Calculus of variations on thin prestressed films

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Contents

Chapter 1
Introduction

This monograph concerns the analytical and geometrical questions emerging from the study of thin elastic films that exhibit residual stress at free equilibria. Prestressed thin films are present in many contexts and applications, ranging from growing tissues, through plastically strained sheets, engineered swelling or shrinking gels, to petals and leaves of flowers, atomically thin graphene layers, etc. While the related questions about the physical basis for shape formation (*morphogenesis*) lie at the intersection of biology, chemistry and physics, fundamentally they have analytical and geometrical character. Indeed, they may be seen as a variation on classical themes: in differential geometry - that of isometrically embedding a shape with a given metric in an ambient space of possibly different dimension; and in calculus of variations - that of minimizing non-convex energy functionals parametrised by a quantity in whose limit the functionals become in some sense degenerate.

Motivation from differential geometry. The field of differential geometry begun through studying of curves and surfaces in $\mathbb{R}^3$. The abstract concept of a Riemannian manifold, formulated in XIXth century, and the natural question whether each such object is simply a subset (submanifold) of some Euclidean space $\mathbb{R}^M$, quickly assumed a position of fundamental conceptual importance. This problem is called the *isometric immersion problem* and it can be formulated as the question of solvability of the following system of partial differential equations:

$$(\nabla u)^T(\nabla u) = g \quad \text{ for } \; u : \mathbb{R}^N \supset \Omega \to \mathbb{R}^M, \tag{1.1}$$

where $g : \Omega \to \mathbb{R}^{N\times N}$ is a given symmetric, positive definite matrix field. A remarkable result, due to Nash, states that any smooth N-dimensional Riemannian manifold (corresponding to g in (1.1)) admits a smooth isometric immersion in $\mathbb{R}^M$ (corresponding to u) for some large dimension $M = M(N)$. On the other hand, the celebrated Nash-Kuiper theorem asserts that any $\mathscr{C}^1$ short immersion of a $\mathscr{C}^1$ metric g into $\mathbb{R}^{N+1}$ (i.e. $u_0 \in \mathscr{C}^1(\Omega,\mathbb{R}^{N+1})$ satisfying $0 < (\nabla u_0)^T\nabla u_0 < g$ in the sense of matrix inequalities), can be uniformly approximated by $\mathscr{C}^1$ solutions to (1.1). Recently, this regularity has been improved to Hölder continuous $u \in \mathscr{C}^{1,\alpha}(\Omega,\mathbb{R}^{N+1})$, with the optimal exponent α being the subject of vigorous ongoing research.

For $N = M$, the isometric immersion problem is linked with the satisfaction or failure of the orientation preservation by u, expressed as:

$$\det \nabla u > 0 \quad \text{in } \Omega. \tag{1.2}$$

Without this restriction, there always exists a Lipschitz $u : \Omega \to \mathbb{R}^N$ solving (1.1). However, under (1.2) the same problem becomes very rigid: a sufficient and necessary condition for the (local) solvability of (1.1) (1.2) is then the vanishing of the Riemann curvature of g at each point of Ω. In the latter context, it is natural to pose the quantitative question: what is the infimum of the averaged pointwise deficit of a map u from being an orientation-preserving isometric immersion of g on Ω? This deficit may be is measured by the following *non-Euclidean energy*:

$$\mathscr{E}_g(u) = \int_\Omega \operatorname{dist}^2\big((\nabla u)g^{-1/2}, \mathrm{SO}(N)\big)\,\mathrm{d}x. \tag{1.3}$$

Indeed, u satisfies (1.1) (1.2) if and only if $\nabla u \in \mathrm{SO}(N)g^{1/2}$ almost everywhere in Ω, which is precisely when $\mathscr{E}_g(u) = 0$. In this monograph, we will be concerned with the following questions: Can one quantify $\inf \mathscr{E}_g$ in relation to g and Ω? What is the structure of minimizers to (1.3), if they exist? In the limit of Ω becoming $(N-1)$-dimensional, what can be said about asymptotic properties of non-Euclidean energies and their minimizers in relation to the Riemann curvatures of g?

Motivation from calculus of variations. The field of calculus of variations originally centered around minimization problems for integral functionals of the form:

$$\mathscr{E}(u) = \int_\Omega W\big(x, u(x), \nabla u(x)\big)\,\mathrm{d}x \quad \text{for all } u : \mathbb{R}^N \supset \Omega \to \mathbb{R}^M, \tag{1.4}$$

where $W : \Omega \times \mathbb{R}^M \times \mathbb{R}^{M\times N} \to \mathbb{R}$ is the given *energy density*, and where u may be subject to various constraints, for example the boundary conditions. The systematic study of existence of minimizers to (1.4), their uniqueness and qualitative properties, begun with Euler and Bernoulli in XVIIth century and rapidly progressed due to seminal contributions by Tonelli, Morrey and De Giorgi in XXth century. These and other related questions are strongly tied to the appropriate convexity (with respect to ∇u) properties of W, which in turn imply the so-called sequential lower semicontinuity of $\mathscr{E}$, necessary to conclude that the minimizing sequences to (1.4) accumulate at its minimizers. This is the celebrated *direct method of calculus of variations* which bypasses solving the potentially complicated Euler-Lagrange equations of (1.4), and allows to treat the aforementioned minimization problem directly.

Unfortunately, this technique does not apply to the functional in (1.3) due to non-convexity. However, for a class of domains Ω that are *thin films*, namely domains whose diameter in certain direction is much smaller than in others, one may consider a family of energies parametrised by the small thickness h of $\Omega^h = \omega \times (-h/2, h/2)$:

$$\mathscr{E}_g^h(u^h) = \int_{\Omega^h} \operatorname{dist}^2\big((\nabla u^h)g^{-1/2}, \mathrm{SO}(N)\big)\,\mathrm{d}x. \tag{1.5}$$

The task is now, rather than to minimize $\mathscr{E}_g^h$ for each particular h, to determine the asymptotic limit of the above minimization problems as $h \to 0$. This can be achieved by the method of *Γ-convergence*, which identifies such "singular limit" energy functional $\mathscr{I}$ with the property that the minimizers and minimum values of (1.5) converge, in a suitable sense, to the minimizers and minimum values of $\mathscr{I}_g$. In this monograph, we will be concerned with the following related questions: What is the optimal mode of convergence in this approach, allowing to recover the most information for the original minimization problems? Can one expect that $\inf \mathscr{E}_g^h \simeq h^\gamma$ as $h \to 0$ and can one determine the optimal scaling exponent γ from the metric g? Which curvatures or components of g play the role in this *dimension reduction* process and how do they contribute to the residual energy $\mathscr{I}_g$?

Motivation from solid mechanics and elasticity. The theory of elasticity is one of the most important fields of continuum mechanics. It studies elastic materials capable of undergoing large deformations, due to the distribution of *local stresses and displacements*, resulting from the application of mechanical or thermal loads. The basic variational model investigated in this monograph pertains to the non-Euclidean version of the *nonlinear elastic energy of deformations*:

$$\mathscr{E}_g(u) = \int_\Omega W\big((\nabla u)g^{-1/2}\big)\,\mathrm{d}x \quad \text{for all } u : \mathbb{R}^3 \supset \Omega \to \mathbb{R}^3, \tag{1.6}$$

where $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is the given energy density, satisfying physically-relevant conditions, including the frame invariance: $W(RF) = W(F)$ valid for all rotations R and all $F \in \mathbb{R}^{3\times 3}$, and the zero-penalty for all rigid motions: $W(R) = 0$. This model postulates formation of a target Riemannian metric g and the induced multiplicative decomposition of the deformation gradient ∇u into an elastic part $(\nabla u)g^{1/2}$ and an inelastic part, carrying the prestress $g^{1/2}$, responsible for the morphogenesis. Equivalently, $\mathscr{E}(u)$ quantifies the total pointwise deviation of ∇u from $g^{1/2}$, modulo rotations that do not cost any energy.

The functional (1.6), corresponding to a range of hyperelastic energies that approximate the behavior of a large class of elastomeric materials, is consistent with microscopic derivations based on statistical mechanics. It has the general form (1.4) and it reduces to the classical nonlinear three-dimensional elasticity when $\mathscr{E}_g(u) = 0$ for some u, with the necessary condition given by the vanishing of the Riemann curvature of g. In the opposite case i.e. for a non-Euclidean g, the infimum of $\mathscr{E}$ in absence of any forces or boundary conditions is strictly positive, pointing to the existence of residual strain. The energy (1.6) reduces also to (a version of) the classical linear elasticity when $g \simeq \mathrm{Id}_3$. The goal is now to answer the following questions: How to determine the minimizing shapes u of the tissue attaining an orientation-preserving configuration closest to being an isometric immersion of g, in terms of an appropriate mechanical theory? Is it possible to quantify the separation of scales arising in *slender structures* and inducing the constraints from the prescription of growth laws? How to pose and resolve the geometric design problems involving prestress, as inverse problems to the minimization of (1.6)?

Overview of the monograph. This monograph consists of three parts. Part I introduces three tools in the mathematical analysis that we will rely on while investigating minimization problems of the energies of the types (1.3) - (1.6). These are: Γ-convergence discussed in chapter 2, *Korn's inequality* in chapter 3, and its nonlinear counterpart which is *Friesecke-James-Müller's inequality* in chapter 4. Our treatment is self-contained and proofs are complete, including Hardy's inequality and the Lusin-type truncation-approximation result in Sobolev spaces, which are of independent interest. We also derive various estimates and properties of the constants appearing in both inequalities, in relation to the dimension of the problem or to the geometry of the domain, including the star-shaped domains and thin films.

In Part II we formulate the modern description of nonlinear elasticity of plates and shells, where the analysis of the scaling of the energy minimizers in terms of the film's thickness leads to the rigorous derivation of a hierarchy of limiting variational theories. These theories are differentiated by shells' responses to external forces: in chapter 5 we derive the Kirchhoff theory (fully nonlinear bending) of elastic plates and shells, while in chapter 6 we turn to the von Kármán (nonlinear) and linear elasticity. In chapter 7 we discuss the linearised Kirchhoff theory (linearised bending) for plates. The aforementioned four plate theories were rigorously derived from the nonlinear elasticity in the fundamental work by Friesecke, James and Müller, relying on the nonlinear rigidity estimate studied in chapter 4.

Chapter 8 provides a heuristic derivation of the *infinite hierarchy of elastic shell models* and explains how it reduces to the finite hierarchy of plates due to the matching and density properties of infinitesimal isometries on two-dimensional domains. Other matching and density properties lead to an even larger collapse of theories: for elliptic shells in chapter 9, and for developable shells in chapter 10. In each case of a particular theory, we give the complete details of Γ-convergence results, including the compactness analysis, the lower bound estimates, constructions of the recovery family, and the induced convergence of minimizers. We frequently take a detour and present a few generalizations: to shells with variable thickness and to shallow shells, where the depth of the midsurface competes with the vanishing thickness.

Part III is the central part of this monograph. There, we continue the discussion of the *dimension reduction* in the context of the prestress-driven response, where the Γ-limiting theories are differentiated by the embeddability properties of the target metrics (rather than by the magnitude of applied forces) and, a-posteriori, by the emergence of isometry constraints on deformations with low regularity. In chapter 11 we derive the Kirchhoff-like theory for prestressed thin films, and in chapter 12 we turn to the von Kármán-like theory. In chapter 13 we show the energy quantisation result, in the sense that only the even powers of films' thickness are viable as the scaling of the energy at minimizers, and all of them are also attained. This leads to the remaining family of the linear elasticity-like theories in the *infinite hierarchy of prestressed films' models*. Along the way, we provide many examples, including applications to liquid crystal elastomers and the relation to experimental observations. There are still unresolved dichotomies between theory and experiments, pertaining to understanding of the role of curvature in determining the mechanical properties of the material, and to the effects of the symmetry and the symmetry breaking solutions, which call for a thorough understanding.

The final chapter 14 analyzes the case of *weak prestress*, using analytical constructions and arguments similar to those in chapter 7 for shallow shells. There, many problems regarding multiplicity, singularities and regularity of the critical points of the obtained models remain open and are hard to tackle. On the other hand, our analysis leads to further questions of rigidity and flexibility of solutions to the weak formulations of related partial differential equations, including the weak Sobolev or Hölder solutions to the Monge-Ampère equation.

Overview of the back matter. The bibliography contains the main references to the material treated in this monograph, updated to 2022. Each chapter ends with the bibliographical notes while the attributions are kept to a minimum in the body of the text. We also provide an extensive index of terminology, containing both the classical notation in analysis and differential geometry, the standard notation in elasticity theory, and the terminology consistently used throughout the text. Finally, we include the alphabetical index of topics, concepts, and named theorems.

Prerequisites. The monograph is self-contained and suitable for beginning graduate students in analysis, with some prior exposure to concepts in differential geometry. No a priori knowledge of elasticity theory is assumed.

This monograph aims at at the systematic and comprehensive treatment of the of the theory of dimension reduction for thin elastic (prestressed or lacking prestress) plates and shells. Starting from the view that shape is the consequence of metric frustration in an ambient space, we explore many surprising connections between the classical Nash embedding problem, its quantitative version via the variational description, the Monge-Ampére equation and the biological morphogenesis. We hope that our text will serve as a friendly introduction to this beautiful and multifaceted topic, as well as suggest and encourage new research directions.

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Chapter 3
Korn's inequality

Among the most important inequalities in the mathematical analysis is *Korn's inequality*. Discovered in the early XXth century in the context of the boundary value problem of linear electrostatics, it is a basic tool to prove existence of solutions of the linearised displacement-traction equations in elasticity. Another area of application is fluid dynamics in presence of Navier's boundary conditions, where Korn's inequality replaces the Poincaré inequality used under the Dirichlet conditions.

The outline of this chapter is as follows. In section 3.1 we introduce Korn's inequality (sometimes called the second Korn's inequality) as the rigidity estimate, quantifying the simple observation that a gradient field that is skew-symmetric must be constant. We show how through an argument by contradiction, it can be deduced from the First Korn's inequality that involves the norm of the vector field itself, in addition to the norm of its gradient. In section 3.2 we divert to deduce a few useful variants of Korn's inequality, in presence of boundary conditions: the homogeneous inequality where Korn's constant equals 2 regardless of the domain (the same result is true on polyhedra under tangential boundary conditions), and Korn's inequality under mixed boundary conditions (Dirichlet and tangential), in which it suffices to look for the approximating skew-symmetric matrix among those matrices that are gradients of affine functions satisfying the same boundary conditions. In particular, if the portion of the boundary corresponding to the Dirichlet condition is nonempty or when the domain has no rotational symmetry, this set is trivial leading to the homogeneous Korn's inequality (albeit with a constant possibly different than 2).

In sections 3.3 and 3.4 we give a proof of First Korn's inequality. The argument relies on the weighted (through the distance from the boundary) reverse Poincaré-inequality and on estimates in star-shaped domains. Since star-shaped domains are the building blocks of Lipschitz domains (as proved in section 3.3), the final result is obtained by decomposition.

In section 3.5 we prove that under tangential boundary conditions, Korn's constant is at least 2. There is however no upper bound, because any Killing field on a given midsurface gives raise to a family of displacements on thin shells that are tangential on the shells' boundaries and whose Korn's constants blow up as the inverse square of shell's thickness. This statement is established in section 3.5 for the case of a curve in $\mathbb{R}^2$, and for the general case a hypersurface in $\mathbb{R}^N$ in section 3.8. In section 3.6 we deduce an approximation result in which the displacement gradient on a thin shell is approximated by a field of skew-symmetric matrices, rather than a constant matrix, with the corresponding Korn's constants uniform in thickness. The construction is a combination of the local application of Korn's inequality and a mollification argument, and it can be repeated in other contexts. In particular, it is of key importance towards the dimension reduction analysis in nonlinear prestressed elasticity, discussed in Parts II and III of this monograph.

In section 3.7 we prove the counterparts of the two Korn's inequalities on hypersurfaces. Rather than redeveloping previous arguments in the Riemannian geometry setting, we apply First Korn's inequality on thin shells and pass to the limit with the vanishing thickness. Then, an argument by contradiction naturally identifies gradients of Killing fields as approximations of the covariant derivatives of arbitrary tangent fields, up to the error quantifying the symmetrized gradients. Killing fields are the infinitesimal generators of isometries on the surface and as such they serve as replacements of the affine maps with skew-symmetric gradients, which are the "linearised rigid motions" on open domains in $\mathbb{R}^N$.

In sections 3.9 and 3.10 we prove that the presence of Killing fields is the only obstruction from the uniformity of Korn's constant on thin shells under tangential boundary conditions. Indeed, for vector fields within any cone that is separated from the displacements derived from Killing fields, Korn's constant is uniform.

3.1 Korn's inequality and First Korn's inequality

In this section we introduce the two versions of Korn's inequality and show how to deduce one, namely the rigidity estimate in Theorem 3.1, from the other which is the First Korn's inequality in Theorem 3.5.

Theorem 3.1. [Korn's inequality]
Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected, Lipschitz domain. Then, for every vector field $v \in H^1(\Omega, \mathbb{R}^N)$ there exists a matrix $A \in \mathrm{so}(N)$ satisfying:

$$\int_\Omega |\nabla v - A|^2 \,\mathrm{d}x \leq C \int_\Omega |\mathrm{sym}\,\nabla v|^2 \,\mathrm{d}x. \tag{3.1}$$

The above constant C depends only on Ω, but not on v.

The inequality (3.1) is an example of a *rigidity estimate*, quantifying the *rigidity statement* below. Namely, the vanishing of the right hand side in (3.1) i.e. having $\mathrm{sym}\nabla v \equiv 0$ in Ω, implies the vanishing of its left hand side i.e. having ∇v constant and skew-symmetric. This simple yet remarkable observation is proved directly:

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected domain. If $v \in H^1(\Omega, \mathbb{R}^N)$ satisfies $\nabla v \in \mathrm{so}(N)$ a.e. in Ω, then:*

$$v(x) = Ax + b \qquad \textit{for some } A \in \mathrm{so}(N),\ b \in \mathbb{R}^N.$$

Proof. The following useful formula, valid in the sense of distributions:

$$\Delta v = 2\,\mathrm{div}\Big(\mathrm{sym}\nabla v - \frac{1}{2}\mathrm{tr}\big(\mathrm{sym}\nabla v\big)\mathrm{Id}_N\Big). \tag{3.2}$$

may be obtained by noting $\mathrm{div}\big(\mathrm{sym}\nabla v\big) = \frac{1}{2}\mathrm{div}\,\nabla v + \frac{1}{2}\mathrm{div}\,(\nabla v)^T = \frac{1}{2}\Delta v + \frac{1}{2}\nabla(\mathrm{div}\,v)$, and that $\mathrm{div}\big((\mathrm{div}\,v)\mathrm{Id}_N\big) = \nabla(\mathrm{div}\,v)$. Consequently, if $\mathrm{sym}\nabla v = 0$ then $\Delta v = 0$ in Ω, so v is harmonic and automatically smooth.

By assumption, there holds $\nabla v = -(\nabla v)^T$. To each row $\{\nabla v^i = -\partial_i v\}_{i=1\dots N}$ of this identity, we apply the N-dimensional curl operator:

$$\mathrm{curl}\,u \doteq \{\partial_k u^j - \partial_j u^k\}_{j,k=1\dots N}.$$

Since $\mathrm{curl}\,\nabla v^i = 0$, it follows that $\nabla \mathrm{curl}\,v = 0$ and so $\mathrm{curl}\,v$ must be constant within the connected domain Ω. It now suffices to note that coefficients of $\mathrm{curl}\,v$ coincide with entries of the matrix $\nabla v = \mathrm{skew}\,\nabla v$. The proof is done. ■

Recall that every matrix in $\mathbb{R}^{N\times N}$ is the orthogonal sum of is symmetric and skew-symmetric parts, so in particular:

$$|\mathrm{sym}\nabla v(x)| = \mathrm{dist}(\nabla v(x), \mathrm{so}(N)) \quad \text{ for all } x \in \Omega.$$

The inequality (3.1) is thus the quantitative version of Lemma 3.2, in the sense that the total pointwise distance of ∇v from $\mathrm{so}(N)$, measured in the $L^2(\Omega)$ norm, yields the deviation of ∇v from being constant skew-symmetric, again measured in $L^2(\Omega)$:

$$\begin{aligned}\mathrm{dist}_{L^2(\Omega)}\big(\nabla v, \mathrm{so}(N)\big) &\doteq \inf_{A\in \mathrm{so}(N)} \Big(\int_\Omega |\nabla v - A|^2\,\mathrm{d}x\Big)^{1/2} \\ &\leq C\|\,\mathrm{dist}(\nabla v, \mathrm{so}(N))\|_{L^2(\Omega)}.\end{aligned} \tag{3.3}$$

It is obvious that the above inequality can also be reversed:

$$\|\,\mathrm{dist}(\nabla v, \mathrm{so}(N))\|_{L^2(\Omega)} \leq \mathrm{dist}_{L^2(\Omega)}\big(\nabla v, \mathrm{so}(N)\big).$$

Hence, Korn's inequality states equivalence of commuting the operations of taking the distance from $\mathrm{so}(N)$ and integrating. One has to be careful though: the right hand side computes the L^2 norm of the pointwise distance in $\mathbb{R}^{N\times N}$, whereas the distance in the left hand side is in the functional space $L^2(\Omega, \mathbb{R}^{N\times N})$. Further, we have:

Remark 3.3. The infimum in (3.3) is the minimum, attained at the unique matrix A:

$$A = \mathrm{skew} \fint_\Omega \nabla v\,\mathrm{d}x.$$

Indeed, let $A \in \mathrm{so}(N)$ be such that $\int_\Omega |\nabla v - A|^2 \, \mathrm{d}x \leq \int_\Omega |\nabla v - (A + \varepsilon B)|^2 \, \mathrm{d}x$ for all $\varepsilon \in \mathbb{R}$ and $B \in \mathrm{so}(N)$. Expanding the right hand side as:

$$\int_\Omega |\nabla v - A|^2 \, \mathrm{d}x + 2\varepsilon \int_\Omega \langle \nabla v - A : B \rangle \, \mathrm{d}x + \varepsilon^2 \int_\Omega |B|^2 \, \mathrm{d}x,$$

and passing to the limit with $\varepsilon \to 0$, we obtain the Euler-Lagrange equations:

$$\Big\langle \int_\Omega \nabla v - A \, \mathrm{d}x : B \Big\rangle = \int_\Omega \langle \nabla v - A : B \rangle \, \mathrm{d}x = 0 \qquad \text{for all } \; B \in \mathrm{so}(N).$$

The matrix $\fint_\Omega \nabla v - A$ is thus orthogonal to $\mathrm{so}(N)$, hence symmetric:

$$\mathrm{skew} \fint_\Omega \nabla v \, \mathrm{d}x - A = \mathrm{skew}\Big(\fint_\Omega \nabla v \, \mathrm{d}x - A \Big) = 0.$$

This yields the claimed result. ■

By combining (3.1) with the Poincaré-Wirtinger inequality, one gets another useful bound (that we already applied in section 2.2):

Corollary 3.4. [Korn-Poincarè inequality]
Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected, Lipschitz domain. Then, for every $v \in H^1(\Omega, \mathbb{R}^N)$ there exists $A \in \mathrm{so}(N)$ and $b \in \mathbb{R}^N$ such that:

$$\|v - (Ax + b)\|_{H^1(\Omega)} \leq C \|\mathrm{sym} \nabla v\|_{L^2(\Omega)}.$$

The constant C depends only on Ω, but not on v.

Korn's inequality in Theorem 3.1 classically follows via an argument by contradiction, from a stronger result called the First Korn's inequality:

Theorem 3.5. [First Korn's inequality]
Let $\Omega \subset \mathbb{R}^N$ be open, bounded, Lipschitz domain. There holds:

$$\int_\Omega |\nabla v|^2 \, \mathrm{d}x \leq C \int_\Omega |v|^2 + |\mathrm{sym} \nabla v|^2 \, \mathrm{d}x, \tag{3.4}$$

for every $v \in H^1(\Omega, \mathbb{R}^N)$, where the constant C depends on Ω, but not on v.

We postpone the proof of (3.4) to section 3.3, and deduce (3.1) right away:

Proof of Theorem 3.1.
Assume that there is no universal constant C for (3.3) to hold. In view of Remark 3.3, this implies existence of a sequence $\{v_n \in H^1(\Omega, \mathbb{R}^N)\}_{n=1}^\infty$, such that:

$$\int_\Omega \left|\nabla v_n - \mathrm{skew} \fint_\Omega \nabla v_n\right|^2 \,\mathrm{d}x > n \int_\Omega |\mathrm{sym}\nabla v_n|^2 \,\mathrm{d}x. \tag{3.5}$$

Denoting $w_n(x) \doteq v_n(x) - \big(\mathrm{skew} \fint_\Omega \nabla v_n\big)x - \fint_\Omega \big(v_n(y) - \big(\mathrm{skew} \fint_\Omega \nabla v_n\big)y\big) \,\mathrm{d}y$ we get:

$$\fint_\Omega w_n \,\mathrm{d}x = 0, \qquad \mathrm{skew} \fint_\Omega \nabla w_n \,\mathrm{d}x = 0. \tag{3.6}$$

Multiplying each v_n by an appropriate constant, we may also ensure that:

$$\int_\Omega |\nabla w_n|^2 \,\mathrm{d}x = 1, \qquad \int_\Omega |\mathrm{sym}\nabla w_n|^2 \,\mathrm{d}x < \frac{1}{n}, \tag{3.7}$$

where the second assertion follows by (3.5). In virtue of Poincaré's inequality, the sequence $\{w_n\}_{n=1}^\infty$ is thus bounded in $H^1(\Omega)$, and so it has a subsequence (that we do not relabel), converging weakly to some $w \in H^1(\Omega, \mathbb{R}^N)$. In particular, the sequence $\{\mathrm{sym}\nabla w_n\}_{n=1}^\infty$ converges weakly in $L^2(\Omega, \mathbb{R}^{N\times N})$ to $\mathrm{sym}\nabla w$.

By (3.7) we see that $\mathrm{sym}\nabla w = 0$ a.s. in Ω, so Lemma 3.2 implies that $w(x) = Ax + b$ for some $A \in \mathrm{so}(N)$ and $b \in \mathbb{R}^N$. On the other hand, passing to the limit in (3.6) implies: $\mathrm{skew} \fint_\Omega \nabla w \,\mathrm{d}x = 0$ and $\fint_\Omega w \,\mathrm{d}x = 0$. In conclusion, $A = 0, b = 0$ and $w = 0$, yielding the following convergences:

$$\int_\Omega |w_n|^2 \,\mathrm{d}x \to 0, \qquad \int_\Omega |\mathrm{sym}\nabla w_n|^2 \,\mathrm{d}x \to 0 \qquad \text{as } n \to \infty.$$

We now apply (3.4) to get:

$$\int_\Omega |\nabla w_n|^2 \,\mathrm{d}x \to 0 \qquad \text{as } n \to \infty.$$

This contradicts the first assertion in (3.7). The proof is done. ■

As a point of comparison, we anticipate that a nonlinear version of (3.1) will be discussed in chapter 4. This celebrated *nonlinear geometric rigidity estimate* (see Theorem 4.1), due to Friesecke, James and Müller, is "nonlinear" in the sense that in both of its sides it quantifies the appropriate distances of the gradient ∇v from the compact manifold $\mathrm{SO}(N)$, rather than from a linear subspace $\mathrm{so}(N) \subset \mathbb{R}^{N\times N}$ as in (3.1). Another connection is that $\mathrm{so}(N)$ is the tangent space to $\mathrm{SO}(N)$ at Id_N, so (3.1) formally follows by collecting the lowest order terms in the expansion of the nonlinear estimate (4.2) close to $\nabla u = \mathrm{Id}_N$. These observations are also inherently related with the concepts of *deformations* and *displacements* in the mathematical description of, respectively, the nonlinear and linear elasticity.

We have used Corollary 3.5 in section 2.2 to prove a Γ-convergence result in the dimension reduction of the linearly elastic plates. The aforementioned Friesecke-James-Müller inequality will be applied in the dimension reduction analysis of the nonlinear prestressed plates and shells in Parts II and III of this monograph.

3.2 Variants of Korn's inequality with different boundary conditions

In this section we prove variants of Korn's inequality valid under specific boundary conditions. The rule of thumb is that, without violating the uniformity of the constant C in (3.1), it suffices to seek the skew-symmetric matrix A in its left hand side, only within constant skew-symmetric gradients of those linear maps that obey the same boundary conditions. This space is, in general, a proper subspace of $\mathrm{so}(N)$ and may even be trivial. Before we make this observation more precise, we prove a classical variant of Korn's inequality where the constant C can be made specific.

Theorem 3.6. [Homogeneous Korn's inequality]
For every open domain $\Omega\subset\mathbb{R}^N$ and every $v\in H_0^1(\Omega,\mathbb{R}^N)$, there holds:

$$\int_\Omega |\nabla v|^2\,\mathrm{d}x \leq 2\int_\Omega |\mathrm{sym}\,\nabla v|^2\,\mathrm{d}x. \tag{3.8}$$

Moreover, the constant 2 above is optimal, for any Ω.

Proof. Without loss of generality, we assume that $v\in\mathscr{C}_0^\infty(\Omega,\mathbb{R}^N)$. Recall the formula (3.2), multiply its both sides by v and integrate by parts on Ω. Thus we arrive at the equality below, implying the claimed bound (3.8):

$$\int_\Omega |\nabla v|^2\,\mathrm{d}x = 2\int_\Omega |\mathrm{sym}\nabla v|^2\,\mathrm{d}x - \int_\Omega |\mathrm{div}\,v|^2\,\mathrm{d}x \leq 2\int_\Omega |\mathrm{sym}\nabla v|^2\,\mathrm{d}x.$$

For the optimality of Korn's constant $C=2$, it suffices to take any divergence-free, compactly supported v, for which the above formula yields equality in (3.8). ■

The same arguments as in the proof of Theorem 3.6 work also under tangential boundary conditions when Ω is a polyhedron:

Example 3.7. Let Ω be a (bounded) polyhedron in $\mathbb{R}^N$. We will show that (3.8) holds for $v\in H^1(\Omega,\mathbb{R}^N)$ satisfying $\langle v,\mathbf{n}\rangle=0$ on $\partial\Omega$. Indeed, (3.2) results in:

$$2\int_\Omega |\mathrm{sym}\nabla v|^2\,\mathrm{d}x = \int_\Omega |\nabla v|^2+|\mathrm{div}\,v|^2\,\mathrm{d}x + \int_{\partial\Omega} \big\langle \big((\nabla v)^T-(\mathrm{div}\,v)\mathrm{Id}_N\big)\mathbf{n}, v\big\rangle\,\mathrm{d}\sigma(x),$$

where we applied integration by parts. The boundary integral equates to:

$$\int_{\partial\Omega} \big\langle(\nabla v)v,\mathbf{n}\big\rangle - (\mathrm{div}\,v)\langle v,\mathbf{n}\rangle\,\mathrm{d}\sigma(x) = \int_{\partial\Omega} \partial_v\langle v,\mathbf{n}\rangle - (\mathrm{div}\,v)\langle v,\mathbf{n}\rangle\,\mathrm{d}\sigma(x) = 0.$$

The last equality above follows as $\mathbf{n}$ is locally constant and v is tangent on $\partial\Omega$. ■

To go beyond the particular geometry in Example 3.7 and also to deal with the mixed boundary conditions, let us denote the following space of linear maps with skew-symmetric gradients on Ω:

$$\mathscr{I}(\Omega) = \big\{Ax+b;\ A \in \mathrm{so}(N),\ b \in \mathbb{R}^N\big\}. \tag{3.9}$$

In section 3.7 we will extend the notion of $\mathscr{I}(\Omega)$ to encompass the space of so-called *Killing vector fields* $\mathscr{I}(S)$, the concept that is central to Korn's inequality on surfaces S and thin shells having S as their midsurface. We anticipate that Killing fields are precisely the generators of one-parameter paths of isometries, which is why they appear in Korn's inequality seen as a linear counterpart of the nonlinear geometric rigidity estimate (4.2) in chapter 4.

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected, Lipschitz domain. Given two open, disjoint (possibly empty) subsets $\Gamma_0, \Gamma_1 \subset \partial\Omega$, denote:*

$$\mathscr{V}_{\Gamma_0,\Gamma_1}(\Omega) = \big\{v \in H^1(\Omega,\mathbb{R}^N);\ \ v=0 \ \text{ on } \Gamma_0,\ \ \langle v, \mathbf{n}\rangle = 0 \ \text{ on } \Gamma_1\big\},$$
$$\mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega) = \mathscr{I}(\Omega) \cap \mathscr{V}_{\Gamma_0,\Gamma_1}(\Omega).$$

Then, for every $v \in \mathscr{V}_{\Gamma_0,\Gamma_1}(\Omega)$ there exists $w \in \mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega)$ with:

$$\int_\Omega |\nabla v - \nabla w|^2\,\mathrm{d}x \le C \int_\Omega |\mathrm{sym}\,\nabla v|^2\,\mathrm{d}x, \tag{3.10}$$

where the constant C above depends only on Ω, Γ_0 and Γ_1.

Proof. We first claim that for all vector fields $v \in \mathscr{V}_{\Gamma_0,\Gamma_1}(\Omega)$ which are $L^2(\Omega)$-orthogonal to the subspace $\mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega)$, the following holds:

$$\int_\Omega |\nabla v|^2\,\mathrm{d}x \le C \int_\Omega |\mathrm{sym}\nabla v|^2\,\mathrm{d}x, \tag{3.11}$$

with a universal constant C as in the statement of the result. To show (3.11), we argue by contradiction as in the proof of Theorem 3.1. Assume existence of a sequence $\{v_n \in \mathscr{V}_{\Gamma_0,\Gamma_1}(\Omega)\}_{n=1}^\infty$ satisfying the same orthogonality condition and such that:

$$\int_\Omega |\nabla v_n|^2\,\mathrm{d}x = 1, \qquad \int_\Omega |\mathrm{sym}\nabla v_n|^2\,\mathrm{d}x \to 0 \quad \text{ as } n \to \infty. \tag{3.12}$$

Passing to a subsequence (that we do not relabel) and modifying each v_n by a constant if necessary, we obtain that v_n converges to a limit field v, weakly in $H^1(\Omega,\mathbb{R}^N)$. Consequently, $\mathrm{sym}\nabla v = 0$ and so $v \in \mathscr{I}(\Omega)$ in virtue of Lemma 3.2. Since the weak convergence implies the strong convergence of traces in $L^2(\partial\Omega,\mathbb{R}^N)$, there also holds $v \in \mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega)$. Finally, since passing to the limit yields: $\int_\Omega \langle v, w\rangle\,\mathrm{d}x = 0$ for all $w \in \mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega)$, one sees that $v = 0$. Hence:

$$\int_\Omega |v_n|^2\,\mathrm{d}x \to 0 \quad \text{ as } n \to \infty.$$

Together with the second assertion in (3.12), the estimate in Theorem 3.5 implies $\int_\Omega |\nabla v_n|\,\mathrm{d}x \to 0$. This contradicts the first assertion in (3.12) and proves (3.11).

Now, given any $v \in \mathscr{V}_{\Gamma_0,\Gamma_1}(\Omega)$, let w be its orthogonal projection on the finite-dimensional subspace $\mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega) \subset L^2(\Omega, R^N)$. Condition (3.11) implies then:

$$\int_\Omega |\nabla v - \nabla w|^2\,\mathrm{d}x = \int_\Omega |\nabla(v-w)|^2\,\mathrm{d}x \leq C\int_\Omega |\mathrm{sym}\nabla v|^2\,\mathrm{d}x.$$

The proof is done. ■

Remark 3.9. With the same analysis as in Remark 3.3, one can identify the unique optimal vector field $w \in \mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega)$ in (3.10). Given $v \in \mathscr{V}_{\Gamma_0,\Gamma_1}(\Omega)$, let:

$$A \doteq \mathbb{P}_{\{\nabla w;\ w\in\mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega)\}} \fint_\Omega \nabla v\,\mathrm{d}x,$$

where $\mathbb{P}$ denotes the orthogonal projection onto the indicated linear subspace of $\mathbb{R}^{N\times N}$. Then there exists $b \in \mathbb{R}^N$ so that $w(x) = Ax + b$ belongs to $\mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega)$ and hence is the desired vector field.

Theorem 3.8 implies Theorem 3.6 (with some uniform constant C rather than the optimal constant 2) in view of the observation below. In fact, (3.11) holds with any nonempty Γ_0, because:

Corollary 3.10. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected, Lipschitz domain. In the context of Theorem 3.8 we have $\mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega) = \{0\}$, in any of the two cases:*

(i) $\Gamma_0 \neq \emptyset$,
(ii) $\Gamma_0 = \emptyset$, $\Gamma_1 = \partial\Omega$ and Ω has no rotational symmetry.

Then, for every $v \in \mathscr{V}_{\Gamma_0,\Gamma_1}(\Omega)$ and with C that depends only on $\Omega, \Gamma_0, \Gamma_1$, there holds:

$$\int_\Omega |\nabla v|^2\,\mathrm{d}x \leq C\int_\Omega |\mathrm{sym}\nabla v|^2\,\mathrm{d}x.$$

Proof. **1.** To prove (i), let $x_0 \in \Gamma_0$ be such that the normal vector $\mathbf{n}(x_0)$ is well defined. Consider the linear subspace:

$$M = \mathrm{span}\{x - x_0;\ x \in \Gamma_0\} \subset \mathbb{R}^N.$$

Then, M contains the (well defined) tangent space to $\partial\Omega$ at x_0, because every $y \in T_{x_0}\partial\Omega$ is generated by some sequence $\{\frac{x_n - x_0}{|x_n - x_0|} \in M\}_{n=1}^\infty$ in the limit of $\Gamma_0 \ni x_n \to x_0$.

Let now $w = Ax + b \in \mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega)$. It is easy to note that $M \subset \ker A$, since:

$$A(x - x_0) = w(x) - w(x_0) = 0 \quad \text{for all } x \in \Gamma_0.$$

Thus $\dim\ker A \geq \dim M \geq N-1$. We further claim that $A = 0$. Indeed, if $y \in \mathbb{R}^N$ was a unit vector in $(\ker A)^\perp$, then not only $\langle Ay, y\rangle = 0$ by skew-symmetry, but also:

$$\langle Ay, z\rangle = -\langle y, Az\rangle = 0 \qquad \text{for all } \; z \in \ker A. \tag{3.13}$$

We see that Ay is orthogonal to $\mathbb{R}^N$ and hence $y \in \ker A$, which is a contradiction. Since $A = 0$, it easily follows that $b = 0$ as well, proving the claim $w = 0$.

2. To show (ii), we take $w = Ax + b \in \mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega)$ and prove below that the flow generated by the tangent vector field $w_{|\partial\Omega}$ must be a rotation. Then, the lack of nontrivial rotations of Ω will directly yield $w = 0$.

Since $A \in \mathrm{so}(n)$ it follows that $\ker A \oplus \mathrm{im} A$ is an orthogonal decomposition of $\mathbb{R}^N$, in virtue of (3.13). Accordingly, we write $b = b^{ker} + Ab_0$ with $b^{ker} \in \ker A$ and some $b_0 \in \mathbb{R}^N$. Consider the translated domain $\Omega_0 = \Omega + b_0$. Since:

$$w(x) = A(x + b_0) + b^{ker} \qquad \text{for all } \; x \in \Omega,$$

it follows that $y \mapsto Ay + b^{ker}$ is a tangent vector field on $\partial\Omega_0$. The flow α which this field generates, is given by:

$$\alpha'(t) = A\alpha(t) + b^{ker}, \qquad \alpha(0) \in \partial\Omega_0,$$

where we may also write $\alpha(t) = \beta(t) + \delta(t)$, with:

$$\begin{cases} \beta'(t) = A\beta(t), & \beta(0) \in \mathrm{im} A \\ \delta'(t) = b^{ker}, & \delta(0) \in \ker A, \quad \beta(0) + \delta(0) = \alpha(0). \end{cases}$$

Notice that $\beta(t)$ remains bounded, because:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\beta(t)|^2 = 2\langle \beta(t), A\beta(t)\rangle = 0,$$

while $\delta(t) = \delta(0) + tb^{ker}$ is unbounded for $b^{ker} \neq 0$. Since $\alpha(t) \in \partial\Omega_0$ for all $t \geq 0$, there must be $b^{ker} = 0$. Hence the flow $\alpha(t) = e^{tA}\beta(0) + \delta(0)$ indeed is a rotation generated by $A \in \mathrm{so}(N)$ on $\partial\Omega_0$. The proof is done. ■

Remark 3.11. (i) From the proof of Corollary 3.10 (ii) it follows that each $w \in \mathscr{I}_{\emptyset,\partial\Omega}(\Omega)$ has the form $w(x) = A(x + b_0)$ with $A \in \mathrm{so}(N)$, $b_0 \in \mathbb{R}^N$. We thus obtain the following characterisation when $\Omega \subset \mathbb{R}^3$:

$$\mathscr{I}_{\emptyset,\partial\Omega}(\Omega) = \begin{cases} \{0\} & \text{if } \; \Omega \text{ has no rotational symmetry} \\ \text{a 1-parameter family} & \text{if } \; \Omega \text{ has one rotational symmetry} \\ \text{a 3-parameter family} & \text{if } \; \Omega = B_r(x). \end{cases}$$

(ii) Let $\Omega = B_1(0) \subset \mathbb{R}^3$. Each $A \in \mathrm{so}(3)$ can be written as $Ax = a \times x$ for some $a \in \mathbb{R}^3$, and hence we obtain: $\mathscr{I}_{\emptyset,\partial\Omega}(B_1(0)) = \{a \times x;\ a \in \mathbb{R}^3\}$. The perpendicularity condition in the proof of Theorem 3.8 then reads:

$$0 = \int_{B_1(0)} \langle a \times x, v(x)\rangle \,\mathrm{d}x = \Big\langle a, \int_{B_1(0)} x \times v(x)\,\mathrm{d}x \Big\rangle \quad \text{ for all } \; a \in \mathbb{R}^3.$$

Consequently, for the class of vector fields v in:

$$\big\{v \in H^1(B_1(0),\mathbb{R}^3);\ \langle v, \mathbf{n}\rangle = 0 \text{ on } \partial B_1(0) \text{ and } \int_{B_1(0)} x \times v(x)\,\mathrm{d}x = 0\big\},$$

there holds the uniform homogeneous estimate (3.11). ■

3.3 Proof of Korn's inequality: preliminary estimates

In this and the next sections, we prove the First Korn's inequality (3.1). We start with two preliminary results:

Lemma 3.12. *Let Ω be an open, bounded, connected, Lipschitz domain, and let $g \in L^2(\Omega,\mathbb{R})$ satisfy $\Delta g = 0$ in Ω. Then:*

$$\int_\Omega |\nabla g|^2 \operatorname{dist}^2(x,\partial\Omega)\,\mathrm{d}x \leq 4\int_\Omega |g|^2\,\mathrm{d}x.$$

Proof. For each small $\varepsilon > 0$, consider the open, Lipschitz domain:

$$\Omega_\varepsilon = \{x \in \Omega;\ \operatorname{dist}(x,\partial\Omega) > \varepsilon\}.$$

Integrating by parts, it follows that:

$$\begin{aligned}\int_{\Omega_\varepsilon} \big(\operatorname{dist}(x,\partial\Omega)-\varepsilon\big)^2|\nabla g|^2\,\mathrm{d}x &= \int_{\Omega_\varepsilon} \operatorname{div}\Big(\big(\operatorname{dist}(x,\partial\Omega)-\varepsilon\big)^2 g\nabla g\Big)\,\mathrm{d}x \\ &\qquad - 2\int_{\Omega_\varepsilon} \big(\operatorname{dist}(x,\partial\Omega)-\varepsilon\big)g\big\langle \nabla_x \operatorname{dist}(x,\partial\Omega),\nabla g\big\rangle\,\mathrm{d}x \\ &= -2\int_{\Omega_\varepsilon} \big(\operatorname{dist}(x,\partial\Omega)-\varepsilon\big)g\big\langle \nabla_x \operatorname{dist}(x,\partial\Omega),\nabla g\big\rangle\,\mathrm{d}x \\ &\leq 2\int_{\Omega_\varepsilon}|g|^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_\varepsilon}\big(\operatorname{dist}(x,\partial\Omega)-\varepsilon\big)^2\big\langle \nabla_x \operatorname{dist}(x,\partial\Omega),\nabla g\big\rangle^2\,\mathrm{d}x,\end{aligned}$$

where the last inequality follows by applying $-2ab \leq a^2+b^2$ with $a = \sqrt{2}g(x)$ and $b = \frac{1}{\sqrt{2}}\big(\operatorname{dist}(x,\partial\Omega)-\varepsilon\big)\big\langle \nabla_x \operatorname{dist}(x,\partial\Omega),\nabla g(x)\big\rangle$, at each $x \in \Omega_\varepsilon$ under the integration sign. Since the function $x \mapsto \operatorname{dist}(x,\partial\Omega)$ has Lipschitz constant 1, we get:

$$\Big|\big\langle \nabla_x \operatorname{dist}(x,\partial\Omega),\nabla g\big\rangle\Big| \leq \big|\nabla_x \operatorname{dist}(x,\partial\Omega)\big|\cdot\big|\nabla g(x)\big| \leq |\nabla g(x)|.$$

It now follows that the last term in the right hand side of the displayed formula can be bounded by the half of the exact term in the left hand side, leading to:

$$\int_{\Omega_\varepsilon} \big(\operatorname{dist}(x,\partial\Omega)-\varepsilon\big)^2|\nabla g|^2\,\mathrm{d}x \leq 4\int_{\Omega_\varepsilon}|g|^2\,\mathrm{d}x.$$

Passing with $\varepsilon \to 0$ while applying Fatou's lemma in the left hand side and the monotone convergence theorem in the right hand side, yields the result. ■

Lemma 3.13. *Let $g \in H^1_{loc}((0,T),\mathbb{R})$ satisfy $\lim_{t\to 0} g(t) = 0$. Then there holds:*

$$\int_0^T |g|^2 \,\mathrm{d}t \leq 4\int_0^T |g'|^2 |T-t|^2 \,\mathrm{d}t.$$

Proof. Consider the function $h \in H^1_{loc}((0,\infty),\mathbb{R})$, given by: $h(t) = g(T-t)\mathbb{1}_{(0,T)}(t)$. This function is absolutely continuous on $(0,\infty)$ and identically equal to zero beyond T. For each $t \in (0,T)$ we have:

$$\begin{aligned} h(t)^2 &= \Big(\int_t^\infty h'(s)\,\mathrm{d}s\Big)^2 = 4t^{-1}\Big(\frac{1}{\int_t^\infty s^{-3/2}\,\mathrm{d}s}\int_t^\infty \big(h'(s)s^{3/2}\big)s^{-3/2}\,\mathrm{d}s\Big)^2 \\ &\leq 4t^{-1}\frac{1}{\int_t^\infty s^{-3/2}\,\mathrm{d}s}\int_t^\infty \big(h'(s)s^{3/2}\big)^2 s^{-3/2}\,\mathrm{d}s \leq 2t^{-1/2}\int_t^\infty h'(s)^2 s^{3/2}\,\mathrm{d}s, \end{aligned}$$

where the inequality above follows by applying Jenssen's inequality to the convex function $x \mapsto x^2$, with the probability measure obtained as the normalisation of the measure $s^{-3/2}\,\mathrm{d}s$ on the interval (t,∞). Integrating on $(0,\infty)$ yields:

$$\begin{aligned} \int_0^T h(t)^2\,\mathrm{d}t &= \int_0^\infty h(t)^2\,\mathrm{d}t \leq 2\int_0^\infty t^{-1/2}\int_t^\infty h'(s)^2 s^{3/2}\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^\infty \Big(\int_0^s t^{-1/2}\,\mathrm{d}t\Big) h'(s)^2 s^{3/2}\,\mathrm{d}s = 4\int_0^\infty h'(s)^2 s^2\,\mathrm{d}s = 4\int_0^T |h'|^2 s^2\,\mathrm{d}s, \end{aligned}$$

in virtue of Fubini's theorem and changing the integration order. Applying the reflection of variables $t \mapsto (T-t)$ results in the claimed bound for g. ■

The main arguments in the proof of Theorem 3.5 will be given in *star-shaped* domains, which by Lemma 3.15 can be seen as building blocks of Lipschitz domains.

Definition 3.14. *We say that an open domain $\Omega \subset \mathbb{R}^N$ is star-shaped with respect to its interior ball $B_r(z) \Subset \Omega$, when:*

$$\{tx + (1-t)\bar{x};\ t \in [0,1]\} \subset \Omega \quad \textit{for all } x \in \Omega,\ \bar{x} \in B_r(z). \tag{3.14}$$

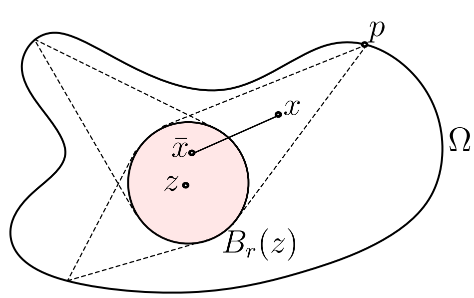

Fig. 3.1 A domain that is star-shaped with respect to an internal ball, see Definition 3.14.

Lemma 3.15. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain.*

(i) if Ω is star-shaped with respect to an internal ball, then it is Lipschitz,
(ii) if Ω is Lipschitz then it can be written as a finite union $\Omega = \{\Omega_i\}_{i=1}^n$ of open domains Ω_i that are each star-shaped with respect to some internal ball.

Proof. **1.** To prove (i), we may without loss of generality assume that $\Omega \subset B_R(0)$ and that it is star-shaped with respect to $B_r(0) \Subset \Omega$. For every $z \in \partial B_r(0)$, define:

$$f(z) = \sup\big\{t > 0;\ (t+r)\frac{z}{r} \in \Omega\big\}.$$

By (3.14), the graph of f over $\partial B_r(0)$ coincides with $\partial\Omega$. We will show that the function f is Lipschitz, which by a simple (local) change of variable will imply the Lipschitz condition on Ω.

Fix $z \in \partial B_r(0)$ and observe that $f(z) \in [a_{min}r, a_{max}r]$ with some uniform constant $a_{min} > 0$ and $a_{max} = \frac{R}{r} - 1$. Further, define the angle $\gamma(z)$ so that: $\cos\gamma(z) = \frac{r}{f(z)+r}$. It is clear that $\gamma(z) \geq \gamma_0$, where $\cos\gamma_0 = \frac{1}{a_{min}+1}$. Take now any pair $z, \bar{z} \in \partial B_r(0)$ satisfying: $\angle(z,\bar{z}) \leq \frac{1}{2}\gamma_0$. In virtue of the mean value theorem and (3.14) we get:

$$f(\bar{z}) - f(z) > \frac{r}{\cos(\gamma(z) - \gamma} - \frac{r}{\cos\gamma(z)} \geq -r|\gamma|\frac{1}{\cos^2\theta},$$

for some angle $\theta \in \big(\gamma(z) - \gamma, \gamma(z)\big)$ for which: $\cos^2\theta > \cos^2\gamma(z) \geq \frac{1}{(\alpha_{max}+1)^2}$. In conclusion and by a symmetric argument, there follows the final bound:

$$|f(\bar{z}) - f(z)| \leq \frac{r}{(\alpha_{max}+1)^2}\gamma.$$

2. To show (ii), it suffices to construct the claimed finite decomposition only covering the boundary layer $\bigcup_{z\in\partial\Omega} B_\varepsilon(z) \cap \bar{\Omega}$ for some $\varepsilon \ll 1$. The remaining interior set can then be covered by balls, which are clearly star-shaped domains.

Let $f : B_{2r}^{N-1}(0) \to \mathbb{R}$ be a Lipschitz function, whose graph coincides with a portion of $\partial\Omega$ and whose subgraph is contained in Ω. We denote the Lipschitz constant of f by L, and take its domain to be a ball $B_{2r}^{N-1}(0) \subset \mathbb{R}^{N-1}$ for some $r > 0$. Without loss of generality we may assume that $\min f > 4r$ and $L > 2$.

Observe that for each $z \in B_{2r}^{N-1}(0)$, the "upside down" cone with its tip at the point $(z, f(z))$, its aperture angle equal γ such that $\tan\gamma = \frac{1}{L}$, and its height $f(z)$, is contained in Ω. Consequently, the open set:

$$\Omega_i = B_r(0) \cup \big\{(z,t) \in \mathbb{R}^N;\ z \in B_r^{N-1}(0),\ 0 < t < f(z)\big\}$$

satisfies the condition (3.14) with respect to $B_r(0)$, which implies that Ω_i is star-shaped with respect to its internal ball $B_{r/2}(0) \Subset \Omega_i$. By construction and compactness of $\bar{\Omega}$, finitely many of the domains of the type Ω_i cover the sufficiently narrow boundary layer of Ω, as claimed. ■

From Lemma 3.13 there follow the key estimates on star-shaped domains:

Theorem 3.16. *Let $\Omega \subset B_R(0) \subset \mathbb{R}^N$ be an open domain that is star-shaped with respect to $B_r(0)$, for some $0 < r < R$. Then, for every $g \in H^1(\Omega, \mathbb{R})$ there holds:*

(i) $\displaystyle\int_\Omega |g|^2 \, \mathrm{d}x \le C\Big(\int_{B_r(0)} |g|^2 \, \mathrm{d}x + \int_\Omega |\nabla g|^2 \mathrm{dist}^2(x, \partial\Omega) \, \mathrm{d}x\Big)$,

(ii) there exists $a \in \mathbb{R}$ such that: $\displaystyle\int_\Omega |g-a|^2 \, \mathrm{d}x \le C \int_\Omega |\nabla g|^2 \mathrm{dist}^2(x, \partial\Omega) \, \mathrm{d}x$,

with constants C depending only on N and R/r.

Proof. **1.** By a simple scaling argument, it suffices to assume that $R = 1$. To prove (i), we first derive an estimate on $\int_{\Omega \setminus B_r(0)} |g|^2 \, \mathrm{d}x$. Choose a smooth cut-off function $\theta : (0, \infty) \to [0, 1]$ with the following properties:

$$\theta_{|(0,r/2)} \equiv 0, \quad \theta_{|(r,\infty)} \equiv 1, \quad \|\theta'\|_{L^\infty} \le \frac{4}{r}.$$

For every $p \in \partial\Omega$ we apply Lemma 3.13 to the function θg on the segment $[0, p]$:

$$\begin{aligned}\int_0^{|p|} |\theta g|^2 \, \mathrm{d}t &\le 4 \int_0^{|p|} \big|(\theta g)'\big|^2 \big||p| - t\big|^2 \, \mathrm{d}t \\ &\le \frac{8}{r^2} \int_0^{|p|} \big(|\theta' g|^2 + |\theta|^2 |\nabla g|^2\big) \, \mathrm{dist}^2\big(t\frac{p}{|p|}, \partial\Omega\big) \, \mathrm{d}t,\end{aligned}$$

where we used the assumption of Ω being start-shaped with respect to $B_r(0)$ in order to conclude that:

$$\frac{\big||p| - |x|\big|}{\mathrm{dist}(x, \partial\Omega)} \le \frac{|p|}{r} \le \frac{1}{r} \qquad \text{for all } \ x \in [0, p].$$

Consequently, it follows that:

$$\int_r^{|p|} |g|^2 \, \mathrm{d}t \le C\Big(\int_{r/2}^r |g|^2 \, \mathrm{d}t + \int_{r/2}^{|p|} |\nabla g|^2 \, \mathrm{dist}^2\big(t\frac{p}{|p|}, \partial\Omega\big) \, \mathrm{d}t\Big),$$

and further, with the constant C that again depends only on r:

$$\int_r^{|p|} t^{N-1} |g|^2 \, \mathrm{d}t \le C\Big(\int_{r/2}^r t^{N-1} |g|^2 \, \mathrm{d}t + \int_{r/2}^{|p|} t^{N-1} |\nabla g|^2 \, \mathrm{dist}^2\big(t\frac{p}{|p|}, \partial\Omega\big) \, \mathrm{d}t\Big).$$

Integrating in spherical coordinates, we finally arrive at (i):

$$\int_{\Omega \setminus B_r(0)} |g|^2 \, \mathrm{d}x \le C\Big(\int_{B_r(0)} |g|^2 \, \mathrm{d}x + \int_\Omega |\nabla g|^2 \, \mathrm{dist}^2\big(x, \partial\Omega\big) \, \mathrm{d}x\Big).$$

2. To prove (ii), we invoke the Poincaré-Wirtinger inequality to get:

$$
\begin{aligned}
\int_{B_{r/2}(0)} \Big|g - \fint_{B_{r/2}(0)} g\Big|^2 \,\mathrm{d}x &\leq Cr^2 \int_{B_{r/2}(0)} |\nabla g|^2 \,\mathrm{d}x \\
&\leq C \int_{B_{r/2}(0)} |\nabla g|^2 \mathrm{dist}^2(x, \partial\Omega) \,\mathrm{d}x,
\end{aligned}
\tag{3.15}
$$

where C depends only on N. Apply now the statement in (i) with the interior ball $B_{r/2}(0)$ replacing $B_r(0)$ and to the function $g - \fint_{B_{r/2}(0)} g$ on Ω. It follows that:

$$
\begin{aligned}
\int_\Omega \Big|g - \fint_{B_{r/2}(0)} g\Big|^2 \,\mathrm{d}x &\leq C\Big(\int_{B_{r/2}(0)} \Big|g - \fint_{B_{r/2}(0)} g\Big|^2 \,\mathrm{d}x + \int_\Omega |\nabla g|^2 \mathrm{dist}^2(x, \partial\Omega) \,\mathrm{d}x\Big) \\
&\leq C \int_\Omega |\nabla \phi|^2 \mathrm{dist}^2(x, \partial\Omega) \,\mathrm{d}x,
\end{aligned}
$$

where in the last inequality we used (3.15). Taking $a = \fint_{B_{r/2}(0)} g \,\mathrm{d}x$ achieves (ii). ■

We now have all the ingredients towards proving Korn's inequality in Theorem 3.5. The proof will be given in the next section, while below we digress and deduce an important corollary from Theorem 3.16:

Theorem 3.17. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected, Lipschitz domain. Then, for every $g \in H^1(\Omega, \mathbb{R})$ there holds:*

(i) $\int_\Omega |g|^2 \,\mathrm{d}x \leq C \int_\Omega \big(|g|^2 + |\nabla g|^2\big) \mathrm{dist}^2(x, \partial\Omega) \,\mathrm{d}x,$

(ii) there exists $a \in \mathbb{R}$ such that: $\int_\Omega |g - a|^2 \,\mathrm{d}x \leq C \int_\Omega |\nabla g|^2 \mathrm{dist}^2(x, \partial\Omega) \,\mathrm{d}x,$

with constants C depending only on Ω. Moreover, these constants can be chosen uniformly for a family of domains Ω which are bilipschitz equivalent with controlled Lipschitz constants.

Proof. **1.** The uniform estimate in (i) follows directly from Theorem 3.16 in view of the finite decomposition statement in Lemma 3.15 (ii). To show the weighted Poincaré-type inequality (ii), for each $\varepsilon > 0$ we consider the domain

$$\Omega_\varepsilon = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > \varepsilon\},$$

and apply (i) to the function $g - \fint_{\Omega_\varepsilon} g$:

$$
\begin{aligned}
&\int_\Omega \Big|g - \fint_{\Omega_\varepsilon} g\Big|^2 \,\mathrm{d}x \\
&\leq C\Big(\varepsilon^2 \int_{\Omega\setminus\Omega_\varepsilon} \Big|g - \fint_{\Omega_\varepsilon} g\Big|^2 \,\mathrm{d}x + \int_{\Omega_\varepsilon} \Big|g - \fint_{\Omega_\varepsilon} g\Big|^2 \,\mathrm{d}x + \int_\Omega |\nabla g|^2 \mathrm{dist}^2(x, \partial\Omega) \,\mathrm{d}x\Big).
\end{aligned}
$$

The constant C above depends only on Ω, so for $\varepsilon \ll 1$ sufficiently small to have $C\varepsilon^2 \leq \frac{1}{2}$, the first term in the right hand side can be absorbed in the left hand side:

$$\int_\Omega \big|g - \fint_{\Omega_\varepsilon} g\big|^2 \,\mathrm{d}x \leq 2C\Big(\int_{\Omega_\varepsilon} \big|g - \fint_{\Omega_\varepsilon} g\big|^2 \,\mathrm{d}x + \int_\Omega |\nabla g|^2 \mathrm{dist}^2(x,\partial\Omega)\,\mathrm{d}x\Big).$$

We now apply the Poincarè-Wirtinger inequality on Ω_ε to get:

$$\int_{\Omega_\varepsilon} \big|g - \fint_{\Omega_\varepsilon} g\big|^2 \,\mathrm{d}x \leq C_\varepsilon \int_{\Omega_\varepsilon} |\nabla g|^2 \,\mathrm{d}x \leq \frac{C_\varepsilon}{\varepsilon^2} \int_{\Omega_\varepsilon} |\nabla g|^2 \mathrm{dist}^2(x,\partial\Omega)\,\mathrm{d}x,$$

where by writing C_ε we indicate the dependence of the constant on ε. Combining the two displayed inequalities, we arrive at (ii).

2. To prove the uniformity of constants, assume that $\varphi : \Omega \to \Omega'$ is a bi-Lipschitz map with Lipschitz constants of φ and φ^{-1} both bounded by some $L > 0$. Given $h \in H^1(\Omega',\mathbb{R})$, define $g \doteq h \circ \varphi \in H^1(\Omega,\mathbb{R})$. Then:

$$\begin{aligned}
&\int_{\Omega'} h^2 \,\mathrm{d}x \leq \|\det\nabla\varphi\|_{L^\infty(\Omega)} \int_\Omega g^2 \,\mathrm{d}x \leq CL^N \int_\Omega g^2 \,\mathrm{d}x,\\
&\int_\Omega \big(g^2 + |\nabla g|^2\big)\,\mathrm{dist}^2(x,\partial\Omega)\,\mathrm{d}x\\
&\qquad\qquad \leq \|\det\nabla(\varphi^{-1})\|_{L^\infty(\Omega')} \int_{\Omega'} \big(h^2 + L^2|\nabla h|^2\big)\,\mathrm{dist}^2(\varphi^{-1}(x),\varphi^{-1}(\partial\Omega'))\,\mathrm{d}x\\
&\qquad\qquad \leq CL^{N+1}(1+L^2) \int_{\Omega'} \big(h^2 + |\nabla h|^2\big)\,\mathrm{dist}^2(x,\partial\Omega')\,\mathrm{d}x,
\end{aligned}$$

where C above depends only on the dimension N. This implies the estimate in (i) on Ω' with the constant that only depends on the constant in (i) on Ω and on L. The argument for (ii) is the same. This ends the proof of the theorem. ■

3.4 Proof of Korn's inequality

In this section, we complete the proof of Theorem 3.5. In the first step, we decompose the given vector field v as the sum of the harmonic part w and the correction u that equals zero on $\partial\Omega$. A simple integration by part yields a desired bound for u. To deal with w, one applies Lemma 3.12 to $g = \mathrm{sym}\nabla w$ and uses the fact that $\nabla^2 w \simeq \nabla(\mathrm{sym}\nabla w)$, in combination with Theorem 3.16 (i). The argument is then lifted from star-shaped domains to arbitrary Lipschitz domains, by Lemma 3.15 (ii).

Theorem 3.18. *Let $\Omega \subset B_R(0) \subset \mathbb{R}^N$ be open and star-shaped with respect to $B_r(0)$, for some $0 < r < R$. Then, there holds:*

$$\int_\Omega |\nabla v|^2 \, \mathrm{d}x \le C\Big(\int_{B_r(0)} |\nabla v|^2 \, \mathrm{d}x + \int_\Omega |\mathrm{sym}\,\nabla v|^2 \, \mathrm{d}x\Big), \tag{3.16}$$

for every $v \in H^1(\Omega,\mathbb{R}^N)$, where C depends only on N and R/r.

Proof. **1.** We decompose v as the sum: $v = u + w$, where $u \in H^1(\Omega,\mathbb{R}^N)$ satisfies:

$$\Delta u = \Delta v \ \text{ in } \ \Omega, \qquad u = 0 \ \text{ on } \ \partial\Omega.$$

Using identity (3.2), it follows that:

$$\Delta u = 2\,\mathrm{div}\Big(\mathrm{sym}\nabla v - \frac{1}{2}\mathrm{tr}\big(\mathrm{sym}\nabla v\big)\mathrm{Id}_N\Big).$$

Integration by parts against u then leads to:

$$\begin{aligned}\int_\Omega |\nabla u|^2 \, \mathrm{d}x &= 2\int_\Omega \Big\langle \nabla u : \mathrm{sym}\nabla v - \frac{1}{2}\mathrm{tr}(\mathrm{sym}\nabla v)\mathrm{Id}_N \Big\rangle \, \mathrm{d}x \\ &\le 2N\,\|\nabla u\|_{L^2(\Omega)}\|\mathrm{sym}\nabla v\|_{L^2(\Omega)},\end{aligned}$$

and further:

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x \le 4N^2 \int_\Omega |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x. \tag{3.17}$$

2. Consider now the harmonic corrector $w = v - u \in H^1(\Omega,\mathbb{R}^N)$, which satisfies:

$$\Delta w = 0 \ \text{ in } \ \Omega, \qquad w = v \ \text{ on } \ \partial\Omega.$$

The application of Lemma 3.12 to each of N^2 components of the harmonic matrix field $\mathrm{sym}\nabla w \in L^2(\Omega,\mathbb{R}^{N\times N})$ yields, upon noting Lemma 3.15 (i):

$$\int_\Omega \big|\nabla(\mathrm{sym}\nabla w)\big|^2 \mathrm{dist}^2(x,\partial\Omega) \, \mathrm{d}x \le C\int_\Omega |\mathrm{sym}\nabla w|^2 \, \mathrm{d}x.$$

We now observe the useful fact that the second partial derivatives are always a linear combination of the first derivatives of the components of symmetric gradient:

$$\begin{aligned}[\nabla^2 w^i]_{jk} = \partial_j[\mathrm{sym}\nabla w]_{ik} + \partial_k[\mathrm{sym}\nabla w]_{ij} - \partial_i[\mathrm{sym}\nabla w]_{jk}& \\ \text{for all } \ i,j,k = 1\ldots N.&\end{aligned} \tag{3.18}$$

The above identity can be checked by a direct calculation. Consequently, the previous bound becomes:

$$\int_\Omega \big|\nabla^2 w\big|^2 \mathrm{dist}^2(x,\partial\Omega) \, \mathrm{d}x \le C\int_\Omega |\mathrm{sym}\nabla w|^2 \, \mathrm{d}x, \tag{3.19}$$

again with a constant C that depends only on N and R/r.

3. Finally, we apply Theorem 3.16 (i) to each component $g = \partial_i w^j$ of ∇w to get:

$$\int_\Omega |\nabla w|^2 \,\mathrm{d}x \le C\Big(\int_{B_r(0)} |\nabla w|^2 \,\mathrm{d}x + \int_\Omega |\nabla^2 w|^2 \operatorname{dist}^2\big(x, \partial\Omega\big) \,\mathrm{d}x \Big). \tag{3.20}$$

Combining (3.17), (3.19) and (3.20) results in:

$$\begin{aligned}
\int_\Omega |\nabla v|^2 \,\mathrm{d}x &\le C\Big(\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega |\nabla w|^2 \,\mathrm{d}x \Big) \\
&\le C\Big(\int_\Omega |\mathrm{sym}\nabla v|^2 \,\mathrm{d}x + \int_{B_r(0)} |\nabla w|^2 \,\mathrm{d}x + \int_\Omega |\nabla^2 w|^2 \operatorname{dist}^2\big(x, \partial\Omega\big) \,\mathrm{d}x \Big) \\
&\le C\Big(\int_\Omega |\mathrm{sym}\nabla v|^2 \,\mathrm{d}x + \int_{B_r(0)} |\nabla w|^2 \,\mathrm{d}x + \int_\Omega |\mathrm{sym}\nabla w|^2 \,\mathrm{d}x \Big) \\
&\le C\Big(\int_\Omega |\mathrm{sym}\nabla v|^2 \,\mathrm{d}x + \int_{B_r(0)} |\nabla v|^2 \,\mathrm{d}x \Big).
\end{aligned}$$

The proof is done. ■

We remark that the same splitting technique used above will also be present in the proof of Friesecke-James-Müller's inequality (4.2) in the next chapter, as well as in the dimension reduction analysis in presence of prestress, in Part III of this monograph. Two corollaries are now in order. In the first one, we already derive the estimate in Theorem 3.5 on star-shaped domains:

Corollary 3.19. [First Korn's inequality on star-shaped domains] *In the setting of Theorem 3.18, for every $v \in H^1(\Omega, \mathbb{R}^N)$ there holds:*

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}x \le C\Big(\int_\Omega |\mathrm{sym}\nabla v|^2 \,\mathrm{d}x + \int_\Omega |v|^2 \,\mathrm{d}x \Big), \tag{3.21}$$

where the constant C depends only on N, R/r and $\operatorname{dist}(B_r(0), \partial\Omega)$.

Proof. Let $\phi \in \mathscr{C}_0^\infty(\Omega, [0,1])$ be some smooth test function satisfying $\phi_{|B_r(0)} \equiv 1$. By Theorem 3.6 applied to $\phi v \in H_0^1(\Omega, \mathbb{R}^N)$, we obtain:

$$\begin{aligned}
\int_{B_r(0)} |\nabla v|^2 \,\mathrm{d}x \le \int_\Omega |\nabla(\phi v)|^2 \,\mathrm{d}x &\le 2\int_\Omega |\mathrm{sym}\nabla(\phi v)|^2 \,\mathrm{d}x \\
&\le 4\Big(\int_\Omega |\mathrm{sym}\nabla v|^2 \,\mathrm{d}x + \int_\Omega |\nabla\phi|^2 |v|^2 \,\mathrm{d}x \Big).
\end{aligned}$$

Since ϕ can be taken radially symmetric with $\|\nabla\phi\|_{L^\infty}$ depending only on the quantity $\operatorname{dist}(B_r(0), \partial\Omega)$, the estimate (3.16) yields the result. ■

Proof of Theorem 3.5

The same estimate (3.21) is evidently valid on any Ω that is a finite union of domains satisfying assumptions of Theorem 3.18. Hence, the proof of Theorem 3.5 is achieved in virtue of Lemma 3.15 (ii). ■

Recall that the argument by contradiction, as in the proof of Theorem 3.1, yields Korn's inequality in the form of the rigidity estimate on those domains on which First Korn's inequality has been established. Below we observe a direct proof of the same statement, which on star-shaped domains gives an information on the dependence of Korn's constant on Ω:

Corollary 3.20. [Korn's inequality on star-shaped domains]
In the setting of Theorem 3.18, for every $v \in H^1(\Omega, \mathbb{R}^N)$ there holds:

$$\int_\Omega |\nabla v - \mathrm{skew} ⨍_\Omega \nabla v|^2 \, \mathrm{d}x \leq C \int_\Omega |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x \tag{3.22}$$

where C depends only on N and R/r.

Proof. Let $v = u + w$ be the decomposition as in the proof of Theorem 3.18. We apply Theorem 3.16 (ii) to the components $g = \partial_i w^j$ of the harmonic matrix field ∇w, to get $B \in \mathbb{R}^{N\times N}$ so that, recalling (3.19):

$$\int_\Omega |\nabla w - B|^2 \, \mathrm{d}x \leq C \int_\Omega |\nabla^2 w|^2 \mathrm{dist}^2(x, \partial\Omega) \, \mathrm{d}x \leq C \int_\Omega |\mathrm{sym}\nabla w|^2 \, \mathrm{d}x. \tag{3.23}$$

Since for every $x \in \Omega$ there holds:

$$\begin{aligned} |B - \mathrm{skew}\, B| = \mathrm{dist}(B, \mathrm{so}(N)) &\leq |B - \nabla w(x)| + \mathrm{dist}(\nabla w(x), \mathrm{so}(N)) \\ &= |B - \nabla w(x)| + |\mathrm{sym}\nabla w(x)|, \end{aligned}$$

it follows that $\int_\Omega |B - \mathrm{skew}\, B|^2 \leq C \int_\Omega |\mathrm{sym}\nabla w|^2 \, \mathrm{d}x$, and consequently:

$$\int_\Omega |\nabla w - \mathrm{skew}\, B|^2 \, \mathrm{d}x \leq C \int_\Omega |\mathrm{sym}\nabla w|^2 \, \mathrm{d}x. \tag{3.24}$$

In conclusion, by (3.17) and (3.24) we get:

$$\|\nabla v - \mathrm{skew}\, B\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)} + \|\nabla w - B\|_{L^2(\Omega)} \leq C\|\mathrm{sym}\nabla v\|_{L^2(\Omega)}, \tag{3.25}$$

and invoking Remark 3.3 completes the proof. ■

3.5 Korn's constant under tangential boundary conditions

We have seen in Theorem 3.6 that under Dirichlet boundary conditions, Korn's constant is universal and equals 2 for all domains Ω. In this section, we deduce that under tangential boundary condition Korn's constant is always at least 2. As shown in the example below there is however no upper bound on it.

Example 3.21. Consider any smooth, closed, nonintersecting curve γ in $\mathbb{R}^2$, which is not rotationally symmetric. Denote by $\gamma \ni z \mapsto \mathbf{n}(z), \tau(z)$ the smooth unit normal and unit tangent vector fields on γ, and recall that $\partial_\tau \mathbf{n} = \kappa\tau$ and $\partial_\tau \tau = -\kappa \mathbf{n}$, where κ is the scalar curvature field on γ. For each $z \in \gamma$ and a small $|t| \ll 1$, let:

$$v(z + t\mathbf{n}(z)) = (1 + t\kappa(z))\tau(z).$$

Then $v_{|\bar{\Omega}^h} \in \mathscr{C}^\infty(\bar{\Omega}^h, \mathbb{R}^2)$ where for each small $h > 0$ we define a thin strip $\Omega^h \subset \mathbb{R}^2$ around γ, by setting:

$$\Omega^h = \big\{x = z + t\mathbf{n}(z);\ z \in \gamma,\ t \in \big(-\frac{h}{2}, \frac{h}{2}\big)\big\}.$$

It is clear that $\langle v, \mathbf{n}\rangle = 0$ on $\partial\Omega^h$, and also we calculate directly:

$$\begin{aligned} &\partial_{\mathbf{n}} v(z + t\mathbf{n}) = \kappa\tau(z), \\ &\partial_\tau v(z + t\mathbf{n}) = \frac{1}{1 + t\kappa}\Big((1 + t\kappa)\partial_\tau\tau(z) + t\partial_\tau\kappa(z)\tau(z)\Big) = -\kappa\mathbf{n}(z) + \frac{t\partial_\tau\kappa}{1 + t\kappa}\tau(z). \end{aligned}$$

Consequently: $\big\langle \partial_{\mathbf{n}} v, \mathbf{n}\big\rangle = 0$, $\big\langle \partial_\tau v, \mathbf{n}\big\rangle + \big\langle \partial_{\mathbf{n}} v, \tau\big\rangle = 0$ and $\big\langle \partial_\tau v, \tau\big\rangle = \frac{t\partial_\tau\kappa}{1+t\kappa}$, yielding:

$$\int_{\Omega^h} |\mathrm{sym}\nabla v|^2\, \mathrm{d}x \le Ch^3, \quad \int_{\Omega^h} |\nabla v|^2\, \mathrm{d}x \ge ch \qquad \text{as } h \to 0. \tag{3.26}$$

By Corollary 3.10, the uniform bound (3.11) holds for all $v \in H^1(\Omega^h, \mathbb{R}^2)$ tangential on the boundary, with some constant $C = C_h$ that depends only on Ω^h. However, (3.26) implies that $C_h \ge \frac{c}{h^2}$ as $h \to 0$. ■

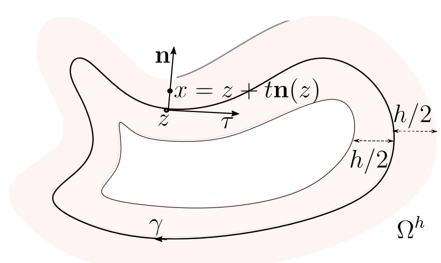

Fig. 3.2 Thin two-dimensional domains in Example 3.21.

We point out that Example 3.21 can be carried out in higher dimensions $N > 2$ as well, upon replacing the tangent vector field τ along a curve γ, by a Killing vector field on a surface S. We refer to Example 3.34 for the related construction.

The following is the main result of this section:

Theorem 3.22. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected, Lipschitz domain. In the context of Theorem 3.8, let $\Gamma_0 = \emptyset$ and $\Gamma_1 = \partial\Omega$. Then, there holds $C \geq 2$ for the constant C in (3.10). Equivalently:*

$$2 \leq \sup\Big\{ \min_{w\in\mathscr{I}_{\emptyset,\partial\Omega}(\Omega)} \int_\Omega |\nabla v - \nabla w|^2 \,\mathrm{d}x;\ v \in H^1(\Omega,\mathbb{R}^N),\ \ \langle v,\mathbf{n}\rangle = 0 \text{ on } \partial\Omega, \\ \text{and } \int_\Omega |\mathrm{sym}\nabla v|^2 \,\mathrm{d}x = 1\Big\}$$

Proof. **1.** Without loss of generality, we may assume that $0 \in \Omega$. Fix some vector field $\bar{v} \in H^1(\mathbb{R}^N,\mathbb{R}^N)$ satisfying $\int_\Omega |\mathrm{sym}\nabla\bar{v}|^2 \,\mathrm{d}x = 1$ and define the sequence $\{\bar{v}_n \in H^1(\mathbb{R}^N,\mathbb{R}^N\}_{n=1}^\infty$ by $\bar{v}_n(x) = n^{N/2-1}\bar{v}(nx)$. Then, there holds:

$$\int_{\mathbb{R}^N} |\nabla\bar{v}_n|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla\bar{v}|^2 \,\mathrm{d}x, \qquad \int_{\mathbb{R}^N} |\mathrm{sym}\nabla\bar{v}_n|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\mathrm{sym}\nabla\bar{v}|^2 \,\mathrm{d}x = 1.$$

Let $\phi \in \mathscr{C}_c^\infty(\Omega,[0,1])$ be a test function, equal identically to 1 in a neighborhood of 0 in Ω. For the modified sequence $\{v_n \doteq \phi\bar{v}_n \in H_0^1(\Omega,\mathbb{R}^N)\}_{n=1}^\infty$, we have:

$$\nabla v_n = \phi\nabla\bar{v}_n + \bar{v}_n \otimes \nabla\phi,$$

where the second term satisfies:

$$\lim_{n\to\infty} \|\bar{v}_n \otimes \nabla\phi\|_{L^2(\Omega)} \leq \lim_{n\to\infty} \|\nabla\phi\|_{L^\infty(\Omega)} n^{-1} \|\bar{v}\|_{L^2(\mathbb{R}^N)} = 0.$$

We now make the following claims:

$$\lim_{n\to\infty} \|\nabla v_n\|_{L^1(\Omega)} = 0, \tag{3.27}$$

$$\lim_{n\to\infty} \|\nabla v_n\|_{L^2(\Omega)} = \|\nabla\bar{v}\|_{L^2(\mathbb{R}^N)}, \quad \lim_{n\to\infty} \|\mathrm{sym}\nabla v_n\|_{L^2(\Omega)} = 1. \tag{3.28}$$

The first convergence in (3.28) follows by noting that:

$$\lim_{n\to\infty} \big\|\phi\nabla\bar{v}_n\big\|^2_{L^2(\Omega)} = \lim_{n\to\infty} \int_{\mathbb{R}^N} \big|\phi\big(\frac{x}{n}\big)\nabla\bar{v}(x)\big|^2 \,\mathrm{d}x = \|\nabla\bar{v}\|^2_{L^2(\Omega)}.$$

The second convergence follows similarly. For the remaining assertion (3.27), we use the result (3.29) in step 2 below to conclude the last equality in:

$$\lim_{n\to\infty} \|\phi\nabla\bar{v}_n\|_{L^1(\Omega)} \leq \lim_{n\to\infty} \|\phi\|_{L^\infty} n^{-N/2} \|\nabla\bar{v}\|_{L^1(n\Omega)} = 0.$$

2. We now prove the following statement, valid for any $g \in L^2(\mathbb{R}^N,\mathbb{R})$:

$$\lim_{R\to\infty} R^{-N/2} \|g\|_{L^1(B_R(0))} = 0. \tag{3.29}$$

Fix $\varepsilon > 0$ and let $R > m$ be two constants, sufficiently large to ensure that:

$$\|g\|_{L^2(\mathbb{R}^N\setminus B_m(0))} < \varepsilon \quad \text{and} \quad \Big(\frac{m}{R}\Big)^{N/2} \|g\|_{L^2(\mathbb{R}^N)} \leq \varepsilon.$$

Then we have, as claimed:

$$\begin{aligned} R^{-N/2}\|g\|_{L^1(B_R)} &= R^{-N/2}\Big(\int_{B_R(0)\setminus B_m(0)} |g|\, \mathrm{d}x + \int_{B_m(0)} |g|\, \mathrm{d}x\Big) \\ &\leq R^{-N/2}|B_R(0)|^{1/2}\varepsilon + R^{-N/2}|B_m(0)|^{1/2}\|g\|_{L^2(\mathbb{R}^N)} \\ &\leq |B_1(0)|^{1/2}\varepsilon + \Big(\frac{m}{R}\Big)^{N/2}|B_1(0)|^{1/2}\|g\|_{L^2(\mathbb{R}^N)} \leq 2|B_1(0)|^{1/2}\varepsilon. \end{aligned}$$

3. Note that, in virtue of Remark 3.9 we obtain:

$$\begin{aligned} \min_{w\in\mathscr{I}_{\Gamma_0,\Gamma_1}(\Omega)} \|\nabla v_n - \nabla w\|_{L^2(\Omega)} &= \Big\|\nabla v_n - \mathbb{P} \fint_\Omega \nabla v_n\, \mathrm{d}x\Big\|_{L^2(\Omega)} \\ &\geq \|\nabla v_n\|_{L^2(\Omega)} - \Big\|\mathbb{P}\fint_\Omega \nabla v_n\, \mathrm{d}x\Big\|_{L^2(\Omega)} \geq \|\nabla v_n\|_{L^2(\Omega)} - |\Omega|^{-1/2}\|\nabla v_n\|_{L^1(\Omega)}. \end{aligned}$$

By (3.27) and (3.28), the right hand side above converges to $\|\nabla\bar{v}\|_{L^2(\mathbb{R}^N)}$ as $n\to\infty$, while the left hand side is bounded by $C^{1/2}\|\mathrm{sym}\nabla v_n\|_{L^2(\Omega)}$, with C being the Korn constant to be estimated. Passing to the limit and using (3.28) again, we conclude:

$$\|\nabla\bar{v}\|_{L^2(\mathbb{R}^N)} \leq C^{1/2}\|\mathrm{sym}\nabla\bar{v}\|_{L^2(\mathbb{R}^N)} = C^{1/2},$$

which yields:

$$C \geq \sup\Big\{\int_{\mathbb{R}^N} |\nabla\bar{v}|^2\, \mathrm{d}x;\ \bar{v}\in H^1(\mathbb{R}^N,\mathbb{R}^N),\ \int_{\mathbb{R}^N} |\mathrm{sym}\nabla\bar{v}|^2\, \mathrm{d}x = 1\Big\}.$$

The proof is done by recalling homogeneous Korn's inequality in Theorem 3.6. ■

The bound in Theorem 3.22 is actually achieved, as shown in Example 3.7. The final observation of this section is that, in general, Korn's constant under tangential boundary condition blows up (at the rate h^{-2}) when the thickness h of a 2-dimensional domain goes to 0. Thin domains are of primary importance in this monograph and will be studied in the context of elasticity, in Parts II and III.

3.6 Approximation theorem and Korn's constant in thin shells

In this section we show how to approximate a displacement gradient ∇v on a *thin shell*, by a field of skew-symmetric matrices rather than by a constant matrix. While Korn's constant in the latter approximation, in general, blows up like $\frac{1}{h^2}$ along the vanishing shell's thickness h (as we have already seen in Example 3.21), the former approximation has the corresponding constants independent of $h \to 0$.

A similar construction, combining a mollification argument with Korn's inequality (3.1) used locally, will be carried out in section 4.5 to obtain the $\mathrm{SO}(N)$-valued approximations of a deformation gradient on thin shells. This approximation, based on the Friesecke-James-Müller nonlinear version of (3.1), will be of key importance in the dimension reduction analysis in Parts II and III of this monograph. In this section, we will also show how a simple application of the same technique yields a uniform Poincaré-Wirtinger type inequality on thin shells.

Let S be a smooth, closed hypersurface (i.e. a compact boundaryless manifold of co-dimension 1) in $\mathbb{R}^N$. We will use the following notation: $\mathbf{n}$ for the outward unit normal to S (seen as the boundary of some bounded domain in $\mathbb{R}^N$), T_zS for the tangent space to S at $z \in S$, and π for the projection onto S along $\mathbf{n}$:

$$\pi(z + t\mathbf{n}(z)) = z \qquad \text{for all } \ z \in S, \ \ |t| \ll 1.$$

For two families of positive, $\mathscr{C}^1$ functions $\{g_1^h, g_2^h : S \to \mathbb{R}\}_{h>0}$, we will consider a family $\{S^h\}_{h>0}$ of thin shells around S, viewed as their *midsurfaces*:

$$S^h = \{x = z + t\mathbf{n}(z); \ z \in S, \ -g_1^h(z) < t < g_2^h(z)\}. \tag{3.30}$$

We have:

Theorem 3.23. *Let $S \subset \mathbb{R}^N$ be a smooth, closed hypersurface. Assume that $\{S^h\}_{h>0}$ is given by (3.30) where $\{g_1^h, g_2^h \in \mathscr{C}^1(S,\mathbb{R})\}_{h>0}$ satisfy:*

$$C_1 h \le g_i^h(z) \le C_2 h, \quad |\nabla g_i^h(z)| \le C_3 h \quad \text{for all } \ z \in S, \ h \ll 1, \tag{3.31}$$

with some positive constants C_1, C_2, C_3 independent of h. Then, for every vector field $v \in H^1(S^h, \mathbb{R}^N)$ there exists a map $A \in H^1(S, \mathrm{so}(N))$ such that:

(i) $\displaystyle\int_{S^h} |\nabla v - A\pi|^2 \,\mathrm{d}x \le C \int_{S^h} |\mathrm{sym}\nabla v|^2 \,\mathrm{d}x,$

(ii) $\displaystyle\int_S |\nabla A|^2 \,\mathrm{d}\sigma(z) \le \frac{C}{h^3} \int_{S^h} |\mathrm{sym}\nabla v|^2 \,\mathrm{d}x.$

The constants C above depend on S and $\{C_i\}_{i=1}^3$, but not on v or $h \ll 1$.

Proof. **1.** For each $z \in S$ define the following sets:

$$D_{z,h} = B_h(z) \cap S, \qquad B_{z,h} = \pi^{-1}(D_{z,h}) \cap S^h,$$

where $B_h(z)$ denotes the ball in $\mathbb{R}^N$. We observe that $B_{z,h}$ is contained in a ball of radius $(C_2 + 1)h$ and and it is star-shaped with respect to a ball of radius $r(C_1, C_2, C_3, S)h$, for h sufficiently small. Hence, an application of Corollary 3.20 on $B_{z,h}$ yields a skew-symmetric matrix $A_{z,h} \in \mathrm{so}(N)$ such that:

$$\int_{B_{z,h}} |\nabla v(x) - A_{z,h}|^2 \, \mathrm{d}x \leq C \int_{B_{z,h}} |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x, \tag{3.32}$$

where C depends only on the quantities indicated in the statement of the result. The goal now is to replace $A_{z,h}$ by $A(z)$ which depends smoothly on z, and thus ultimately replace $A_{z,h}$ by $A(\pi z)$ in (3.32). The desired estimate on S^h will then follow by summing over a finite family of cylinders $D_{z,h}$.

To this end, let $\theta : [0,1) \to [0,2]$ be a smooth cut-off function that is compactly supported, constant in a neighbourhood of 0, and satisfying $\int_0^1 \theta = 1$. Define:

$$\eta_z(x) = \frac{\theta(|\pi x - z|/h)}{\int_{S^h} \theta(|\pi x - z|/h) \, \mathrm{d}x} \quad \text{ for all } \ z \in S, \ x \in S^h.$$

Then η_z is supported on $B_{z,h}$ and moreover we have:

$$\int_{S^h} \eta_z(x) \, \mathrm{d}x = 1, \quad |\eta_z| \leq \frac{C}{h^N}, \quad |\nabla_z \eta_z| \leq \frac{C}{h^{N+1}}. \tag{3.33}$$

Finally, we define the skew-symmetric matrix field A as the average:

$$A(z) = \int_{S^h} \eta_z(x) \, \mathrm{skew}\nabla v(x) \, \mathrm{d}x.$$

2. Since $A(z) - A_{z,h} = \int_{S^h} \eta_z(x) \, \mathrm{skew}\big(\nabla v(x) - A_{z,h}\big) \, \mathrm{d}x$, the Cauchy-Schwarz inequality together with (3.32) and (3.33) yield:

$$|A(z) - A_{z,h}|^2 \leq \Big(\int_{S^h} \eta_z(x) |\nabla v(x) - A_{z,h}| \, \mathrm{d}x \Big)^2 \leq \frac{C}{h^N} \int_{B_{z,h}} |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x. \tag{3.34}$$

To estimate ∇A we use that: $\int_{S^h} \nabla_z \eta_z(x) \, \mathrm{d}x = \nabla_z \big(\int_{S^h} \eta_z(x) \, \mathrm{d}x \big) = 0$, to get: $\nabla A(z) = \int_{S^h} (\nabla_z \eta_z) \, \mathrm{skew}\nabla v \, \mathrm{d}x = \int_{S^h} (\nabla_z \eta_z) \, \mathrm{skew}(\nabla v - A_{z,h}) \, \mathrm{d}x$. Consequently:

$$|\nabla A(z)|^2 \leq \int_{B_{z,h}} |\nabla_z \eta_z|^2 \, \mathrm{d}x \cdot \int_{B_{z,h}} \big|\nabla v - A_{z,h}\big|^2 \, \mathrm{d}x \leq \frac{C}{h^{N+2}} \int_{B_{z,h}} |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x, \tag{3.35}$$

in virtue of (3.32) and (3.33). Similarly, for all $z' \in D_{z,h}$ there holds:

$$|\nabla A(z')|^2 \leq \frac{C}{h^{N+2}} \int_{B_{z',h}} |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x \leq \frac{C}{h^{N+2}} \int_{2B_{z,h}} |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x, \tag{3.36}$$

where $2B_{z,h} = \pi^{-1}(D_{z,2h}) \cap S^h$. From this, by the fundamental theorem of calculus:

$$|A(z'') - A(z)|^2 \leq \frac{C}{h^N} \int_{2B_{z,h}} |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x \quad \text{ for all } \ z'' \in D_{z,h}.$$

In combination with (3.32) and (3.34) the above yields:

$$\int_{B_{z,h}} |\nabla v(x) - A(\pi x)|^2 \, \mathrm{d}x \leq C \int_{2B_{z,h}} |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x. \tag{3.37}$$

3. We now cover S^h with a family of sets $\{B_{z_i,h}\}_{i=1}^{n(h)}$ with the property that the covering number of the derived family $\{2B_{z_i,h}\}_{i=1}^{n(h)}$ is independent of h. An argument for the existence of such a covering goes as follows. The surface S is contained in the finite union of balls $\bigcup_{i=1}^{n(h)} B_{h/2}(k_i)$ where $k_i \in (\frac{h}{2}\mathbb{Z})^N$. Fix a one-to-one map $k_i \mapsto z_i \in S \cap B_{h/2}(k_i)$, so that $S^h = \bigcup_i B_{z_i,h}$. Then, if $x \in 2B_{z_i,h}$ there must be $\pi(x) \in B_{2h}(z_i)$, so that $|k_i - \pi(x)| \leq |k_i - x_i| + |\pi(x) - z_i| \leq 5h/2$. Therefore $k_i \in B_{5h/2}(x) \cap (\frac{h}{2}\mathbb{Z})^n$. The cardinality of this last set is bounded by 10^N, which serves as an upper bound on the covering number for the family $\{2B_{z_i,h}\}_{i=1}^{n(h)}$.

Summing (3.37) over $i = 1 \dots n$ proves (i). Integrating (3.36) on $D_{z,h}$ we get:

$$\int_{D_{z,h}} |\nabla A(z')|^2 \, \mathrm{d}\sigma(z') \leq \frac{C}{h^3} \int_{2B_{z,h}} |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x,$$

which yields (ii) by using the same covering argument above. ■

As a corollary, we readily deduce:

Theorem 3.24. *Let $S \subset \mathbb{R}^N$ be a smooth, closed hypersurface and assume (3.31). Then, for every vector field $v \in H^1(S^h, \mathbb{R}^N)$ defined on S^h in (3.30) with $h \ll 1$, there exists $A_0 \in \mathrm{so}(N)$ such that:*

$$\int_{S^h} |\nabla v - A_0|^2 \, \mathrm{d}x \leq \frac{C}{h^2} \int_{S^h} |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x.$$

The constant C above depends only on S and $\{C_i\}_{i=1}^3$ in (3.31).

Proof. Let $A : S \to \mathrm{so}(N)$ be as in Theorem 3.23 and define:

$$A_0 \doteq ⨍_S A(z) \, \mathrm{d}\sigma(z) \in \mathrm{so}(N).$$

Applying additionally the Poincaré inequality on S, we obtain:

$$\begin{aligned}\int_{S^h} |\nabla v - A_0|^2 \, \mathrm{d}x &\leq C\Big(\int_{S^h} |\nabla v - A\pi|^2 \, \mathrm{d}x + h\int_S |A(z) - A_0|^2 \, \mathrm{d}\sigma(z)\Big) \\ &\leq C\Big(\int_{S^h} |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x + h\int_S |\nabla A|^2 \, \mathrm{d}\sigma(z)\Big) \leq \frac{C}{h^2}\int_{S^h} |\mathrm{sym}\nabla v|^2 \, \mathrm{d}x.\end{aligned}$$

This ends the proof. ■

The decomposition and mollification argument as in Theorem 3.23 can be applied towards a useful uniform Poincaré inequality in thin shells:

Theorem 3.25. *Let $S \subset \mathbb{R}^N$ be a smooth, closed hypersurface and assume (3.31). Then, for every $g \in H^1(S^h, \mathbb{R})$, there exists $a \in \mathbb{R}$ so that:*

$$\int_{S^h} |g-a|^2 \,\mathrm{d}x \leq C \int_{S^h} |\nabla g|^2 \,\mathrm{d}x.$$

The constant C above depends only on S and $\{C_i\}_{i=1}^3$ in (3.31).

Proof. Let $D_{z,h}, B_{z,h}, \eta_x$ be as in the proof of Theorem 3.23. We will show the result for the constant $a = ⨍_S \tilde{a}(z) \,\mathrm{d}\sigma(z)$, where we define:

$$\tilde{a}(z) = \int_{S^h} \eta_z(x) g(x) \,\mathrm{d}x \qquad \text{for all } z \in S.$$

First, by Lemma 3.16 (ii), the local estimate (3.32) can be replaced by:

$$\int_{B_{z,h}} |g - a_{z,h}|^2 \,\mathrm{d}x \leq Ch^2 \int_{B_{z,h}} |\nabla u|^2 \,\mathrm{d}x.$$

Repeating the calculations leading to (3.34) and (3.35), it follows that:

$$\begin{aligned} |\tilde{a}(z) - a_{z,h}|^2 &\leq Ch^{2-N} \int_{B_{z,h}} |\nabla g|^2 \,\mathrm{d}x, \\ |\nabla \tilde{a}(z')|^2 &\leq Ch^{-N} \int_{2B_{z,h}} |\nabla g|^2 \,\mathrm{d}x \qquad \text{for all } z' \in D_{z,h}, \end{aligned}$$

which imply, exactly as in (3.37):

$$\int_{S^h} |g - \tilde{a}\pi|^2 \,\mathrm{d}x \leq Ch^2 \int_{S^h} |\nabla g|^2 \,\mathrm{d}x, \qquad \int_S |\nabla \tilde{a}|^2 \,\mathrm{d}\sigma(z) \leq Ch^{-1} \int_{S^h} |\nabla g|^2 \,\mathrm{d}x.$$

Using the standard Poincaré inequality on surfaces, we thus get:

$$\begin{aligned} \int_{S^h} |g-a|^2 \,\mathrm{d}x &\leq C\Big(\int_{S^h} |g - \tilde{a}\pi|^2 \,\mathrm{d}x + h \int_S |\tilde{a}(z) - a|^2 \,\mathrm{d}\sigma(z) \Big) \\ &\leq C\Big(h^2 \int_{S^h} |\nabla g|^2 \,\mathrm{d}x + h \int_S |\nabla \tilde{a}|^2 \,\mathrm{d}\sigma(z) \Big) \leq C \int_{S^h} |\nabla g|^2 \,\mathrm{d}x. \end{aligned}$$

The proof is done. ∎

Theorems 3.24 and 3.25 directly imply the following Korn-Poincaré inequality:

Corollary 3.26. *Let $S \subset \mathbb{R}^N$ be a smooth, closed hypersurface and assume (3.31). Then, for every $v \in H^1(S^h, \mathbb{R}^N)$, defined on S^h in (3.30), there exists affine map with skew-symmetric gradient $w \in \mathscr{I}(S^h)$, satisfying:*

$$\|v - w\|_{H^1(S^h)} \leq \frac{C}{h} \|\mathrm{sym} \nabla v\|_{L^2(S^h)}.$$

The constant C above depends only on S and $\{C_i\}_{i=1}^3$ in (3.31).

We close this section by a trace theorem, resulting by applying the scaled versions of the usual trace theorem to each neighbourhood $B_{z,h}$ in the proof of Theorem 3.23:

Lemma 3.27. *Let $S \subset \mathbb{R}^N$ be a smooth, closed hypersurface and assume (3.31). Then, for every $g \in H^1(S^h, \mathbb{R})$ there holds:*

$$\|g\|_{L^2(S)} + \|g\|_{L^2(\partial S^h)} \leq C\Big(\frac{1}{h^{1/2}}\|g\|_{L^2(S^h)} + h^{1/2}\|\nabla g\|_{L^2(S^h)}\Big),$$

where in the left hand side we have norms of traces of g on S and ∂S^h. The constant C is independent of g or $h \ll 1$.

3.7 Killing vector fields and Korn's inequality on surfaces

The purpose of this section is to prove the counterparts of Korn's inequalities in Theorems 3.5 and 3.1, where the open domain $\Omega \subset \mathbb{R}^N$ is replaced by a $N-1$ dimensional surface $S \subset \mathbb{R}^N$. Rather than carrying out proofs (for now) in the more general context of Riemannian geometry, we will apply the previous results on thin shells around S and recover Korn's inequality on S in the vanishing limit of the shell's thickness. In addition to the independent interest of these results, the analysis below will be essential for continuing the discussion in section 3.9.

Let S be a smooth, closed hypersurface in $\mathbb{R}^N$. We use the notation $\mathbf{n}$, T_zS and π as in section 3.6. For a vector field $v \in H^1(S, \mathbb{R}^N)$, we denote by $\mathrm{sym}\nabla v$ the symmetric part of the (covariant) gradient of (the tangent component of) v, in:

$$\langle \mathrm{sym}\nabla v(z)\eta, \tau\rangle = \frac{1}{2}\big(\langle \partial_\eta v(z), \tau\rangle + \langle \partial_\tau v(z), \eta\rangle\big) \qquad \text{for all } z \in S, \quad \tau, \eta \in T_zS.$$

By $\partial_\tau v(z)$ we denote the derivative of v in the tangent direction τ, i.e. if $\gamma : (-\varepsilon, \varepsilon) \to S$ is a $\mathscr{C}^1$ curve with $\gamma(0) = z$ and $\gamma'(0) = \tau$, then $\partial_\tau v(z) = (v \circ \gamma)'(0)$. By $\Pi(z) = \nabla \mathbf{n}(z) : T_zS \to T_zS$ we denote the shape operator (which is the negative second fundamental form) on S.

We have the following counterpart to Theorem 3.5:

Theorem 3.28. [First Korn's inequality on surfaces]
Let S be a smooth, closed hypersurface in $\mathbb{R}^N$. There holds:

$$\int_S |\nabla v|^2 \,\mathrm{d}\sigma(z) \leq C \int_S |v|^2 + |\mathrm{sym}\nabla v|^2 \,\mathrm{d}\sigma(z), \tag{3.38}$$

for all vector fields $v \in H^1(S, \mathbb{R}^N)$ tangent to S, i.e. satisfying $\langle v(z), \mathbf{n}(z)\rangle = 0$ for a.e. $z \in S$. The constant C above depends only on S but not on v.

Proof. **1.** Consider the extension of v on the thin neighbourhood of S^{h_0} of S:

$$\begin{aligned}&\tilde{v}(z+t\mathbf{n}(z)) = (\mathrm{Id}+t\Pi(z))^{-1}v(z) \qquad \text{for all } \ x\in S^{h_0},\\ &\text{where } \ S^{h_0} = \big\{x = z+t\mathbf{n}(z);\ z\in S,\ |t|<\frac{h_0}{2}\big\}, \qquad h_0 \ll 1.\end{aligned} \tag{3.39}$$

We have $\tilde{v}\in H^1(S^{h_0},\mathbb{R}^N)$ and for every $x = z+t\mathbf{n}(z)\in S^{h_0}$ and $\tau\in T_zS$ there holds:

$$\begin{aligned}\partial_\tau\tilde{v}(z) = \Big\{\nabla\left[(\mathrm{Id}+t\Pi(z))^{-1}\right](\mathrm{Id}+t\Pi(z))^{-1}\tau\Big\}v(z)\\ +(\mathrm{Id}+t\Pi(z))^{-1}\nabla v(z)(\mathrm{Id}+t\Pi(z))^{-1}\tau.\end{aligned} \tag{3.40}$$

The first component above is bounded by $C|tu(x)|$. Taking the scalar product of the second term with any $\eta\in T_xS$ gives: $\langle(\mathrm{Id}+t\Pi(z))^{-1}\eta, \nabla v(z)(\mathrm{Id}+t\Pi(z))^{-1}\tau\rangle$. Consequently, we obtain:

$$\begin{aligned}&\langle \mathrm{sym}\nabla\tilde{v}(x)\tau,\eta\rangle = \big\langle(\mathrm{Id}+t\Pi(z))^{-1}\eta, \mathrm{sym}\nabla v(z)(\mathrm{Id}+t\Pi(z))^{-1}\tau\big\rangle\\ &\qquad\qquad\qquad\qquad + \langle Z(t,z), v(z)\rangle\\ &|Z(t,z)|\le C.\end{aligned} \tag{3.41}$$

On the other hand, $\langle\mathbf{n}(z),\tilde{v}(x)\rangle = 0$, so for any $\tau\in T_zS$:

$$\begin{aligned}\langle\partial_\tau\tilde{v}(x),\mathbf{n}(z)\rangle &= -\big\langle\Pi(z)(\mathrm{Id}+t\Pi(z))^{-1}\tau,\tilde{v}(x)\big\rangle\\ &= -\big\langle(\mathrm{Id}+t\Pi(z))^{-1}\Pi(z)(\mathrm{Id}+t\Pi(z))^{-1}v(z),\tau\big\rangle = \langle\tau,\partial_{\mathbf{n}}\tilde{v}(x)\rangle.\end{aligned}$$

Hence it follows that:

$$\begin{aligned}&\langle\mathrm{sym}\nabla\tilde{v}(x)\tau,\mathbf{n}\rangle = -\big\langle(\mathrm{Id}+t\Pi(z))^{-1}\Pi(z)(\mathrm{Id}+t\Pi(z))^{-1}v(z),\tau\big\rangle,\\ &\langle\mathrm{sym}\nabla\tilde{v}(x)\mathbf{n},\mathbf{n}\rangle = 0.\end{aligned} \tag{3.42}$$

2. We now invoke first Korn's inequality on the open bounded smooth S^{h_0}, to get:

$$\int_{S^{h_0}}|\nabla\tilde{v}|^2\,\mathrm{d}x \le C\int_{S^{h_0}}|\tilde{v}|^2+|\mathrm{sym}\nabla\tilde{v}|^2\,\mathrm{d}x,$$

where C depends only on S and the chosen parameter h_0. By (3.40) and noting that:

$$\langle\partial_\tau v,\mathbf{n}\rangle = -\langle\Pi\tau, v\rangle,$$

we further obtain:

$$\int_{S^{h_0}}|\nabla\tilde{v}|^2\,\mathrm{d}x \ge c_1\int_S|\nabla v|^2\,\mathrm{d}\sigma(z) - c_2\int_S|v|^2\,\mathrm{d}\sigma(z),$$

again with some uniform positive constants c_1, c_2. Further, by (3.41) and (3.42):

$$\int_{S^{h_0}}|\mathrm{sym}\nabla\tilde{v}|^2\,\mathrm{d}x \le C\int_S|v|^2+|\mathrm{sym}\nabla v|^2\,\mathrm{d}\sigma(z).$$

The three above displayed inequalities imply (3.38), in view of the elementary bound $\int_{S^{h_0}} |\tilde{v}|^2 \,\mathrm{d}x \leq C \int_S |v|^2 \,\mathrm{d}\sigma(z)$. ■

In order to formulate second Korn's inequality on S, we necessitate the displacements whose gradients replace the constant skew matrices $A \in \mathrm{so}(N)$ in (3.1).

Definition 3.29. *We say that a (smooth) vector field $w : S \to \mathbb{R}^N$ is a* Killing vector field *on S, provided that: $w(z) \in T_z S$ and* $\mathrm{sym}\nabla w = 0$ *for all $z \in S$. The linear space (and the Lie algebra) of all Killing fields on S will be denoted by $\mathscr{I}(S)$.*

The Killing fields are infinitesimal generators of isometries on S, in the sense that for every fixed t the map $S \ni z \mapsto \Phi(t,z) \in S$ is an isometry, where Φ is the flow of:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(s,z) = w(\Phi(t,z)), \qquad \Phi(0,z) = z.$$

The smoothness assumption in Definition 3.29 is not a restriction, because of the following counterpart of Lemma 3.2:

Lemma 3.30. *(i) Let $w \in H^1(S,\mathbb{R}^N)$ be a tangent field satisfying* $\mathrm{sym}\nabla w = 0$ *almost everywhere on S. Then $w \in \mathscr{I}(S)$.*
(ii) The space $\mathscr{I}(S)$ has finite dimension.

Proof. To show smoothness of w in (i), recall the extension $\tilde{w} \in H^1(S^{h_0},\mathbb{R}^N)$ given by the formula in (3.39). By (3.41), (3.42) we see that $\mathrm{sym}\nabla\tilde{w}$ has the improved regularity H^1 and hence in virtue of (3.2) we get: $\Delta\tilde{w} \in L^2(S^{h_0},\mathbb{R})$. The result is a consequence of the elliptic regularity and a bootstrap argument.

The finite dimensionality assertion in (ii) follows from the equivalence of the L^2 and the H^1 norms on $\mathscr{I}(S)$, in view of (3.38) which yields:

$$\|\nabla w\|_{L^2(S)} \leq C\|w\|_{L^2(S)} \quad \text{for all } w \in \mathscr{I}(S). \tag{3.43}$$

For otherwise the space $(\mathscr{I}(S), \|\cdot\|_{H^1})$ would have a countable Hilbert (orthonormal) basis $\{e_n\}_{n=1}^\infty$. The sequence $\{e_n\}$ must then converge to 0 as $n \to \infty$, weakly in $H^1(S,\mathbb{R}^N)$. But this implies that $\lim_{n\to\infty}\|e_n\|_{L^2(S,\mathbb{R}^N)} = 0$, which by the norms equivalence gives the same convergence in $H^1(S,\mathbb{R}^3)$, and a contradiction. ■

As in the proof of Theorem 3.1, an argument by contradiction now yields:

Theorem 3.31. [Korn-Poincaré's inequality on surfaces]
Let S be a smooth, closed hypersurface in $\mathbb{R}^N$. For every tangent vector field $v \in H^1(S,\mathbb{R}^N)$ there exists a Killing field $w \in \mathscr{I}(S)$ such that:

$$\|v - w\|_{H^1(S)} \leq C\|\mathrm{sym}\nabla v\|_{L^2(S)}$$

and the constant C depends only on S.

Proof. Consider the orthogonal complement $\mathscr{I}(S)^\perp$ of $\mathscr{I}(S)$ in the Hilbert space of $H^1(S,\mathbb{R}^N)$ regular tangent vector fields on S. Both spaces are closed (with respect to both weak and strong convergences in $H^1(S,\mathbb{R}^N)$). We will show that:

$$\|v\|_{H^1(S)} \leq C\|\mathrm{sym}\nabla v\|_{L^2(S)} \qquad \text{for all } \ v \in \mathscr{I}(S)^\perp,$$

which clearly implies the result in the Theorem. We argue by contradiction. If the above was not true, there would exist a sequence $\{v_n \in \mathscr{I}(S)^\perp\}_{n=1}^\infty$ such that:

$$\|v_n\|_{H^1(S)} = 1, \qquad \|\mathrm{sym}\nabla v_n\|_{L^2(S)} \to 0 \quad \text{ as } \ n \to \infty.$$

Without loss of generality, or passing to a subsequence if necessary, $\{v_n\}_{n\to\infty}$ converges weakly to some $v \in \mathscr{I}(S)^\perp$. Moreover, $\mathrm{sym}\nabla v = 0$ by the second condition above, so Lemma 3.30 implies that $v \in \mathscr{I}(S)$. Since the spaces $\mathscr{I}(S)$ and $\mathscr{I}(S)^\perp$ are orthogonal, there must be $v = 0$, and hence $\{v_n\}_{n\to\infty}$ converges to 0 (strongly) in $L^2(S,\mathbb{R}^N)$. This contradicts the normalisation $\|v_n\|_{H^1(S)} = 1$, in virtue of (3.38). ■

A few remarks on the linear space $\mathscr{I}(S)$ are of interest:

Remark 3.32. (i) The bound (3.43) together with an estimate of the uniform constant C (that depends on S), can be recovered directly using the following identity, valid for all Killing vector fields $w \in \mathscr{I}(S)$:

$$\Delta_S\big(\frac{1}{2}|w|^2\big) = \big|\widetilde{\nabla} w\big|^2 - \mathrm{Ric}(w,w). \tag{3.44}$$

Here Δ_S is the Laplace-Beltrami operator on S, $\widetilde{\nabla} w = (\nabla w)_{tan}$ is the covariant derivative of w on S, and Ric stands for the Ricci curvature form on S. To calculate $\mathrm{Ric}(w,w)$ in our particular setting, note that by Gauss' Teorema Egregium, the Riemann curvature 4-tensor on S satisfies:

$$\langle R(\tau,\eta)\xi,\vartheta\rangle = \langle \Pi(z)\tau,\vartheta\rangle\langle \Pi(z)\eta,\xi\rangle - \langle \Pi(z)\tau,\xi\rangle\langle \Pi(z)\eta,\vartheta\rangle,$$

for all $z \in S$ and $\tau,\eta,\xi,\vartheta \in T_zS$. Thus, the Ricci curvature 2-tensor being the appropriate trace of R, we obtain for all $z \in S$ and $\eta,\xi \in T_zS$:

$$\begin{aligned}\mathrm{Ric}(\eta,\xi) &= \mathrm{tr}\,(\tau \mapsto R(\tau,\eta)\xi) = \mathrm{tr}\,\Pi(z)\langle \Pi(z)\eta,\xi\rangle - \langle \Pi(z)\xi,\Pi(z)\eta\rangle \\ &= \big\langle \big((\mathrm{tr}\,\Pi(z))\Pi(z) - \Pi(z)^2\big)\eta,\xi\big\rangle.\end{aligned} \tag{3.45}$$

Integrating (3.44) on S and using (3.45) we arrive at:

$$\|\widetilde{\nabla} w\|^2_{L^2(S)} = \int_S \big\langle \big((\mathrm{tr}\,\Pi(z))\Pi(z) - \Pi(z)^2\big)w(z),w(z)\big\rangle \,\mathrm{d}\sigma(z). \tag{3.46}$$

To calculate the L^2 norm of the full gradient ∇u on S, we use:

$$\|\nabla w\|^2_{L^2(S)} - \|\widetilde{\nabla} w\|^2_{L^2(S)} = \int_S \sum_{i=1}^{n-1} \langle \partial_{\tau_i} w, \mathbf{n}\rangle^2 \,\mathrm{d}\sigma(z)$$
$$= \int_S \sum_{i=1}^{n-1} \langle w(z), \Pi(z)\tau_i\rangle^2 \,\mathrm{d}\sigma(z) = \int_S |\Pi(z)w|^2 \,\mathrm{d}\sigma(z).$$

Hence we arrive at:

$$\int_S |\nabla w|^2 \,\mathrm{d}\sigma(z) = \int_S \mathrm{tr}\,\Pi(z)\big\langle \Pi(z)w(z), w(z)\big\rangle \,\mathrm{d}\sigma(z) \qquad \text{for all } w \in \mathscr{I}(S), \quad (3.47)$$

which clearly implies (3.43).

(ii) Notice that in the special case of a 2×2 matrix Π, when $N = 3$ and S is a 2d surface in $\mathbb{R}^3$, the Cayley-Hamilton theorem implies:

$$(\mathrm{tr}\,\Pi)\Pi - \Pi^2 = (\det \Pi)\mathrm{Id}_2,$$

and so (3.46) becomes:

$$\|\widetilde{\nabla} w\|^2_{L^2(S)} = \int_S \det \Pi(z)|w|^2 \,\mathrm{d}\sigma(z)$$

In this case $\det \Pi(z)$ coincides with the Gaussian curvature of S at z.

(iii) An equivalent way of obtaining the formula (3.47), but without using the language of Riemannian geometry, is to look at the extension of w:

$$\tilde{w}(z + t\mathbf{n}(z)) = w(z) \qquad \text{for all } x = z + t\mathbf{n}(z) \in S^{h_0}.$$

Since $\partial_{\mathbf{n}}\tilde{w} = 0$ and $\langle \tilde{w}, \mathbf{n}\rangle = 0$ on the boundary of S^{h_0}, by (3.2) one has:

$$\int_{S^{h_0}} |\nabla \tilde{w}|^2 \,\mathrm{d}x = -2\int_{S^{h_0}} \langle \mathrm{div}\,\mathrm{sym}\nabla\tilde{w}, \tilde{w}\rangle \,\mathrm{d}x - \int_{S^{h_0}} |\mathrm{div}\,\tilde{w}|^2 \,\mathrm{d}x. \qquad (3.48)$$

Calculating $\langle \mathrm{sym}\nabla\tilde{w}, \tilde{w}\rangle$ in terms of $\Pi(z)$, dividing both sides of (3.48) by $2h_0$ and passing to the limit with $h_0 \to 0$, one recovers (3.47) directly. ■

We conclude this section by the following Korn-type inequality on 2d surfaces:

Lemma 3.33. *Let $S \subset \mathbb{R}^3$ be a smooth, closed hypersurface in $\mathbb{R}^3$. Assume that $A \in \mathscr{C}^{0,1}(S, \mathbb{R}^{2\times 2})$ satisfies $\det A \neq 0$ on S. Then there holds, for every tangent vector field $v \in H^1(S, \mathbb{R}^3)$, with C that depends only on S and A, but not on v:*

$$\|\nabla v\|_{L^2(S)} \leq C\big(\|v\|_{L^2(S)} + \|(\nabla v)_{tan} - \langle(\nabla v)_{tan} : A\rangle A\|_{L^2(S)}\big).$$

Proof. It suffices to prove the claimed bound locally, and hence below we replace S by a single patch, with a boundary that is a Lipschitz curve. Take $J \in \mathscr{C}^{0,1}(\bar{S}, \mathrm{so}(2))$ to be any skew-symmetric matrix field with nonvanishing determinant. Define $\tilde{v} = JA^{-1}v$ and note the decomposition:

$$\nabla v = AJ^{-1}(\nabla\tilde{v})_{tan} + \nabla(AJ^{-1})JA^{-1}v. \tag{3.49}$$

To estimate $\nabla\tilde{v}$, we use (the local version of) Theorem 3.28:

$$\|\nabla\tilde{v}\|_{L^2(S)} \leq C(\|v\|_{L^2(S)} + \|\mathrm{sym}\nabla\tilde{v}\|_{L^2(S)}). \tag{3.50}$$

Further, $\mathrm{sym}\nabla\tilde{v} = \mathrm{sym}\big((\nabla\tilde{v})_{tan} - \langle(\nabla v)_{tan} : A\rangle J\big)$ as J is skew-symmetric. Thus:

$$\begin{aligned}\|\mathrm{sym}\nabla\tilde{v}\|_{L^2(S)} &\leq \|(\nabla\tilde{v})_{tan} - \langle(\nabla v)_{tan} : A\rangle J\|_{L^2(S)} \\ &\leq C\|AJ^{-1}(\nabla\tilde{v})_{tan} - \langle(\nabla v)_{tan} : A\rangle A\|_{L^2(S)} \\ &\leq C\|(\nabla v)_{tan} - \langle(\nabla v)_{tan} : A\rangle A\|_{L^2(S)} + C\|v\|_{L^2(S)},\end{aligned}$$

in view of (3.49). Combining (3.49), (3.50) and the above completes the proof. ■

3.8 Blowup of Korn's constant in thin shells

In this section, we present an extension of the argument in Example 3.21 showing that in general, the uniform constant $C = C_h$ in (3.1) posed on the thin film $\Omega = S^h$ with the mid-surface S, blows up quadratically: $C_h \geq \frac{c}{h^2}$ as $h \to 0$.

Example 3.34. Given two smooth positive functions $g_1, g_2 : S \to \mathbb{R}$, we now consider the family $\{S^h\}_{h>0}$ of thin shells around S:

$$S^h = \{x = z + t\mathbf{n}(z);\ z \in S,\ -hg_1(z) < t < hg_2(z)\}.$$

By $\mathbf{n}^h$ we denote for the outward unit normal to ∂S^h. Define the subspace of $\mathscr{I}(S)$:

$$\mathscr{I}_{g_1,g_2}(S) = \{w \in \mathscr{I}(S);\ \langle w(z), \nabla(g_1+g_2)(z)\rangle = 0 \ \text{ for all } z \in S\},$$

consisting of those Killing fields w which satisfy: $\lim_{h\to 0}\frac{1}{h}\langle w(z), (\mathbf{n}^h_+ + \mathbf{n}^h_-)\rangle = 0$, where $\mathbf{n}^h_+$ and $\mathbf{n}^h_-$ denote, respectively, the outward unit normals to S^h at its boundary points $z + hg_2(z)$ and $z - hg_1(z)$.

For $w \in \mathscr{I}_{g_1,g_2}(S) \setminus \{0\}$ we now construct a family $\{v^h \in H^1(S^h, \mathbb{R}^N)\}_{h\to 0}$ with:

$$\langle v^h, \mathbf{n}^h\rangle = 0 \quad \text{ on } \partial S^h, \tag{3.51}$$

for which $C = C_h \geq \frac{c}{h^2}$ as $h \to 0$ in (3.1), even though $\mathscr{I}_{\emptyset,\partial S^h}(S^h) = \{0\}$ for all $h \ll 1$. This is the case, for example, when S has no rotational symmetry, see Remark 3.11 and Theorem 3.8.

1. Namely, for all $z \in S$ and all $t \in (-hg_1(z), hg_2(z))$, we set:

$$v^h(z + t\mathbf{n}(z)) = \Big(\mathrm{Id} + t\Pi(z) + h\mathbf{n}(z) \otimes \nabla g_2(z)\Big)w(z) \tag{3.52}$$

and calculate directly:

$$\begin{aligned} v^h(z+t\mathbf{n}(z)) = &\frac{hg_1(z)+t}{h(g_1(z)+g_2(z))}\Big(\mathrm{Id}+hg_2(z)\Pi(z)+h\mathbf{n}(z)\otimes\nabla g_2(z)\Big)w(z)\\ &+\frac{hg_2(z)-t}{h(g_1(z)+g_2(z))}\Big(\mathrm{Id}-hg_1(z)\Pi(z)-h\mathbf{n}(z)\otimes\nabla g_1(z)\Big)w(z). \end{aligned}$$

The above means that each v^h is a linear interpolation between the push-forward of the vector field w from S onto the external boundary ∂^+S^h and the push-forward onto the internal boundary ∂^-S^h of ∂S^h. Indeed, derivatives of the maps in:

$$S\ni z\mapsto z\pm hg_i(z)\mathbf{n}(z)\in\partial^\pm S^h$$

are given through:

$$\mathrm{Id}\pm hg_i(z)\Pi(z)\pm h\mathbf{n}(z)\otimes\nabla g_i(z).$$

In particular, we see that (3.51) holds.

2. Write now $v^h=\tilde{w}+(v^h-\tilde{w})$, with:

$$\tilde{w}(x)=\big(\mathrm{Id}+t\Pi(z)\big)w(z),$$

and estimate components of $\nabla\tilde{w}$ and $\mathrm{sym}\nabla\tilde{w}$. For all $z\in S$ and $\tau\in T_zS$, there holds:

$$\begin{aligned} &\partial_{\mathbf{n}}\tilde{w}(x)=\Pi(z)w(z),\\ &\partial_\tau\tilde{w}(x)=\frac{t\partial\Pi(z)}{\partial((\mathrm{Id}+t\Pi(z))^{-1}\tau)}w(z)+(\mathrm{Id}+t\Pi(z))\nabla w(z)(\mathrm{Id}+t\Pi(z))^{-1}\tau. \end{aligned}\tag{3.53}$$

Since $\langle\mathbf{n},\tilde{w}\rangle=0$ and since $\Pi(x)$ commutes with $(\mathrm{Id}+t\Pi(x))^{-1}$, we get:

$$\begin{aligned} &\langle\partial_\tau\tilde{w},\mathbf{n}\rangle+\langle\partial_{\mathbf{n}}\tilde{w},\tau\rangle=-\langle\partial_\tau\mathbf{n},\tilde{w}\rangle+\langle\partial_{\mathbf{n}}\tilde{w},\tau\rangle\\ &\quad=-\big\langle\Pi(z)(\mathrm{Id}+t\Pi(z))^{-1}\tau,(\mathrm{Id}+t\Pi(z))w(z)\big\rangle+\langle\Pi(z)w(z),\tau\rangle=0,\\ &\langle\partial_{\mathbf{n}}\tilde{w},\mathbf{n}\rangle=0. \end{aligned}\tag{3.54}$$

To estimate $\langle\mathrm{sym}\nabla\tilde{w}(x)\tau,\eta\rangle$ for $\tau,\eta\in T_zS$, note that:

$$\begin{aligned} \Big|&\langle(\mathrm{Id}+t\Pi(z))\nabla w(z)(\mathrm{Id}+t\Pi(z))^{-1}\tau,\eta\rangle\\ &-\langle(\mathrm{Id}+t\Pi(z))^{-1}\nabla w(z)(\mathrm{Id}+t\Pi(z))^{-1}\tau,\eta\rangle\Big|\leq Ct|\nabla w(z)|, \end{aligned}$$

because $|(\mathrm{Id}+t\Pi(z))-(\mathrm{Id}+t\Pi(z))^{-1}|\leq Ct$ with C that, as usual, denotes any positive constant independent of h. Since $(\mathrm{Id}+t\Pi(z))^{-1}\tau\in T_zS$, we obtain:

$$|\langle\mathrm{sym}\nabla\tilde{w}(x)\tau,\eta\rangle|\leq Ct(|w(z)|+|\nabla w(z)|).\tag{3.55}$$

Further: $|\nabla(v^h-\tilde{w})(x)|\leq Ch$, and by (3.54) and (3.55): $|\mathrm{sym}\nabla\tilde{w}|\leq Ch$ on S^h.

3. Consequently, it follows that:

$$\int_{S^h} |\mathrm{sym}\nabla v^h|^2 \, \mathrm{d}x \leq Ch^3.$$

On the other hand, inspecting the terms in ∇v^h and recalling that $w \neq 0$ so that $\nabla w \neq 0$ as well, we see that:

$$\int_{S^h} |\nabla v^h|^2 \, \mathrm{d}x \geq \frac{1}{2} \int_{S^h} |\nabla w|^2 \, \mathrm{d}x - c_2 h^3 \geq c_1 h.$$

The two last inequalities yield the claim. ∎

3.9 Uniformity of Korn's constant under tangential boundary conditions in thin shells

We have shown in Example 3.34 that Korn's constants C in (3.1) may converge to infinity as the thickness h of a thin shell S^h converges to 0. In this section, we prove that this blow-up, under tangential boundary condition is only due to the presence of Killing vector fields. In particular, if the Killing fields are treated as the kernel of the rigidity estimate, then the corresponding constants C_h on S^h are uniform in h.

As in section 3.6, we consider a family $\{S^h\}_{h>0}$ of thin shells around a smooth, closed hypersurface $S \subset \mathbb{R}^N$, given by:

$$S^h = \{x = z + t\mathbf{n}(z);\ z \in S,\ -g_1^h(z) < t < g_2^h(z)\}, \tag{3.56}$$

whose boundary is determined by some positive functions $\{g_1^h, g_2^h \in \mathscr{C}^1(S,\mathbb{R})\}_{h>0}$.

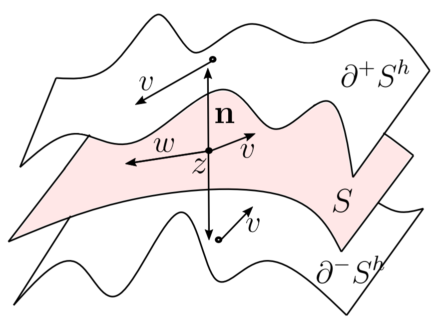

Fig. 3.3 The midsurface S and the lower and upper boundaries in Theorem 3.35.

By $\mathbf{n}^h$ we denote the outward unit normal to ∂S^h, while $\mathbf{n}$ is the unit normal to S. Recall the following spaces of Killing fields on S:

$$\begin{aligned} &\mathscr{I}(S) = \big\{w \in H^1(S,\mathbb{R}^N);\ w(z) \in T_z S \ \text{ and } \ \mathrm{sym}\nabla w(z) = 0 \ \text{ for all } \ z \in S\big\}, \\ &\mathscr{I}_{g_1,g_2}(S) = \big\{w \in \mathscr{I}(S);\ \langle w(z), \nabla(g_1+g_2)(z)\rangle = 0 \quad \text{for all } \ z \in S\big\}. \end{aligned} \tag{3.57}$$

We have the following first main result, which can be seen as the homogeneous and thickness-independent version of the Korn-Poincaré inequality in Corollary 3.26:

Theorem 3.35. *Let $S \subset \mathbb{R}^N$ be a smooth, closed hypersurface and let the boundary functions of $\{S^h\}_{h>0}$ in (3.56) satisfy, with constants $C_1, C_2, C_3 > 0$:*

$$C_1 h \leq g_i^h(z) \leq C_2 h, \quad |\nabla g_i^h(z)| \leq C_3 h \qquad \text{for all } z \in S, \quad h \ll 1. \tag{3.58}$$

Let $\alpha \in [0,1)$. Then, for all $v \in H^1(S^h, \mathbb{R}^N)$ satisfying one of the conditions:

$$\begin{aligned} &\langle v, \mathbf{n}^h \rangle = 0 \quad \text{on } \partial^+ S^h = \{z + g_2^h(z)\mathbf{n}(z);\ z \in S\}, \\ \text{or} \quad &\langle v, \mathbf{n}^h \rangle = 0 \quad \text{on } \partial^- S^h = \{z - g_1^h(z)\mathbf{n}(z);\ z \in S\}, \end{aligned} \tag{3.59}$$

and also satisfying:

$$\Big| \int_{S^h} \langle v(x), w(\pi(x)) \rangle \,\mathrm{d}x \Big| \leq \alpha \|v\|_{L^2(S^h)} \|w\pi\|_{L^2(S^h)} \quad \text{for all } w \in \mathscr{I}(S), \tag{3.60}$$

there holds, with C independent of v and $h \ll 1$:

$$\|\nabla v\|_{H^1(S)} \leq C \|\mathrm{sym}\nabla v\|_{L^2(S^h)}.$$

As shown in the second result, replacing condition (3.58) by a more restrictive requirement (3.61) below, one can prove uniform Korn's inequality for a larger class of vector fields, namely those forming a cone and satisfying the angle condition (3.59) with the subspace $\mathscr{I}_{g_1,g_2}(S)$ rather than the whole $\mathscr{I}(S)$. We note that (3.61) implies (3.58) upon taking $C_1 \doteq \frac{1}{2}\min\{g_i(z);\ z \in S,\ i = 1,2\}$, $C_2 \doteq 2\max_i \|g_i\|_{L^\infty(S)}$ and $C_3 \doteq \max_i \|\nabla g_i\|_{L^\infty(S)} + 1$. Our second main result is:

Theorem 3.36. *Let $S \subset \mathbb{R}^N$ be a smooth, closed hypersurface and let the boundary functions of $\{S^h\}_{h>0}$ in (3.56) satisfy, with constants $C_1, C_2, C_3 > 0$:*

$$\frac{1}{h} g_i^h \to g_i \quad \text{in } \mathscr{C}^1(S, \mathbb{R}) \quad \text{as } h \to 0, \quad \text{for } i = 1,2. \tag{3.61}$$

Let $\alpha \in [0,1)$. Then, for all $v \in H^1(S^h, \mathbb{R}^N)$ satisfying $\langle v, \mathbf{n}^h \rangle = 0$ on ∂S^h and:

$$\Big| \int_{S^h} \langle v(x), w(\pi(x)) \rangle \,\mathrm{d}x \Big| \leq \alpha \|v\|_{L^2(S^h)} \|w\pi\|_{L^2(S^h)} \text{ for all } v \in \mathscr{I}_{g_1,g_2}(S), \tag{3.62}$$

there holds, with C independent of v and $h \ll 1$:

$$\|\nabla v\|_{H^1(S)} \leq C \|\mathrm{sym}\nabla v\|_{L^2(S^h)}.$$