_{L^p(R_a)}. \tag{3.16}$$

Define:

$$q(x, \tilde{x}) = q_1(x) + \sum_{i=1}^{n-1} \int_{-b}^{\tilde{x}_i} h^{i+1}(x, \tilde{x}_1 \dots \tilde{x}_{i-1}, s, \tilde{x}_{i+1} \dots \tilde{x}_{n-1})\,\mathrm{d}s,$$

where we use the convention $\tilde{x} = (\tilde{x}_1 \dots \tilde{x}_{n-1}) \in \Omega'$, and the function q_1 satisfies on $[-a-1, a+1]$:

$$q_1'(x) = \frac{1}{|\Omega'|} \int_{\Omega'} h^1(x, \cdot).$$

We clearly have:

$$\begin{aligned} \frac{\partial q}{\partial x}(x, \tilde{x}) &= \frac{1}{|\Omega'|} \int_{\Omega'} h^1(x, \cdot) + \sum_{i=1}^{n-1} \int_{-b}^{\tilde{x}_i} \frac{\partial h^{i+1}}{\partial x}\,\mathrm{d}s, \\ \forall i : 1 \dots n-1 \quad \frac{\partial q}{\partial \tilde{x}_i}(x, \tilde{x}) &= h^{i+1}(x, \tilde{x}) + \sum_{j \neq i} \int_{-b}^{\tilde{x}_i} \frac{\partial h^{i+1}}{\partial x}\,\mathrm{d}s. \end{aligned}$$

Hence:

$$||h - \nabla q||_{L^p(R'_a)} \leq C||\nabla h||_{L^p(R'_a)},$$

where C depends only on the Poincare' constant of Ω' and the magnitude of $|b|$. In view of (3.16) the result follows. ■

Lemma 3.7. *For every $p \geq 2$ we have:*

(*i*) $||u||_{W^{1,2}(D_a)} \leq C||\nabla T||_{L^2(R_a)}$,

(*ii*) $||\nabla T||_{L^p} + ||u||_{L^p(D_a)} \preceq ||u||_{L^\infty}^{\frac{p-1}{p}}$.

Proof. We integrate (2.6) against u and note that by the Cauchy-Schwartz inequality:

$$\int_{D_a} |\nabla u|^2 \leq ||T\vec{\rho} - \nabla q||_{L^2(D_a)} \cdot ||u||_{L^2(D_a)}, \tag{3.17}$$

where q is as in Lemma 3.6. Now, the Poincare' inequality applied to the crossection of D_a implies:

$$||u||_{L^2(D_a)} \leq C||\nabla u||_{L^2(D_a)},$$

which together with (3.17), Lemma 3.6 and (2.9) yields (*i*).

Now, (*ii*) follows from Lemma 3.3, Lemma 3.5, Lemma 3.6 and the interpolation inequality:

$$||u||_{L^p(D_a)} \leq ||u||_{L^2}^{\frac{2}{p}} \cdot ||u||_{L^\infty}^{1-\frac{2}{p}}.$$

■

The following bound which is an extension of Lemma 3.7 (*i*) for exponents $p > 2$ follows for example from [G] or can be found in [K]. We give an alternative, to our knowledge new, proof of this fundamental estimate in Appendix A.

Lemma 3.8. *For every $p \geq 2$, there holds:*

$$||u||_{W^{3,p}(D_a)} \leq C(||\nabla T||_{L^p(R_a)} + ||u||_{L^p(D_a)}).$$

Proof of Theorem 3.1. For $p > n$, we have:

$$||u||_{L^\infty} \leq C||u||_{W^{1,p}} \preceq ||u||_{L^\infty}^{\frac{p-1}{p}},$$

where we have used Lemma 3.8 and Lemma 3.7 (*ii*). This clearly implies that $||u||_{L^\infty} \leq C$. Therefore, by Lemma 3.8, Lemma 3.7 (*ii*), Lemma 3.3 and Lemma 3.5, the same uniform bound holds for the quantities:

$$||u||_{W^{3,p}},\ ||\nabla T||_{L^p},\ |c|,\ \int_{R_a} f(T),$$

and consequently also for $||u||_{\mathcal{C}^{2,\alpha}}$. By standard elliptic estimates applied to (2.6) we obtain $||T||_{\mathcal{C}^{1,\alpha}} \leq C$. ■

Note that up to Lemma 3.8, all the uniform estimates remain valid also for the system (1.2). For the discussion of the finite Prandtl number case, see Appendix B.

4. A proof of Theorem 2.1

Fix $a > 0$ and let $c \in \mathbf{R}$, $T \in \mathcal{C}^{1,\alpha}(R_a)$ and $\tau \in [0,1]$. Extend first T on D_a as in (2.8) and consider the Stokes problem:

$$\begin{aligned} -\Delta u + \nabla p &= \tau T\vec{\rho} && \text{in } D_a, \\ \text{div } u &= 0 && \text{in } D_a, \\ u &= 0 && \text{on } \partial D_a. \end{aligned}$$

By the standard regularity results [L] we have $u \in \mathcal{C}^{2,\alpha}(D_a)$. Let now Z be the solution to:

$$\begin{aligned} -cZ_x + \tau u \cdot \nabla Z - \Delta Z &= \tau f(T) && \text{in } R_a, \\ Z(-a,\tilde{x}) = 1,\ Z(a,\tilde{x}) &= 0 && \text{for } \tilde{x} \in \Omega, \\ \frac{\partial Z}{\partial \vec{n}}(x,\tilde{x}) &= 0 && \text{for } x \in [-a,a] \text{ and } \tilde{x} \in \partial\Omega, \end{aligned}$$

which is $\mathcal{C}^{1,\alpha}$ regular [GT]. We now set:

$$K(c,T,\tau) := (c - \theta_0 + \max\{T(x,\tilde{x});\ x \in [0,a],\ \tilde{x} \in \Omega\}, Z). \tag{4.1}$$

Now, the operator $K : \mathbf{R} \times \mathcal{C}^{1,\alpha}(R_a) \times [0,1] \longrightarrow \mathbf{R} \times \mathcal{C}^{1,\alpha}(R_a)$ is well defined, continuous and compact [GT, L]. Also, by Theorem 3.1 its fixed points (c, T), such that $K(c, T, \tau) = (c, T)$ for some $\tau \in [0, 1]$, are uniformly bounded:

$$|c| + ||u||_{\mathcal{C}^{2,\alpha}(D_a)} \leq C.$$

It is clear that C is also independent of τ. Therefore, the Leray-Schauder degree $\deg(\mathrm{Id} - K(\cdot, \cdot, \tau), B_{2C}(0), 0)$ is well defined and equal to $\deg(\mathrm{Id} - K(\cdot, \cdot, 0), B_{2C}(0), 0)$. To conclude that $K(\cdot, \cdot, 1)$ has a fixed point (which is, by definition, a solution to (2.4) - (2.7)) it is hence enough to see that

$$\deg(\mathrm{Id} - K(\cdot, \cdot, 0), B_{2C}(0), 0) \neq 0. \tag{4.2}$$

To prove (4.2), notice that $K(c, T, 0) = (c - \theta_0 + \max\{T(x, \tilde{x});\ x \in [0, a],\ \tilde{x} \in \Omega\}, Z)$ where

$$Z(x, \tilde{x}) = \phi^c(x) = \frac{e^{-cx} - e^{-ca}}{e^{ca} - e^{-ca}}$$

is the solution (when $c \neq 0$) to:

$$\phi^c_{xx} + c\phi^c_x = 0 \quad \text{in } [-a, a], \qquad \phi^c(-a) = 1,\ \phi^c(a) = 0.$$

As in [BCR] we see that $K(\cdot, \cdot, 0)$ is homotopic to the map

$$\mathcal{F}(c, T) = (c - \theta_0 + \max_{x \in [0,a]} \phi^c(x), \phi^{c^*}),$$

where c^* is the unique number so that

$$\max_{x \in [0,a]} \phi^{c^*}(x) = \phi^{c^*}(0) = \theta_0.$$

The homotopy can be taken to be compact and without fixed points on a sufficiently large ball $B_C(0)$ in $\mathbf{R} \times \mathcal{C}^{1,\alpha}(R_a)$.

By the homotopy invariance of the Leray-Schauder degree of compact perturbations of identity we conclude that:

$$\deg(\mathrm{Id} - K(\cdot, \cdot, 0), B_{2C}(0), 0) = \deg(\mathrm{Id} - \mathcal{F}, B_R(0), 0) = 1,$$

as the degree of the map $(\mathrm{Id} - \mathcal{F})(c, T) = (\theta_0 - \phi^c(0), T - \phi^{c^*})$ is the product of degrees of each component, all of them equal to 1. This proves (4.2) and hence also Theorem 2.1.

5. Identification of the limit and a proof of Theorem 1.1

In this section we make precise and prove Theorem 1.1. We first observe that the sequence of solutions (c^a, T^a, u^a, p^a) to (2.5) - (2.11) has a converging subsequence, as the length of the compact domains increases: $a \to +\infty$. We then prove that the limit is a solution to (2.4) - (2.7).

Theorem 5.1. *There exist $c > 0$, $T, u \in \mathcal{C}^{2,\alpha}(D)$, $p \in \mathcal{C}^{1,\alpha}_{loc}(D)$ satisfying (2.5) - (2.7). Moreover we have: $\nabla T \in L^2(D)$, $u \in W^{3,2}(D)$ and:*

$$T(D) \subset [0,1] \quad \text{and} \quad \max_{x \geq 0, y \in \Omega} T(x,y) = \theta_0, \tag{5.1}$$

$$\frac{\partial T}{\partial \vec{n}} = 0 \text{ and } u = 0 \quad \text{on } \partial D, \tag{5.2}$$

$$\lim_{x \to \pm\infty} ||u(x,\cdot)||_{L^\infty(\Omega)} = \lim_{x \to \pm\infty} ||\nabla u(x,\cdot)||_{L^\infty(\Omega)} = \lim_{x \to \pm\infty} ||\nabla T(x,\cdot)||_{L^\infty(\Omega)} = 0. \tag{5.3}$$

$$\int_D f(T) \in (0,\infty). \tag{5.4}$$

$$\lim_{x \to +\infty} ||T(x,\cdot)||_{L^\infty(\Omega)} = 0, \tag{5.5}$$

The left limit $\theta_- \in (0,\theta_0] \cup \{1\}$ exists:

$$\lim_{x \to -\infty} ||T(x,\cdot) - \theta_-||_{L^\infty(\Omega)} = 0. \tag{5.6}$$

If moreover

$$f(T) \leq C_\Omega [(T-\theta_0)_+]^n, \tag{5.7}$$

where $C_\Omega > 0$ is a constant, depending only on ρ^{n-2} and on the geometry of Ω, then $\theta_- = 1$.

Towards a proof of Theorem 5.1, notice first that by a bootstrap argument we clearly have:

$$||T^a||_{\mathcal{C}^{2,\alpha}(D_{a-1})} \leq C.$$

Therefore we may choose a sequence $a_n \to \infty$ such that $c_n := c^{a_n}$ converges to some $c \in \mathbf{R}$ and $T_n := T^{a_n}$, $u_n := u^{a_n}$ converge in $\mathcal{C}^{2,\alpha}_{loc}(D)$ (for a smaller α which we for simplicity denote with the same symbol) to some $T, u \in \mathcal{C}^{2,\alpha}(D)$. Also, $p_n := p^{a_n}$ converges in $\mathcal{C}^{1,\alpha}_{loc}(D)$ to some $p \in \mathcal{C}^{1,\alpha}_{loc}(D)$. Obviously, c, T, u, p must satisfy (2.5) - (2.7) and (5.2), while Lemma 3.2 implies (5.1). Since $\nabla T \in L^2(D) \cap \mathcal{C}^{1,\alpha}(D)$ and $u \in W^{3,2}(D) \cap \mathcal{C}^{2,\alpha}(D)$, we obtain (5.3). By (2.12), we have:

$$\int_D f(T) < +\infty. \tag{5.8}$$

The main difficulty is now to identify the limits of $T(x,\cdot)$. This will be done through a sequence of lemmas whose proofs extensively use 'the sliding method'.

Lemma 5.2. *The propagation speed is positive: $c > 0$.*

Proof. The proof is made of various ingredients of different proofs in [BCR], section 3.

1. Notice that there exists $\tilde{x} \in \Omega$ such that there simultaneously hold:

$$\int_{-a_n}^{a_n} |\nabla T_n(\cdot,\tilde{x})|^2 \leq \frac{3}{|\Omega|}\int_{R_{a_n}} |\nabla T_n|^2, \tag{5.9}$$

$$\int_{-a_n}^{a_n} f(T_n(\cdot,\tilde{x})) \leq \frac{3}{|\Omega|}\int_{R_{a_n}} f(T_n). \tag{5.10}$$

Indeed, consider the set:

$$\Omega' = \left\{\tilde{x} \in \Omega;\ \int_{-a_n}^{a_n} |\nabla T_n(\cdot,\tilde{x})|^2 > \frac{3}{|\Omega|}\int_{R_{a_n}} |\nabla T_n|^2\right\}.$$

We have:

$$3\frac{|\Omega'|}{|\Omega|}\int_{R_{a_n}} |\nabla T_n|^2 \leq \int_{\Omega'}\int_{-a_n}^{a_n} |\nabla T_n|^2 \leq \int_{R_{a_n}} |\nabla T_n|^2,$$

which implies:

$$|\Omega'| \leq |\Omega|/3.$$

With exactly the same argument one proves that the set of $\tilde{x} \in \Omega$ violating (5.10) is as well not bigger than a third of Ω. The claim follows.

2. We will deduce that:

$$\int_{R_{a_n}} f(T_n) \geq C, \tag{5.11}$$

for some constant $C > 0$ independent of n.

For a given n, fix a small $\epsilon > 0$ and let $-a_n < x_1 < x_2 < 0$ be such that $T_n(x_1,\tilde{x}) = 1-\epsilon$, $T_n(x_2,\tilde{x}) = \theta_0+\epsilon$ and $T_n([x_1,x_2],\tilde{x}) \subset [\theta_0+\epsilon, 1-\epsilon]$. Using the Cauchy-Schwartz inequality and the fact that $f(T_n(\cdot,\tilde{x}))$ must be bounded away from 0 on $[x_1,x_2]$ we obtain:

$$\begin{aligned}
\int_{x_1}^{x_2} |\nabla T_n(\cdot,\tilde{x})|^2 &\geq \frac{1}{|x_1-x_2|}\cdot\left|\int_{x_1}^{x_2}\frac{\partial T_n}{\partial x}(\cdot,\tilde{x})\right|^2 = \frac{C}{|x_1-x_2|},\\
\int_{x_1}^{x_2} f(T_n(\cdot,\tilde{x})) &\geq C|x_1-x_2|.
\end{aligned}$$

Combining the above with (5.9) and (5.10) we obtain that

$$\int_{R_{a_n}} f(T_n)\cdot\int_{R_{a_n}} |\nabla T_n|^2 \geq C,$$

which in view of (2.12) yields (5.11).

3. Now define $\Phi_n(x,\tilde{x}) = T_n(x+a_n,\tilde{x})$, $\zeta_n(x,\tilde{x}) = u_n(x+a_n,\tilde{x})$ on domains R_{a_n} and D_{a_n} "shifted" to the left by the distance a_n. Since Φ_n, ζ_n obviously satisfy the same uniform bounds as T_n and u_n, they converge (uniformly on compact sets, together with their derivatives) to some Φ and ζ defined on $(-\infty,0]\times\Omega$, where they satisfy:

$$-c\Phi_x+\zeta\cdot\nabla\Phi=\Delta\Phi, \tag{5.12}$$

and $\Phi(0,\cdot)=0$, $\Phi_x(0,\cdot)\leq 0$. The absence of the nonlinearity in (5.12) is due to (3.2) which gives:

$$\Phi(\cdot,\cdot)\in[0,\theta_0]. \tag{5.13}$$

Moreover ζ and Φ have the same limit properties at $-\infty$ as u and T in (5.3). Therefore taking x large and negative and integrating (5.12) on $[x,0]\times\Omega$ we obtain:

$$\int_\Omega \Phi_x(0,\cdot)\geq -\frac{C}{2}+c\int_\Omega \Phi(x,\cdot) \tag{5.14}$$

4. Now integrating (2.5) on R_{a_n} we have:

$$c_n|\Omega|\geq \int_\Omega \frac{\partial T_n}{\partial x}(a_n,\cdot)+\int_{R_{a_n}} f(T_n)$$

Note that $\partial T_n/\partial x(a_n,\cdot)=\partial\Phi_n/\partial x(0,\cdot)$ and pass to the limit with $n\to\infty$, using (5.11) and (5.14). This gives:

$$c|\Omega|\geq \frac{C}{2}+c\int_\Omega \Phi(x,\cdot).$$

Therefore we have:

$$c\int_\Omega (1-\Phi)(x,\cdot)>0,$$

which in view of (5.13) proves the lemma. ■

Lemma 5.3. *For some $\theta_-,\theta_+\in[0,\theta_0]\cup\{1\}$ there must be:*

$$\lim_{x\to\pm\infty}||T(x,\cdot)-\theta_\pm||_{L^\infty(\Omega)}=0.$$

Proof. We argue by contradiction. Let $\lim_{n\to\infty}T(x_n,\tilde{x}_n)=\theta_1$, $\lim_{n\to\infty}T(y_n,\tilde{y}_n)=\theta_2$ with some $\theta_1\neq\theta_2$ and $\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=+\infty$. Integrating (2.5) on $S_n:=[x_n,y_n]\times\Omega$ and using the boundary conditions we obtain:

$$\begin{aligned}&-c\int_\Omega [T(y_n,\cdot)-T(x_n,\cdot)]\\ &\qquad =-\int_\Omega [(Tu^1-T_x)](y_n,\cdot)-(Tu^1-T_x)](x_n,\cdot)]+\int_{S_n} f(T),\end{aligned} \tag{5.15}$$

where as usual u^1 refers to the horizontal component of the velocity vector u. Notice that by (5.3) and (5.8), the right hand side in (5.15) converges to 0 as $n\to\infty$. At the same time, the left hand side converges to $-c|\Omega|(\theta_2-\theta_1)$ where we used (5.3). In view of Lemma 5.2 we conclude that indeed $\theta_1=\theta_2$.

In view of (5.8), we have $f(\theta_-) = f(\theta_+) = 0$ which concludes the proof. ■

Lemma 5.4. *The temperature limits satisfy: $\theta_- > \theta_+$.*

Proof. Taking $x_n \to -\infty$, $y_n \to +\infty$ in (5.15) and using (5.3) we get:

$$c|\Omega|(\theta_- - \theta_+) = \int_D f(T). \tag{5.16}$$

Therefore $\theta_- \geq \theta_+$. Assume that $\theta_- = \theta_+$. By Lemma 5.2 this implies that $f(T) \equiv 0$ and hence by the maximum principle T must be constant. By (5.1) we conclude that $T \equiv \theta_0$.

Integrate now (2.5) on $[0, a_n] \times \Omega$:

$$c_n|\Omega|\theta_0 \leq \int_\Omega u_n^1 T_n(0, \cdot) - \int_\Omega \frac{\partial T_n}{\partial x}(0, \cdot),$$

pass with $n \to \infty$ and obtain:

$$c|\Omega|\theta_0 \leq \theta_0 \int_\Omega u^1(0, \cdot). \tag{5.17}$$

Now by (5.3):

$$-\int_\Omega u^1(x, \cdot) = \int_{[x,+\infty)\times\Omega} \operatorname{div} u = 0, \tag{5.18}$$

which contradicts (5.17) and Lemma 5.2. ■

Notice that Lemma 5.4, Lemma 5.2 and (5.16) imply that

$$\int_D f(T) > 0$$

which together with (5.8) proves (5.4).

Lemma 5.5. *For every $\epsilon > 0$ there exists A such that for all sufficiently large n:*

$$|T_n(x, y)| \leq \epsilon \qquad \forall (x, y) \in [A, a_n] \times \Omega.$$

In particular (5.5) holds.

Proof. We argue by contradiction. Let $\epsilon_0 > 0$ and a sequence $(x_n, \tilde{x}_n)$ be such that $\lim x_n = +\infty$ and:

$$T_n(x_n, \tilde{x}_n) \geq \epsilon_0, \tag{5.19}$$

Without loss of generality we consider two cases.

Case 1. Assume that:

$$\lim(a_n - x_n) = y_0.$$

Define $\Phi_n(x, \tilde{x}) = T_n(x + a_n - y_0, \tilde{x})$, $\zeta_n(x, \tilde{x}) = u_n(x + a_n - y_0, \tilde{x})$. As in step 3 of the proof of Lemma 5.2 we observe that Φ_n, ζ_n converge (uniformly on compacts, together with their derivatives) to Φ and ζ satisfying (5.12), (5.13) and $\Phi(y_0,\cdot)=0$. Now reasoning exactly as in the proof of Lemma 5.3 we see that Φ converges uniformly to some limit $\Phi_-\in[0,\theta_0]$, as $x\to-\infty$. Integrating (5.12) on $(-\infty,y_0]\times\Omega$ we get:

$$c|\Omega|\Phi_- = \int_\Omega \Phi_x(y_0,\cdot)\le 0,$$

which by Lemma 5.2 implies that $\Phi_-=0$. But now, by maximum principle $\Phi\equiv 0$, which contradicts (5.19).

Case 2. Assume that

$$(a_n - x_n)\to+\infty. \tag{5.20}$$

Define $\Phi_n(x,\tilde{x}) = T_n(x+x_n,\tilde{x})$ and $\zeta_n(x,\tilde{x}) = u_n(x+x_n,\tilde{x})$, satisfying on $\tilde{R}_n = [0,a_n-x_n]\times\Omega$:

$$-c_n(\Phi_n)_x + \zeta_n\cdot\nabla\Phi_n - \Delta\Phi_n = 0.$$

Multiplying the last equation by Φ_n and integrating over $\tilde{R}_n$ we obtain:

$$c_n\int_{\tilde{R}_n}\frac{\partial|\Phi_n|^2}{\partial x} - \int_{\partial\tilde{R}_n}|\Phi_n|^2\zeta_n\cdot\vec{n} + 2\int_{\partial\tilde{R}_n}\Phi_n\frac{\partial\Phi_n}{\partial\vec{n}} = 2\int_{\tilde{R}_n}|\nabla\Phi_n|^2,$$

where $\vec{n}$ is the outward normal to $\partial\tilde{R}_n$. Using the boundary conditions, this yields:

$$\int_\Omega\Big[(\zeta_n^1 - c_n)|\Phi_n|^2 - 2\Phi_n\partial_x\Phi_n\Big](0,\cdot)\ge 0 \tag{5.21}$$

where the superscript in ζ_n^1 refers to the horizontal component of the vector ζ_n.

As before, Φ_n and ζ_n satisfy the same uniform bounds as T_n and u_n and hence converge to some Φ and ζ defined by virtue of (5.20) on whole D and still satisfying (5.12). By an argument as in the proof of Lemma 5.3 we see that Φ must have left and right limits $\Phi_\pm$ as $x\to\pm\infty$. Integrating (5.12) there follows $c(\Phi_- - \Phi_+) = 0$ which by Lemma 5.2 implies $\Phi_- = \Phi_+$. Therefore, by the maximum principle Φ must be constant and say, equal to Φ_0. By (5.19) there must be $\Phi_0\ge\epsilon_0$. Passing to the limit in (5.21), we obtain:

$$|\Phi_0|^2\int_\Omega(\zeta^1 - c)(0,\cdot)\ge 0. \tag{5.22}$$

On the other hand, since div $\zeta = 0$ we have $\int_\Omega\zeta^1(0,\cdot)=0$, as in the calculation (5.18). Finally, (5.22) becomes:

$$-h|\Omega||\Phi_0|^2\ge 0,$$

which contradicts the positivity of c in Lemma 5.2. ∎

Notice now that Lemmas 5.3, 5.4 and (5.5) imply (5.6) and in particular $\theta_->0$.

The following lemma gives a sufficient condition for the left limit of $T=\lim_{n\to\infty}T_n$ to be 1.

Lemma 5.6. *There exists a constant $C_\Omega > 0$, depending only on ρ^{n-2} and the geometry of Ω, such that if (5.7) holds, then $\theta_- = 1$.*

Proof. In the course of the proof, C will denote any positive constant depending only on ρ^{n-2} and on the geometry of Ω, that is its dimension $n-1$, diameter, Poincare' and Sobolev constants, etc.

For every $x \in \mathbf{R}$ denote $M(x) = \max_{\tilde{x}\in\Omega} T(x,\tilde{x})$, $m(x) = \min_{\tilde{x}\in\Omega} T(x,\tilde{x})$. Notice first that $m(x)$ is nonincreasing. This can be easily proved for each t_n on R_n, using the maximum principle. Passing with n to ∞, we obtain the same result in the limit.

We now argue by contradiction. If $\theta_- \le \theta_0$ then $m(x) \le \theta_0$ for every $x \in \mathbf{R}$ and therefore:

$$\int_D [(T-\theta_0)_+]^n \le |\Omega| \cdot \int_{-\infty}^{+\infty} |M(x) - m(x)|^n \, \mathrm{d}x. \tag{5.23}$$

By Sobolev and Poincare' inequalities on Ω we get:

$$\begin{aligned} M(x) - m(x) &= M(x) - \frac{1}{|\Omega|}\int_\Omega T(x,\cdot) + \frac{1}{|\Omega|}\int_\Omega T(x,\cdot) - m(x) \\ &\le 2 \parallel T(x,\cdot) - \frac{1}{|\Omega|}\int_\Omega T(x,\cdot) \parallel_{L^\infty(\Omega)} \le C \parallel T(x,\cdot) - \frac{1}{|\Omega|}\int_\Omega T(x,\cdot) \parallel_{W^{1,n}(\Omega)} \\ &\le C \parallel \nabla T(x,\cdot) \parallel_{L^n(\Omega)} . \end{aligned}$$

Therefore, by (5.23):

$$\int_D [(T-\theta_0)_+]^n \le C \int_D |\nabla T|^n. \tag{5.24}$$

Now, revisiting the proof of Lemma 3.5 we see that:

$$\begin{aligned} \int_D |\nabla T|^p &\le C(1 + |c| + ||u||_{L^\infty}) \cdot \int_D |\nabla T|^{p-1} \\ &\quad + C\left(\int_D f(T)\right)^{\frac{1}{p-1}} \cdot \left(\int_D |\nabla T|^{p-1}\right)^{\frac{p-2}{p-1}} \\ &\le C(1 + |c| + ||u||_{L^\infty}) \int_D |\nabla T|^{p-1} + C\int_D f(T). \end{aligned}$$

By (2.12) we have $|c| + ||u||_{L^\infty} \le C$ and hence:

$$\int_D |\nabla T|^n \le C\int_D |\nabla T|^2 + C\int_D f(T). \tag{5.25}$$

On the other hand, integrating the temperature equation in (2.5) - (2.7) on D, we obtain:

$$c\theta_-|\Omega| = \int_D f(T), \qquad \int_D |\nabla T|^2 + \frac{1}{2}c\theta_-^2|\Omega| = \int f(T)T,$$

which implies:

$$\int_D |\nabla T|^2 = \int_D \left(T - \frac{\theta_-}{2}\right) f(T).$$

Now combining (5.24), (5.25) and (5.7) we obtain:

$$3\int_D f(T) \le \frac{1}{C}\int_D [(T-\theta_0)_+]^n \le \int_D \left(T - \frac{\theta_-}{2}\right) f(T) + \int_D f(T) \le 2\int_D f(T),$$

if only C_Ω is chosen to be smaller than $1/3C$ above. Therefore $f(T) \equiv 0$ and, as in the proof of Lemma 5.5 we deduce that T must be constant. This contradicts Lemma 5.2. ∎

Remark 5.7. If $\theta_- \le \theta_0$, then integrating the equation:

$$-\Delta(T-\theta_0) + u\cdot\nabla(T-\theta_0) - c(T-\theta_0)_x = f(T)$$

on D, against $(T-\theta_0)_+$, we obtain:

$$\int_D |\nabla(T-\theta_0)_+|^2 = \int_D f(T)(T-\theta_0)_+.$$

On the other hand, when $n = 2$, the Poincare inequality implies:

$$\int_D [(T-\theta_0)_+]^2 \le |\Omega|^2 \int_D |\nabla(T-\theta_0)_+|^2,$$

as for each $x \in \mathbf{R}$, the function $(T-\theta_0)_+(x,\cdot)$ has a zero in Ω (as in Lemma 5.6). Thus, for $n = 2$, we obtain $\theta_- = 1$ under a conditon that for some $\epsilon > 0$:

$$\forall T \in [\theta_0, 1] \quad f(T) \le \frac{1}{|\Omega|^2}(T-\theta_0)_+ \quad \text{and} \quad \forall T \in (\theta_0, \theta_0+\epsilon) \quad f(T) < \frac{1}{|\Omega|^2}(T-\theta_0)_+.$$

This condition is weaker than (5.7).

We believe that under the requirement of (5.7) or, similarly, under some smallness requirement on Ω, the left limit $\theta_- = 1$ should follow from $f(T) \not\equiv 0$ and $\theta_+ = 0$, for any traveling wave solution of (1.1) (2.4). Theorem 5.6 and Remark 5.7 prove it for the wave (T, u) obtained in our limiting procedure. We believe that, actually, this wave is unique, under the mentioned conditions.

6. Appendix A: a proof of Lemma 3.8

In this section we want to give a proof of the local Sobolev up-to-the boundary estimates for strong solutions of the stationary Stokes problem in a bounded, smooth and simply connected domain $U \subset \mathbf{R}^n$:

$$\begin{aligned} \Delta u + \nabla p &= g \qquad \text{in } U,\\ \text{div } u &= 0 \qquad \text{in } U,\\ u &= 0 \qquad \text{on } \partial U. \end{aligned} \tag{6.1}$$

Namely, for every $p \geq 1$ and every integer $k \geq 0$ there holds:

$$||u||_{W^{k+3,p}(Q)} \leq C\left(||\nabla g||_{W^{k,p}(2Q)} + ||u||_{W^{1,p}(2Q)}\right), \tag{6.2}$$

where for a given $x \in \overline{U}$ and $\epsilon > 0$ sufficiently small we define: $Q = U \cap B(x, \epsilon)$ and $2Q = U \cap B(x, 2\epsilon)$. The constant C in (6.2) depends only on the geometry of the domain, and in the particular setting of this paper, it is uniform in a. Therefore the proof of Lemma 3.8 follows directly, through an interpolation inequality, which allows to exchange the Sobolev norm of u in the right hand side of (6.2) with its L^p norm.

To prove (6.2) we will first write an equivalent to (6.1) elliptic problem, coupled in the equations as well as in the boundary conditions. We then verify the Lopatinsky-Shapiro conditions at the boundary and deduce the estimate (6.2) from the classical theory in [ADN]. When $n = 3$, the equivalent system can be found, roughly speaking, by taking curl of the equation in (6.1) (see Remark 6.3). For higher dimensions, the natural generalization is to use the exterior derivative and the language of differential forms [GMS].

Given smooth vector fields $u, g : U \longrightarrow \mathbf{R}^n$ and a function $p : U \longrightarrow \mathbf{R}$, we will naturally interpret them as respectively 1- and 0- differential forms on U:

$$u, f \in \Omega^1(U), \qquad p \in \Omega^0(U).$$

By $\Delta : \Omega^k \longrightarrow \Omega^k$ we denote the Laplace-Beltrami operator:

$$\Delta = d\delta + \delta d$$

where $d : \Omega^k \longrightarrow \Omega^{k+1}$ is the exterior derivative of forms and $\delta : \Omega^k \longrightarrow \Omega^{k-1}$ is the codifferential operator. Recall that (in the flat metrics of $\mathbf{R}^n$) the components of the differential form $\Delta\alpha$ are given by the usual Laplacian of the components of α, for any $\alpha \in \Omega^k$. The trace of α on ∂U, and the normal and the tangential parts of trace are denoted by, respectively:

$$\mathrm{tr}\ \alpha, \ \mathrm{ntr}\ \alpha, \ \mathrm{ttr}\ \alpha.$$

The system (6.1) can be now written as:

$$\begin{aligned} \Delta u + dp &= g \qquad && \text{in } \Omega^1(U), \\ \delta u &= 0 && \text{in } \Omega^1(U), \\ \mathrm{tr}\ u &= 0 && \text{on } \partial U. \end{aligned} \tag{6.3}$$

Denoting $\omega = du \in \Omega^2$, the above system of equations clearly implies:

$$\begin{aligned} \Delta u - \delta\omega &= 0 \qquad && \text{in } \Omega^1(U), \\ \Delta\omega &= dg && \text{in } \Omega^2(U), \\ \mathrm{tr}\ u &= 0 && \text{on } \partial U, \\ \mathrm{ntr}\ (du - \omega) &= 0 && \text{on } \partial U, \\ \mathrm{ntr}\ (d\omega) &= 0 && \text{on } \partial U. \end{aligned} \tag{6.4}$$

The second equality above follows by applying the operator d to the first equation in (6.3) and recalling that $dd = 0$.

Lemma 6.1. *The systems (6.1), (6.3) and (6.4) are equivalent.*

Proof. It is enough to prove that if u, ω satisfy (6.4) then (6.3) must hold, for some $p \in \Omega^0$. This will follow by a sequence of steps, showing that:

$$d\omega = 0, \tag{6.5}$$
$$\omega = du, \tag{6.6}$$
$$\text{div } u = 0, \tag{6.7}$$
$$\Delta u - g \in im(d). \tag{6.8}$$

To prove (6.5) notice first that the differential form $\alpha = d\omega$ satisfies:

$$\begin{aligned} \Delta\alpha = 0 &\qquad \text{in } U, \\ \text{ntr } \alpha = 0 &\qquad \text{on } \partial U. \end{aligned} \tag{6.9}$$

By the Hodge-Morrey theorem [GMS], the above problem has a unique solution in the class of forms perpendicular (with respect to the L^2 product $\langle\cdot,\cdot\rangle$ of forms) to $ker(d) \cap ker(\delta) \cap ker(\text{ntr})$. This class clearly contains $im(d)$, because the product:

$$\langle d\omega, \beta\rangle = -\langle\omega, \delta\beta\rangle + \int_{\partial U} \text{ttr } \omega \cdot \text{ntr } \beta,$$

is 0, if $\delta\beta = 0$ and ntr $\beta = 0$. This proves (6.5).

Now, $\alpha = du - \omega$ satisfies (6.9) as well and moreover in view of (6.5) and the simple connectedness of U, again we have $\alpha \in im(d)$. Therefore, the same argument as above implies $\alpha = 0$, proving (6.6).

By (6.6) and the first equation in (6.4) we deduce $\delta u \in ker(d)$. This means that div u is constant in U. By Stokes' theorem and the first boundary condition in (6.4) we conclude (6.7).

Finally, (6.8) follows from (6.6), the second equation in (6.4) and the simple connectedness of U. ∎

We will now concentrate on the elliptic system (6.4). At a first glance, we notice that the number of its boundary conditions is $n + \binom{n}{2} - \binom{n-1}{2} + \binom{n}{3} - \binom{n-1}{3} = n + \binom{n}{2}$ which equals the number of unknowns and the number of equations. Therefore, we may hope for the well-posedness of (6.4). Indeed, we have:

Lemma 6.2. *The system (6.4) satisfies the Lopatinsky-Shapiro conditions and (6.2) holds.*

Proof. We will use the Euclidean metric in U, so that:

$$u(x) = \sum_{i=1}^{n} u^i(x)\ dx_i, \quad \omega(x) = \sum_{1\leq i<j\leq n} \omega^{i,j}(x)\ dx_i \wedge dx_j,$$

Using the notation of [ADN], the square operator matrix L of dimension $d = \binom{n}{2} + n$, and its adjugate matrix L^{adj} are given in the block form:

$$L = \begin{bmatrix} \Delta \cdot Id_{\binom{n}{2}} & 0 \\ -A & \Delta \cdot Id_n \end{bmatrix}, \qquad L^{adj} = \Delta^{d-2} \begin{bmatrix} \Delta \cdot Id_{\binom{n}{2}} & 0 \\ A & \Delta \cdot Id_n \end{bmatrix}. \tag{6.10}$$

The coefficients of the above matrices are polynomials in the variables $x_1, \ldots, x_n$, and in particular $\Delta = x_1^2 + \ldots + x_n^2$. The appropriate monomials appearing in matrix A can be derived from the formula:

$$\delta\omega = \sum_{i=1}^{n} \left(\sum_{j \neq i} \text{sgn } (i-j) \frac{\partial \omega^{(i,j)}}{\partial x_j} \right) \, dx_i,$$

where $\omega^{(j,i)} = \omega^{j,i}$ if $j < i$ or $\omega^{i,j}$ if $j > i$.

The first $\binom{n}{2}$ rows of the matrix L correspond to the second equation is (6.4), that is to the operator $\Delta\omega$; to these rows we assign weights $s = 0$. The first $\binom{n}{2}$ columns correspond to the components of the differential form ω; to these columns we assign weights $t = 2$. The following n rows correspond to the linear operator $\Delta u - \delta\omega$; we impose weights $s = -1$. The following n columns correspond to the components of u, we assign weights $t = 3$. Clearly, with this choice of weights we have: $L' = L$.

We want to verify the algebraic version of the Lopatinsky-Shapiro conditions (the complementing condition), at a boundary point $P \in \partial U$ and relative to any tangent vector Θ, perpendicular to the unit normal $\vec{n}_P$ to ∂U at P. First of all, notice that because of the coordinate invariance of the system (6.4), we may without loss of generality assume that

$$P = 0, \qquad \vec{n}_P = dx_1, \qquad \Theta = dx_2.$$

With these simplifications, the rectangular matrix A is given by:

$$A = \begin{array}{|ccc|ccc|ccc|l} \hline -1 & 0 & \cdots & & 0 & & & 0 & & u^1 \\ \hline & & & & 0 & & & & & u^2 \\ \cline{4-6} & x_1 \cdot Id_{n-1} & & & Id_{n-2} & & & 0 & & u^3 \\ & & & & & & & & & \vdots \\ & & & & & & & & & u^n \\ \hline \omega^{1,2} & \ldots & \omega^{1,n} & \omega^{2,3} & \ldots & \omega^{2,n} & \omega^{3,4} & \ldots & \omega^{n-1,n} & \end{array},$$

where the shaded minors are all 0.

We need now to derive the boundary operator matrix B, calculate the product $B' \cdot L^{adj}$, write the elements as the latter matrix as polynomials in $\tau = x_1$ and check that its rows are linearly independent modulo the complex polynomial:

$$M^+(\tau) = (\tau - i)^d.$$

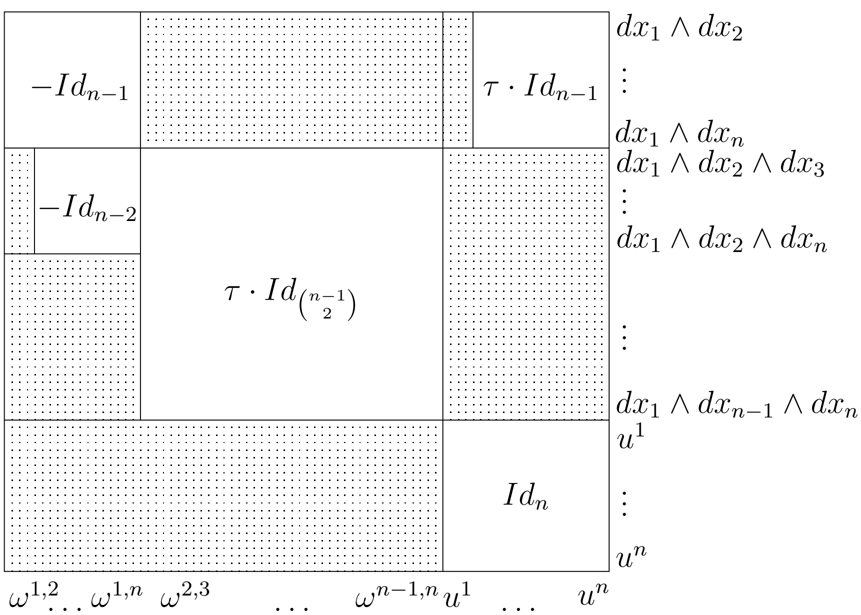
This is because the polynomial $\det L(\tau) = \Delta^d(\tau) = (\tau^2+1)^d$ has i as the only root with positive imaginary part, and it is of multiplicity d.

The first $n-1$ rows of B will correspond to the operator ntr $(du - \omega)$ and its coefficients at $dx_1 \wedge dx_i$, for $1 < i \leq n$; we assign them the weights $r = -2$. The following $\binom{n-1}{2}$ rows, to which we assign weights $r = -1$, will correspond to the coefficients of ntr $(d\omega)$ at $dx_1 \wedge dx_i \wedge dx_j$, for $1 < i < j \leq n$. The last n rows will correspond to $u = 0$ on ∂U and coefficients $u^1, \dots, u^n$ of the differential form u; to these rows we assign weights $r = -3$. Since $u = 0$ on ∂U, we have:

$$
\begin{aligned}
(\text{ntr } (du))^{1,i} &= \frac{\partial u^i}{\partial x_1} - \frac{\partial u^1}{\partial x_i} = \frac{\partial u^i}{\partial x_1}, \quad \forall\, 1 < i \leq n, \\
(\text{ntr } (d\omega))^{1,i,j} &= \frac{\partial \omega^{i,j}}{\partial x_1} - \frac{\partial \omega^{1,j}}{\partial x_i} + \frac{\partial \omega^{1,i}}{\partial x_j} \quad \forall\, 1 < i < j \leq n.
\end{aligned}
$$

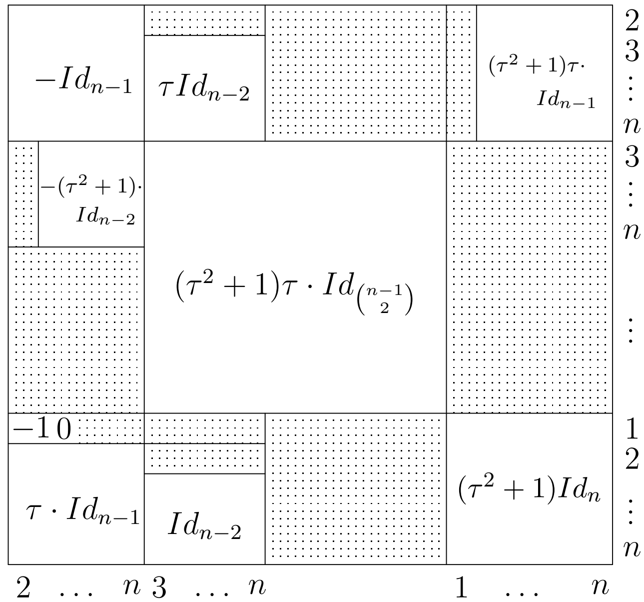
By an elementary calculation we thus see that the matrices $B = B'$ and $B \cdot L^{adj}$ have the following block structure:

$B(\tau) =$



,

$$B \cdot L^{adj}(\tau) = (\tau^2+1)^{d-2} \cdot$$



,

where again the shaded minors are all 0.

We now want to reduce each element of the last matrix modulo M^+ and evaluate it at $\tau = 0$. It is clear that if the matrix W composed of such numbers is invertible, then the Lopatinsky-Shapiro conditions hold as well.

The elements of W are calculated in an elementary way. For example, for the reduction of the polynomial $\Delta^{d-2}(\tau) = (\tau^2+1)^{d-2}$, we notice that if a polynomial $r(\tau)$ of degree $< d$ satisfies:

$$\Delta^{d-2}(\tau) = (\tau - i)^{d-2}(\tau + i)^{d-2} = (\tau - i)^d \cdot q(\tau) + r(\tau),$$

then necessarily $(\tau+i)^{d-2} - (\tau - i)^2 \cdot q(\tau) = a\tau + b$ for some $a, b \in \mathbf{C}$. We obtain:

$$\begin{aligned} ai + b &= (2i)^{d-2}, \\ a &= \frac{\mathrm{d}}{\mathrm{d}\tau_{|\tau=0}} \left[(\tau+i)^{d-2} - (\tau-i)^2 \cdot q(\tau)\right] = (2i)^{d-3}(d-2), \end{aligned}$$

and thus:

$$r(0) = (\tau - i)^{d-2} \cdot \left((\tau+i)^{d-2} - (\tau-i)^2 q(\tau)\right)_{|\tau=0} = (-i)^{d-2} b = 2^{d-3}(4-d),$$

which we symbolically denote: $\Delta^{d-2} \longrightarrow w_1 = 2^{d-3}(4-d)$. Similarly, for other elements of the matrix $B \cdot L^{adj}$ we have:

$$\begin{aligned} \tau \cdot \Delta^{d-2} &\longrightarrow w_2 = 2^{d-3} i(2-d), \\ (\tau^2+1)\tau \cdot \Delta^{d-2} &\longrightarrow w_3 = 2^{d-1} i, \\ (\tau^2+1) \cdot \Delta^{d-2} &\longrightarrow w_4 = 2^{d-1}. \end{aligned}$$

The matrix W is now obtained from $B \cdot L^{adj}$ by replacing each of its elements as indicated above. Since every w_i is nonzero and also $-w_1 - iw_2 \neq 0$, we see (after performing row operations on W) that the invertibility of W is equivalent to the invertibility of the following square matrix (of dimension $2(n-2)$):

$$\tilde{W} = \begin{bmatrix} (-w_1 - iw_2) \cdot Id_{n-2}, & (w_2 - iw_1) \cdot Id_{n-2} \\ -w_4 \cdot Id_{n-2} & w_3 \cdot Id_{n-2} \end{bmatrix}.$$

One checks directly that $\det \tilde{W} \neq 0$, which proves the validity of the complementing condition.

Consequently and in agreement with our choice of weights s, t, r, we obtain the following a-priori bound:

$$||u||_{W^{k+3,p}(Q)} + ||\omega||_{W^{k+2,p}(Q)} \leq C \left(||dg||_{W^{k,p}(2Q)} + ||u||_{L^p(2Q)} + ||\omega||_{L^p(2Q)} \right),$$

which clearly implies (6.2). ■

Remark 6.3. When $n = 3$, the system (6.4) is equivalent to (after taking into account the possible change of sign in the components of $\omega = \text{curl } u$):

$$\begin{aligned} \Delta u + \text{curl } \omega &= 0 && \text{in } U, \\ \Delta \omega &= \text{curl } g && \text{in } U, \\ u &= 0 && \text{on } \partial U, \\ t(\text{curl } u - \omega) &= 0 && \text{on } \partial U, \\ \text{div } \omega &= 0 && \text{on } \partial U, \end{aligned}$$

where for a vector field v, by $t(v)$ we denote its two tangential components on ∂U. The above system follows from (6.1) by taking curl of the first equation and recalling the formula: curl curl $= -\Delta + \nabla\text{div}$.

7. Appendix B: a remark concerning the finite Prandtl number case

It is clear that all the uniform bounds in section 3, up to Lemma 3.8, remain valid as well for the compactified version of the full problem (1.2), that is (2.5) - (2.11) where the equation (2.6) is replaced by:

$$\frac{1}{\sigma}(-cu_x + u \cdot \nabla u) - \Delta u + \nabla p = T\vec{\rho}. \tag{7.1}$$

The presence of c in (7.1) should allow to improve the bound $|c| \preceq ||u||_{L^\infty}$ in Lemma 3.3, which follows from equation (2.5) alone.

In this section we want to remark that an improvement of the propagation speed bound is crucial for extending Theorem 1.1 to the Navier-Stokes-Boussinesq system (1.2) in dimension $n \geq 3$.

When $n = 2$, the bound in Lemma 3.3 is enough, as shown in [CLR]. Roughly speaking, this follows from the bounds $||u||_{L^\infty} \preceq ||u||_{W^{1,2+\epsilon}}$ and $||u||_{W^{1,2}} \preceq ||u||_{L^\infty}^{1/2}$.

Using interpolation inequalities we are still able to close the estimates because the exponent of $||u||_{L^\infty}$ in the latter inequality is much smaller than the exponent of $||u||_{L^\infty}$ in the former one.

In general, we only have $||u||_{L^\infty} \preceq ||u||_{W^{1,n+\epsilon}}$. To make the argument as above work for $n > 2$, we would need a uniform L^p ($p > 2$) estimate on u solving (7.1) and in particular the estimate should be uniform in c and a.

Remark 7.1. Consider the problem:

$$\begin{aligned} -\Delta u - cu_x &= g \qquad \text{in } [0,2a] \times \Omega, \\ u &= 0 \qquad \text{on } \partial([0,2a] \times \Omega). \end{aligned} \tag{7.2}$$

We are interested in an estimate of the form:

$$||\nabla u||_{L^p} \leq C||g||_{L^p}$$

with C independent from a and c. We will show that it is not possible unless $p = 2$.

Let $v \neq 0$ be an eigenfunction, solving:

$$\begin{aligned} -\Delta u &= \lambda v \qquad \text{in } \Omega, \\ v &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

for some given eigenvalue $\lambda > 0$ of $-\Delta$ on Ω. Define $u(x, \tilde{x}) = q(x) \cdot v(\tilde{x})$, which solves (7.2) with $g(x, \tilde{x}) = -v(\tilde{x})$ iff q solves the following ODE:

$$\begin{aligned} &-q'' - cq' + \lambda q + 1 = 0, \\ &q(0) = q(2a) = 0. \end{aligned}$$

The function $r(x) = 1 + \lambda q(x)$ is of the form $r(x) = Ae^{s_1 x} + Be^{s_2 x}$ where $s_1 \approx \lambda/c$ and $s_2 \approx -c$ as $c \to \infty$. The constants A and B can be obtained through the boundary conditions on q and one sees that they satisfy: $A \approx e^{-2\lambda a/c}$ and $B = 1 - A$.

Assume now that $a = c$ and calculate:

$$\left(\lambda^p \int_0^{2a} |q'(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \geq \left(\int_0^1 |r'(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \geq -A||s_1 e^{s_1 x}||_{L^p[0,1]} + B||s_2 e^{s_2 x}||_{L^p[0,1]}.$$

Now the first norm in the right hand side of the above converges to 0 as $c \to \infty$ while the second one is bounded from below by $(c^{p-1}/(p4^{p-1})^{1/p}$, for large c. Therefore we have:

$$\begin{aligned} ||\nabla u||_{L^p} &\geq ||u_x||_{L^p} = ||v||_{L^p(\Omega)} \cdot \left(\int_0^{2a} |q'(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \geq C||v||_{L^p(\Omega)} \cdot c^{\frac{p-1}{p}}, \\ ||g||_{L^p} &= ||v||_{L^p(\Omega)} \cdot (2c)^{\frac{1}{p}}, \end{aligned}$$

where C depends on λ and p but not on c. This proves the claim.

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