MORPHOGENESIS BY GROWTH AND NON-EUCLIDEAN ELASTICITY: SCALING LAWS AND THIN FILM MODELS

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Abstract. The purpose of this paper is to report on the recent development concerning the analysis and the rigorous derivation of thin film models for structures exhibiting residual stress at free equilibria. This phenomenon has been observed in different contexts: growing leaves, torn plastic sheets and specifically engineered polymer gels. The study of wavy patterns in these contexts suggest that the sheet endeavors to reach a non-attainable equilibrium and hence assumes a non-zero stress rest configuration.

1. Elastic energy of a growing tissue and the non-Euclidean elasticity

This paper concerns the elastic structures which exhibit non-zero strain at free equilibria. Many growing tissues (leaves, flowers or marine invertebrates) attain complicated configurations during their free growth. Recent work has focused on some of the related questions by using variants of thin plate theory [1, 5, 4, 23]. However, the theories used are not all identical and some of them arbitrarily ignore certain terms and boundary conditions without prior justification. This suggests that it might be useful to rigorously derive an asymptotic theory for the shape of a residually strained thin lamina to clarify the role of the assumptions used while shedding light on the errors associated with the use of the approximate theory that results. Recently, such rigorous derivations were carried out [8, 17, 19, 21] in the context of standard nonlinear elasticity for thin plates and shells.

The purpose of this paper is to present these results in a concise manner, departing from the 3d incompatible elasticity theory conjectured to explain the mechanism for the spontaneous formation of non-Euclidean metrics. Namely, recall that a smooth Riemannian metric on a simply connected domain can be realized as the pull-back metric of an orientation preserving deformation if and only if the associated Riemann curvature tensor vanishes identically. When this condition fails, one seeks a deformation yielding the closest metric realization. It is conjectured that the same principle plays a role in the developmental processes of naturally growing tissues, where the process of growth provides a mechanism for the spontaneous formation of non-Euclidean metrics and consequently leads to complicated morphogenesis of the thin film exhibiting waves, ruffles and non-zero residual stress.

Below, we set up a variational model describing this phenomenon by introducing the non-Euclidean version of the nonlinear elasticity functional, and establish its Γ-convergence under the proper scaling. Heuristically, a sequence of functionals F_n is said to Γ-converge to a limit functional F if the the minimizers of F_n, if converging, have a minimizer of F as a limit.

Consider a sequence of thin 3d films $\Omega^h = \Omega\times(-h/2, h/2)$, viewed as the reference configurations of thin elastic tissues. Here, $\Omega \subset \mathbb{R}^2$ is an open, bounded and simply connected set which we

Date: May 9, 2013.
1991 *Mathematics Subject Classification.* 74K20, 74B20.
Key words and phrases. non-Euclidean plates, nonlinear elasticity, Gamma convergence, calculus of variations.

refer to as the mid-plate of thin films under consideration. Each Ω^h is now assumed to undergo a growth process, described instantaneously by a (given) smooth tensor:

$$a^h = [a^h_{ij}] : \Omega^h \longrightarrow \mathbb{R}^{3\times 3} \quad \text{such that} \quad \det a^h(x) > 0.$$

According to the formalism in [25], the following multiplicative decomposition:

$$\nabla u = F a^h \tag{1.1}$$

is postulated for the gradient of any deformation $u : \Omega^h \longrightarrow \mathbb{R}^3$. The tensor $F = \nabla u (a^h)^{-1}$ corresponds to the elastic part of u, and accounts for the reorganization of Ω^h in response to the growth tensor a^h. The validity of decomposition (1.1) into an elastic and inelastic part requires that it is possible to separate out a reference configuration, and thus this formalism is most relevant for the description of processes such as plasticity, swelling and shrinkage in thin films, or plant morphogenesis.

The elastic energy of u depends now only on F:

$$I^h_W(u) = \frac{1}{h}\int_{\Omega^h} W(F)\ \mathrm{d}x = \frac{1}{h}\int_{\Omega^h} W(\nabla u (a^h)^{-1})\ \mathrm{d}x, \qquad \forall u \in W^{1,2}(\Omega^h, \mathbb{R}^3). \tag{1.2}$$

We remark that although our results are valid for thin laminae that might be residually strained by a variety of means, we only consider the one-way coupling of growth to shape and ignore the feedback from shape back to growth (plasticity, swelling, shrinkage etc). However, it seems fairly easy to include this coupling once the basic coupling mechanisms are known.

In (1.2), the energy density $W : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}_+$ is a nonlinear function, assumed to be $\mathcal{C}^2$ in a neighborhood of $SO(3)$ and assumed to satisfy the following conditions of normalization, frame indifference and nondegeneracy:

$$\begin{aligned} \exists c > 0 \quad \forall F \in \mathbb{R}^{3\times 3} \quad \forall R \in SO(3) \quad & W(R) = 0, \quad W(RF) = W(F) \\ & W(F) \geq c\ \mathrm{dist}^2(F, SO(3)). \end{aligned} \tag{1.3}$$

The reason for using a nonlinear elasticity model (rather than the more familiar linear elasticity) is that, as our analysis shows, the resulting deformations u^h when $h \to 0$, are expected to be of order $\mathcal{O}(1)$, even though their gradients are locally $\mathcal{O}(h)$ close to rigid rotations.

We now compare the above approach with the 'target metric' formalism [5, 20]. On each Ω^h one assumes to be given a smooth Riemannian metric $g^h = [g^h_{ij}]$. A deformation u of Ω^h is then an orientation preserving realization of g^h, when:

$$(\nabla u)^T \nabla u = g^h \ \text{ and } \ \det \nabla u > 0,$$

or equivalently, by polar decomposition theorem:

$$\nabla u(x) \in \mathcal{F}^h(x) = \left\{ R\sqrt{g^h}(x);\ R \in SO(3) \right\} \qquad \text{a.e. in } \Omega^h. \tag{1.4}$$

It is hence instructive to study the following energy, bounding from below $I^h_W(u)$:

$$\tilde{I}^h_{dist}(u) = \frac{1}{h}\int_{\Omega^h} \mathrm{dist}^2(\nabla u(x), \mathcal{F}^h(x))\ \mathrm{d}x \qquad \forall u \in W^{1,2}(\Omega^h, \mathbb{R}^3). \tag{1.5}$$

The functional $\tilde{I}^h_{dist}$ measures the average pointwise deviation of the deformation u from being an orientation preserving realization of g^h. Note that $\tilde{I}^h_{dist}$ is comparable in magnitude with I^h_W, for $W = \mathrm{dist}^2(\cdot, SO(3))$. Also, observe that the intrinsic metric of the material is transformed by a^h to the target metric $g^h = (a^h)^T a^h$ and, for isotropic W, only the symmetric positive definite part of a^h given by $\sqrt{g^h}$ plays the role in determining the deformed shape.

2. The residual stress and a result on its scaling

Note that one could define the energy as the difference between the pull-back metric of a deformation u and the given metric: $I^h_{str}(u) = \int |(\nabla u)^T \nabla u - g^h|^2 \,\mathrm{d}x$. However, such 'stretching' functional is not appropriate from the variational point of view, because there always exists $u \in W^{1,\infty}$ such that $I^h_{str}(u) = 0$. Further, if the Riemann curvature tensor R^h associated to g^h does not vanish identically, say $R^h_{ijkl}(x) \neq 0$, then u has a 'folding structure' [9]; it cannot be orientation preserving (or reversing) in any open neighborhood of x.

As proved in [20], the functionals I^h_W, $\tilde{I}^h_W$ below and $\tilde{I}^h_{dist}$ have strictly positive infima for non-flat g^h, which points to the existence of non-zero stress at free equilibria (in the absence of external forces or boundary conditions):

Theorem 2.1. *For each fixed h, the following two conditions are equivalent:*

(i) *The Riemann curvature tensor $R^h_{ijkl} \not\equiv 0$,*

(ii) $\inf \left\{ \tilde{I}^h_{dist}(u);\ u \in W^{1,2}(\Omega^h, \mathbb{R}^3) \right\} > 0.$

Several interesting questions further arise in the study of the proposed energy functionals. A first one is to determine the scaling of the infimum energy in terms of the vanishing thickness $h \to 0$. Another is to find the limiting zero-thickness theories under obtained scaling laws.

In [20], we considered the case where g^h is given by a tangential Riemannian metric $[g_{\alpha\beta}]$ on Ω, and is independent of the thickness variable:

$$g^h = g(x', x_3) = \begin{bmatrix} \begin{bmatrix} g_{\alpha\beta}(x') \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} \qquad \forall x' \in \Omega, \quad x_3 \in (-h/2, h/2). \tag{2.1}$$

The above particular choice of the metric is motivated by the results of [14]. The experiment presented therein consisted in fabricating programmed flat disks of gels having a non-constant monomer concentration which induces a 'differential shrinking factor'. The disk was then activated in a temperature raised above a critical threshold, whereas the gel shrunken with a factor proportional to its concentration. This process defined a new target metric on the disk, of the form (2.1) and radially symmetric. Consequently, the metric induced hence a 3d configuration in the initially planar plate; one of the most remarkable features of this deformation is the onset of some transversal oscillations (wavy patterns), which broke the radial symmetry.

Following our point of view, note that if $[g_{\alpha\beta}]$ in (2.1) has non-zero Gaussian curvature $\kappa_{[g_{\alpha\beta}]}$, then each $R^h \not\equiv 0$. In [20], we observed the following:

Theorem 2.2. $[g_{\alpha\beta}]$ *has an isometric immersion $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ if and only if:*

$$h^{-2} \inf \tilde{I}^h_{dist} \leq C$$

(for a uniform constant C). Also, $\kappa_{[g_{\alpha\beta}]} \not\equiv 0$ if and only if, with a uniform positive constant c:

$$h^{-2} \inf \tilde{I}^h_{dist} \geq c > 0.$$

The existence (or lack of thereof) of local or global isometric immersions of a given 2d Riemannian manifold into $\mathbb{R}^3$ is a longstanding problem in differential geometry, its main feature being finding the optimal regularity. By a classical result of Kuiper [15], a $\mathcal{C}^1$ isometric embedding into $\mathbb{R}^3$ can be obtained by means of convex integration (see also [9]). This regularity is far from $W^{2,2}$, where information about the second derivatives is also available. On the other hand, a smooth isometry exists for some special cases, e.g. for smooth metrics with uniformly positive or negative

Gaussian curvatures on bounded domains in $\mathbb{R}^2$ (see [11], Theorems 9.0.1 and 10.0.2). Counterexamples to such theories are largely unexplored. By [13], there exists an analytic metric $[g_{\alpha\beta}]$ with nonnegative Gaussian curvature on 2d sphere, with no $\mathcal{C}^3$ isometric embedding. However such metric always admits a $\mathcal{C}^{1,1}$ embedding (see [10] and [12]). For a related example see also [24].

3. The prestrained Kirchhoff model

Consider now a class of more general 3d non-Euclidean elasticity functionals:

$$\tilde{I}_W^h(u) = \int_{\Omega^h} W(x', \nabla u(x))\, \mathrm{d}x, \tag{3.1}$$

where the inhomogeneous stored energy density $W : \Omega \times \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}_+$ satisfies the given below conditions of frame invariance, normalization, growth and regularity, as in (1.3): with respect to the energy well $\mathcal{F}^h$ given in (1.4), relative to $g^h = g$ as in (2.1). Note that $\mathcal{F}^h(x) = \mathcal{F}(x')$ is independent of h and of x_3.

(i) $W(x', RF) = W(x', F)$ for all $R \in SO(3)$,
(ii) $W(x', \sqrt{g}(x')) = 0$,
(iii) $W(x', F) \geq c\ \mathrm{dist}^2(F, \mathcal{F}(x'))$, with some uniform constant $c > 0$,
(iv) W has regularity $\mathcal{C}^2$ in some neighborhood of the set $\{(x', F);\ x' \in \Omega, F \in \mathcal{F}(x')\}$.

The properties (i) – (iii) are assumed to hold for all $x \in \Omega$ and all $F \in \mathbb{R}^{3\times 3}$.

The following two results provide the description of the limiting behavior of the energies $\tilde{I}_W^h$ as $h \to 0$. Namely, we prove that any sequence of deformations u^h with $\tilde{I}_W^h(u^h) \leq Ch^2$, converges to a $W^{2,2}$ regular isometric immersion y of the metric $[g_{\alpha\beta}]$. Conversely, every y with these properties can be recovered as a limit of u^h whose energy scales like h^2. The Γ-limit [3] of the energies is a curvature functional on the space of all $W^{2,2}$ realizations y of $[g_{\alpha\beta}]$ in $\mathbb{R}^3$:

$$\frac{1}{h^2}\tilde{I}_W^h \xrightarrow{\Gamma} \mathcal{I}_2(y) \quad \text{where} \quad \mathcal{I}_2(y) = \frac{1}{24}\int_\Omega \tilde{\mathcal{Q}}_2(x')\Big(\sqrt{[g_{\alpha\beta}]}^{-1}(\nabla y)^T \nabla \vec{n}\Big)\, \mathrm{d}x', \tag{3.2}$$

Here $\vec{n}$ is the unit normal to the image surface $y(\Omega)$, while $\tilde{\mathcal{Q}}_2(x')$ are the following quadratic forms, nondegenerate and positive definite on the symmetric 2×2 tensors:

$$\tilde{\mathcal{Q}}_3(x')(F) = \nabla^2 W(x', \cdot)_{|\sqrt{g}(x')}(F, F), \quad \tilde{\mathcal{Q}}_2(x')(F_{2\times 2}) = \min\{\tilde{\mathcal{Q}}_3(x')(\tilde{F});\ \tilde{F}_{2\times 2} = F_{2\times 2}\}.$$

We use the following notational convention: for a matrix F, its $n \times m$ principle minor is denoted by $F_{n\times m}$ and the superscript ^{T} refers to the transpose of a matrix or an operator.

Theorem 3.1. *Assume that a given sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ satisfies:*

$$\tilde{I}_W^h(u^h) \leq Ch^2, \tag{3.3}$$

where $C > 0$ is a uniform constant. Then, for some sequence of constants $c^h \in \mathbb{R}^3$, the following holds for the renormalized deformations $y^h(x', x_3) = u^h(x', hx_3) - c^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$:

(i) *y^h converge, up to a subsequence, in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to $y(x', x_3) = y(x')$ and $y \in W^{2,2}(\Omega, \mathbb{R}^3)$.*
(ii) *The matrix field $Q(x')$ with columns $Q(x') = \Big[\partial_1 y(x'), \partial_2 y(x'), \vec{n}(x')\Big] \in \mathcal{F}(x')$, for a.e. $x' \in \Omega$. Here:*

$$\vec{n} = \frac{\partial_1 y \times \partial_2 y}{|\partial_1 y \times \partial_2 y|} \tag{3.4}$$

is the (well defined) normal to the image surface $y(\Omega)$. Consequently, y realizes the mid-plate metric: $(\nabla y)^T \nabla y = [g_{\alpha\beta}]$.

(iii) *We have the lower bound:* $\liminf_{h\to 0} \frac{1}{h^2}\tilde{I}_W^h(u^h) \geq \mathcal{I}_2(y)$, *where* $\mathcal{I}_2$ *is given in (3.2).*

We further prove that the lower bound in (iii) above is optimal, in the following sense. Let $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ be a Sobolev regular isometric immersion of the given mid-plate metric, that is $(\nabla y)^T \nabla y = [g_{\alpha\beta}]$. The normal vector $\vec{n} \in W^{1,2}(\Omega, \mathbb{R}^3)$ is then given by (3.4) and it is well defined because $|\partial_1 y \times \partial_2 y| = (\det g)^{1/2} > 0$. We have:

Theorem 3.2. *For every isometric immersion* $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ *of* $[g_{\alpha\beta}]$, *there exists a sequence of recovery deformations* $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ *such that the assertion (i) of Theorem 3.1 hold, together with:*

$$\lim_{h\to 0} \frac{1}{h^2}\tilde{I}_W^h(u^h) = \mathcal{I}_2(y)$$

Assume now a slightly more general case of plates with slowly varying thickness, that is when:

$$\Omega^h = \{(x', x_3);\ x \in \Omega,\ -hq_1(x') < x_3 < hq_2(x')\}$$

with some positive $\mathcal{C}^1$ functions $q_1, q_2 : \Omega \longrightarrow (0, \infty)$. In this setting, the same results as in Theorem 3.1 and 3.2 have been reproved in [22], with the limiting functional:

$$\mathcal{I}_2^{q_1,q_2}(y) = \frac{1}{24}\int_\Omega \big(q_1(x') + q_2(x')\big)^3 \tilde{\mathcal{Q}}_2(x')\Big(\sqrt{[g_{\alpha\beta}]}^{-1}(\nabla y)^T \nabla \vec{n}\Big)\ \mathrm{d}x'.$$

For classical elasticity ($g^h = \mathrm{Id}$) of shells with mid-surface of arbitrary geometry and slowly oscillating boundaries as above, the analysis has been previously carried out in [18].

An important reference in the context of Theorems 3.1 and 3.2 (for flat films) is [26], containing the derivation of Kirchhoff plate theory for heterogeneous multilayers from 3d nonlinear energies given through an inhomogeneous density in $\int W(x_3/h, \nabla u)$.

4. A rigidity estimate

As a crucial ingredient of the proof of compactness in Theorem 3.1, we present a generalization of the nonlinear rigidity estimate obtained [7] in the Euclidean setting, extended to the non-Euclidean metrics in [20]. Note that in case $g^h = \mathrm{Id}$, the infimum of I_{dist}^h in (1.5) is naturally 0 and is attained only by the rigid motions. In [7], the authors proved an optimal estimate of the deviation in $W^{1,2}$ of a deformation u (on Ω^h), from rigid motions, in terms of the energy $I_{dist}^h(u)$. In our setting, since there is no realization of $I_{dist}^h(u) = 0$ if the Riemann curvature of the metric g^h is non-zero, we choose to estimate the deviation of the deformation from a linear map at the expense of an extra term, proportional to the gradient of the metric.

Theorem 4.1. *Let* $\mathcal{U}$ *be an open, bounded subset of* $\mathbb{R}^n$ *and let* g *be a smooth (up to the boundary) metric on* $\mathcal{U}$. *For every* $u \in W^{1,2}(\mathcal{U}, \mathbb{R}^n)$ *there exists* $Q \in \mathbb{R}^{n\times n}$ *such that:*

$$\int_{\mathcal{U}} |\nabla u(x) - Q|^2\ \mathrm{d}x \leq C\left(\int_{\mathcal{U}} \mathrm{dist}^2\Big(\nabla u, SO(n)\sqrt{g}(x)\Big)\ \mathrm{d}x + \|\nabla g\|^2_{L^\infty}(\mathrm{diam}\ \mathcal{U})^2|\mathcal{U}|\right),$$

where the constant C *depends on* $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$, *and on the domain* $\mathcal{U}$. *The dependence on* $\mathcal{U}$ *is uniform for a family of domains which are bilipschitz equivalent with controlled Lipschitz constants.*

For an embeddable metric g (i.e. whose $R_{ijkl} \equiv 0$) a related result has been obtained in [2]; namely an estimate of the deviation of (orientation preserving) deformation u from the realizations of g in terms of the L^1 stretching energy $\int |(\nabla u)^T \nabla u - g|$.

5. A hierarchy of scalings

Given a sequence of growth tensors a^h (say, each close to Id) defined on Ω^h, the general objective is now to analyze the behavior of the minimizers of the corresponding energies I_W^h as $h \to 0$. By Theorem 2.1, the infimum: $m_h = \inf\left\{I_W^h(u);\ u \in W^{1,2}(\Omega^h, \mathbb{R}^3)\right\}$ must be strictly positive whenever the Riemann tensor of the metric $g^h = (a^h)^T a^h$ does not vanish identically on Ω^h. This condition for g^h, under suitable scaling properties, can be translated into a first order curvature condition (5.1) below. In a first step (Theorem 5.1) we established [16] a lower bound on m_h in terms of a power law: $m_h \geq ch^\beta$, for all values of β greater than a critical β_0 in (5.2). This critical exponent depends on the asymptotic behavior of the perturbation $a^h - \mathrm{Id}$ in terms of the thickness h. Under existence conditions for certain classes of isometries, it can be shown that actually $m_h \sim h^{\beta_0}$.

Theorem 5.1. *For a given sequence of growth tensors a^h define their variations:*

$$Var(a^h) = \|\nabla_{tan}(a^h{}_{|\Omega})\|_{L^\infty(\Omega)} + \|\partial_3 a^h\|_{L^\infty(\Omega^h)}$$

together with the scaling in h:

$$\omega_1 = \sup\left\{\omega;\ \lim_{h\to 0}\frac{1}{h^\omega} Var(a^h) = 0\right\}.$$

Assume that: $\|a^h\|_{L^\infty(\Omega^h)} + \|(a^h)^{-1}\|_{L^\infty(\Omega^h)} \leq C$ and $\omega_1 > 0$.

Further, assume that for some $\omega_0 \geq 0$, there exists the limit:

$$\epsilon_g(x') = \lim_{h\to 0}\frac{1}{h^{\omega_0}} \fint_{-h/2}^{h/2} a^h(x',t) - \mathrm{Id}\ \mathrm{d}t \qquad \text{in } L^2(\Omega, \mathbb{R}^{3\times 3}).$$

which moreover satisfies:

$$\mathrm{curl}^T \mathrm{curl}\ (\epsilon_g)_{2\times 2} \not\equiv 0, \tag{5.1}$$

and that $\omega_0 < \min\{2\omega_1, \omega_1 + 1\}$.

Then, for every β with:

$$\beta > \beta_0 = \max\{\omega_0 + 2, 2\omega_0\}, \tag{5.2}$$

there holds: $\limsup_{h\to 0} \dfrac{1}{h^\beta} \inf I_0^h = +\infty.$

We expect it should be possible to rigorously derive a hierarchy of prestrained limiting theories, differentiated by the embeddability properties of the target metrics, encoded in the scalings of (the components of) their Riemann curvature tensors. This is in the same spirit as the different scalings of external forces lead to a hierarchy of nonlinear elastic plate theories displayed by Friesecke, James and Müller in[8]. For shells, that are thin films with mid-surface or arbitrary (non-flat) geometry, an infinite hierarchy of models was proposed, by means of asymptotic expansion in [21], and it remains in agreement with all the rigorously obtained results [6, 17, 18, 19].

6. The prestrained von Kármán model

Towards studying the dynamical growth problem (that is, incorporating the feedback from the minimizing shape u^h at the prior time-step, to growth tensor a^h at the current time-step) in [16] we considered the growth tensor:

$$a^h(x', x_3) = \mathrm{Id} + h^2\epsilon_g(x') + hx_3\gamma_g(x'), \tag{6.1}$$

with given matrix fields $\epsilon_g, \gamma_g : \overline{\Omega} \longrightarrow \mathbb{R}^{3\times 3}$. Note that the assumptions of Theorem 5.1 do not hold, since in the present case $\omega_0 = 2\omega_1 = \omega_1 + 1 = 2$.

We proved that, in this setting $\inf I_W^h \leq Ch^4$, while the lower bound $h^{-4} \inf I_W^h \geq c > 0$ is equivalent to:

$$\operatorname{curl}((\gamma_g)_{2\times 2}) \not\equiv 0 \quad \text{or} \quad 2\operatorname{curl}^T\operatorname{curl}(\epsilon_g)_{2\times 2} + \det(\gamma_g)_{2\times 2} \not\equiv 0, \tag{6.2}$$

which are the (negated) linearized Gauss-Codazzi equations corresponding to the metric $I = \mathrm{Id} + h^2(\epsilon_g)_{2\times 2}$ and the second fundamental form $II = \frac{1}{2}h(\gamma_g)_{2\times 2}$ on Ω. Equivalently, the above conditions guarantee that the highest order terms in the expansion of the Riemann curvature tensor components R_{1213}, R_{2321} and R_{1212} of $g^h = (a^h)^T a^h$ do not vanish. Also, either of them implies that $\inf \mathcal{I}_4 > 0$ (see definition below), which yields the lower bound on $\inf I_W^h$.

The Γ-limit of the rescaled energies is, in turn, expressed in terms of the out-of-plane displacement $v \in W^{2,2}(\Omega, \mathbb{R})$ and in-plane displacement $w \in W^{1,2}(\Omega, \mathbb{R}^2)$:

$$\begin{aligned} &\frac{1}{h^4} I_W^h \xrightarrow{\Gamma} \mathcal{I}_4 \quad \text{where} \\ &\mathcal{I}_4(w,v) = \frac{1}{24}\int_\Omega \mathcal{Q}_2\big(\nabla^2 v + \frac{1}{2}(\gamma_g)_{2\times 2}\big) + \frac{1}{2}\int_\Omega \mathcal{Q}_2\big(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v \otimes \nabla v - \frac{1}{2}(\epsilon_g)_{2\times 2}\big), \end{aligned} \tag{6.3}$$

with the quadratic nondegenerate form $\mathcal{Q}_2$, acting on matrices $F \in \mathbb{R}^{2\times 2}$:

$$\mathcal{Q}_2(F) = \min\{\mathcal{Q}_3(\tilde{F});\ \tilde{F} \in \mathbb{R}^{3\times 3}, \tilde{F}_{2\times 2} = F\} \text{ and } \mathcal{Q}_3(\tilde{F}) = D^2 W(\mathrm{Id})(\tilde{F} \otimes \tilde{F}).$$

The two terms in (6.3) measure: the first order in h change of II, and the second order change in I, under the deformation $id + hve_3 + h^2 w$ of Ω. Moreover, any sequence of deformations u^h with $I_W^h(u^h) \leq Ch^4$ is, asymptotically, of this form.

More precisely, we proved in [16]:

Theorem 6.1. *Let the growth tensor a^h be as in (6.1). Assume that the energies of a sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ satisfy:*

$$I_W^h(u^h) \leq Ch^4, \tag{6.4}$$

where W fulfills (1.3). Then there exist proper rotations $\bar{R}^h \in SO(3)$ and translations $c^h \in \mathbb{R}^3$ such that for the normalized deformations:

$$y^h(x', x_3) = (\bar{R}^h)^T u^h(x', hx_3) - c^h : \Omega^1 \longrightarrow \mathbb{R}^3$$

the following holds.

(i) *$y^h(x', x_3)$ converge in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x'.*

(ii) *The scaled displacements:*

$$V^h(x') = \frac{1}{h}\fint_{-1/2}^{1/2} y^h(x', t) - x' \, \mathrm{d}t \tag{6.5}$$

converge (up to a subsequence) in $W^{1,2}(\Omega, \mathbb{R}^3)$ to the vector field of the form $(0, 0, v)^T$, with the only non-zero out-of-plane scalar component: $v \in W^{2,2}(\Omega, \mathbb{R})$.

(iii) *The scaled in-plane displacements $h^{-1}V_{tan}^h$ converge (up to a subsequence) weakly in $W^{1,2}(\Omega, \mathbb{R}^2)$ to an in-plane displacement field $w \in W^{1,2}(\Omega, \mathbb{R}^2)$.*

(iv) *Recalling the definition (6.3), there holds:*

$$\liminf_{h\to 0} \frac{1}{h^4} I_W^h(u^h) \geq \mathcal{I}_4(w, v),$$

The limsup part of the Γ-convergence statement in the below Theorem 6.2 establishes that for any pair of displacements (w, v) in suitable limit spaces, one can construct a sequence of 3d deformations of thin plates Ω^h which approximately yield the energy $\mathcal{I}_4(w, v)$. The form of such *recovery sequence* delivers an insight on how to reconstruct the 3d deformations out of the data on the mid-plate Ω. In particular, comparing the present von Kármán growth model with the classical model ([8], Section 6.1) we observe the novel warping effect in the non-tangential growth.

Theorem 6.2. *Assume the setting of Theorem 6.1. For every $w \in W^{1,2}(\Omega, \mathbb{R}^3)$ and every $v \in W^{2,2}(\Omega, \mathbb{R})$, there exists a sequence of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ such that the following holds:*

(i) *The sequence $y^h(x', x_3) = u^h(x', hx_3)$ converge in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x'.*

(ii) $V^h(x') = h^{-1} \fint_{-h/2}^{h/2} (u^h(x', t) - x')\, \mathrm{d}t$ *converge in $W^{1,2}(\Omega, \mathbb{R}^3)$ to $(0, 0, v)^T$.*

(iii) $h^{-1}V^h_{tan}$ *converge in $W^{1,2}(\Omega, \mathbb{R}^2)$ to w.*

(iv) *Recalling the definition (6.3) one has:*

$$\lim_{h\to 0} \frac{1}{h^4} I^h_W(u^h) = \mathcal{I}_4(w, v).$$

The main consequences of the Γ-convergence results above are as follows:

Corollary 6.3. *Assume the setting of Theorem 6.1. Then:*

(i) *There exist uniform constants $C, c \geq 0$ such that for every h there holds:*

$$c \leq \frac{1}{h^4} \inf I^h_W \leq C. \tag{6.6}$$

If moreover (6.2) holds then one may have $c > 0$.

(ii) *There exists at least one minimizing sequence $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3$ for I^h_W:*

$$\lim_{h\to 0} \Big(\frac{1}{h^4} I^h_W(u^h) - \frac{1}{h^4} \inf I^h_W\Big) = 0. \tag{6.7}$$

For any such sequence the convergences (i), (ii) and (iii) of Theorem 6.1 hold and the limit (w, v) is a minimizer of $\mathcal{I}_4$.

(iii) *For any minimizer (w, v) of $\mathcal{I}_4$, there exists a minimizing sequence u^h, satisfying (6.7) together with (i), (ii), (iii) and (iv) of Theorem 6.2.*

7. The prestrained von Kármán equations

For elastic energy W satisfying (1.3) which is additionally isotropic:

$$\forall F \in \mathbb{R}^{3\times 3} \quad \forall R \in SO(3) \qquad W(FR) = W(F), \tag{7.1}$$

one can see [8] that the quadratic forms in $\mathcal{I}_4$ are given explicitly as:

$$\mathcal{Q}_3(F) = 2\mu|\mathrm{sym}\ F|^2 + \lambda|\mathrm{tr}\ F|^2, \qquad \mathcal{Q}_2(F_{2\times 2}) = 2\mu|\mathrm{sym}\ F_{2\times 2}|^2 + \frac{2\mu\lambda}{2\mu+\lambda}|\mathrm{tr}\ F_{2\times 2}|^2, \tag{7.2}$$

for all $F \in \mathbb{R}^{3\times 3}$. Here, *tr* stands for the trace of a quadratic matrix, and μ and λ are the Lamé constants, satisfying: $\mu \geq 0$, $3\lambda + \mu \geq 0$.

Under these conditions, the Euler-Lagrange equations of the limiting functional $\mathcal{I}_4$ are equivalent, under a change of variables which replaces the in-plane displacement w by the Airy stress potential Φ, to the new system proposed in [23]:

$$\Delta^2\Phi = -S(K_G + \lambda_g), \qquad B\Delta^2 v = [v, \Phi] - B\Omega_g,$$

where $S = \mu(3\lambda+2\mu)/(\lambda+\mu)$ is the Young's modulus, K_G the Gaussian curvature, $B = S/(12(1-\nu^2))$ the bending stiffness, and $\nu = \lambda/(2(\lambda+\mu))$ the Poisson ratio given in terms of the Lamé constants λ and μ. The corrections due to the prestrain are:

$$\lambda_g = \mathrm{curl}^T\mathrm{curl}\ (\epsilon_g)_{2\times 2}, \qquad \Omega_g = \mathrm{div}^T\mathrm{div}\,((\gamma_g)_{2\times 2} + \nu\ \mathrm{cof}\ (\gamma_g)_{2\times 2})$$

More precisely:

Theorem 7.1. *Assume (1.3) and (7.1). Then the leading order displacements in a thin tissue which tries to adapt itself to an internally imposed metric $g^h = (a^h)^T a^h$ with a^h as in (6.1) satisfy:*

$$\begin{aligned} \Delta^2\Phi &= -S\Big(\det\nabla^2 v + \mathrm{curl}^T\mathrm{curl}(\epsilon_g)_{2\times 2}\Big),\\ B\Delta^2 v &= \mathrm{cof}\nabla^2\Phi : \nabla^2 v - B\ \mathrm{div}^T\mathrm{div}\Big((\gamma_g)_{2\times 2} + \nu\ \mathrm{cof}(\gamma_g)_{2\times 2}\Big). \end{aligned}$$

together with the (free) boundary conditions on $\partial\Omega$:

$$\begin{aligned} &\Phi = \partial_{\vec n}\Phi = 0,\\ &\tilde\Psi : (\vec n\otimes\vec n) + \nu\ \tilde\Psi : (\tau\otimes\tau) = 0,\\ &(1-\nu)\partial_\tau\Big(\tilde\Psi : (\vec n\otimes\tau)\Big) + \mathrm{div}\Big(\tilde\Psi + \nu\ \mathrm{cof}\tilde\Psi\Big)\vec n = 0. \end{aligned}$$

Here $\vec n$ denotes the normal, τ the tangent to $\partial\Omega$, while:

$$\tilde\Psi = \nabla^2 v + \mathrm{sym}(\gamma_g)_{2\times 2},$$

The in-plane displacement w can be recovered from the Airy stress potential Φ and the out-of-plane displacement v, uniquely up to rigid motions, by means of:

$$\mathrm{cof}\nabla^2\Phi = 2\mu\Big(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v\otimes\nabla v - \mathrm{sym}(\epsilon_g)_{2\times 2}\Big) + \frac{2\mu\lambda}{2\mu+\lambda}\Big(\mathrm{div}\ w + \frac{1}{2}|\nabla v|^2 - \mathrm{tr}(\epsilon_g)_{2\times 2}\Big)\mathrm{Id}.$$

Notice that in the particular case when $(\mathrm{sym}\ \kappa_g)_{2\times 2} = 0$ on $\partial\Omega$, the two last boundary conditions become:

$$\begin{aligned} &\partial^2_{\vec n\vec n}v + \nu\Big(\partial^2_{\tau\tau}v - K\partial_{\vec n}v\Big) = 0\\ &(2-\nu)\partial_\tau\partial_{\vec n}\partial_\tau v + \partial^3_{\vec n\vec n\vec n}v + K\Big(\Delta v + 2\partial^2_{\vec n\vec n}v\Big) = 0, \end{aligned}$$

where K stands for the (scalar) curvature of $\partial\Omega$, so that $\partial_\tau\tau = K\vec n$. If additionally $\partial\Omega$ is a polygonal, then the above equations simplify to equations (5) in [23].

Acknowledgments. M.L. was partially supported by grants NSF DMS-0707275 and DMS-0846996, and by the Polish MN grant N N201 547438.

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