# EQUIVALENCE OF THE SUPPORTING SPHERES CONDITION AND THE $C^{1,1}$ REGULARITY OF THE BOUNDARY

MARTA LEWICKA AND YUVAL PERES

Abstract. We prove equivalence of the $C^{1,1}$ regularity of a bounded domain $\Omega \subset \mathbb{R}^N$, with the two-sided, uniform radius supporting spheres condition. We relate the maximal radius of the supporting spheres to the optimal global Lipschitz constant (in the Euclidean metric of the ambient space $\mathbb{R}^N$) of the unit normal vector on $\partial\Omega$. These results are not surprising and indeed rather folklore in Analysis, however their elementary and self-contained proofs below are, to our best knowledge, new. We also offer an extension to $L^p$ spaces, $p \in (1, \infty)$.

## 1. Introduction

In this note, we propose a simple proof of the equivalence of two geometric boundary regularity conditions that are often used in Analysis and PDEs. Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected, nonempty set. The first condition is called the *uniform (two-sided) supporting sphere condition*:

$$\text{(S)} \quad \left[ \begin{array}{l} \text{For some radius } r > 0 \text{ we have: for every } x_0 \in \partial\Omega \text{ there exist } a, b \in \mathbb{R}^N \text{ with:} \\ \qquad B_r(a) \subset \Omega, \qquad B_r(b) \subset \mathbb{R}^N \setminus \bar{\Omega} \quad \text{and} \quad |x_0 - a| = |x_0 - b| = r. \end{array} \right.$$

The second condition is the $C^{1,1}$ *regularity of* $\partial\Omega$:

$$\text{(C)} \quad \left[ \begin{array}{l} \text{For every } x_0 \in \partial\Omega \text{ there exist } \rho, h > 0 \text{ and a rigid map } T : \mathbb{R}^N \to \mathbb{R}^N \text{ with } T(x_0) = 0, \\ \text{along with a } C^{1,1} \text{ function } \phi : \mathbb{R}^{N-1} \supset B_\rho(0) \to (-h, h) \text{ with } \phi(0) = 0,\ \nabla\phi(0) = 0, \text{ such} \\ \text{that the following holds. Consider the cylinder } \mathcal{C} = B_\rho(0) \times (-h, h) \subset \mathbb{R}^N, \text{ then:} \\ \qquad \mathcal{C} \cap T(\Omega) = \{(x', x_N) \in \mathcal{C};\ x_N < \phi(x')\}, \\ \qquad \mathcal{C} \cap T(\partial\Omega) = \{(x', x_N) \in \mathcal{C};\ x_N = \phi(x')\}. \end{array} \right.$$

We recall that $T$ is a rigid map when it is a composition of rotation and translation: $T(x) = Ax + b$ for some $A \in SO(N)$ and $b \in \mathbb{R}^N$. A function $\phi$ is said to be of class $C^{1,1}$ if it is differentiable and its gradient is Lipschitz continuous, in the domain where $\phi$ is defined.

The geometric meaning of both conditions (S) and (C) is explained in Figure 1.1. Condition (C) states that, locally, $\partial\Omega$ coincides with the graph of a $C^{1,1}$ function $\phi$. In particular, the external unit normal vector $\vec{n} : \partial\Omega \to \mathbb{R}^N$ is then well defined and given in the local coordinates of $\phi$ (for simplicity we assume here that $T(x) = x$) by:

$$\vec{n}(x', \phi(x')) = \frac{(-\nabla\phi(x'), 1)}{\sqrt{|\nabla\phi(x')|^2 + 1}}$$

It is easy to observe that $\vec{n}$ is Lipschitz, as it is a composition of two Lipschitz functions: the projection on $x_N = 0$ and the vector field given in the right hand side of the above formula. Our goal is to give an elementary, self-contained proof of the following result:

*Date*: April 24, 2018.

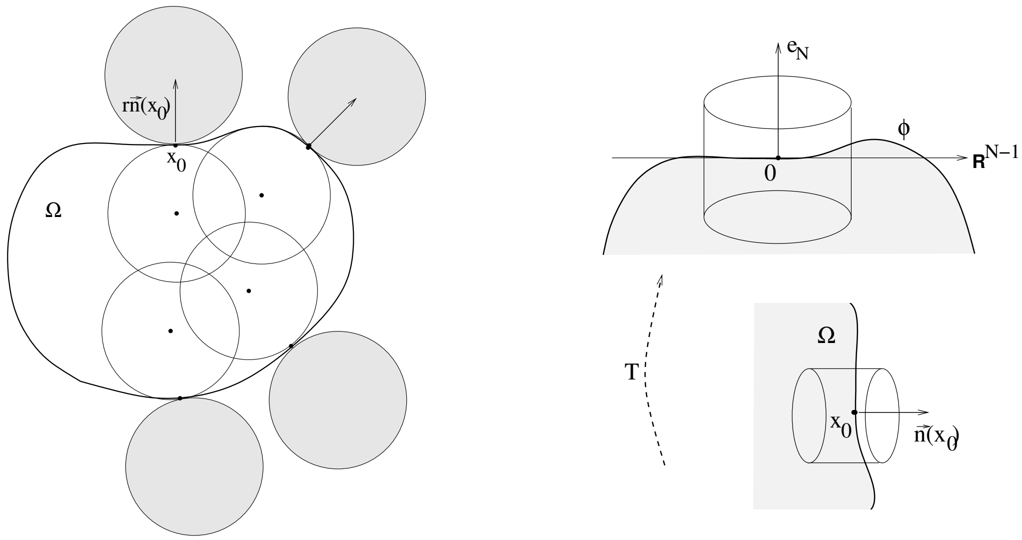


FIGURE 1.1. The uniform two-sided supporting spheres condition (S), and the boundary $C^{1,1}$ regularity condition (C).

**Theorem 1.** *Conditions (S) and (C) are equivalent and the global Lipschitz constant of $\vec{n}$ can be taken as the inverse of the uniform supporting spheres radius:*

$$|\vec{n}(x_0) - \vec{n}(y_0)| \leq \frac{1}{r}|x_0 - y_0| \qquad \text{for all } x_0, y_0 \in \partial\Omega. \tag{1.1}$$

The estimate (1.1) is sharp and achieved for $\Omega = B_r(0)$ where we have $\vec{n}(x_0) = \frac{1}{r}x_0$. It is also consistent with the well known fact that if $\phi \in C^2$, $\phi(0) = 0$, $\nabla\phi(0) = 0$, then the surface patch given by the graph of $\phi$ has, at $x_0 = 0$, an external supporting sphere of radius $r$, for any $r > 0$ satisfying $\frac{1}{r} \geq \lambda_{max}$, where $\lambda_{max}$ is the maximal eigenvalue of the symmetric matrix $\nabla^2\phi(0)$. More generally, the two-sided supporting sphere radius $r$ at a point $x_0$ of a $C^2$ surface $S$ is the inverse of the largest (in absolute value) eigenvalue of the second fundamental form of $S$ at $x_0$.

In Theorem 1 we show that this (local) optimal statement persists globally, in connection with the Lipschitz constant of $\vec{n}$, rather than that of $\nabla\phi$. Indeed, the smooth "thin neck" set in Figure 1.2 has small Lipschitz constant of each $\nabla\phi$ (and thus small local Lipschitz constant of the unit normal vector $\vec{n}$), but it does not allow the internal supporting sphere radius exceed a prescribed $0 < \delta \ll 1$. On the other hand, the global Lipschitz constant of $\vec{n}$ in this example must be at least $\frac{|\vec{n}(x_0)-\vec{n}(y_0)|}{|x_0-y_0|} = \frac{2}{2\delta}$, so there is no contradiction with (1.1).

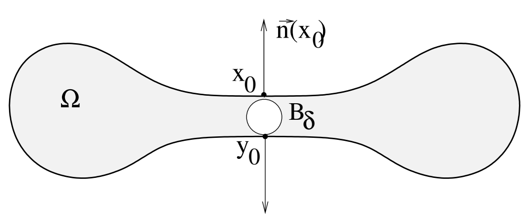


FIGURE 1.2. For each $\phi$ in the local construction in (C), we have $\|\nabla\phi\|_{\mathcal{C}^0} \leq 1$, but the maximal internal ball at $x_0$ has only a small radius $\delta \ll 1$.

For completeness, we mention that an entirely equivalent notion of $C^{1,1}$ regularity of $\partial\Omega$ is:

(C') $\left[\begin{array}{l} \text{For every } x_0 \in \partial\Omega \text{ there exist } \rho, h > 0 \text{ and a } C^{1,1} \text{ diffeomorphism } \Phi : \mathcal{C} \to U \text{ between} \\ \text{cylinder } \mathcal{C} = B_\rho(0) \times (-h, h) \subset \mathbb{R}^N \text{ and an open neighbourhood } U \subset \mathbb{R}^N \text{ of } x_0 \text{, so that:} \\ \qquad \Phi\big((x', x_N) \in \mathcal{C};\ x_N < 0\big) = U \cap \Omega, \\ \qquad \Phi\big((x', x_N) \in \mathcal{C};\ x_N = 0\big) = U \cap \partial\Omega, \end{array}\right.$

We recall that $\Phi$ is a $C^{1,1}$ diffeomorphism when it is invertible and when both $\Phi$ and $\Phi^{-1}$ have regularity $C^{1,1}$. We leave proving the equivalence of (C) and (C') as a (not hard) exercise.

Condition (C') is at the heart of the technique of "straightening the boundary". This technique is familiar to analysts and it relies on reducing an argument (e.g. constructing extension operator, deriving estimates on a solution to some PDE) needed in a proximity of a boundary point of a domain, to the much simpler case of a flat boundary to the half-space. The two cases are then related via the diffeomorphism $\Phi$ with controlled derivatives (of course, sometimes one needs more than just one derivative!). On the other hand, it is often more straightforward to deal with the geometric condition (S) rather than with charts and requirements on $\phi$ in (C) or $\Phi$ in (C').

We close the discussion by observing that the key ingredient in the proof of (S) implying (1.1), called the "four ball lemma", remains valid in the Lebesgue spaces $L^p$, $p \in (1, \infty)$. This will be derived in Section 5. As a consequence, we obtain that if a bounded domain $\Omega \subset L^p$ satisfies (S), then its unit normal vector is Hölder continuous, with exponent $2/p$ for $p \geq 2$ and $p/2$ for $p \leq 2$.

To our best knowledge, the proofs in this note are new, though equivalence of (S) and (C) in the finite-dimensional case (without the sharp bound (1.1) and at the expense of more complicated calculations) has been shown in [2]. Similarly, Section 1.2 in [4] discusses the relation of the tangent paraboloids (rather than spheres) condition and second order differentiability, yielding (locally and without specifying the constants) equivalence of (S) and (C).

## 2. The four ball lemma

The key ingredient of the proof of Theorem 1 is a geometrical lemma about four balls in $\mathbb{R}^N$. The balls have the same radius $r > 0$ and they come in two couples (see Figure 2.1), with balls in each couple tangent to each other. It turns out that if we change the pairings and ensure that the two balls in each newly formed pair are disjoint, then the directions perpendicular to the tangency planes differ at most by the distance between the tangency points. More precisely:

**Lemma 2 (The four ball lemma).** *Let $x, u, v \in \mathbb{R}^N$ with $|u| = |v| = r > 0$. Assume that $B_r(x+u) \cap B_r(-x-v) = \emptyset$ and $B_r(x-u) \cap B_r(-x+v) = \emptyset$. Then $|u - v| \leq 2|x|$.*

*Proof.* Consider the vector $z = x + \frac{u+v}{2}$ and observe that:

$$\frac{u-v}{2} = (x+u) - z = (-x-v) + z \in \bar{B}_{|z|}(x+u) \cap \bar{B}_{|z|}(-x-v),$$

so that the first disjointness assumption gives:

$$r^2 \leq |z|^2 = \left|x + \frac{u+v}{2}\right|^2.$$

Exchange $x$ with $-x$ and $u$ with $v$, and apply the second disjointness assumption to obtain:

$$r^2 \leq \left|x - \frac{u+v}{2}\right|^2.$$

Finally, summing the two above inequalities and using (twice) the parallelogram identity, we get:

$$\begin{aligned}2r^2 \leq \Big|x+\frac{u+v}{2}\Big|^2 + \Big|x-\frac{u+v}{2}\Big|^2 &= 2|x|^2 + 2\Big|\frac{u+v}{2}\Big|^2 = 2|x|^2 + \frac{1}{2}|u+v|^2 \\ &= 2|x|^2 + \frac{1}{2}\big(2|u|^2+2|v|^2-|u-v|^2\big).\end{aligned} \tag{2.1}$$

Recalling that $|u| = |v| = r$, this results in $|u-v|^2 \leq 4|x|^2$. The claim is proved. ■

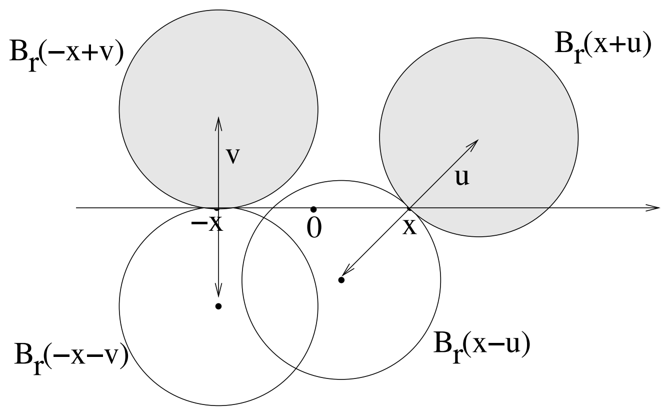


FIGURE 2.1. The four balls in Lemma 2: the "vertical" couples are tangential, the "diagonal" couples are disjoint.

A similar result remains valid for domains $\Omega \subset L^p$ in Lebesgue spaces $L^p(Z)$, $p \in (1, \infty)$, as will be shown in Lemma 8 in Section 5.

## 3. A proof of Theorem 1: (C) implies (S)

In this and the next Sections we propose a proof of Theorem 1. Its first implication follows through a straightforward calculus argument below; in particular it does not require Lemma 2.

**Lemma 3.** *Assume condition (C) at a given boundary point $x_0 = 0 \in \partial\Omega$, where $T = id$. Then:*

$$|\phi(x')| \leq \frac{\max_{\bar{B}_{|x'|}(0)} |\nabla\phi|^2 + 1}{2r}|x'|^2 \qquad \text{for all } x' \in B_\rho(0), \tag{3.1}$$

*for every $r > 0$ satisfying (1.1). In particular, $x_0$ has some supporting balls $B_{\delta_0}(0, \delta_0)$, $B_{\delta_0}(0, -\delta_0)$.*

*Proof.* Since $T = id$, the bound (1.1) implies, in view of $x_0 = 0$ and $\vec{n}(x_0) = (0, 1)$:

$$|\vec{n}(x', \phi(x')) - (0,1)|^2 \leq \frac{1}{r^2}\big(|x'|^2 + |\phi(x')|^2\big).$$

Consequently, for all $x' \in B_\rho(0)$ there holds:

$$|\nabla\phi(x')|^2 \leq \frac{|\nabla\phi(x')|^2+1}{r^2}\big(|x'|^2+|\phi(x')|^2\big) \leq \frac{\big(\max_{\bar{B}_{|x'|}(0)} |\nabla\phi|^2+1\big)^2}{r^2}|x'|^2, \tag{3.2}$$

where we have used the fact that $\phi(0) = 0$ and $\nabla\phi(0) = 0$ to get:

$$|\phi(x')| = \Big|\int_0^1 \langle\nabla\phi(tx'), x'\rangle \ \mathrm{d}t\Big| \leq |x'| \int_0^1 |\nabla\phi(tx')| \ \mathrm{d}t.$$

Applying (3.2) in the right hand side above, we derive (3.1) and deduce that the graph of $\phi$ is contained between some parabolas $x' \mapsto \pm C|x'|^2$. It easily follows that for $\delta_0 = \frac{1}{2C}$:

$$(3.3) \qquad |(x', \phi(x')) - (0, \pm\delta_0)|^2 \geq \frac{1}{C}|\phi(x')| + |\phi(x') \mp \delta_0|^2 \geq |\phi(x')|\big(\frac{1}{C} - 2\delta_0\big) + \delta_0^2 = \delta_0^2,$$

hence the balls $B_{\delta_0}(0, \delta_0)$ and $B_{\delta_0}(0, -\delta_0)$ are supporting at $x_0$. ■

**Lemma 4.** *Assume that $\Omega$ satisfies (C). Then (S) holds, for every radius $r > 0$ satisfying (1.1).*

*Proof.* Assume (C) and (1.1). For a boundary point $x_0 \in \partial\Omega$, let $\phi$ be as described in condition (C), where without loss of generality we take $T = id$, so that we may use Lemma 3. We argue by contradiction. If $B_r(0, r)$ was not supporting, then $B_r(0, r) \cap \Omega \neq \emptyset$ and further:

$$\bar{r} \doteq \inf \big\{\delta \geq \delta_0;\ \bar{B}_\delta(0, \delta) \cap \partial\Omega \neq \{0\}\big\} < r.$$

Take a sequence of radii: $\delta_n \searrow \bar{r}$ as $n \to \infty$, and a sequence of points: $y_n \in \bar{B}_{\delta_n}(0, \delta_n) \cap \big(\partial\Omega \setminus \{0\}\big)$. Without loss of generality, $y_n \to y_0 \in \bar{B}_{\bar{r}}(0, \bar{r}) \cap \partial\Omega$. By the minimality of $\bar{r}$, there must be:

$$y_0 \in \partial B_{\bar{r}}(0, \bar{r}) \cap \partial\Omega.$$

Since the inward normal $-\vec{n}(y_0)$ coincides with the normal to $\partial B_{\bar{r}}(0, \bar{r})$ at $y_0$ and since the same is valid at $x_0 = 0$, we get the following equality, which in virtue of (1.1) implies:

$$\Big|\frac{1}{\bar{r}}y_0 - \frac{1}{\bar{r}}x_0\Big| = \big|\vec{n}(x_0) - \vec{n}(y_0)\big| \leq \frac{1}{r}|x_0 - y_0|.$$

This results in the contradictory statement $\bar{r} \geq r$, provided that we prove $y_0 \neq x_0$. To this end, it suffices to show that $\{y_n\}$ must be bounded away from 0. Let $\sigma > 0$ satisfy:

$$\big(\sup_{\bar{B}_\sigma(0)} |\nabla\phi|^2 + 1\big)\sigma^2 \leq 2r\bar{r} \leq 2r\delta_n \qquad \text{for all } n.$$

Using (3.1) we refine the bound in (3.3) on $x' \in B_\sigma(0)$, to:

$$\begin{aligned} |(x', \phi(x')) - (0, \delta_n)|^2 = |x'|^2 + |\delta_n - \phi(x')|^2 &\geq |x'|^2 + \Big|\delta_n - \frac{\sup_{\bar{B}_\sigma(0)} |\nabla\phi|^2 + 1}{2r}|x'|^2\Big|^2 \\ &\geq |x'|^2 + \delta_n^2 - \frac{\delta_n}{r}\big(\sup_{\bar{B}_\sigma(0)} |\nabla\phi|^2 + 1\big)|x'|^2. \end{aligned}$$

Note that for all large $n$, the ratio $\frac{\delta_n}{r}$ is smaller than and bounded away from 1. It follows that for sufficiently small $\sigma > 0$ there also holds: $\frac{\delta_n}{r}\big(\sup_{\bar{B}_\sigma(0)} |\nabla\phi|^2 + 1\big) < 1$ for all $n$. Consequently:

$$|(x', \phi(x')) - (0, \delta_n)|^2 \geq \delta_n^2 \qquad \text{for all } x' \in B_\sigma(0) \quad \text{and all } n,$$

where equality only takes place at $x' = 0$. This implies that $|y_n'| \geq \sigma$ and hence $|y_0| \geq \sigma$ as well, as claimed. We conclude that the ball $B_r(0, r)$ is external supporting. The proof that $B_r(0, -r)$ is an internal supporting ball follows in the same manner. ■

## 4. A proof of Theorem 1: (S) implies (C)

The reverse implication uses Lemma 2 in several places. In particular, we have:

**Lemma 5.** *Assume that $\Omega$ satisfies (S). For each $x_0 \in \partial\Omega$, define $p(x_0) = \frac{b-a}{|b-a|}$. Then:*

$$|p(x_0) - p(y_0)| \leq \frac{1}{r}|x_0 - y_0| \qquad \textit{for all } \ x_0, y_0 \in \partial\Omega.$$

*Proof.* We first observe that the function $p : \partial\Omega \to \mathbb{R}^N$ is indeed well defined, in view of (S). Applying, if needed, a rigid transformation that maps a given $x_0, y_0 \in \partial\Omega$ to some symmetric points $x$ and $-x$, we now use Lemma 2 to $u = rp(x_0)$, $v = rp(y_0)$ and conclude that $|rp(x_0) - rp(y_0)| \leq |x_0 - y_0|$. This proves the claim. ∎

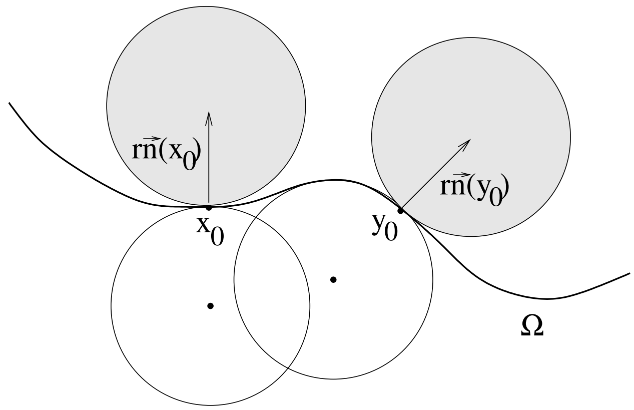


FIGURE 4.1. The supporting balls at two boundary points $x_0$ and $y_0$. The "external" supporting balls are shaded.

**Lemma 6.** *Assume that $\Omega$ satisfies (S). Then (C) holds, together with (1.1).*

*Proof.* By applying some rigid transformation $T$, we may without loss of generality assume that $x_0 = 0$ and $p(0) = e_N$, with the vector field $p$ given in Lemma 5. We now claim that choosing $h \gg \delta > 0$ sufficiently small, for each $x' \in B_\delta(0) \subset \mathbb{R}^{N-1}$ there is a unique $\phi(x') \in \mathbb{R}$, satisfying:

$$(x', \phi(x')) \in \partial\Omega \cap \mathcal{C}, \qquad \text{where } \ \mathcal{C} = B_\delta(0) \times (-h, h).$$

Indeed, when $\delta < \frac{r}{2}$ then the line $x' + \mathbb{R}e_N$ intersects both supporting balls at 0, so for a small fixed $h$ and sufficiently small $\delta$, there is at least one $\phi(x')$ with the indicated property. To prove uniqueness, take $\bar{\delta}$ small enough for Lemma 5 to guarantee that the angle between $p(y_0)$ and $e_N$ is less that $r\frac{\pi}{4}$ for all $y_0 \in \partial\Omega \cap B_{\bar{\delta}}(0)$. Then, if $y_0, \bar{y}_0 \in \partial\Omega \cap B_{\bar{\delta}}(0)$ satisfy $\bar{y}_0 - y_0 = te_N$ for some $t > 0$, it follows that $\bar{y}_0$ belongs to the external supporting ball at $y_0$, contradicting that $\bar{y}_0 \in \partial\Omega$. It now suffices to ensure that $\mathcal{C} \subset B_{\bar{\delta}}(0)$ by taking $h, \delta \ll 1$.

We have thus defined the function $\phi$ whose graph locally coincides with $\partial\Omega$. By (S), $\phi$ must be continuous, differentiable and also:

$$\vec{n}(x', \phi(x')) = \frac{(-\nabla\phi(x'), 1)}{\sqrt{|\nabla\phi(x')|^2 + 1}} = p(x', \phi(x')) \qquad \text{for all } x' \in \partial B_\delta(0).$$

The displayed formula and Lemma 5 imply that, for every $j = 1, \ldots, N-1$, the function:

$$\partial_i\phi(x') = \frac{\langle p(x', \phi(x')), e_j\rangle}{\langle p(x', \phi(x')), e_N\rangle}$$

is continuous on $B_\delta(0)$ as the quotient of two continuous functions, whose denominator is bounded away from 0. It now follows that functions involved in the above quotient are actually Lipschitz, so $\nabla\phi$ must be Lipschitz as well. This ends the proof of $\phi \in C^{1,1}$ and the proof of the Lemma. ∎

From the proof of Lemma 4 and Lemma 5 one can easily deduce the following geometric statement, which is yet another equivalent version of (S):

**Corollary 7.** *(S) is equivalent to the* Lipschitz regularity of the normal vector *condition:*

(N) *For every $x_0 \in \partial\Omega$, the normal vector $\vec{n}(x_0)$ is well defined and it satisfies (1.1).*

## 5. An extension of the four ball lemma to the Lebesgue spaces

We now discuss an extension of our main results to infinitely dimensional Banach spaces. We focus on the Lebesgue spaces $L^p = L^p(Z)$ where $Z$ is an arbitrary measure space; we denote by $\|u\|$ the $L^p$ norm of a given $u \in L^p$. Observe that the four ball lemma (Lemma 2) remains valid in any inner product space, i.e. in a space where one can use the parallelogram identity. Hence, Lemma 2 holds true in $L^2$. On the other hand, for $p \neq 2$ we still get the following:

**Lemma 8 (The four ball lemma in $L^p$).** *Let $x, u, v \in L^p$ with $\|u\| = \|v\| = r > 0$. Assume that $B_r(x+u) \cap B_r(-x-v) = \emptyset$ and $B_r(x-u) \cap B_r(-x+v) = \emptyset$. Then:*

$$\begin{aligned} &\|u-v\|^p \leq 2^{p-1}p(p-1)r^{p-2}\|x\|^2 && \text{for } p \in [2,\infty) \\ &\|u-v\|^2 \leq \frac{8}{p(p-1)} r^{2-p}\|x\|^p && \text{for } p \in (1,2]. \end{aligned} \tag{5.1}$$

*Proof.* By the same reasoning as in the first part of the proof of Lemma 2, we obtain:

$$r \leq \Big\|x + \frac{u+v}{2}\Big\| \qquad \text{and} \qquad r \leq \Big\|x - \frac{u-v}{2}\Big\|.$$

We will now replace the parallelogram identity argument in Lemma2, with a somewhat more involved estimate, derived separately in the two cases indicated in (5.1).

When $p \geq 2$, we use first the 2-uniform smoothness inequality [1, Proposition 3], followed by the first Clarkson's inequality [3, page 95] and recall that $\|u\| = \|v\| = r$, to get:

$$\begin{aligned} 2r^2 &\leq \Big\|x + \frac{u+v}{2}\Big\|^2 + \Big\|x - \frac{u+v}{2}\Big\|^2 \leq 2(p-1)\|x\|^2 + 2\Big\|\frac{u+v}{2}\Big\|^2 \\ &\leq 2(p-1)\|x\|^2 + 2\Big(\frac{1}{2}\|u\|^p + \frac{1}{2}\|v\|^p - \Big\|\frac{u+v}{2}\Big\|^p\Big)^{2/p} \\ &= 2(p-1)\|x\|^2 + 2\Big(r^p - \Big\|\frac{u+v}{2}\Big\|^p\Big)^{2/p}. \end{aligned} \tag{5.2}$$

Further, concavity of the function $a \mapsto a^{2/p}$ implies that $(a-b)^{2/p} \leq a^{2/p} - \frac{2}{p}a^{(2/p)-1}b$ whenever $a, b, a-b \geq 0$. Applying this bound to $a = r^p$ and $b = \|\frac{u-v}{2}\|^p$ in (5.2), yields:

$$2r^2 \leq 2(p-1)\|x\|^2 + 2\Big(r^2 - \frac{2}{p}r^{2-p}\Big\|\frac{u-v}{2}\Big\|^p\Big),$$

which directly results in (5.1) in case $p \in [2,\infty)$.

For $p \leq 2$, we denote the conjugate exponent by $q = \frac{p}{p-1}$ and use the second Clarkson's inequality [3, page 97], followed by the 2-uniform smoothness inequality [1, Proposition 3] to get:

$$\begin{aligned} 2r^q &\leq \Big\|x + \frac{u+v}{2}\Big\|^q + \Big\|x - \frac{u+v}{2}\Big\|^q \leq 2\Big(\|x\|^p + \Big\|\frac{u+v}{2}\Big\|^p\Big)^{q/p} = 2\Big(\|x\|^p + \Big(\Big\|\frac{u+v}{2}\Big\|^2\Big)^{p/2}\Big)^{q/p} \\ &\leq 2\Big(\|x\|^p + \Big(\frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - (p-1)\Big\|\frac{u-v}{2}\Big\|^2\Big)^{p/2}\Big)^{q/p} \\ &= 2\Big(\|x\|^p + \Big(r^2 - (p-1)\Big\|\frac{u-v}{2}\Big\|^2\Big)^{p/2}\Big)^{q/p}. \end{aligned}$$

As before, concavity of the function $a \mapsto a^{p/2}$ applied to $a = r^2$ and $b = (p-1)\|\frac{u-v}{2}\|^2$, yields:

$$2r^q \leq 2\Big(|x|^p + r^p - \frac{(p-1)p}{2} r^{p-2}\Big\|\frac{u-v}{2}\Big\|^2\Big)^{q/p}.$$

Dividing both sides of the above inequality by $2r^q$ we obtain:

$$1 \leq \frac{|x|^p}{r^p} + 1 - \frac{(p-1)p}{2} r^{-2}\Big\|\frac{u-v}{2}\Big\|^2$$

which directly gives (5.1) in case $p \in (1, 2]$. ■

By the same reasoning as in Lemma 5, there follows Hölder's regularity of the normal vector to domains satisfying the uniform supporting sphere condition:

**Corollary 9.** *Let $\Omega \subset L^p$ be a nonempty, open, bounded and connected domain that satisfies the condition (S). Then the unit normal vector, defined for each $x_0 \in \partial\Omega$ by $p(x_0) = \frac{b-a}{\|b-a\|}$, is Hölder continuous with exponent:* $\min\{2/p, p/2\}$. *More precisely, we have:*

$$\|p(x_0) - p(y_0)\| \leq C_p \cdot \begin{cases} \dfrac{1}{r^{2-2/p}} \|x_0 - y_0\|^{2/p} & \text{for } p \in [2, \infty) \\ \dfrac{1}{r^{p/2}} \|x_0 - y_0\|^{p/2} & \text{for } p \in (1, 2], \end{cases}$$

*for all $x_0, y_0 \in \partial\Omega$, and with a constant $C_p > 0$ that depends only on $p \in (1, \infty)$.*

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Marta Lewicka, University of Pittsburgh, Department of Mathematics, 139 University Place, Pittsburgh, PA 15260

Yuval Peres, Microsoft Research, One Microsoft Way, WA 98052
*E-mail address*: lewicka@pitt.edu, peres@microsoft.com