WHICH DOMAINS HAVE TWO-SIDED SUPPORTING UNIT SPHERES AT EVERY BOUNDARY POINT?

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Abstract. We prove the quantitative equivalence of two important geometrical conditions, pertaining to the regularity of a domain $\Omega \subset \mathbb{R}^N$. These are: (i) the uniform two-sided supporting sphere condition, and (ii) the Lipschitz continuity of the outward unit normal vector. In particular, the answer to the question posed in our title is: "Those domains whose unit normal is well defined and has Lipschitz constant one." We also offer an extension to infinitely dimensional spaces L^p, $p \in (1, \infty)$.

1. Introduction

In this note, we prove the quantitative equivalence of two geometric boundary regularity conditions, that are often used in Analysis and PDEs: the uniform supporting sphere condition and the Lipschitz continuity of the unit normal vector. This latter condition is often referred to as the $C^{1,1}$ regularity of the domain Ω. In particular, the answer to the question in the title: "Which domains have two-sided supporting unit spheres at every boundary point?" is: "Those whose outward unit normal vector is well defined and has Lipschitz constant one."

Given $r > 0$, we say that $\Omega \subset \mathbb{R}^N$ satisfies the *two-sided supporting r-sphere condition*, if:

(S_r) For every $x_0 \in \partial\Omega$ there exist $a, b \in \mathbb{R}^N$ such that:

$$B_r(a) \subset \Omega, \qquad B_r(b) \subset \mathbb{R}^N \setminus \bar{\Omega} \quad \text{and} \quad |x_0 - a| = |x_0 - b| = r.$$

We also say that Ω satisfies the condition of *$1/r$-Lipschitz continuity of the normal vector*, if:

(L_r) The boundary $\partial\Omega$ is C^1 regular and the outward unit normal $\vec{n} : \partial\Omega \to \mathbb{R}^N$ obeys:

$$|\vec{n}(x_0) - \vec{n}(y_0)| \leq \frac{1}{r}|x_0 - y_0| \qquad \text{for all } x_0, y_0 \in \partial\Omega.$$

It is easy to note that for $\Omega = B_r(0)$ where (S_r) holds trivially, the property (L_r) holds as well. It turns out that this observation can be generalized, as stated in our main result below:

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be a domain (i.e. an open, connected set). For any $r > 0$, conditions (S_r) and (L_r) are equivalent.*

Recall that $\partial\Omega$ being C^1 regular, signifies that it is locally a graph of a C^1 function, namely:

(C^1) For every $x_0 \in \partial\Omega$ there exist $\rho, h > 0$ and a rigid map $T : \mathbb{R}^N \to \mathbb{R}^N$ with $T(x_0) = 0$, along with a C^1 function $\phi : \mathbb{R}^{N-1} \supset \bar{B}_\rho(0) \to (-h, h)$ such that $\phi(0) = 0$, $\nabla\phi(0) = 0$, and the following holds. Consider the cylinder $\mathcal{C} = B_\rho(0) \times (-h, h) \subset \mathbb{R}^N$, then:

$$\mathcal{C} \cap T(\Omega) = \{(x', x_N) \in \mathcal{C};\ x_N < \phi(x')\},$$
$$\mathcal{C} \cap T(\partial\Omega) = \{(x', x_N) \in \mathcal{C};\ x_N = \phi(x')\}.$$

Saying that T is a rigid map means that it is a composition of a rotation and a translation: $T(x) = Ax + b$ for some $A \in SO(N)$ and $b \in \mathbb{R}^N$. Further, a function ϕ is said to be of class C^1 if it is differentiable and its gradient is continuous in the domain where ϕ is defined. The geometric meaning of (S_r) and (C^1) is sketched in Figure 1.1. We note in passing that under condition (C^1), the outward unit normal $\vec{n}$ is always well defined and given in the local coordinates ϕ (for simplicity we assume here that $T(x) = x$) by:

$$\vec{n}(x', \phi(x')) = \frac{(-\nabla\phi(x'), 1)}{\sqrt{|\nabla\phi(x')|^2 + 1}}.$$

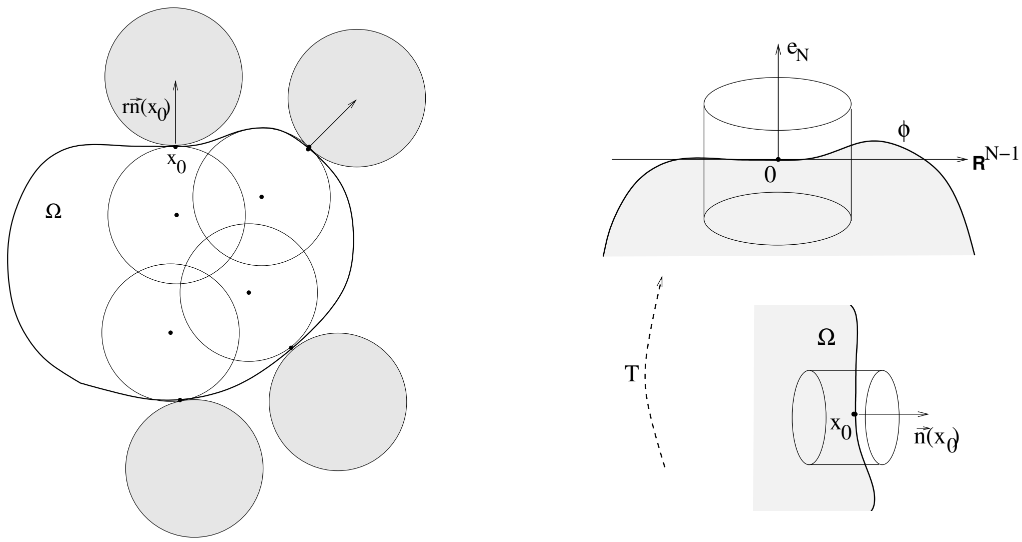

FIGURE 1.1. The two-sided supporting r-spheres condition (S_r), and the boundary regularity definitions (C^1) and ($\mathrm{C}^{1,1}$).

Recall that for a domain $\Omega \subset \mathbb{R}^2$ whose boundary coincides with a C^2 simple curve γ, parametrized by the arc-length s, the radius of curvature at a point $\gamma(s_0)$ is: $|\frac{\mathrm{d}\vec{n}}{\mathrm{d}s}(s_0)|^{-1}$. Similarly in higher dimensions, the radius of curvature at $x_0 \in \partial\Omega$ is the reciprocal of the Lipschitz constant of $\vec{n}$ on a neighbourhood of x_0, in the limit when the neighbourhood shrinks to the point $\{x_0\}$ itself. In Theorem 1 we argue that this local statement persists globally, in connection with the (global) Lipschitz constant of $\vec{n}$ rather than the (locally defined) curvature. Indeed, the smooth "thin neck" set in Figure 1.2 has small curvature (i.e. small local Lipschitz constant of $\vec{n}$ and, equivalently, small $\|\nabla\phi\|_{C^0}$), but it does not allow the radius of the internal supporting sphere to exceed a prescribed $0 < \delta \ll 1$. On the other hand, the global Lipschitz constant of $\vec{n}$ in this example must be at least $\frac{|\vec{n}(x_0)-\vec{n}(y_0)|}{|x_0-y_0|} = \frac{2}{2\delta}$, so there is no contradiction with Theorem 1.

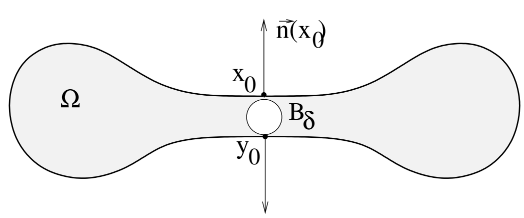

FIGURE 1.2. The normal vector $\vec{n}$ has local Lipschitz constant less than 1, but the maximal internal ball at x_0 has only a small radius $\delta \ll 1$.

The main result will be proved in Sections 2–4. In parallel, we will also deduce a consequence of Theorem 1, previously shown in [4, Section 1.2] and [2], via more complex calculations:

Corollary 2. *A bounded domain* $\Omega \subset \mathbb{R}^N$ *satisfies* (S_r)*, for some* $r > 0$*, if and only if there holds:*

$(\mathrm{C}^{1,1})$ $\Big[$ *The statement in* (C^1) *is valid with* $C^{1,1}$ *regular functions* ϕ*, i.e. with each* ϕ *having its gradient Lipschitz continuous.*

In Section 5 we will then adapt a key ingredient of the proofs of Theorem 1 and Corollary 2, called the "four ball lemma", to the setting of the Lebesgue spaces L^p, $p \in (1, \infty)$. As a result, we obtain that if a domain $\Omega \subset L^p$ satisfies (S_r) for some $r > 0$, then its outward unit normal vector is Hölder continuous, with exponent $2/p$ for $p \geq 2$ and $p/2$ for $p \leq 2$. In the last Section 6 we will gather some final remarks.

2. The four ball lemma

The key ingredient in the proof of Theorem 1 is a geometrical lemma about four balls in $\mathbb{R}^N$. The balls have the same radius $r > 0$ and they come in two couples (see Figure 2.1), with balls in the same couple tangent to each other. It turns out that if we change the pairings and ensure that the two balls in each newly formed pair are disjoint, then the directions perpendicular to the tangency planes differ at most by the distance between the tangency points. A similar result remains also valid for domains $\Omega \subset L^p(Z)$, $p \in (1, \infty)$, as will be shown in Lemma 9 in Section 5.

Lemma 3 (The four ball lemma). *Let* $x, u, v \in \mathbb{R}^N$ *with* $|u| = |v| = r > 0$*. Assume that* $B_r(x + u) \cap B_r(-x - v) = \emptyset$ *and* $B_r(x - u) \cap B_r(-x + v) = \emptyset$*. Then* $|u - v| \leq 2|x|$*.*

Proof. Define the vector $z = x + \frac{u+v}{2}$ and observe that:

$$\frac{u - v}{2} = (x + u) - z = (-x - v) + z \in \bar{B}_{|z|}(x + u) \cap \bar{B}_{|z|}(-x - v),$$

so that the first disjointness assumption yields:

$$r^2 \leq |z|^2 = \Big|x + \frac{u + v}{2}\Big|^2.$$

Exchange now x with $-x$ and u with v, and apply the second disjointness assumption to obtain:

$$r^2 \leq \Big|x - \frac{u + v}{2}\Big|^2.$$

Finally, summing the two above inequalities and using (twice) the parallelogram identity, we get:

$$\begin{aligned} 2r^2 \leq \Big|x + \frac{u + v}{2}\Big|^2 + \Big|x - \frac{u + v}{2}\Big|^2 &= 2|x|^2 + 2\Big|\frac{u + v}{2}\Big|^2 = 2|x|^2 + \frac{1}{2}|u + v|^2 \\ &= 2|x|^2 + \frac{1}{2}\big(2|u|^2 + 2|v|^2 - |u - v|^2\big), \end{aligned} \tag{2.1}$$

which results in $|u - v|^2 \leq 4|x|^2$ in view of $|u| = |v| = r$. The claim is proved. ■

As an immediate consequence, we derive the following:

Lemma 4. *Assume that a domain* $\Omega \subset \mathbb{R}^N$ *satisfies* (S_r)*. For each* $x_0 \in \partial\Omega$*, define* $\vec{p}(x_0) = \frac{b-a}{|b-a|}$*. Then:*

$$|\vec{p}(x_0) - \vec{p}(y_0)| \leq \frac{1}{r}|x_0 - y_0| \qquad \textit{for all} \;\; x_0, y_0 \in \partial\Omega.$$

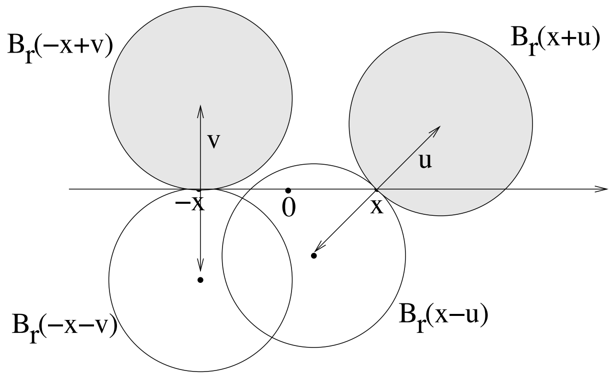

FIGURE 2.1. The four balls in Lemma 3: the "vertical" couples are tangential, the "diagonal" couples are disjoint.

Proof. We first observe that the function $\vec{p} : \partial\Omega \to \mathbb{R}^N$ is indeed well defined, in view of (S_r). Applying, if needed, a rigid transformation that maps a given $x_0, y_0 \in \partial\Omega$ to some symmetric points x and $-x$, we now use Lemma 3 to $u = r\vec{p}(x_0)$, $v = r\vec{p}(y_0)$ and conclude that $|r\vec{p}(x_0) - r\vec{p}(y_0)| \leq |x_0 - y_0|$. This proves the claim (see Figure 2.2). ∎

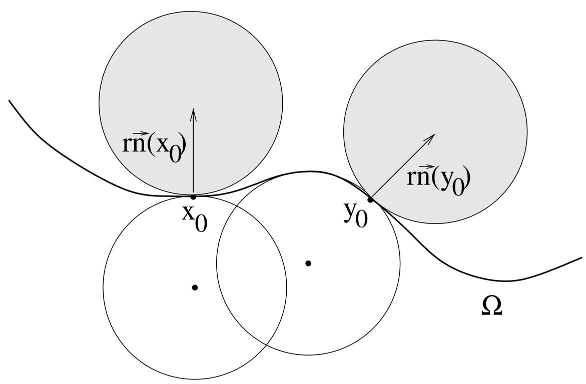

FIGURE 2.2. The supporting balls at two boundary points x_0 and y_0. The "external" supporting balls are shaded.

3. A PROOF OF THEOREM 1 AND COROLLARY 2: (S_r) IMPLIES (L_r) AND $(\mathrm{C}^{1,1})$

It is geometrically clear that, given (S_r), the normal vector $\vec{n}(x_0)$ must coincide with the, previously introduced, normalized shift between the two supporting balls at $x_0 \in \partial\Omega$:

$$\vec{p}(x_0) = \frac{b-a}{|b-a|},$$

and thus the statement in Lemma 4 is essentially that of (S_r) implies (L_r) in Theorem 1. Below, we verify this statement formally, together with the parallel implication in Corollary 2.

Lemma 5. *Assume that a domain $\Omega \subset \mathbb{R}^N$ satisfies (S_r). Then $\vec{p} = \vec{n}$ on $\partial\Omega$ and $(\mathrm{C}^{1,1})$ holds.*

Proof. By applying some rigid transformation T, we may without loss of generality assume that $x_0 = 0$ and $\vec{p}(0) = e_N$. We now claim that choosing $h \gg \delta > 0$ sufficiently small, for each

$x' \in B_\delta(0) \subset \mathbb{R}^{N-1}$ there exists a unique $\phi(x') \in \mathbb{R}$, satisfying:

$$(x', \phi(x')) \in \partial\Omega \cap \mathcal{C}, \qquad \text{where } \mathcal{C} = B_\delta(0) \times (-h, h).$$

Indeed, consider the two supporting balls $B_r(-re_N)$ and $B_r(re_N)$ at 0; when $\delta < \frac{r}{2}$ then the line $x' + \mathbb{R}e_N$ intersects both of them, so for a small fixed h and sufficiently small δ, there is at least one $\phi(x')$ with the indicated property. To prove uniqueness, take $\bar\delta < \frac{r}{2}$ small enough for Lemma 4 to guarantee that the angle between $\vec{p}(y_0)$ and e_N is less than $\frac{\pi}{4}$ for all $y_0 \in \partial\Omega \cap B_{\bar\delta}(0)$. Then, if $y_0, \bar y_0 \in \partial\Omega \cap B_{\bar\delta}(0)$ satisfy $\bar y_0 - y_0 = te_N$ for some $t \in (0, r)$, it follows that $\bar y_0$ belongs to the external supporting ball at y_0, contradicting the fact that $\bar y_0 \in \partial\Omega$. It now suffices to ensure that $\mathcal{C} \subset B_{\bar\delta}(0)$ by taking $h, \delta \ll 1$.

We have thus defined the function ϕ whose graph locally coincides with $\partial\Omega$. By similar arguments as above, (S_r) implies that ϕ must be continuous, differentiable and also:

$$\vec{n}(x', \phi(x')) = \frac{(-\nabla\phi(x'), 1)}{\sqrt{|\nabla\phi(x')|^2 + 1}} = \vec{p}(x', \phi(x')) \qquad \text{for all } x' \in \partial B_\delta(0).$$

The above formula and Lemma 4 give that, for every $j = 1, \ldots, N-1$, the function:

$$\partial_j\phi(x') = \frac{\langle \vec{p}(x', \phi(x')), e_j\rangle}{\langle \vec{p}(x', \phi(x')), e_N\rangle}$$

is continuous on $B_\delta(0)$ as the quotient of two continuous functions, whose denominator is bounded away from 0. It now follows that functions involved in the above quotient are actually Lipschitz, so $\nabla\phi$ must be Lipschitz as well. This ends the proof of $\phi \in C^{1,1}$ and the proof of the Lemma. ■

4. A proof of Theorem 1 and Corollary 2: both (L_r) and $(\mathrm{C}^{1,1})$ imply (S_r)

We now complete the proofs of our main results.

Lemma 6. *Let $\Omega \subset \mathbb{R}^N$ be a domain. At a given $x_0 = 0 \in \partial\Omega$, let ϕ represent the local coordinates of $\partial\Omega$ as in (C^1), and with $T = id$, so that $\phi(0) = 0$ and $\nabla\phi(0) = 0$. Then:*

$$|\phi(x')| \leq \frac{\max_{\bar B_{|x'|}(0)} |\nabla\phi|^2 + 1}{2r}|x'|^2 \qquad \text{for all } x' \in B_\rho(0) \tag{4.1}$$

is valid, in the following two cases:

(i) *condition (L_r) holds,*

(ii) *condition $(\mathrm{C}^{1,1})$ holds and $\frac{1}{r}$ bounds the Lipschitz constant of $\nabla\phi$ from above.*

Consequently, in both cases, x_0 has the supporting balls $B_{\delta_0}(0, \delta_0)$, $B_{\delta_0}(0, -\delta_0)$ with the radius:

$$\delta_0 = \min\Big\{\frac{r}{\max_{\bar B_\rho(0)} |\nabla\phi|^2 + 1}, \rho, \frac{h}{2}\Big\}.$$

Proof. Since $\vec{n}(x_0) = e_N$, condition (L_r) implies:

$$|\vec{n}(x', \phi(x')) - e_N|^2 \leq \frac{1}{r^2}\big(|x'|^2 + |\phi(x')|^2\big).$$

Consequently, in case (i), for all $x' \in B_\rho(0)$ there holds:

$$|\nabla\phi(x')|^2 \leq \frac{|\nabla\phi(x')|^2 + 1}{r^2}\big(|x'|^2 + |\phi(x')|^2\big) \leq \frac{\big(\max_{\bar B_{|x'|}(0)} |\nabla\phi|^2 + 1\big)^2}{r^2}|x'|^2, \tag{4.2}$$

where we have used the fact that $\phi(0) = 0$ and $\nabla\phi(0) = 0$ to get:

$$|\phi(x')| = \Big|\int_0^1 \langle \nabla\phi(tx'), x'\rangle \,\mathrm{d}t\Big| \leq |x'| \int_0^1 |\nabla\phi(tx')| \,\mathrm{d}t. \tag{4.3}$$

Clearly, in case (ii) we have: $|\nabla\phi(x')| \leq \frac{1}{r}|x'|$, so (4.2) holds then as well. Applying now (4.2) in the right hand side of (4.3), we derive (4.1).

To prove the final statement, we first note from (4.1) that the graph of ϕ is contained between two parabolas $x' \mapsto \pm\frac{1}{2\delta}|x'|^2$, where $\delta = \frac{r}{\max_{\bar{B}_\rho(0)}|\nabla\phi|^2+1}$. It hence easily follows that:

$$|(x', \phi(x')) - (0, \pm\delta)|^2 \geq 2\delta|\phi(x')| + |\phi(x') \mp \delta|^2 \geq 2\delta|\phi(x')| + \big(\delta^2 - 2\delta|\phi(x')|\big) = \delta^2. \tag{4.4}$$

Decreasing the radius, if necessary, to the indicated value $\delta_0 \leq \delta$, we obtain that the balls $B_{\delta_0}(0, \delta_0)$ and $B_{\delta_0}(0, -\delta_0)$ are supporting at x_0. ■

We readily deduce:

Lemma 7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, satisfying* $(\mathrm{C}^{1,1})$*. Then* (S_r) *holds with some $r > 0$.*

Proof. For each $x_0 \in \partial\Omega$, denote by $\mathcal{C}(x_0) = B_\rho(0) \times (-h, h) \subset \mathbb{R}^N$ the cylinder with radius $\rho = \rho(x_0)$ and height $h = h(x_0)$, to which the definition (C^1) applies. We first observe that fixing a sufficiently small $\eta = \eta(x_0)$, every point $y_0 \in \partial\Omega \cap B_\eta(x_0)$ has the property that the halved cylinder $\mathcal{C}_{1/2}(x_0) = B_{\frac{\rho}{2}}(0) \times (-\frac{h}{2}, \frac{h}{2})$ can also be taken as its corresponding $\mathcal{C}$. In particular, since $(T_{y_0})^{-1}\big(\mathcal{C}_{1/2}(x_0)\big) \subset (T_{x_0})^{-1}\big(\mathcal{C}(x_0)\big)$ and the graph of ϕ_{y_0} is a rigid motion of a part of the graph of ϕ_{x_0}, it follows that the radius δ_0 of the supporting balls guaranteed in Lemma 6, is the same for every $y_0 \in \partial\Omega \cap B_\eta(x_0)$. By compactness, $\partial\Omega$ may be covered by finitely many balls in the family $\{B_\eta(x_0)\}_{x_0\in\partial\Omega}$. Taking the smallest of such constructed radii $\{\delta = \delta_0(x_0)\}$ proves (S_r). ■

The following final argument completes the proof of Theorem 1:

Lemma 8. *Assume that a domain $\Omega \subset \mathbb{R}^N$ satisfies* (L_r)*. Then* (S_r) *holds.*

Proof. For a boundary point $x_0 \in \partial\Omega$, let ϕ be as described in condition (C^1), where without loss of generality we take $T = id$, so that Lemma 6 may be used. We argue by contradiction. If $B_r(re_N)$ was not supporting, then $B_r(re_N) \cap \Omega \neq \emptyset$ and further:

$$\bar{r} \doteq \inf\big\{\delta \geq \delta_0;\ \bar{B}_\delta(0, \delta) \cap \partial\Omega \neq \{0\}\big\} < r.$$

Take a sequence of radii: $\delta_n \searrow \bar{r}$ as $n \to \infty$, and a sequence of points: $y_n \in \bar{B}_{\delta_n}(\delta_n e_N) \cap \big(\partial\Omega \setminus \{0\}\big)$. Without loss of generality, $y_n \to y_0 \in \bar{B}_{\bar{r}}(\bar{r}e_N) \cap \partial\Omega$. By the minimality of $\bar{r}$, there must be:

$$y_0 \in \partial B_{\bar{r}}(\bar{r}e_N) \cap \partial\Omega.$$

Since the inward normal $-\vec{n}(y_0)$ coincides with the normal to $\partial B_{\bar{r}}(\bar{r}e_N)$ at y_0 and since the same is valid at $x_0 = 0$, we get the following equality, which in virtue of (S_r) implies:

$$\Big|\frac{1}{\bar{r}}y_0 - \frac{1}{\bar{r}}x_0\Big| = \big|\vec{n}(x_0) - \vec{n}(y_0)\big| \leq \frac{1}{r}|x_0 - y_0|.$$

This results in the contradictory statement $\bar{r} \geq r$, provided that we show $y_0 \neq x_0$. To this end, it suffices to argue that $\{y_n\}$ must be bounded away from 0. Let $\sigma > 0$ satisfy:

$$\big(\sup_{\bar{B}_\sigma(0)} |\nabla\phi|^2 + 1\big)\sigma^2 \leq 2r\bar{r} \leq 2r\delta_n \qquad \text{for all } n.$$

Using (4.1) we refine the bound in (4.4) for $x' \in B_\sigma(0)$, to:

$$\begin{aligned}|(x', \phi(x')) - \delta_n e_N|^2 = |x'|^2 + |\delta_n - \phi(x')|^2 &\geq |x'|^2 + \Big|\delta_n - \frac{\sup_{\bar{B}_\sigma(0)}|\nabla\phi|^2 + 1}{2r}|x'|^2\Big|^2 \\ &\geq |x'|^2 + \delta_n^2 - \frac{\delta_n}{r}\big(\sup_{\bar{B}_\sigma(0)}|\nabla\phi|^2 + 1\big)|x'|^2.\end{aligned}$$

Note that for all large n, the ratio $\frac{\delta_n}{r}$ is smaller than and bounded away from 1. It follows that for sufficiently small $\sigma > 0$ there also holds: $\frac{\delta_n}{r}\big(\sup_{\bar{B}_\sigma(0)} |\nabla\phi|^2 + 1\big) < 1$ for all n. Consequently:

$$|(x', \phi(x')) - \delta_n e_N|^2 \geq \delta_n^2 \qquad \text{for all } x' \in B_\sigma(0) \quad \text{and all } n,$$

where equality only takes place at $x' = 0$. This implies that $|y_n'| \geq \sigma$ and hence $|y_0| \geq \sigma$ as well, as claimed. We conclude that the ball $B_r(re_N)$ is external supporting. The proof that $B_r(-re_N)$ is an internal supporting ball follows in the same manner. ■

5. An extension of the four ball lemma to the Lebesgue spaces

We now discuss an extension of our results to infinitely dimensional Banach spaces $L^p = L^p(Z)$, where Z is an arbitrary measure space. We denote by $\|u\|$ the L^p norm of a given $u \in L^p$. Observe that the "four ball lemma" (Lemma 3) remains valid in any inner product space, i.e. in a space where one can use the parallelogram identity. Thus, it is valid in L^2, whereas for $p \neq 2$ we still get the following:

Lemma 9 (The four ball lemma in L^p). *Let $x, u, v \in L^p$ with $\|u\| = \|v\| = r > 0$. Assume that $B_r(x+u) \cap B_r(-x-v) = \emptyset$ and $B_r(x-u) \cap B_r(-x+v) = \emptyset$. Then:*

$$\begin{aligned} &\|u - v\|^p \leq 2^{p-1} p(p-1) r^{p-2} \|x\|^2 \qquad &&\text{for } p \in [2, \infty) \\ &\|u - v\|^2 \leq \frac{8}{p(p-1)} r^{2-p} \|x\|^p \qquad &&\text{for } p \in (1, 2]. \end{aligned} \tag{5.1}$$

Proof. By the same reasoning as in the first part of the proof of Lemma 3, we obtain:

$$r \leq \Big\| x + \frac{u+v}{2} \Big\| \qquad \text{and} \qquad r \leq \Big\| x - \frac{u-v}{2} \Big\|.$$

We will now replace the parallelogram identity argument in Lemma 3, with a somewhat more involved estimate, derived separately in the two cases indicated in (5.1).

When $p \geq 2$, we use first the 2-uniform smoothness inequality [1, Proposition 3], followed by the first Clarkson's inequality [3, page 95] and recall that $\|u\| = \|v\| = r$, to get:

$$\begin{aligned} 2r^2 &\leq \Big\| x + \frac{u+v}{2} \Big\|^2 + \Big\| x - \frac{u+v}{2} \Big\|^2 \leq 2(p-1)\|x\|^2 + 2\Big\| \frac{u+v}{2} \Big\|^2 \\ &\leq 2(p-1)\|x\|^2 + 2\Big(\frac{1}{2}\|u\|^p + \frac{1}{2}\|v\|^p - \Big\| \frac{u+v}{2} \Big\|^p \Big)^{2/p} \\ &= 2(p-1)\|x\|^2 + 2\Big(r^p - \Big\| \frac{u+v}{2} \Big\|^p \Big)^{2/p}. \end{aligned} \tag{5.2}$$

Further, concavity of the function $a \mapsto a^{2/p}$ implies that $(a-b)^{2/p} \leq a^{2/p} - \frac{2}{p} a^{(2/p)-1} b$ whenever $a, b, a-b \geq 0$. Applying this bound to $a = r^p$ and $b = \|\frac{u-v}{2}\|^p$ in (5.2), yields:

$$2r^2 \leq 2(p-1)\|x\|^2 + 2\Big(r^2 - \frac{2}{p} r^{2-p} \Big\| \frac{u-v}{2} \Big\|^p \Big),$$

which directly results in (5.1) in case $p \in [2, \infty)$.

For $p \leq 2$, we denote the conjugate exponent by $q = \frac{p}{p-1}$ and use the second Clarkson's inequality [3, page 97], followed by the 2-uniform smoothness inequality [1, Proposition 3] to get:

$$\begin{aligned} 2r^q &\leq \Big\|x + \frac{u+v}{2}\Big\|^q + \Big\|x - \frac{u+v}{2}\Big\|^q \leq 2\Big(\|x\|^p + \Big\|\frac{u+v}{2}\Big\|^p\Big)^{q/p} = 2\Big(\|x\|^p + \Big(\Big\|\frac{u+v}{2}\Big\|^2\Big)^{p/2}\Big)^{q/p} \\ &\leq 2\Big(\|x\|^p + \Big(\frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - (p-1)\Big\|\frac{u-v}{2}\Big\|^2\Big)^{p/2}\Big)^{q/p} \\ &= 2\Big(\|x\|^p + \Big(r^2 - (p-1)\Big\|\frac{u-v}{2}\Big\|^2\Big)^{p/2}\Big)^{q/p}. \end{aligned}$$

As before, concavity of the function $a \mapsto a^{p/2}$ applied to $a = r^2$ and $b = (p-1)\|\frac{u-v}{2}\|^2$, yields:

$$2r^q \leq 2\Big(|x|^p + r^p - \frac{(p-1)p}{2} r^{p-2}\Big\|\frac{u-v}{2}\Big\|^2\Big)^{q/p}.$$

Dividing both sides of the above inequality by $2r^q$ we obtain:

$$1 \leq \frac{|x|^p}{r^p} + 1 - \frac{(p-1)p}{2} r^{-2}\Big\|\frac{u-v}{2}\Big\|^2$$

which directly gives (5.1) in case $p \in (1, 2]$. ■

Similarly to Lemma 4, there follows Hölder's regularity of the unit normal to domains satisfying the two-sided uniform supporting sphere condition:

Corollary 10. *Let $\Omega \subset L^p$ be a domain satisfying (S_r). Then the outward unit normal vector, defined for each $x_0 \in \partial\Omega$ by $\vec{p}(x_0) = \frac{b-a}{\|b-a\|}$, is Hölder continuous with exponent:* $\min\{2/p, p/2\}$. *Namely:*

$$\|\vec{p}(x_0) - \vec{p}(y_0)\| \leq C_p \cdot \begin{cases} \dfrac{1}{r^{2-2/p}}\|x_0 - y_0\|^{2/p} & \text{for } p \in [2, \infty) \\ \dfrac{1}{r^{p/2}}\|x_0 - y_0\|^{p/2} & \text{for } p \in (1, 2], \end{cases}$$

holds for all $x_0, y_0 \in \partial\Omega$, with a constant $C_p > 0$ that depends only on $p \in (1, \infty)$.

6. Concluding remarks

Theorem 1 is consistent with the fact that if $\phi \in C^2$, $\phi(0) = 0$, $\nabla\phi(0) = 0$, then the surface patch given by the graph of ϕ has, at $x_0 = 0$, an external supporting sphere of radius r, for any $r > 0$ satisfying $\frac{1}{r} \geq \lambda_{max}$ where λ_{max} is the maximal eigenvalue of the symmetric matrix $\nabla^2\phi(0)$. More generally, the two-sided supporting sphere radius r at a point x_0 of a C^2 surface S is the inverse of the largest (in absolute value) eigenvalue of the second fundamental form of S at x_0.

Observe also that condition (S_r) is more restrictive than the merely assuming existence of the two-sided supporting spheres at each boundary point, even for a bounded domain (see Figure 6.1).

For completeness, we mention that an entirely equivalent notion of $C^{1,1}$ regularity of $\partial\Omega$ is:

$(\mathrm{C}^{1,1}_\Phi)$ For every $x_0 \in \partial\Omega$ there exist $\rho, h > 0$ and a $C^{1,1}$ diffeomorphism $\Phi : \mathcal{C} \to U$ between the cylinder $\mathcal{C} = B_\rho(0) \times (-h, h) \subset \mathbb{R}^N$ and an open neighbourhood $U \subset \mathbb{R}^N$ of x_0, such that:

$$\Phi\big((x', x_N) \in \mathcal{C};\ x_N < 0\big) = U \cap \Omega,$$
$$\Phi\big((x', x_N) \in \mathcal{C};\ x_N = 0\big) = U \cap \partial\Omega,$$

We recall that Φ is a $C^{1,1}$ diffeomorphism when it is invertible and when both Φ and Φ^{-1} have regularity $C^{1,1}$. Condition $(\mathrm{C}^{1,1}_\Phi)$ is at the heart of the technique of "straightening the boundary". This technique is familiar to analysts and relies on reducing an argument (e.g. constructing an

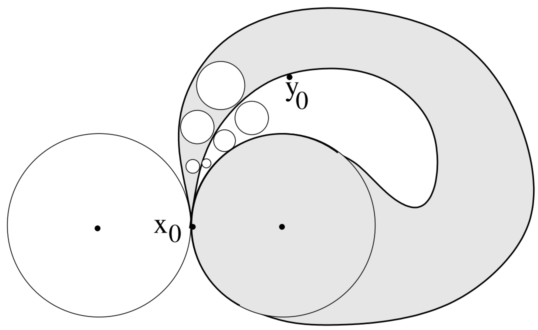

FIGURE 6.1. A bounded domain Ω satisfying the nonuniform two-sided supporting spheres condition. The maximal radii converge to 0 as $y_0 \to x_0$.

extension operator, deriving estimates on a solution to some PDEs, etc) needed in a proximity of a boundary point of a domain, to the simpler case of flat boundary to the half-space. The two cases are then related via the diffeomorphism Φ with controlled derivatives (sometimes more than just one derivative is needed!). On the other hand, it is often more straightforward to deal with the geometric condition (S_r) rather than with requirements on ϕ in ($\mathrm{C}^{1,1}$) or Φ in ($\mathrm{C}^{1,1}_\Phi$).

To our best knowledge, result as in Theorem 1 has first appeared in an unpublished technical report [7], referenced in [6]. The proof is based on a geometrical argument as in the proof of Lemma 3. The equivalence of the two-sided supporting sphere condition and the $C^{1,1}$ regularity of the boundary, has been also shown in [4] and in [2] via more complex calculations. The one-sided supporting sphere condition, called the "positive reach" was studied in [5], however it does not suffice to lead to $C^{1,1}$ regularity.

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