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VISUALIZATION OF THE CONVEX INTEGRATION SOLUTIONS TO THE MONGE-AMPÈRE EQUATION

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ABSTRACT. In this article, we implement the algorithm based on the convex integration result proved in [13] and obtain visualizations of the first iterations of the Nash-Kuiper scheme, approximating the anomalous solutions to the Monge-Ampère equation in two dimensions.

1. INTRODUCTION

The purpose of this paper is to implement the technique of convex integration to obtain and analyze the visualizations of the Hölder regular $\mathcal{C}^{1,\alpha}$ solutions to the Monge-Ampère equation:

$$\mathcal{D}et\,\nabla^2 v \doteq -\frac{1}{2}\mathrm{curl}\,\mathrm{curl}\,(\nabla v\otimes\nabla v) = f \qquad \text{in } \Omega\subset\mathbb{R}^2. \tag{1.1}$$

We rely on the results of the paper [13], where the Nash-Kuiper iteration was shown to be adaptable to prove flexibility of the very weak solutions to (1.1) in the regularity regime $0<\alpha<\frac{1}{7}$. More precisely, the following holds:

Theorem 1.1. [13] *Let $f\in L^{7/6}(\Omega)$ on an open, bounded, simply connected domain $\Omega\subset\mathbb{R}^2$. Fix an exponent $\alpha\in(0,\frac{1}{7})$. Then the set of functions $v\in\mathcal{C}^{1,\alpha}(\bar\Omega)$ that satisfy (1.1) in the sense of distributions, is dense in the space $\mathcal{C}^0(\bar\Omega)$. Namely, for every $v_0\in\mathcal{C}^0(\bar\Omega)$ there exists a sequence $v_n\in\mathcal{C}^{1,\alpha}(\bar\Omega)$ converging uniformly to v_0 and solving (1.1). When $f\in L^p(\Omega)$ and $p\in(1,\frac{7}{6})$, the same result is true for any exponent $\alpha\in(0,1-\frac{1}{p})$.*

We note that by a straightforward approximation argument, if $v\in W^{2,2}$ then $\mathcal{D}et\,\nabla^2 v=\det\nabla^2 v$ a.e. in Ω, where $\nabla^2 v\in L^2(\Omega)$ stands for the standard Hessian matrix field of v. It has been shown [12, 18] that in this case, condition $f\geq c>0$ in Ω (respectively, $f\equiv 0$ in Ω) implies that v must be $\mathcal{C}^1$ and globally convex or concave (respectively, developable). Likewise [13, 14], the flexibility result of Theorem 1.1 has its rigidity counterpart: if $v\in\mathcal{C}^{1,\alpha}(\bar\Omega)$ with $\alpha>\frac{2}{3}$ solves (1.1) for the right hand side that is positive Dini continuous (respectively, $f\equiv 0$ in Ω) then v is a convex Alexandrov solution to $\det\nabla^2 v=f$ (respectively, v is developable).

For a better understanding of the above statements and their relation to nonlinear elasticity of thin films, we refer to [13]. We now recall the connection between solutions to (1.1) and the problem of isometric immersion of Riemannian metrics. Consider the one-parameter family of metrics $h_\epsilon=\mathrm{Id}_2+2\epsilon^2 A$ on Ω, where $A:\Omega\to\mathbb{R}^{2\times 2}_{sym}$ is a perturbation tensor that satisfies:

$$-\mathrm{curl}\,\mathrm{curl}\,A=f, \tag{1.2}$$

for example one can take: $A=-\Delta^{-1}(f)\mathrm{Id}_2$. Consider also the pull-back metrics $(\nabla\phi_\epsilon)^T\nabla\phi_\epsilon$ of the family of deformations $\phi_\epsilon=id+\epsilon v e_3+\epsilon^2 w:\Omega\to\mathbb{R}^3$. Computing the Gauss curvature of h_ϵ:

$$\kappa(h_\epsilon)=-\epsilon^2\mathrm{curl}\,\mathrm{curl}\,A+o(\epsilon^2)$$

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and the curvature of the pull-back metrics: $\kappa((\nabla\phi_\varepsilon)^T\nabla\phi_\varepsilon) = -\epsilon^2 \mathrm{curl}\,\mathrm{curl}\,\big(\frac{1}{2}\nabla v\otimes\nabla v + \mathrm{sym}\,\nabla w\big) + o(\epsilon^2)$, we see that (1.1) can be interpreted as requesting the out-of-plane displacement v to be amenable to matching by a higher order in-plane displacement w, to achieve the prescribed Gaussian curvature f, at its highest order term. Indeed, on a simply connected domain the kernel of the differential operator "curl curl" consists of the fields of the form $\mathrm{sym}\,\nabla w$. Thus, equation (1.1) is equivalent to:

$$\frac{1}{2}\nabla v\otimes\nabla v + \mathrm{sym}\,\nabla w = A \qquad \text{in } \Omega, \tag{1.3}$$

where A satisfies (1.2) and $w : \Omega \to \mathbb{R}^2$ is an auxiliary variable (a Lagrange multiplier). In the same spirit, equation (1.3) can be interpreted as the "infinitesimal isometry" constraint, namely: the family h_ϵ should coincide, up to terms of order ϵ^2, with the pull-back metrics $(\nabla\phi_\epsilon)^T\nabla\phi_\epsilon$ defined above.

In order to construct $\mathcal{C}^{1,\alpha}$ solutions of (1.3) with v approximating a given continuous v_0 (by density, it suffices to work with v_0 smooth), it has been shown in [13] that a Nash-Kuiper convex integration method can be employed. This method [7, 11] modifies an initial "short infinitesimal isometry" (v_0, w_0), i.e. a couple for whom (1.3) is satisfied with an inequality (hence a subsolution) rather than the equality, towards an exact solution, in successive small steps. Technical similarities with the isometric immersion construction [16, 17, 11, 4] are based on the presence of the quadratic terms $(\nabla\phi_\epsilon)^T\nabla\phi_\epsilon$ and $\nabla v\otimes\nabla v$ in both PDEs. For parallel application of the convex integration technique to constructing the energy-dissipative solutions of the Euler equations and to the recent resolution of the Onsager's conjecture, we refer to [5, 6, 3, 8].

As we have mentioned, Theorem 1.1 follows from the main result in [13] below:

Theorem 1.2. [13] *Let $\Omega\subset\mathbb{R}^2$ be an open and bounded domain. Let $v_0\in\mathcal{C}^1(\bar\Omega)$, $w_0\in\mathcal{C}^1(\bar\Omega,\mathbb{R}^2)$ and $A\in\mathcal{C}^{0,\beta}(\bar\Omega,\mathbb{R}^{2\times2}_{sym})$, for some $\beta\in(0,1)$, be such that:*

$$\exists c_0>0 \qquad A - \big(\frac{1}{2}\nabla v_0\otimes\nabla v_0 + \mathrm{sym}\,\nabla w_0\big) \geq c_0\mathrm{Id}_2 \qquad \text{in } \bar\Omega. \tag{1.4}$$

Then, for every exponent $\alpha\in\big(0,\min\big\{\frac{1}{7},\frac{\beta}{2}\big\}\big)$, there exist sequences: $v_n\in\mathcal{C}^{1,\alpha}(\bar\Omega)$ which converges uniformly to v_0, and $w_n\in\mathcal{C}^{1,\alpha}(\bar\Omega,\mathbb{R}^2)$ which converges uniformly to w_0, satisfying:

$$\frac{1}{2}\nabla v_n\otimes\nabla v_n + \mathrm{sym}\,\nabla w_n = A \qquad \text{in } \bar\Omega. \tag{1.5}$$

In this paper, we implement the proof of Theorem 1.2 (for f and v_0 as in Theorem 1.1 and specially prepared A and w_0) into an explicit algorithm, leading to the visualizations of the anomalous solutions to the Monge-Ampère equation (1.1). These are solutions failing to possess the indicated geometrical structural properties valid in the rigid regime $v\in W^{2,2}$ or $v\in\mathcal{C}^{1,2/3+}$:

- nonconvex solutions, approximating a saddle $v_0(x,y)=x^2-y^2$, when $f\equiv 1$,
- nondevelopable solutions, approximating a convex paraboloid $v_0(x,y)=x^2+y^2$, when $f\equiv 0$,
- solutions uniformly close to $v_0(x,y)=x^2+y^2$, when $f\equiv -1$.

We remark that numerical implementation of convex integration construction for the isometric immersion problem has been reported in [1, 2], leading among others to the remarkable images of flat torus embedded in $\mathbb{R}^3$.

The paper is organized as follows. In sections 2, 4 and 5 we revisit the proof of Theorem 1.2 and explicitly trace all the constants appearing in the arguments in [13] for an optimal performance of the numerical algorithm. Without loss of generality, we will assume that $0 \in \Omega$. In sections 3 and 6 we implement the indicated technique and visualise the first few approximations of anomalous solutions to (1.1) in case of f being positive, negative and equivalently equal 0, respectively. The implementation is done in MATLAB, where for the detailed Hölder continuous approximations in section 6 we use of the Python package mpmath [15], allowing the user to define floating point arithmetics up to an arbitrary precision.

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2. The $\mathcal{C}^1$ convergence and construction of corrugations

We start with a weaker version of Theorem 1.2. Define three unit vectors:

$$\eta_1 = (1,0), \quad \eta_2 = \frac{1}{\sqrt{5}}(1,2), \quad \eta_3 = \frac{1}{\sqrt{5}}(1,-2).$$

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain. Given $v_0 \in \mathcal{C}^\infty(\bar\Omega)$, $w_0 \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^2)$ and $A \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^{2\times 2}_{sym})$, we have the following. For any $\epsilon > 0$ there exist $v \in \mathcal{C}^1(\bar\Omega)$ and $w \in \mathcal{C}^1(\bar\Omega, \mathbb{R}^2)$ that satisfy:*

$$\|v - v_0\|_0 \leq \epsilon \quad \textit{and} \quad \frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\nabla w = A. \tag{2.1}$$

Moreover, if with some smooth positive functions $\phi_k \in \mathcal{C}^\infty(\bar\Omega)$ there holds:

$$A - \big(\frac{1}{2}\nabla v_0 \otimes \nabla v_0 + \mathrm{sym}\nabla w_0\big) = \sum_{k=1}^{3} \phi_k \eta_k \otimes \eta_k, \qquad \phi_k \geq d_0 > 0 \quad \textit{in } \bar\Omega. \tag{2.2}$$

then it is possible to have the field w in (2.1) additionally satisfy: $\|w - w_0\|_0 \leq \epsilon$. The vector field (v,w) is obtained as a $\mathcal{C}^1(\bar\Omega)$-limit of smooth fields $v_k \in \mathcal{C}^\infty(\bar\Omega)$ and $w_k \in \mathcal{C}(\bar\Omega, \mathbb{R}^2)$ such that $\|v_k - v_0\|_0 \leq \epsilon$ and $\|w_k - w_0\|_0 \leq \epsilon$ for all $k \geq 1$.

As noted below, condition (2.2) implies positive definiteness of the left hand side: $D_0 \doteq A - \big(\frac{1}{2}\nabla v_0 \otimes \nabla v_0 + \mathrm{sym}\nabla w_0\big) \geq d_0 \mathrm{Id}_2$ in $\bar\Omega$. We now recall the convex integration construction which allows to decrease such positive definite defect, in each of the $\eta_k \otimes \eta_k$ directions, through oscillatory perturbations in v and w. Let η be a unit vector and ϕ a smooth positive function on $\bar\Omega$. Given two fields v and w, we want to adjust them to $\tilde v$ and $\tilde w$ so that:

$$\Big(A - \big(\frac{1}{2}\nabla \tilde v \otimes \nabla \tilde v + \mathrm{sym}\nabla \tilde w\big)\Big) - \Big(A - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\nabla w + \phi\eta\otimes\eta\big)\Big) \sim 0. \tag{2.3}$$

Firstly, define $\tilde v \doteq v + \frac{1}{\lambda} f(x, \lambda x\cdot\eta)$ by adding oscillations of amplitude $\frac{1}{\lambda}$ and frequency λ, with $\lambda \to \infty$ in the ultimate limiting process. Secondly, decompose perturbation of w in the two directions ∇v and η, and write $\tilde w \doteq w + \frac{1}{\lambda} g(x, \lambda x\cdot\eta)\nabla v + \frac{1}{\lambda} h(x, \lambda x\cdot\eta)\eta$. Each of $f(x,t)$, $g(x,t)$, and $h(x,t)$ should be bounded and 1-periodic in the variable t. The expansion:

$$\begin{aligned}\frac{1}{2}\nabla \tilde v \otimes \nabla \tilde v + \mathrm{sym}\nabla \tilde w =& \frac{1}{2}\nabla v \otimes \nabla v + \frac{1}{2}(\partial_t f)^2 \eta\otimes\eta + (\partial_t f)\mathrm{sym}(\eta\otimes\nabla v) \\ & + \mathrm{sym}\nabla w + (\partial_t g)\mathrm{sym}(\eta\otimes\nabla v) + (\partial_t h)\eta\otimes\eta + O\big(\frac{1}{\lambda}\big),\end{aligned}$$

yields (2.3) provided that $f = -g$ and $\frac{1}{2}(\partial_t f)^2 + \partial_t h = \phi$. The actual choice of f, g and h is given in Lemma 2.2.

In the subsequent proof of Theorem 2.1, we will aim at keeping the frequencies $\lambda = \lambda_k$ minimal, facilitating a good numerical implementation. Three modifications with respect to proofs in [13] will be done in order to improve the estimates on λ_k:

(i) all the global inequalities will be specified to pointwise estimates;
(ii) we will allow the error threshold parameter δ to depend on x rather than be constant;
(iii) we will set distinct rates of convergence for the errors D_k and the approximations v_k in Proposition 2.5.

We first make the following simple observation:

Lemma 2.2. *For a positive function $a \in \mathcal{C}^\infty(\bar\Omega)$, the functions $V, W \in \mathcal{C}^\infty(\bar\Omega \times \mathbb{R})$ defined by:*

$$V(x,t) \doteq \frac{a(x)}{\pi}\sin(2\pi t), \qquad W(x,t) \doteq -\frac{a(x)^2}{4\pi}\sin(4\pi t),$$

are 1-periodic in t, and they satisfy in $\bar\Omega \times \mathbb{R}$:

$$\frac{1}{2}(\partial_t V)^2 + \partial_t W = a^2, \tag{2.4}$$

$$\begin{aligned} & |V| \leq \frac{a}{\pi}, \quad |\partial_t V| \leq 2a, \quad |\nabla_x V| \leq \frac{|\nabla a|}{\pi}, \quad |\nabla_x^2 V| \leq \frac{|\nabla^2 a|}{\pi}, \\ & |W| \leq \frac{a^2}{4\pi}, \quad |\partial_t W| \leq a^2, \quad |\nabla_x W| \leq \frac{a|\nabla a|}{2\pi}. \end{aligned} \tag{2.5}$$

We then obtain, consistently with (2.3):

Proposition 2.3. *Let $v \in \mathcal{C}^\infty(\bar\Omega)$, $w \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^2)$ and let $a \in \mathcal{C}^\infty(\bar\Omega)$ be a positive function. For a unit vector $\eta \in \mathbb{R}^2$ and a frequency $\lambda > 0$, define $v_\lambda \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R})$, $w_\lambda \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^2)$ through:*

$$\begin{aligned} & v_\lambda(x) \doteq v(x) + \frac{1}{\lambda} V(x, \lambda x \cdot \eta), \\ & w_\lambda(x) \doteq w(x) - \frac{1}{\lambda} V(x, \lambda x \cdot \eta)\nabla v(x) + \frac{1}{\lambda} W(x, \lambda x \cdot \eta)\eta. \end{aligned} \tag{2.6}$$

Then we have the following pointwise estimates, valid in $\bar\Omega$:

$$\begin{aligned} & \Big| \big(\frac{1}{2}\nabla v_\lambda \otimes \nabla v_\lambda + \mathrm{sym}\nabla w_\lambda\big) - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\nabla w + a^2 \eta \otimes \eta\big)\Big| \\ & \qquad\qquad \leq \frac{1}{\lambda}\big(\frac{a|\nabla a|}{2\pi} + \frac{a|\nabla^2 v|}{\pi}\big) + \frac{1}{2\lambda^2\pi^2}|\nabla a|^2, \end{aligned} \tag{2.7}$$

$$|v_\lambda - v| \leq \frac{a}{\lambda\pi}, \qquad |w_\lambda - w| \leq \frac{a}{\lambda\pi}\big(|\nabla v| + \frac{a}{4}\big), \tag{2.8}$$

$$|\nabla v_\lambda - \nabla v| \leq \frac{|\nabla a|}{\lambda\pi} + 2a, \tag{2.9}$$

$$\text{and} \quad |\nabla w_\lambda - \nabla w| \leq 2a|\nabla v| + a^2 + \frac{1}{\lambda}\big(\frac{1}{\pi}|\nabla v||\nabla a| + \frac{a}{\pi}|\nabla^2 v| + \frac{1}{2\pi}a|\nabla a|\big),$$

$$|\nabla^2 v_\lambda - \nabla^2 v| \leq \frac{|\nabla^2 a|}{\lambda\pi} + 4|\nabla a| + 4\lambda\pi a. \tag{2.10}$$

Proof. To show (2.8), we use (2.5) to estimate:

$$|v_\lambda - v| = |\frac{1}{\lambda}V| \leq \frac{a}{\lambda\pi}, \qquad |w_\lambda - w| = \frac{1}{\lambda}|V\nabla v - W\eta| \leq \frac{a}{\lambda\pi}\big(|\nabla v| + \frac{a}{4}\big).$$

Similarly, (2.9) and (2.10) follow, in view of $|a\otimes b| = |a|\cdot|b|$:

$$\begin{aligned}
|\nabla v_\lambda - \nabla v| &\leq \frac{1}{\lambda}|\nabla_x V| + |\partial_t V| \leq \frac{|\nabla a|}{\lambda\pi} + 2a,\\
|\nabla w_\lambda - \nabla w| &= \Big|\frac{1}{\lambda}\nabla v\otimes\nabla_x V + (\partial_t V)\nabla v\otimes\eta + \frac{1}{\lambda}V\nabla^2 v - \frac{1}{\lambda}\eta\otimes\nabla_x W - (\partial_t W)\eta\otimes\eta\Big|\\
&\leq 2a|\nabla v| + a^2 + \frac{1}{\lambda}\big(\frac{1}{\pi}|\nabla v||\nabla a| + \frac{a}{\pi}|\nabla^2 v| + \frac{1}{2\pi}a|\nabla a|\big),\\
|\nabla^2 v_\lambda - \nabla^2 v| &\leq \frac{1}{\lambda}|\nabla_x^2 V| + 2|\nabla_x\partial_t V| + \lambda|\partial_t^2 V| \leq \frac{|\nabla^2 a|}{\lambda\pi} + 4|\nabla a| + 4\lambda\pi a.
\end{aligned}$$

Lastly, we observe:

$$\begin{aligned}
\frac{1}{2}\nabla v_\lambda\otimes\nabla v_\lambda + \mathrm{sym}\nabla w_\lambda = {}& \frac{1}{2}\nabla v\otimes\nabla v + \frac{1}{2}(\partial_t V)^2\eta\otimes\eta + (\partial_t W)\eta\otimes\eta + \mathrm{sym}\nabla w\\
&+ \frac{1}{\lambda}\big((\partial_t V)\mathrm{sym}(\nabla_x V\otimes\eta) - V\nabla^2 v + \mathrm{sym}(\nabla_x W\otimes\eta)\big)\\
&+ \frac{1}{2\lambda^2}\nabla_x V\otimes\nabla_x V.
\end{aligned} \tag{2.11}$$

By(2.4) and $(\partial_t V)(\nabla_x V) + \nabla_x W = \frac{1}{2\pi}\sin(4\pi t)a\nabla a$, we arrive at:

$$\begin{aligned}
\Big|\big(\frac{1}{2}\nabla v_\lambda\otimes\nabla v_\lambda &+ \mathrm{sym}\nabla w_\lambda\big) - \big(\frac{1}{2}\nabla v\otimes\nabla v + \mathrm{sym}\nabla w + a^2\eta\otimes\eta\big)\Big|\\
&\leq \frac{1}{\lambda}\Big|\mathrm{sym}\Big(\big((\partial_t V)(\nabla_x V) + \nabla_x W\big)\otimes\eta\Big) - V\nabla^2 v\Big| + \frac{1}{2\lambda^2}\Big|\nabla_x V\otimes\nabla_x V\Big|\\
&\leq \frac{1}{\lambda}\big(\frac{a|\nabla a|}{2\pi} + \frac{a|\nabla^2 v|}{\pi}\big) + \frac{1}{2\lambda^2\pi^2}|\nabla a|^2,
\end{aligned}$$

concluding the proof of (2.7). ∎

Lemma 2.4. *Matrices $\{\eta_k\otimes\eta_k\}_{k=1}^3$ form a basis of $\mathbb{R}^{2\times2}_{sym}$. If $B = [b_{ij}]_{i,j=1,2} = \sum_{k=1}^3 \phi_k\eta_k\otimes\eta_k$, then:*

(i) *$\phi_1 = b_{11} - \frac{1}{4}b_{22}$, $\phi_2 = \frac{5}{8}(b_{22} + 2b_{12})$, $\phi_3 = \frac{5}{8}(b_{22} - 2b_{12})$.*
(ii) *$\sum_{k=1}^3 \phi_k = \mathrm{Tr}\,B$ and $|\phi_k| \leq \frac{5\sqrt{3}}{8}|B|$ for all $k = 1\dots3$.*
(iii) *If $\phi_k \geq d > 0$ for all $k = 1\dots3$, then $B \geq d\,\mathrm{Id}_2$.*
(iv) *Let $\tilde{B} = B + \alpha\, diag\{\frac{\sqrt{2}+9}{4}, (\sqrt{2} + \frac{9}{5})\}$, with $\alpha \geq |B|$. Then: $|\tilde{B}| \leq 5.15\cdot\alpha$ and $\tilde{B} = \sum_{k=1}^3 \tilde{\phi}_k\eta_k\otimes\eta_k$ with $\tilde{\phi}_k \geq \frac{\alpha}{2}$.*

Proof. The three indicated rank-one matrices are linearly independent by a straightforward calculation. The formula in (i) follows directly as well and implies, in view of the Cauchy-Schwartz inequality: $|\phi_1| \leq \frac{\sqrt{17}}{4}|B|$. In the same manner, we have:

$$|\phi_2| \leq \frac{5}{8}|b_{22} + b_{12} + b_{21}| \leq \frac{5\sqrt{3}}{8}\big(b_{22}^2 + b_{12}^2 + b_{21}^2\big)^{1/2} \leq \frac{5\sqrt{3}}{8}|B|,$$

and likewise $|\phi_3| \leq \frac{5\sqrt{3}}{8}|B|$, so (ii) follows. For (iii), observe that $\sum_{k=1}^3 \phi_k \eta_k \otimes \eta_k = \sum_{k=1}^3 (\phi_k - d)\eta_k \otimes \eta_k + d\ \mathrm{diag}\big\{\frac{7}{5}, \frac{8}{5}\big\} \geq d\ \mathrm{Id}_2$. In the setting of (iv), since $b_{kk} \leq |B|$ for $k = 1, 2$ and $b_{12} \leq \frac{\sqrt{2}}{2}|B|$, we get:

$$\tilde{\phi}_1 = b_{11} + \frac{\sqrt{2}+9}{4}|B| - \frac{1}{4}\big(b_{22} + (\sqrt{2} + \frac{9}{5})\big)|B| \geq -|B| - \frac{1}{4}|B| + \frac{9}{5}|B| \geq \frac{1}{2}|B|,$$
$$\tilde{\phi}_{2,3} = \frac{5}{8}\big(b_{22} + (\sqrt{2} + \frac{9}{5})|B| \pm 2b_{12}\big) \geq \frac{1}{2}|B|.$$

Finally: $|\tilde{B}| \leq |B| + \Big(\big(\frac{\sqrt{2}+9}{4}\big)^2 + \big(\sqrt{2} + \frac{9}{5}\big)^2\Big)^{1/2}|B| \leq 5.15 \cdot |B|$. ∎

Proposition 2.5. *Let $v \in \mathcal{C}^\infty(\bar{\Omega})$, $w \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$ and $A \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{2\times 2}_{sym})$ satisfy:*

$$D \doteq A - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\nabla w\big) = \sum_{k=1}^3 \phi_k \eta_k \otimes \eta_k, \qquad \phi_k \geq d \quad \text{in } \bar{\Omega}, \tag{2.12}$$

with some $\phi_k = \phi_k(x)$ and a constant $d > 0$. Let $\xi > 0$ be such that:

$$\xi \leq \min_{x \in \bar{\Omega}} |D(x)|. \tag{2.13}$$

Fix $\epsilon > 0$; then there exist $\tilde{v} \in \mathcal{C}^\infty(\bar{\Omega})$, $\tilde{w} \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$ and a constant $\tilde{d} > 0$ such that:

$$\tilde{D} \doteq A - \big(\frac{1}{2}\nabla \tilde{v} \otimes \nabla \tilde{v} + \mathrm{sym}\nabla \tilde{w}\big) = \sum_{k=1}^3 \tilde{\phi}_k \eta_k \otimes \eta_k, \qquad \tilde{\phi}_k \geq \tilde{d} \quad \text{in } \bar{\Omega}, \tag{2.14}$$

$$\|\tilde{D}\|_0 \leq \frac{3}{4}\xi, \qquad \|\tilde{v} - v\|_0 \leq \epsilon, \qquad \|\tilde{w} - w\|_0 \leq C\epsilon\big(\|\nabla v\|_0 + \|D\|_0^{1/2}\big), \tag{2.15}$$

$$\|\nabla \tilde{v} - \nabla v\|_0 \leq C\|D\|_0^{1/2}, \qquad \|\nabla \tilde{w} - \nabla w\|_0 \leq C\big(\|D\|_0^{1/2}\|\nabla v\|_0 + \|D\|_0\big), \tag{2.16}$$

where C is a universal constant.

Proof. **1.** We construct the intermediate fields $\{(v_k, w_k)\}_{k=1\ldots 3}$, from the given $(v_0, w_0) \doteq (v, w)$ to the requested $(\tilde{v}, \tilde{w}) \doteq (v_3, w_3)$. Define smooth, positive functions $\delta, a_k : \bar{\Omega} \to \mathbb{R}$:

$$\delta(x)|D(x)| = \frac{\xi}{2}, \qquad a_k(x)^2 = (1 - \delta(x))\phi_k(x) \qquad \text{for } \ k = 1 \ldots 3,$$

so that: $\delta D = D - \sum_{k=1}^3 a_k^2 \eta_k \otimes \eta_k$. Clearly, (2.13) guarantees the bound:

$$\delta \leq \frac{1}{2} \quad \text{in } \ \bar{\Omega}. \tag{2.17}$$

Given $(v_{k-1}, w_{k-1}) \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^3)$, the successive corrections v_k and w_k are now constructed by applying Proposition 2.3 to $v = v_{k-1}$, $w = w_{k-1}$, $a = a_k$, $\eta = \eta_k$ and an appropriate $\lambda = \lambda_k \geq 1$

determined below in (2.21) and (2.23). Observe that:

$$\begin{aligned}\tilde{D} &= D + \big(\frac{1}{2}\nabla v\otimes\nabla v + \mathrm{sym}\nabla w\big) - \big(\frac{1}{2}\nabla\tilde{v}\otimes\nabla\tilde{v} + \mathrm{sym}\nabla\tilde{w}\big)\\ &= D - \sum_{k=1}^{3} a_k^2\eta_k\otimes\eta_k\\ &\qquad - \sum_{k=1}^{3}\Big(\big(\frac{1}{2}\nabla v_k\otimes\nabla v_k + \mathrm{sym}\nabla w_k\big) - \big(\frac{1}{2}\nabla v_{k-1}\otimes\nabla v_{k-1} + \mathrm{sym}\nabla w_{k-1} + a_k^2\eta_k\otimes\eta_k\big)\Big)\\ &= \delta D - \sum_{k=1}^{3} B_k,\end{aligned}$$

where (2.7) yields the following pointwise bound on the matrix-valued fields $\{B_k\}_{k=1}^3$:

$$|B_k| \leq \frac{a_k|\nabla a_k|}{2\pi\lambda_k} + \frac{a_k|\nabla^2 v_{k-1}|}{\pi\lambda_k} + \frac{|\nabla a_k|^2}{2\pi^2\lambda_k^2} \quad \text{in } \bar{\Omega}. \tag{2.18}$$

2. To prove positivity of the decomposition in (2.14), we set:

$$\tilde{d} = \frac{\xi d}{4\|D\|_0} = \frac{d}{2}\min_{x\in\bar{\Omega}}\delta(x)$$

and use Lemma 2.4 to: $\sum_{k=1}^3(\tilde{\phi}_k - \delta\phi_k)\eta_k\otimes\eta_k = \tilde{D} - \delta D = -\sum_{k=1}^3 B_k$ in $\bar{\Omega}$. Namely, by Lemma 2.4 (ii) it follows that for $k = 1\ldots 3$ we have:

$$\tilde{\phi}_k \geq \delta\phi_k - \frac{5\sqrt{3}}{8}\big|\sum_{i=1}^{3} B_i\big| \geq \frac{\delta\phi_k}{2} \geq \frac{\delta d}{2} \geq \tilde{d} \qquad \text{in } \bar{\Omega}, \tag{2.19}$$

where the second inequality above is valid when:

$$\frac{5\sqrt{3}}{8}|B_i| \leq \frac{\delta\phi_k}{6} \qquad \text{in } \bar{\Omega}, \qquad \text{for } i,k : 1\ldots 3. \tag{2.20}$$

Note that the first estimate in (2.15) holds then as well, again in view of Lemma 2.4 (ii):

$$|\tilde{D}| \leq \delta|D| + \big|\sum_{i=1}^{3} B_i\big| \leq \frac{\xi}{2} + \frac{4}{5\sqrt{3}}\delta\phi_i \leq \frac{\xi}{2} + \frac{\delta}{2}|D| = \frac{3}{4}\xi \quad \text{in } \bar{\Omega}.$$

For the validity of (2.20), we choose $\{\lambda_k\}_{k=1}^3$ so that:

$$\begin{aligned}&\frac{5\sqrt{3}}{8}\frac{a_i|\nabla a_i|}{2\pi\lambda_i} \leq \frac{\delta\phi_k}{18} \quad \text{and} \quad \frac{5\sqrt{3}}{8}\frac{a_i|\nabla^2 v_{i-1}|}{\pi\lambda_i} \leq \frac{\delta\phi_k}{18}\\ &\qquad\qquad \text{and} \quad \frac{5\sqrt{3}}{8}\frac{|\nabla a_i|}{2\pi^2\lambda_i^2} \leq \frac{\delta\phi_k}{18} \quad \text{in } \bar{\Omega}, \qquad \text{for } i,k : 1\ldots 3.\end{aligned} \tag{2.21}$$

3. Observe that, by Lemma 2.4 and the definition of a_k, we get:

$$\sum_{k=1}^{3} a_k \leq \sqrt{3}\big(\sum_{k=1}^{3} a_k^2\big)^{1/2} = \sqrt{3}\big((1-\delta)\mathrm{Tr}\ D\big)^{1/2} \leq 3|D|^{1/2} \quad \text{in } \bar{\Omega}. \tag{2.22}$$

By (2.8), there follows the second inequality in (2.15):

$$|\tilde v - v| \leq \sum_{k=1}^3 \frac{a_k}{\lambda_k \pi} \leq \frac{3|D|^{1/2}}{\pi \min\limits_{k=1\dots 3}\{\lambda_k\}} \leq \epsilon \quad \text{in } \bar\Omega,$$

if only we assume the following condition:

$$\lambda_k \geq \frac{\|D\|_0^{1/2}}{\epsilon} \quad \text{for } k = 1\dots 3. \tag{2.23}$$

Note that the third inequality in (2.21) and the bound on ϕ_k in terms of $|D|$ yield:

$$\frac{|\nabla a_k|}{\pi\lambda_k} \leq \Big(\frac{16}{5\sqrt{3}}\frac{\delta\phi_k}{18}\Big)^{1/2} \leq \Big(\frac{\delta|D|}{9}\Big)^{1/2} \leq \frac{1}{3}|D|^{1/2} \quad \text{in } \bar\Omega. \tag{2.24}$$

Thus, by (2.9) and (2.22) we conclude the first bound in (2.16):

$$|\nabla\tilde v - \nabla v| \leq \sum_{k=1}^3 |\nabla v_k - \nabla v_{k-1}| \leq \sum_{k=1}^3 \big(\frac{|\nabla a_k|}{\lambda_k\pi} + 2a_k\big) \leq 7|D|^{1/2} \quad \text{in } \bar\Omega.$$

Similarly: $|\nabla v_k| \leq |\nabla v_0| + 7|D|_0^{1/2}$ for all $k = 1\dots 3$. Using the second inequality in (2.8) together with (2.23) and (2.22), we obtain the third bound in (2.15):

$$\begin{aligned}|\tilde w - w| &\leq \sum_{k=1}^3 \frac{a_k}{\lambda_k\pi}\big(|\nabla v_{k-1}| + \frac{a_k}{4}\big) \leq \sum_{k=1}^3 \frac{a_k}{\lambda_k\pi}\big(|\nabla v| + 7|D|^{1/2} + \frac{a_k}{4}\big) \\ &\leq \frac{|D|^{1/2}}{\min\limits_{k=1\dots 3}\{\lambda_k\}}\big(|\nabla v| + 8|D|^{1/2}\big) \leq \epsilon\big(|\nabla v| + 8|D|^{1/2}\big) \quad \text{in } \bar\Omega,\end{aligned}$$

whereas (2.9) and (2.24) are used for the final estimate in (2.16):

$$\begin{aligned}|\nabla\tilde w - \nabla w| &\leq \sum_{k=1}^3 \Bigg(2a_k|\nabla v_{k-1}| + a_k^2 + \frac{2|\nabla v_{k-1}||\nabla a_k| + a_k|\nabla a_|}{2\pi\lambda_k} + \frac{2a_i|\nabla^2 v_{i-1}|}{2\pi\lambda_k}\Bigg) \\ &\leq \sum_{k=1}^3 \Bigg(2a_k(|\nabla v| + 7|D|^{1/2}) + a_k^2 + \frac{2|\nabla a_k|(|\nabla v| + 7|D|^{1/2}) + a_k|\nabla a_k|}{2\pi\lambda_k} + \frac{a_k|\nabla^2 v_{k-1}|}{\pi\lambda_k}\Bigg) \\ &\leq 7|D|^{1/2}\big(|\nabla v| + 7|D|^{1/2}\big) + 4|D| \quad \text{in } \bar\Omega.\end{aligned}$$

Above, the term $\frac{a_k|\nabla^2 v_{k-1}|}{\pi\lambda_k}$ has been bounded by $\frac{1}{2}|D|_0$ in view of (2.21). ■

Proof of Theorem 2.1.

1. Assume first that (2.2) holds. We will construct a sequence of smooth approximations $\{(v_k, w_k)\}_{k=0}^\infty$ which converge in $\mathcal{C}^1$ to the required solution in (2.1). Starting with v_0, w_0, we define recursively v_{k+1} and w_{k+1} by applying Proposition 2.5 to $v_k \in \mathcal{C}^\infty(\bar\Omega)$ and $w_k \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^2)$, where we denote the corresponding defect $D_k \doteq A - (\frac{1}{2}\nabla v_k \otimes \nabla v_k + \mathrm{sym}\nabla w_k)$, and request that the construction parameters $\epsilon_k, \xi_k > 0$ satisfy:

$$\sum_{k=0}^\infty \epsilon_k \leq \beta\epsilon, \qquad \sum_{k=0}^\infty \xi_k^{1/2} \leq 1 \quad \text{and} \quad \xi_k \leq \min_{x\in\bar\Omega} |D_k(x)| \quad \text{for all } k \geq 0, \tag{2.25}$$

for an appropriately small constant $\beta \leq 1$. By (2.14), each D_k can be decomposed in the basis $\{\xi_i \otimes \xi_i\}_{i=1}^3$ with strictly positive coefficients. Further, (2.15) implies:

$$\|v_k - v_0\|_0 \leq \sum_{i=0}^{k-1} \|v_{i+1} - v_i\|_0 \leq \sum_{i=0}^{k-1} \epsilon_i \leq \epsilon, \tag{2.26}$$

while by (2.16) and (2.15) we obtain:

$$\|\nabla v_{k+m} - \nabla v_k\|_0 \leq \sum_{i=k}^{k+m-1} \|\nabla v_{i+1} - \nabla v_i\|_0 \leq C \sum_{i=k}^{k+m-1} \|D_i\|_0^{1/2} \leq C \sum_{i=k}^{k+m-1} \xi_{i-1}^{1/2}.$$

Consequently, the sequence $\{v_k\}_{k=0}^\infty$ is Cauchy in $\mathcal{C}^1(\bar\Omega)$ and thus it converges to some $v \in \mathcal{C}^1(\bar\Omega)$. By (2.26), the first statement of (2.1) follows. In particular, the norms $\|\nabla v_k\|_0$ are uniformly bounded (by $C(\|\nabla v_0\|_0 + \|D_0\|_0 + 1)$, which allows to compute:

$$\begin{aligned}
\|w_k - w_0\|_0 &\leq \sum_{i=0}^{k-1} \|w_{i+1} - w_i\|_0 \leq C \sum_{i=0}^{k-1} \epsilon_i \big(\|\nabla v_i\|_0 + \|D_i\|_0^{1/2}\big) \\
&\leq C\big(\|\nabla v_0\|_0 + \|D_0\|_0 + \|D_0\|_0^{1/2} + 1\big) \sum_{i=0}^{k-1} \epsilon_i, \\
\|\nabla w_{k+m} - \nabla w_k\|_0 &\leq \sum_{i=k}^{k+m-1} \|\nabla w_{i+1} - \nabla w_i\|_0 \leq C \sum_{i=k}^{k+m-1} \|D_i\|_0^{1/2} \big(\|\nabla v_i\|_0 + \|D_i\|_0^{1/2}\big) \\
&\leq C\big(\|\nabla v_0\|_0 + \|D_0\|_0 + \|D_0\|_0^{1/2} + 1\big) \sum_{i=k}^{k+m-1} \xi_{i-1}^{1/2},
\end{aligned}$$

where C is a universal constant. Hence, $\{w_k\}_{k=0}^\infty$ is Cauchy and converges in $\mathcal{C}^1(\bar\Omega, \mathbb{R}^2)$ to some $w \in \mathcal{C}^1(\bar\Omega, \mathbb{R}^2)$, satisfying:

$$\|A - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\nabla w\big)\|_0 = \lim_{k\to\infty} \|D_k\|_0 = 0.$$

This proves the second statement in (2.1), by (2.15) and (2.25). Also, it is clear that taking $\beta = C^{-1}\big(\|\nabla v_0\|_0 + \|D_0\|_0 + \|D_0\|_0^{1/2} + 1\big)^{-1}$, there follows $\|w_k - w_0\|_0 \leq \epsilon$ and so $\|w - w_0\|_0 \leq \epsilon$.

2. In the absence of the positivity of the decomposition (2.2), we set $\tilde w_0(x,y) = w_0(x,y) - \|D_0\|_0\big(\frac{\sqrt2+9}{4}x, (\sqrt2 + \frac95)y\big)$. By Lemma 2.4 (iv), the decomposition of the modified defect:

$$\tilde D_0 \doteq A - \big(\frac12 \nabla v_0 \otimes \nabla v_0 + \mathrm{sym}\,\nabla \tilde w_0\big) = D_0 + \|D_0\|_0 \cdot \mathrm{diag}\Big\{\frac{\sqrt2+9}{4}, \sqrt2 + \frac95\Big\} = \sum_{k=1}^3 \phi_k \eta_k \otimes \eta_k,$$

obeys $\phi_k \geq \frac{\|D_0\|_0}{2}$ for $k = 1\ldots3$. Applying the first part of the proof to v_0, $\tilde w_0$ and $\tilde D_0$ yields $v \in \mathcal{C}^1(\bar\Omega)$ and $w \in \mathcal{C}^1(\bar\Omega, \mathbb{R}^2)$ with the desired properties. ■

3. A numerical implementation of Theorem 2.1

The purpose of this section is to obtain images of the first few approximation steps in the convex integration construction of the solutions to the Monge-Ampère equation (1.1). We will consider two case scenarios: (i) the right hand side $f(x,y)=1$ and the subsolution $v_0(x,y)=x^2-y^2$ is non-convex; (ii) the right hand side $f(x,y)=-1$ and the subsolution $v_0(x,y)=x^2+y^2$ is strictly convex on the domain $\Omega=(-0.5,0.5)\times(-0.5,0.5)$. We set the threshold $\epsilon=0.1$ in seeking a solution v with $\|v-v_0\|_0<0.1$ as in Theorem 2.1.

We were able to exhibit three consecutive corrugations. Since the frequencies $\{\lambda_k\}$ quickly become very large, fine meshes needed to be used, requiring a great computing power. On the other hand, a priori estimates (2.21) and (2.23) could be efficiently validated; indeed these estimates helped to experimentally reduce the values of each λ_k.

For the first approximation we sampled on a square grid with the initial step size 0.001, followed by the step size of the order $h\sim\frac{0.1}{\lambda_k}$. With this choice of h we were so far able to perform two steps of convex integration in both examples below. The derivatives on the mesh are:

$$\frac{\partial f}{\partial x}(x,\cdot)=\frac{1}{12h}\big(f(x-2h,\cdot)-8f(x-h,\cdot)+8f(x+h,\cdot)-f(x+2h,\cdot)\big)+O(h^5).$$

We assigned the values of the initial auxiliary variable w_0 to make the initial defect $D_0=A-\big(\frac{1}{2}\nabla v_0\otimes\nabla v_0+\operatorname{sym}\nabla w_0\big)$ diagonal, with the initial positive definiteness constant $d=0.4\leq\phi_k$ for all $k=1\ldots 3$. At each step, the defect D was calculated explicitly and then decomposed into $D=\sum_{k=1}^3\phi_k\eta_k\otimes\eta_k$ using the formulas from Lemma 2.4. We set $\delta=0.5$ and followed the indicated construction. By (2.16), our numerical implementation reduced the norm of defect to $\frac{3}{4}$ of the previous defect norm every three convex integration steps.

Define the coefficients:

$$\tilde a_k=\sqrt{\frac{\phi_k}{2}}$$

and the auxiliary matrices $\tilde B_k$ by:

$$\tilde B_k=\big(\frac{1}{2}\nabla v_k\otimes\nabla v_k+\operatorname{sym}\nabla w_k\big)-\big(\frac{1}{2}\nabla v_{k-1}\otimes\nabla v_{k-1}+\operatorname{sym}\nabla w_{k-1}+\tilde a_k^2\eta_k\otimes\eta_k\big).$$

From (2.18) we obtain bounds on $\tilde B_k$. To have $\|\tilde D\|_0\leq\frac{1}{2}\|D\|_0+\sum_{k=1}^3\|B_k\|_0\leq\frac{3}{4}\|D\|_0$ we request:

$$\|\tilde B_k\|_0\leq\frac{1}{12}\|D\|_0\quad\text{for all }k=1\ldots 3,\tag{3.1}$$

while in order to get $\tilde\phi_k\geq\tilde d=\frac{d}{4}=0.1$, we proceed as in (2.19) and introduce the condition:

$$|\tilde B_k|<\frac{8}{15\sqrt{3}}\big(\frac{1}{2}\phi_i-0.1\big)\quad\text{in }\bar\Omega,\qquad\text{for all }k,i=1\ldots 3.\tag{3.2}$$

Conditions (3.1) and (3.2) were used to determine the frequencies λ_k, where in estimating the three terms in each $\tilde B_k$ according to (2.18), we bound $|\nabla^2 v_{k-1}|$ using (2.10) with the previous choice of λ_{k-1}. Once the three choices of λ_k were made, we performed a verification that the new v and w obtained with the described above modification step remained indeed within $\epsilon=0.1$ error from the original fields.

The values of λ_1 thus obtained were small enough to allow for the numerical executing the first step of convex integration. We defined the fields v_1 and w_1 as in Proposition 2.3, calculated $\tilde B_1$, its decomposition into $\tilde B_1=\sum_{k=1}^3\phi_k\eta_k\otimes\eta_k$, and verified the two conditions (3.1) and (3.2). This operation has been repeated until the smallest possible λ_1 was found. We then used this value and the resulting a priori estimates to obtain the second frequency λ_2, small enough to apply the second step of convex integration. Thus we obtained the fields v_2 and w_2, and, by the same procedure described above, the third optimal frequency λ_3 together with v_3 and w_3. The final step size h needed to be reduced to 0.0001 in order to allow 10 mesh points for each corrugation.

Example 3.1. We approximate $v_0(x,y) = x^2 - y^2$ with a solution v to:

$$\mathcal{D}et\,\nabla^2 v = 1.$$

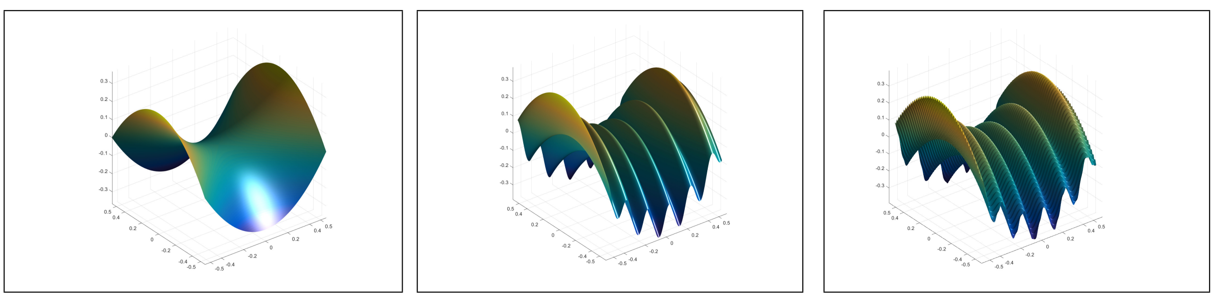
Take $w_0(x,y) = (xy^2, x^2y)$ and $A(x,y) = \big(5 - \frac{x^2+y^2}{4}\big)\mathrm{Id}_2$, so that: $-\mathrm{curl}\,\mathrm{curl}\,A(x,y) = 1$. Then:

$$\frac{1}{2}\nabla v_0(x,y) \otimes \nabla v_0(x,y) + \mathrm{sym}\nabla w_0(x,y) = \mathrm{diag}\big\{2x^2 + y^2, x^2 + 2y^2\big\}$$

and the corresponding defect is diagonal and positive definite:

$$D(x,y) = \mathrm{diag}\big\{5 - \frac{9x^2+5y^2}{4}, 5 - \frac{5x^2+9y^2}{4}\big\}$$

The function v_0 and its two subsequent corrugations are shown in Figure 3.1. In Figure 3.2 we show a detailed picture of the second corrugation; the red portion of the graph indicates the area on which we applied the third corrugation shown in Figure 3.3.



(A) The subsolution v_0 on $\bar{\Omega}$ (B) One corrugation (C) Two corrugations

FIGURE 3.1. Construction in Example 3.1

Example 3.2. We approximate $v_0(x,y) = x^2 + y^2$ with a solution v to:

$$\mathcal{D}et\,\nabla^2 v = -1.$$

In this example we take $w_0(x,y) = (-xy^2, -x^2y)$ and $A(x,y) = \big(5 + \frac{x^2+y^2}{4}\big)\mathrm{Id}_2$, satisfying $-\mathrm{curl}\,\mathrm{curl}\,A(x,y) = -1$ and resulting in the diagonal, positive definite defect:

$$D(x,y) = \big\{5 - \frac{7x^2-5y^2}{4}, 5 + \frac{5x^2-7y^2}{4}\big\}.$$

We plot three images starting from v_0 and subsequently adding the first and second corrugations in Figure 3.4. As before, we provide a more detailed picture of the second and third corrugations in Figures 3.5 and 3.6.

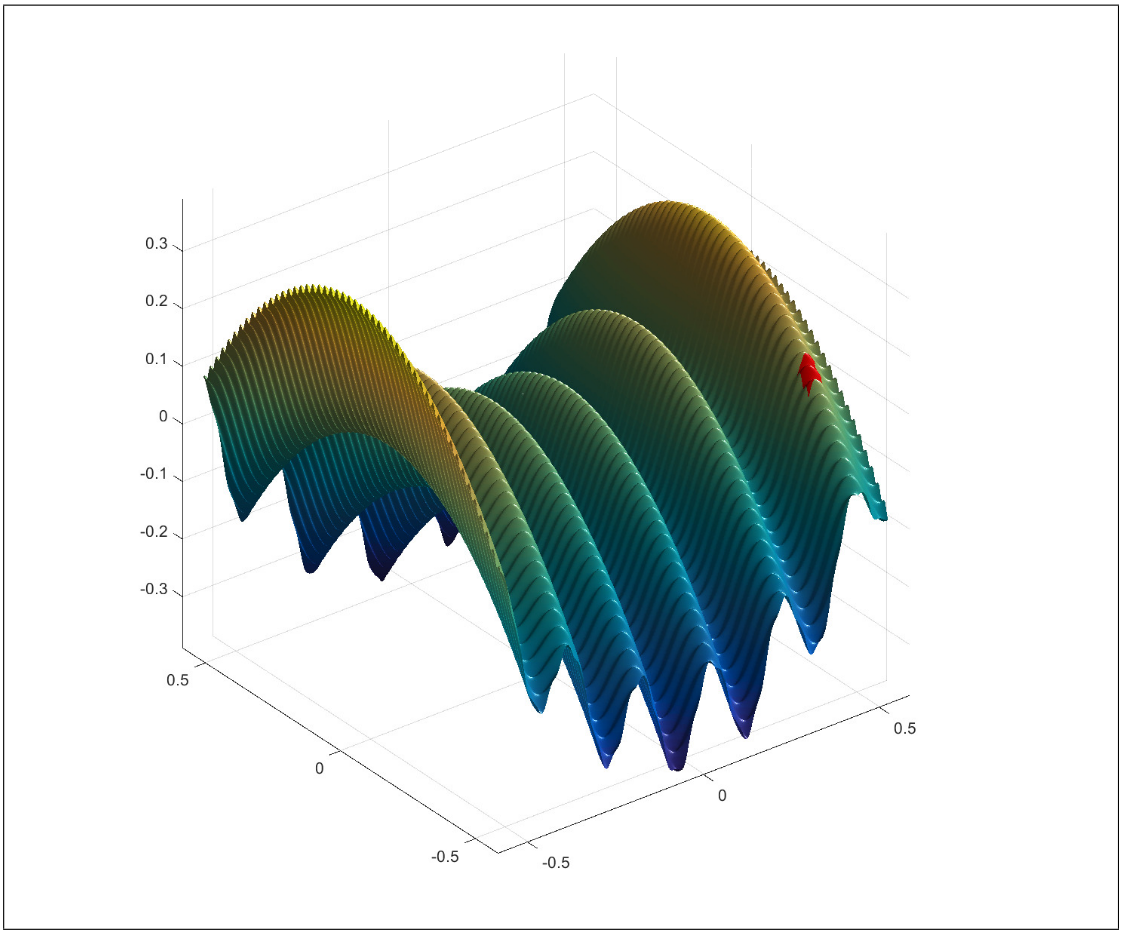


FIGURE 3.2. Two corrugations in Example 3.1. The red detail is shown in Figure 3.3

We conclude the discussion with a table listing some of the numerics results and implementation choices. The values of $\{\lambda_k\}_{k=1}^3$ were obtained experimentally. The values $\|v - v_0\|_0$ give an upper estimate of the uniform distance of v obtained through three steps of convex integration from the initial subsolution v_0. The value $(\|\tilde{B}_1\|_0 + \|\tilde{B}_2\|_0)/\|D\|_0$ which does not take the third corrugation into account, needs to be below $2 \cdot \frac{1}{12} = \frac{1}{6}$. The contribution of the last corrugation is guaranteed to be less than $\frac{1}{12}\|D\|_0$ through the a priori estimates. Lastly, $\min \phi_k$ are the minima of each of the coefficients in $\bar{\Omega}$, in the defect computed after two corrugations. The a priori estimates once again guarantee that the third step will not make these less than the required error value 0.1.

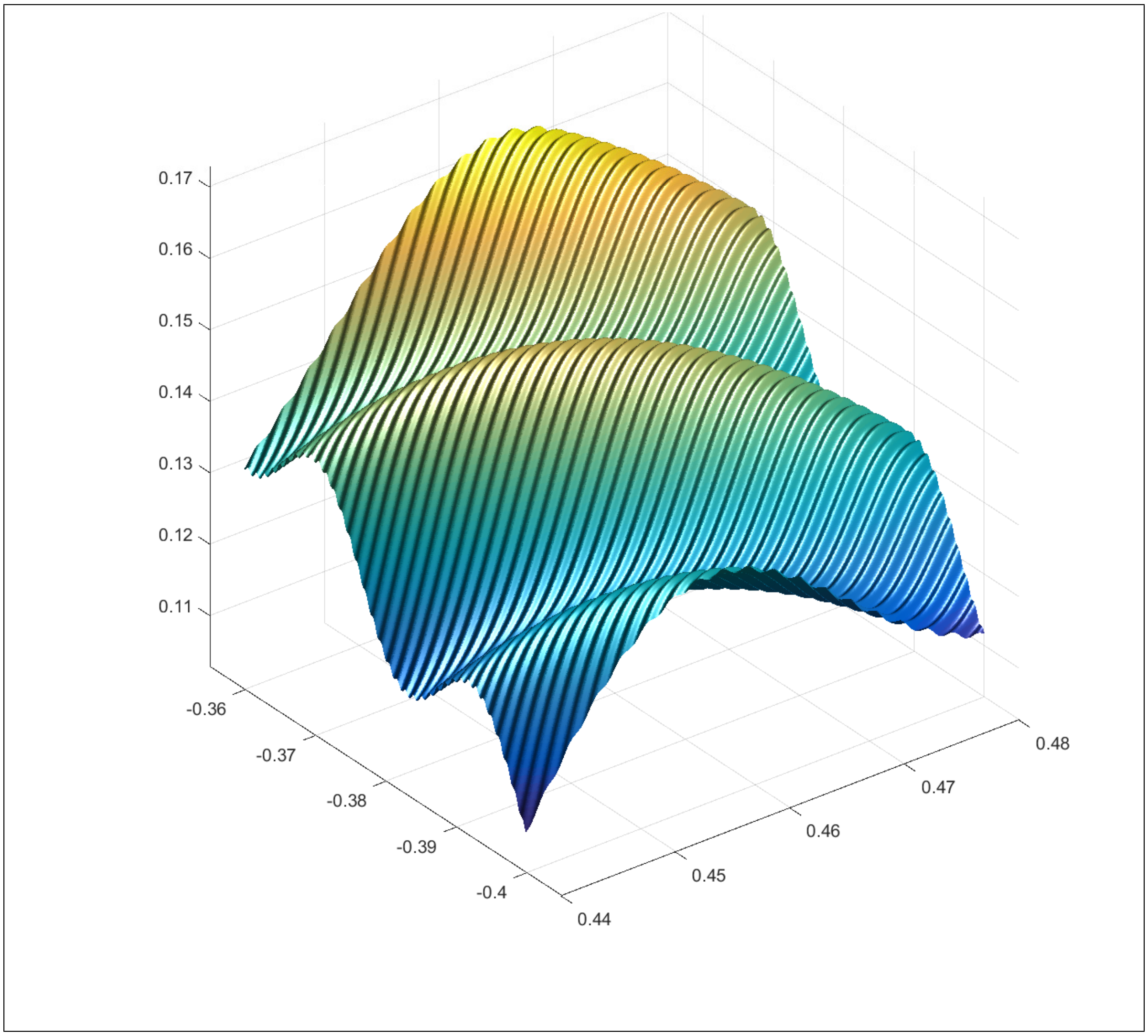
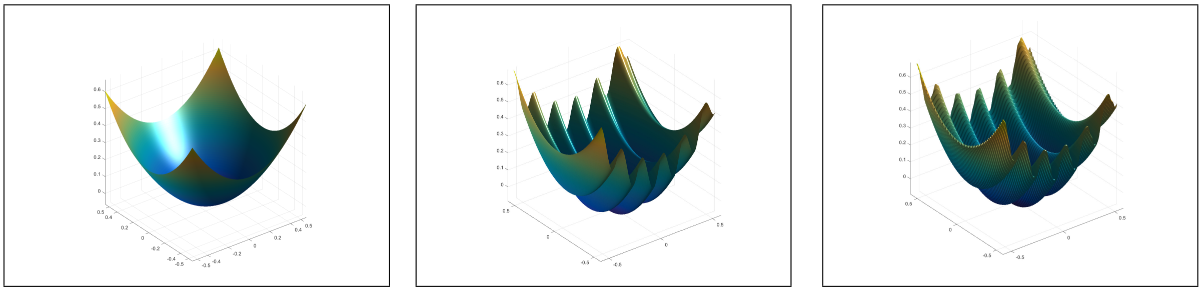


FIGURE 3.3. The detail of the three corrugations in Example 3.1

	Example 3.1	Example 3.2
$f(x,y)$	1	-1
$v_0(x,y)$	x^2-y^2	x^2+y^2
$w_0(x,y)$	(xy^2, yx^2)	$(-xy^2, -yx^2)$
λ_1	5	5
λ_2	50	57
λ_3	1000	1100
$\|v-v_0\|_0$	0.0995	0.999
$(\|\tilde{B}_1\|_0+\|\tilde{B}_2\|_0)/\|D\|_0$	0.1339	0.1246
$\min\phi_1$	0.79	0.94
$\min\phi_2$	1.14	1.29
$\min\phi_3$	1.14	1.28



(A) Original function v_0 (B) One corrugation (C) Two corrugations

FIGURE 3.4. Construction in Example 3.2

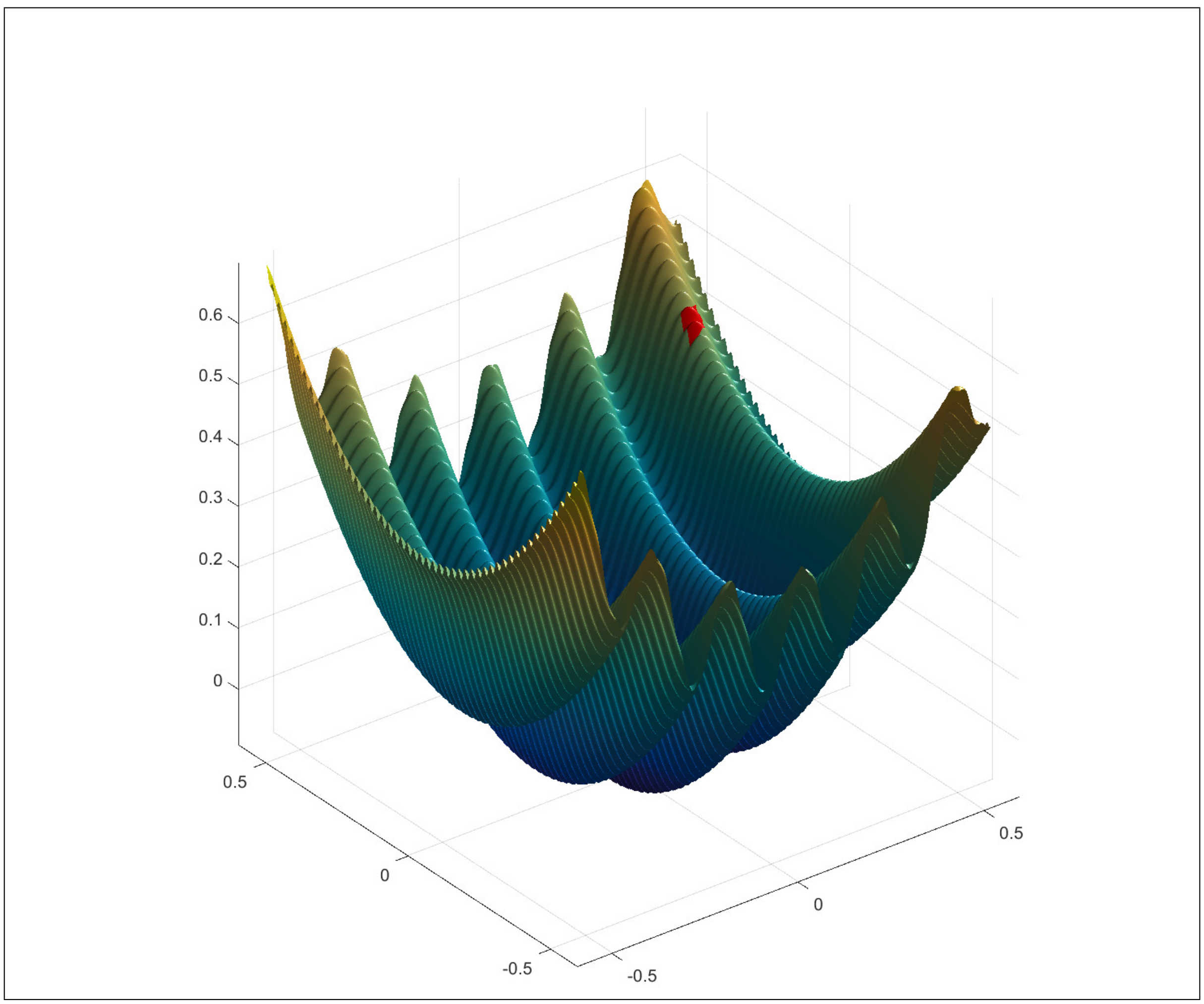


FIGURE 3.5. Two corrugations in Figure 3.4. The red detail shown in Figure 3.6

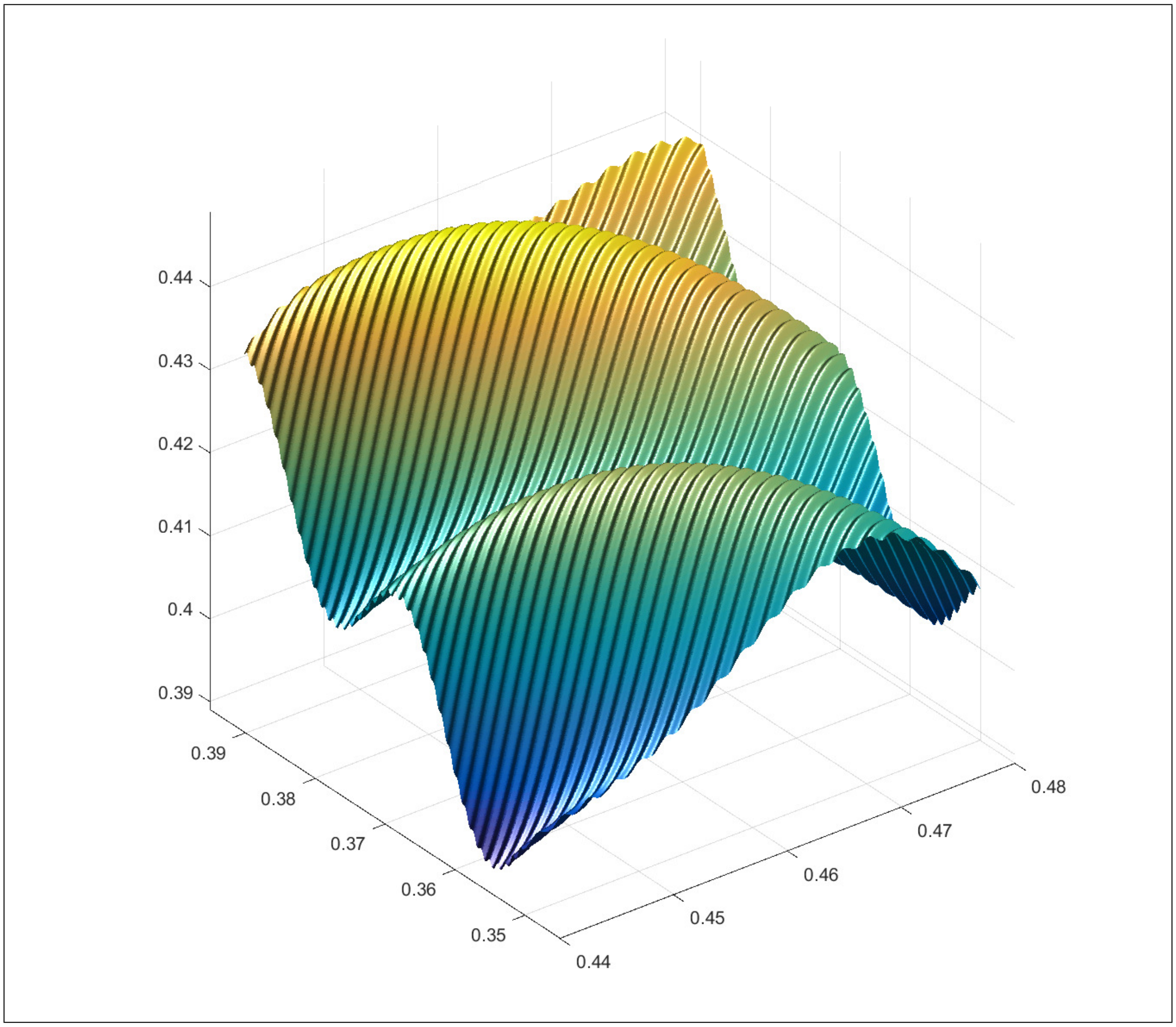


FIGURE 3.6. The detail of the three corrugations in Example 3.2

4. THE $\mathcal{C}^{1,\alpha}$ APPROXIMATION

In this and the next section we give a proof of Theorem 1.1 using a constructive, numerically implementable algorithm. The details of the implementation and the resulting visualizations will be presented in section 6. The construction follows the proof in [13], we however make some important modifications and compute all the relevant constants explicitly. We begin with a choice of a standard mollifier and some preliminary estimates.

Let $\phi \in \mathcal{C}_c^\infty(B(0,1))$ be the following radially symmetric function:

$$\varphi(x) = \begin{cases} \frac{1}{A}\exp\big(-\frac{1}{1-|x|^2}\big) & |x| \leq 1 \\ 0 & |x| \geq 1, \end{cases} \tag{4.1}$$

where $A = \int_{B_1(0)} \exp(-\frac{1}{1-|x|^2})\,\mathrm{d}x = \pi\cdot(1/e + Ei(-1))$ so that $\int_{\mathbb{R}^2}\varphi = 1$. The constant A, given in terms of the exponential integral Ei, may be approximated to any degree of precision. We will use $A\in[0.46, 0.47]$ in the estimates below, but when implementing the algorithm numerically we will evaluate A more precisely.

Lemma 4.1. *Taking φ as in (4.1), we have:*

$$\|\varphi\|_{L^1(\mathbb{R}^2)} = 1, \quad \|\nabla\varphi\|_{L^1(\mathbb{R}^2)} \leq 3.1, \quad \|\nabla^2\varphi\|_{L^1(\mathbb{R}^2)} \leq 15.9, \quad \|\nabla^3\varphi\|_{L^1(\mathbb{R}^2)} \leq 210. \tag{4.2}$$

Denote: $\varphi_l(x) = \frac{1}{l^2}\varphi\big(\frac{x}{l}\big)$ for all $l\in(0,1)$. Then, for every $f,g\in\mathcal{C}^0(\mathbb{R}^2)$ there holds:

$$\|\nabla^{k+j}(f*\varphi_l)\|_0 \leq \frac{1}{l^k}\|\nabla^k\varphi\|_{L^1(\mathbb{R}^2)}\|\nabla^j f\|_0 \quad \text{for all} \ \ k,j\geq 0, \tag{4.3}$$

$$\begin{aligned} &\|f*\varphi_l - f\|_0 \leq \frac{1}{2}l^2\|\nabla^2 f\|_0, \qquad \|\nabla(f*\varphi_l - f)\|_0 \leq l\|\nabla^2 f\|_0, \\ &\|\nabla^2(f*\varphi_l - f)\|_0 \leq 2\|\nabla^2 f\|_0. \end{aligned} \tag{4.4}$$

Moreover, for all $\alpha\in(0,1]$ there holds:

$$\|f*\varphi_l - f\|_0 \leq l^\alpha[f]_\alpha, \qquad \|\nabla(f*\varphi_l)\|_0 \leq (3.1)l^{\alpha-1}[f]_\alpha, \tag{4.5}$$

and further:

$$\begin{aligned} &\|(fg)*\varphi_l - (f*\varphi_l)(g*\varphi_l)\|_0 \leq 2l^{2\alpha}[f]_\alpha[g]_\alpha, \\ &\|\nabla\big((fg)*\varphi_l - (f*\varphi_l)(g*\varphi_l)\big)\|_0 \leq (9.3)l^{2\alpha-1}[f]_\alpha[g]_\alpha, \\ &\quad\|\nabla^2\big((fg)*\varphi_l - (f*\varphi_l)(g*\varphi_l)\big)\|_0 \leq (67)l^{2\alpha-2}[f]_\alpha[g]_\alpha, \\ &\quad\|\nabla^3\big((fg)*\varphi_l - (f*\varphi_l)(g*\varphi_l)\big)\|_0 \leq (925.8)l^{2\alpha-3}[f]_\alpha[g]_\alpha. \end{aligned} \tag{4.6}$$

All norms above are taken on the whole domain $\mathbb{R}^2$.

Proof. The estimates (4.2) follow by calculating the indicated integrals in polar coordinates and then evaluating the 1-dimensional integrals numerically. The bound (4.3) results from: $\|\nabla^k\varphi_l\|_{L^1(\mathbb{R}^2)} = l^{-k}\|\nabla^k\varphi\|_{L^1(\mathbb{R}^2)}$ for every $k\geq 0$, whereas to get (4.4) we use Taylor's expansion of f at a given $x\in\mathbb{R}^2$. Finally, writing $h = (fg)*\varphi_l - (f*\varphi_l)(g*\varphi_l)$, we observe as in Lemma 2.1 [4] that:

$$\begin{aligned} &\|\nabla h\|_0 \leq 3l^{2\alpha}\|\nabla\varphi_l\|_{L^1(\mathbb{R}^2)}[f]_\alpha[g]_\alpha \\ &\|\nabla^2 h\|_0 \leq l^{2\alpha}\big(3\|\nabla^2\varphi_l\|_{L^1(\mathbb{R}^2)} + 2\|\nabla\varphi_l\|^2_{L^1(\mathbb{R}^2)}\big)[f]_\alpha[g]_\alpha \\ &\|\nabla^3 h\|_0 \leq l^{2\alpha}\big(3\|\nabla^3\varphi_l\|_{L^1(\mathbb{R}^2)} + 6\|\nabla^2\varphi_l\|^2_{L^1(\mathbb{R}^2)}\|\nabla\varphi_l\|_{L^1(\mathbb{R}^2)}\big)[f]_\alpha[g]_\alpha, \end{aligned}$$

which proves (4.6) in view of (4.2). ■

The next result is a modification of Proposition 2.3 so we omit its proof.

Proposition 4.2. *Given $v\in\mathcal{C}^3(\bar\Omega)$, $w\in\mathcal{C}^2(\bar\Omega,\mathbb{R}^2)$, a nonnegative function $a\in\mathcal{C}^3(\bar\Omega)$ and a unit vector $\eta\in\mathbb{R}^2$, let $\delta,l\in(0,1)$ be two parameter constants satisfying:*

$$\|\nabla^m a\|_0 \leq \frac{\delta}{l^m} \quad \text{for} \ \ m=0\dots 3 \quad \text{and} \quad \|\nabla^{m+1}v\|_0 \leq \frac{\delta}{l^m} \quad \text{for} \ \ m=1,2. \tag{4.7}$$

Then for any frequency $\lambda \geq 1/l$*, the approximations* $v_\lambda \in \mathcal{C}^3(\bar\Omega)$ *and* $w_\lambda \in \mathcal{C}^2(\bar\Omega, \mathbb{R}^2)$ *defined in (2.6), satisfy:*

$$\Big|\big(\frac{1}{2}\nabla v_\lambda \otimes \nabla v_\lambda + \mathrm{sym}\nabla w_\lambda\big) - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\nabla w + a^2\eta\otimes\eta\big)\Big| \leq \frac{\delta^2}{\lambda l}, \tag{4.8}$$

$$\begin{aligned} &\|v_\lambda - v\|_0 \leq (0.4)\frac{\delta}{\lambda}, \quad \|\nabla(v_\lambda - v)\|_0 \leq (2.4)\delta,\\ &\|\nabla^2(v_\lambda - v)\|_0 \leq (16.9)\delta\lambda, \quad \|\nabla^3(v_\lambda - v)\|_0 \leq (123)\delta\lambda^2\\ &\quad \|w_\lambda - w\|_0 \leq (0.4)\frac{\delta}{\lambda}(1+\|\nabla v\|_0), \quad \|\nabla(w_\lambda - w)\|_0 \leq (2.4)\delta(1+\|\nabla v\|_0),\\ &\quad \|\nabla^2(w_\lambda - w)\|_0 \leq (21.9)\delta\lambda(1+\|\nabla v\|_0). \end{aligned} \tag{4.9}$$

The following is the "stage" of the Hölder approximation construction, consisting of iterating three convex integration steps detailed in Proposition 2.5, with an additional mollification at each step in order to control the second derivative norms. The statement and the proof are similar to Proposition 5.2 in [13], but we avoid the extension argument and consequently make the universal constants explicit to allow for a numerical implementation.

Proposition 4.3 (Proposition 5.2 [13])**.** *For an open bounded domain* $\Omega \subset \mathbb{R}^2$*, we denote:* $\Omega_r \doteq \Omega + B_r(0)$ *for some* $r \in (0,1)$*. Given three functions* $v \in \mathcal{C}^2(\bar\Omega_r)$*,* $w \in \mathcal{C}^2(\bar\Omega_r, \mathbb{R}^2)$ *and* $A \in \mathcal{C}^{0,\beta}(\bar\Omega_r, \mathbb{R}^{2\times 2}_{sym})$ *of Hölder regularity* $\beta \in (0,1)$*, assume that for* $\delta_0 < (5.4)\cdot 10^{-16}$ *we have:*

$$D \doteq A - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\nabla w\big), \quad 0 < \|D\|_{\mathcal{C}^0(\bar\Omega_r)} \leq \delta_0. \tag{4.10}$$

Then, for every two constants M, σ *which satisfy:*

$$M > \max\Big\{\frac{\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}}{r}, \|\nabla^2 v\|_{\mathcal{C}^0(\bar\Omega_r)}, \|\nabla^2 w\|_{\mathcal{C}^0(\bar\Omega_r)}, 1\Big\} \quad \textit{and} \quad \sigma > 1, \tag{4.11}$$

there exist $\tilde v \in \mathcal{C}^2(\bar\Omega)$ *and* $\tilde w \in \mathcal{C}^2(\bar\Omega, \mathbb{R}^2)$ *such that with:* $\tilde D \doteq A - \big(\frac{1}{2}\nabla\tilde v \otimes \nabla \tilde v + \mathrm{sym}\nabla\tilde w\big)$ *we have:*

$$\|\tilde D\|_0 \leq \frac{\|A\|_{\mathcal{C}^{0,\beta}(\bar\Omega_r)}}{M^\beta}\|D\|^{\beta/2}_{\mathcal{C}^0(\bar\Omega_r)} + \frac{(1.9)10^{15}}{\sigma}\|D\|_{\mathcal{C}^0(\bar\Omega_r)}, \tag{4.12}$$

$$\begin{aligned} &\|\tilde v - v\|_0 \leq \frac{(1.8)10^7}{M}\|D\|_{\mathcal{C}^0(\bar\Omega_r)}\\ &\qquad \textit{and} \quad \|\tilde w - w\|_0 \leq \Big(\frac{(1.8)10^7}{M} + (12.6)\mathrm{diam}(\Omega_r)\Big)\|D\|_{\mathcal{C}^0(\bar\Omega_r)}(1+\|\nabla v_0\|_{\mathcal{C}^0(\bar\Omega_r)}), \end{aligned} \tag{4.13}$$

$$\begin{aligned} &\|\nabla(\tilde v - v)\|_0 \leq (1.1)10^8\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}\\ &\qquad\qquad \textit{and} \quad \|\nabla(\tilde w - w)\|_0 \leq (1.1)10^8(1+\|\nabla v\|_{\mathcal{C}^0(\bar\Omega_r)})\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}, \end{aligned} \tag{4.14}$$

$$\|\nabla^2\tilde v\|_0 \leq (7.3)10^8 M\sigma^3 \quad \textit{and} \quad \|\nabla^2\tilde w\|_0 \leq (9.5)10^8(1+\|\nabla v\|_{\mathcal{C}^0(\bar\Omega_r)})M\sigma^3. \tag{4.15}$$

The norms $\|\cdot\|_0$ *above signify* $\|\cdot\|_{\mathcal{C}^0(\bar\Omega)}$*.*

Proof. The proof proceeds in three parts: mollification of v, w, A to control higher derivatives, modification of w to ensure the positive decomposition of the defect in the basis $\{\eta_k\otimes\eta_k\}_{k=1}^3$, and application of three consecutive steps of convex integration to reduce the defect.

1. Mollification. For all $l < r$, define the following functions on $\bar\Omega$ through mollification with the standard kernel φ as in (4.1):

$$\mathfrak{v} \doteq v * \varphi_l, \quad \mathfrak{w} \doteq w * \varphi_l, \quad \mathfrak{A} \doteq A * \varphi_l, \quad \text{where} \quad l \doteq \frac{\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}}{M} < r < 1.$$

Denote: $\mathfrak{D} \doteq \mathfrak{A} - \big(\frac{1}{2}\nabla\mathfrak{v}\otimes\nabla\mathfrak{v} + \mathrm{sym}\nabla\mathfrak{w}\big)$. We use Lemma 4.1 and assumption (4.11) to obtain the uniform error bounds below, where the relevant norms of quantities $\mathfrak{v}, \mathfrak{w}, \mathfrak{A}, \mathfrak{D}$ are taken on $\bar\Omega$, while other norms are taken on the superset $\bar\Omega_r$:

$$\begin{aligned} &\|\mathfrak{v} - v\|_0, \ \|\mathfrak{w} - w\|_0 \leq \frac{l}{2}\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}, \quad \|\nabla(\mathfrak{v} - v)\|_0, \ \|\nabla(\mathfrak{w} - w)\|_0 \leq \|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}, \\ &\|\mathfrak{A} - A\|_0 \leq l^\beta [A]_{\beta,\bar\Omega_r}, \\ &\|\nabla^m\mathfrak{D}\|_0 \leq \|\nabla^m(D*\varphi_l)\|_0 + \frac{1}{2}\|\nabla^m\big((\nabla v * \varphi_l)\otimes(\nabla v*\varphi_l) - (\nabla v\otimes\nabla v)*\varphi_l\big)\|_0. \end{aligned} \tag{4.16}$$

In view of (4.2), (4.6) and (4.11), the last bound above is specified to:

$$\begin{aligned} &\|\mathfrak{D}\|_0 \leq \|D\|_0 + l^2[\nabla v]_1^2 \leq 2\|D\|_{\mathcal{C}^0(\bar\Omega_r)}, \\ &\|\nabla\mathfrak{D}\|_0 \leq \frac{3.1}{l}\|D\|_0 + (4.7)l[\nabla v]_1^2 \leq (7.8)\frac{1}{l}\|D\|_{\mathcal{C}^0(\bar\Omega_r)}, \\ &\|\nabla^2\mathfrak{D}\|_0 \leq \frac{15.9}{l^2}\|D\|_0 + (33.5)[\nabla v]_1^2 \leq (49.4)\frac{1}{l^2}\|D\|_{\mathcal{C}^0(\bar\Omega_r)}, \\ &\|\nabla^3\mathfrak{D}\|_0 \leq \frac{210}{l^3}\|D\|_0 + (462.9)\frac{1}{l}[\nabla v]_1^2 \leq (672.9)\frac{1}{l^3}\|D\|_{\mathcal{C}^0(\bar\Omega_r)}. \end{aligned} \tag{4.17}$$

We also get, by (4.3):

$$\begin{aligned} &\|\nabla^2\mathfrak{v}\|_0 \leq \|\nabla^2 v\|_{\mathcal{C}^0(\bar\Omega_r)} \leq \frac{1}{l}\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}, \\ &\|\nabla^3\mathfrak{v}\|_0 \leq \frac{1}{l}\|\nabla\varphi\|_{L^1(\mathbb{R}^2)}\|\nabla^2 v\|_{\mathcal{C}^0(\bar\Omega_r)} \leq (3.1)\frac{1}{l^2}\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}. \end{aligned} \tag{4.18}$$

Finally, Lemma 4.1 yields:

$$\|\nabla^2\mathfrak{w}\|_0 \leq \|\nabla^2 w\|_{\mathcal{C}^0(\bar\Omega_r)} \leq M. \tag{4.19}$$

2. Modification and decomposition. To ensure that the deficit may be decomposed with positive coefficients, we define:

$$\tilde{\mathfrak{w}} \doteq \mathfrak{w} - (\|D\|_{\mathcal{C}^0(\bar\Omega_r)} + \|\mathfrak{D}\|_0)\Big(\frac{\sqrt{2}+9}{4}x, (\sqrt{2}+\frac{9}{5})y\Big), \qquad \tilde{\mathfrak{D}} \doteq \mathfrak{A} - \big(\frac{1}{2}\nabla\mathfrak{v}\otimes\nabla\mathfrak{v} + \mathrm{sym}\nabla\tilde{\mathfrak{w}}\big).$$

By (4.16) there follow the bounds:

$$\begin{aligned} &\|\nabla(\tilde{\mathfrak{w}} - \mathfrak{w})\|_0 \leq (4.2)(\|D\|_{\mathcal{C}^0(\bar\Omega_r)} + \|\mathfrak{D}\|_0) \leq (12.6)\|D\|_{\mathcal{C}^0(\bar\Omega_r)}, \\ &\|\nabla^2(\tilde{\mathfrak{w}} - \mathfrak{w})\|_0 = 0. \end{aligned} \tag{4.20}$$

Note that $\tilde{\mathfrak{D}} - \mathfrak{D} = \big(\|D\|_{\mathcal{C}^0(\bar\Omega_r)} + \|\mathfrak{D}\|_0\big)\mathrm{diag}\big(\frac{\sqrt{2}+9}{4}, \sqrt{2}+\frac{9}{5}\big)$ and therefore: $\nabla^m\tilde{\mathfrak{D}} = \nabla^m\mathfrak{D}$ for all $m \geq 1$. Further, this construction guarantees that in the decomposition $\tilde{\mathfrak{D}} = \sum_{k=1}^3 \phi_k\eta_k\otimes\eta_k$ on $\bar\Omega$, in view of Lemma 2.4 (iv) we get: $\phi_k \geq (\|D\|_{\mathcal{C}^0(\bar\Omega_r)} + \|\mathfrak{D}\|_0)/2$. We now find the bounds on the first norms of the smooth positive functions $a_k \doteq \sqrt{\phi_k}$. We begin by noting that $\min_{x\in\bar\Omega} a_k(x) \geq$

$\frac{\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}}{\sqrt{2}}$. In view of (4.20), (4.17) and since $\nabla a_k = \frac{\nabla\phi_k}{2a_k}$ and $\nabla^2 a_k = \frac{\nabla^2\phi_k}{2a_k} - \frac{\nabla a_k\otimes\nabla a_k}{a_k}$, Lemma 2.4 yields:

$$\begin{aligned}
\|a_k\|_0 &\leq \Big(\frac{5\sqrt{3}}{8}\|\tilde{\mathfrak{D}}\|_0\Big)^{1/2} \leq (4.1)\|\bar D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)},\\
\|\nabla a_k\|_0 &\leq \frac{\|\nabla\phi_k\|_0}{2\min_{x\in\bar\Omega} a_k(x)} \leq \frac{5\sqrt{3}}{8\sqrt{2}}\frac{\|\nabla\tilde{\mathfrak{D}}\|_0}{\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}} \leq 6\frac{1}{l}\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)},\\
\|\nabla^2 a_k\|_0 &\leq \frac{\|\nabla^2\phi_k\|_0 + 2\|\nabla a_k\|_0^2}{2\min_{x\in\bar\Omega} a_k(x)} \leq \frac{5\sqrt{3}}{8\sqrt{2}}\frac{\|\nabla^2\tilde{\mathfrak{D}}\|_0}{\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}} + \sqrt{2}\frac{\|\nabla a_k\|_0^2}{\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}} \leq (88.8)\frac{1}{l^2}\|D\|_0^{1/2}, \qquad (4.21)\\
\|\nabla^3 a_k\|_0 &\leq \frac{\|\nabla^3\phi_k\|_0 + 6\|\nabla a_k\|_0\|\nabla^2 a_k\|_0}{2\min_{x\in\bar\Omega} a_k(x)} \leq \frac{5\sqrt{3}}{8\sqrt{2}}\frac{\|\nabla^3\tilde{\mathfrak{D}}\|_0}{\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}} + 3\sqrt{2}\frac{\|\nabla a_k\|_0\|\nabla^2 a_k\|_0}{\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}}\\
&\leq (2775.6)\frac{1}{l^3}\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}.
\end{aligned}$$

3. Iteration of convex integration. We set $v_0 \doteq \mathfrak{v}$, $w_0 \doteq \tilde{\mathfrak{w}}$ restricted to $\bar\Omega$ and then define recursively $v_k \in \mathcal{C}^3(\bar\Omega)$, $w_k \in \mathcal{C}^2(\bar\Omega, \mathbb{R}^2)$ for $k = 1, 2, 3$ by applying Proposition 4.2. We apply it to v_{k-1}, w_{k-1}, with a_k from the decomposition of $\tilde{\mathfrak{D}}$ in the basis given by $\{\eta_k\}_{k=1}^3$. Lastly, we set the parameters:

$$l_k \doteq \frac{l}{\sigma^{k-1}} < 1, \quad \lambda_k \doteq \frac{1}{l_{k+1}} > \frac{1}{l_k},$$

and the non-decreasing triple $\{\delta_k\}_{k=1}^3$ with the initial choice:

$$\delta_1 \doteq \max_{m=1,2}\big\{l^m\|\nabla^{m+1}\mathfrak{v}\|_0\big\} + \max_{m=0\dots3,\ k=1\dots3}\{l^m\|\nabla^m a_k\|_0\}. \qquad (4.22)$$

The construction is complete by setting $\tilde v \doteq v_3$ and $\tilde w \doteq w_3$, and claiming that these satisfy the error bounds (4.12)-(4.15).

First, we check that the assumptions of Proposition 4.2 hold at every step. The condition that $l_k \in (0,1)$ is easily verified as $l < 1$ and $\sigma > 1$. By (4.21) we get $l^m\|\nabla^{m+1}\mathfrak{v}\|_0 \leq (3.1)\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}$ for $m = 1, 2$ and $l^m\|\nabla^m a_k\|_0 \leq (2775.6)\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)}$ for $m, k = 0\dots3$, which yield:

$$\delta_1 \leq (2778.7)\|D\|^{1/2}_{\mathcal{C}^0(\bar\Omega_r)} < 1$$

by (4.10). The first condition in (4.7) is clearly satisfied for all k by (4.22), given that $l_k < l$ and $\delta_1 \leq \delta_k$. Further, by induction on k and using (4.9), we obtain:

$$\begin{aligned}
\|\nabla^{m+1}v_k\|_0 &\leq \|\nabla^{m+1}v_{k-1}\|_0 + \|\nabla^{m+1}v_k - \nabla^{m+1}v_{k-1}\|_0 \leq \frac{\delta_k}{l_k^m} + (123)\delta_k\lambda_k^m\\
&\leq \delta_k\frac{124}{l_{k+1}^m} \leq \frac{\delta_{k+1}}{l_{k+1}^m} \qquad \text{for all } \ m, k = 1, 2,
\end{aligned}$$

if only $\delta_{k+1} = (124)\delta_k$ for $k = 1, 2$. This ensures the second condition in (4.7) in view of (4.22), provided that $(2778.7)\cdot(124)^2\delta_0^{1/2} < 1$ to get $\delta_1, \delta_2, \delta_3 < 1$. This last inequality is implied by the original bound on δ_0. We omit the verification of estimates (4.12), (4.13) (note that here we use the condition $0 \in \Omega$), (4.14) and (4.15) as they follow directly as in [13]. ■

5. A proof of Theorem 1.1

We start by showing the main approximation result needed in Theorem 1.1, that is a version of the result in Theorem 1.2. It consists of iterating the "stages" construction, with the sole restrictive assumption on the smallness of the initial deficit.

Proposition 5.1. *Let $\Omega \subset \Omega_r \subset \mathbb{R}^2$ be open bounded sets, where $\Omega_r = \Omega + B_r(0)$ for $r > 0$. Let:*

$$\delta_0 < \min\Big\{\frac{r}{2}, (5.4)10^{-16}\Big\}.$$

Given three functions: $v \in \mathcal{C}^2(\bar\Omega_r)$, $w \in \mathcal{C}^2(\bar\Omega_r, \mathbb{R}^2)$ and $A \in \mathcal{C}^{0,\beta}(\bar\Omega_r, \mathbb{R}^{2\times 2}_{sym})$ of Hölder regularity $\beta \in (0,1)$, assume that:

$$D \doteq A - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\nabla w\big), \qquad 0 < \|D\|_{\mathcal{C}^0(\bar\Omega_r)} \leq \delta_0.$$

Then, for every exponent $\alpha \in (0, \min\{\frac{1}{7}, \frac{\beta}{2}\})$ one can find $\bar v \in \mathcal{C}^{1,\alpha}(\bar\Omega)$, $\bar w \in \mathcal{C}^{1,\alpha}(\bar\Omega, \mathbb{R}^2)$ such that: $\frac{1}{2}\nabla \bar v \otimes \nabla \bar v + \mathrm{sym}\nabla \bar w = A$, and:

$$\|v - \bar v\|_0 < (0.21)\delta_0, \qquad \|w - \bar w\|_0 < \big(0.5 + (63.4)\cdot \mathrm{diam}\,\Omega_r\big)(1 + \|\nabla v_0\|_{\mathcal{C}^0(\bar\Omega_r)})\delta_0.$$

Proof. **1.** Define a decreasing sequence of open domains $\{\Omega^k\}_{k=0}^\infty$ by setting:

$$\Omega^0 \doteq \Omega_r, \qquad \Omega^k \doteq \Omega_{r - \delta_0 \sum_{i=1}^k 2^{-i}} \quad \text{for } k \geq 1,$$

so that: $\Omega^k = \Omega^{k+1} + B_{2^{-(k+1)}\delta_0}(0)$. Let $v_0 \doteq v$ and $w_0 \doteq w$ on $\bar\Omega^0$. Given $k \geq 0$ and $v_k \in \mathcal{C}^2(\bar\Omega^k)$, $w_k \in \mathcal{C}^2(\bar\Omega^k, \mathbb{R}^2)$, resulting in the nonzero deficit:

$$D_k \doteq A - \big(\frac{1}{2}\nabla v_k \otimes \nabla v_k + \mathrm{sym}\,\nabla w_k\big),$$

we will construct the functions $v_{k+1} \in \mathcal{C}^2(\bar\Omega^{k+1})$, $w_{k+1} \in \mathcal{C}^2(\bar\Omega^{k+1}, \mathbb{R}^2)$ by applying Proposition 4.3 to the sets $\Omega^{k+1} \subset \Omega^k$, the matrix field $A_{|\bar\Omega^k}$ and the parameters σ_k, M_k chosen according to the procedure indicated below. First, choose the exponent $s \in (0,1)$ to satisfy:

$$\frac{6\alpha}{1-\alpha} < s < \frac{6\beta}{2-\beta}, \tag{5.1}$$

Existence of such s in guaranteed by $\alpha \in (0, \min\{\frac{1}{7}, \frac{\beta}{2}\})$. Second, set the constant:

$$\mathfrak{C} \doteq (20.9)10^8\big(1 + \|\nabla v_0\|_{\mathcal{C}(\bar\Omega^0)}\big) \tag{5.2}$$

and let $\{\sigma_k\}_{k=0}^\infty$ be an increasing sequence of positive numbers, converging to some σ_{max} and satisfying:

$$\sigma_k^s \geq \frac{16}{9}, \quad \sigma_k^{1-s} > (3.7)10^{15} \quad \text{for all } k \geq 0 \qquad \text{and} \qquad (\sigma_{max})^{\frac{s}{2}(1-\alpha)-3\alpha} > \mathfrak{C}^\alpha. \tag{5.3}$$

We note that it is enough to take $\sigma_k = \sigma_{max}$ for all k, but having σ_k as small as possible is advantageous for the numerical calculations. Third, the constants $\{M_k\}_{k=0}^\infty$ are defined by:

$$\begin{aligned} & M_k \doteq M_0 \mathfrak{C}^k \prod_{j=0}^{k-1} \sigma_j^3 \qquad \text{for all } k \geq 1 \\ & \text{and} \qquad M_0 \doteq \begin{cases} N 2^{\frac{1}{\beta}} (\sigma_{max})^{\frac{1}{\beta}} \|A\|_{\mathcal{C}^{0,\beta}(\bar\Omega^0)}^{\frac{1}{\beta}} \|D_0\|_{\mathcal{C}^{0,\beta}(\bar\Omega^0)}^{\frac{1}{2}-\frac{1}{\beta}} & \text{if } A \neq 0 \\ N & \text{if } A = 0, \end{cases} \end{aligned} \tag{5.4}$$

for a large constant $N \geq 1$ below (that is evaluated numerically in the following implementation). In particular, there holds:

$$M_0 \geq 2\delta_0^{-\frac{1}{2}}. \tag{5.5}$$

2. We now inductively prove that:

$$\|D_k\|_{\mathcal{C}^0(\bar{\Omega}^k)} \leq \frac{\|D_0\|_{\mathcal{C}^0(\bar{\Omega}^0)}}{\prod_{j=0}^{k-1}\sigma_j^s} \qquad \text{for all } k \geq 1. \tag{5.6}$$

For $k = 1$, Proposition 4.3 gives:

$$\frac{\|D_1\|_{\mathcal{C}^0(\bar{\Omega}^1)}}{\|D_0\|_{\mathcal{C}^0(\bar{\Omega}^0)}} \leq \frac{\|A\|_{\mathcal{C}^{0,\beta}(\bar{\Omega}^0)}\|D_0\|_{\mathcal{C}^0(\bar{\Omega}^0)}^{\frac{\beta}{2}-1}}{M_0^\beta} + \frac{(1.9)10^{15}}{\sigma_0}.$$

The second term in the right hand side above is majored by $\frac{1}{2\sigma_0^s}$ by (5.3), whereas the first term is also bounded by: $\frac{1}{2N^\beta\sigma_0^s} \leq \frac{1}{2\sigma_0^s}$ in view of (5.4). For $k \geq 1$, we similarly observe that:

$$\frac{\|D_{k+1}\|_{\mathcal{C}^0(\bar{\Omega}^{k+1})}}{\|D_k\|_{\mathcal{C}^0(\bar{\Omega}^k)}} \leq \frac{\|A\|_{\mathcal{C}^{0,\beta}(\bar{\Omega}^0)}\|D_k\|_{\mathcal{C}^0(\bar{\Omega}^k}^{\frac{\beta}{2}-1}}{M_k^\beta} + \frac{(1.9)10^{15}}{\sigma_k} \leq \frac{1}{\sigma_k^s},$$

where the bound on the second term follows as in the case $k = 0$, while to estimate the first term:

$$\frac{\|A\|_{\mathcal{C}^{0,\beta}(\bar{\Omega}^0)}\|D_k\|_{\mathcal{C}^0(\bar{\Omega}^k)}^{\frac{\beta}{2}-1}}{M_k^\beta} \leq \frac{1}{2N\sigma_{max}^s\mathfrak{C}^{k\beta}\sigma_k^{3\beta}\prod_{j=0}^{k-1}\sigma_j^{3\beta+s(\frac{\beta}{2}-1)}} \leq \frac{1}{2\sigma_k^s},$$

we used the inductive assumption, (5.4) and (5.1), implying that $3\beta + s(\frac{\beta}{2} - 1) > 0$. This ends the proof of (5.6).

Observe also that by (5.6), (5.3) and the assumed bound on δ_0 we get, for all $k \geq 0$:

$$\begin{aligned} 1 + \|\nabla v_k\|_{\mathcal{C}^0(\bar{\Omega}^k)} &\leq 1 + \|\nabla v_0\|_{\mathcal{C}^0(\bar{\Omega}^0)} + (1.1)10^8\sum_{j=0}^\infty \|D_k\|_{\mathcal{C}^0(\bar{\Omega}^k)}^{\frac{1}{2}} \\ &\leq 1 + \|\nabla v_0\|_{\mathcal{C}^0(\bar{\Omega}^0)} + (1.1)10^8\|D_0\|_{\mathcal{C}^0(\bar{\Omega}^0)}^{\frac{1}{2}}\Big(\sum_{j=0}^\infty \frac{1}{\prod_{i=0}^{j-1}\sigma_i^s}\Big)^{\frac{1}{2}} \\ &\leq 1 + \|\nabla v_0\|_{\mathcal{C}^0(\bar{\Omega}^0)} + (1.1)10^8 \cdot \delta_0^{\frac{1}{2}}\sum_{j=0}^\infty (\frac{3}{4})^j \\ &\leq 1 + \|\nabla v_0\|_{\mathcal{C}^0(\bar{\Omega}^0)} + 1.2 \leq (2.2)\big(1 + \|\nabla v_0\|_{\mathcal{C}^0(\bar{\Omega}^0)}\big). \end{aligned} \tag{5.7}$$

3. Clearly, when $D_k = 0$, we stop the recursive process and set $(\bar{v}, \bar{w}) \doteq (v_k, w_k)$ with the claimed error estimates established in the next step below. We now proceed by validating the assumptions of Proposition 4.3 in case $D_k \neq 0$.

The bound $\|D_k\|_{\mathcal{C}^0(\bar{\Omega}^k)} \leq \delta_0$ is a consequence of (5.6) and $\sigma_j \geq 1$ for all $j \geq 0$. Similarly:

$$\frac{2^{k+1}}{\delta_0}\|D_k\|_{\mathcal{C}^0(\bar{\Omega}^k)} \leq M_k$$

follows by (5.5) for $k=0$, whereas for $k\geq 1$ we additionally use (5.6) in:

$$\frac{2^{k+1}}{\delta_0}\|D_k\|_{\mathcal{C}^0(\bar\Omega^k)} \leq \frac{2^{k+1}}{\delta_0}\|D_0\|_{\mathcal{C}^0(\bar\Omega^0)} \leq \frac{2^{k+1}}{\delta_0^{\frac12}} \leq 2^k M_0 \leq M_k.$$

The estimates: $\|\nabla^2 v_0\|_{\mathcal{C}^0(\bar\Omega^0)}, \|\nabla^2 w_0\|_{\mathcal{C}^0(\bar\Omega^0)} \leq M_0$ are valid if N is chosen large enough. Finally, for all $k\geq 1$ we have:

$$\begin{aligned}
\|\nabla^2 v_k\|_{\mathcal{C}^0(\bar\Omega^k)} &\leq (7.3)10^8 M_{k-1}\sigma_{k-1}^3 \leq M_k,\\
\|\nabla^2 w_k\|_{\mathcal{C}^0(\bar\Omega^k)} &\leq (9.5)10^8 M_{k-1}\sigma_{k-1}^3\big(1+\|\nabla v_{k-1}\|_{\mathcal{C}(\bar\Omega^{k-1})}\big)\\
&\leq (20.9)10^8 M_{k-1}\sigma_{k-1}^3\big(1+\|\nabla v_0\|_{\mathcal{C}(\bar\Omega^0)}\big) \leq M_k,
\end{aligned}$$

in virtue of (5.4), (5.7), (5.2) and Proposition 4.3.

4. We now show that the sequences $\{v_k\}_{k=0}^\infty$, $\{w_k\}_{k=0}^\infty$ converge in $\mathcal{C}^{1,\alpha}(\bar\Omega)$ to some $\bar v, \bar w$. By (5.6) this will imply that $A - \big(\frac12\nabla\bar v\otimes\nabla\bar v + \mathrm{sym}\nabla\bar w\big) = \lim_{k\to\infty} D_k = 0$ together with the desired estimates on $\|\bar v - v\|_0$ and $\|\bar w - w\|_0$. Indeed:

$$\begin{aligned}
\|v_k - v_0\|_0 &\leq \sum_{i=1}^{k-1}\|v_{i+1}-v_i\|_0 \leq \sum_{i=0}^{\infty}\frac{(1.8)10^7}{M_i}\|D_i\|_{\mathcal{C}^0(\bar\Omega^i)}\\
&\leq (1.8)10^7\Big(\frac{\|D_0\|_{\mathcal{C}^0(\bar\Omega^0)}}{M_0} + \sum_{i=1}^\infty \frac{\|D_0\|_{\mathcal{C}^0(\bar\Omega^0)}}{M_i\prod_{j=0}^{i-1}\sigma_j^s}\Big) \leq (1.8)10^7\frac{\|D_0\|_{\mathcal{C}^0(\bar\Omega^0)}}{M_0}\Big(1+\sum_{i=1}^\infty\frac{1}{\mathfrak{C}^i}\Big)\\
&= (1.8)10^7\frac{\delta_0}{M_0}\frac{\mathfrak{C}}{\mathfrak{C}-1} \leq (1.8)10^7\frac{\delta_0^{\frac12}}{2}\cdot\delta_0 \leq (0.21)\delta_0,
\end{aligned}$$

by Proposition 4.3, (5.6), (5.4). Automatically, the bound above implies that $\{v_k\}_{k=0}^\infty$ is Cauchy in $\mathcal{C}^0(\bar\Omega)$. The similar statements for $\{w_k\}_{k=0}^\infty$ follow by (5.7) in:

$$\begin{aligned}
\|w_k - w_0\|_0 &\leq \sum_{i=1}^{k-1}\|w_{i+1}-w_i\|_0 \leq \sum_{i=0}^\infty\Big(\frac{(1.8)10^7}{M_i} + (12.6)\cdot\mathrm{diam}\,\Omega^0\Big)\|D_i\|_{\mathcal{C}^0(\bar\Omega^i)}\big(1+\|\nabla v_i\|_{\mathcal{C}^0(\bar\Omega^i)}\big)\\
&\leq (2.2)\big(1+\|\nabla v_0\|_{\mathcal{C}^0(\bar\Omega^0)}\big)\Big((12.6)(\mathrm{diam}\,\Omega^0)\sum_{i=0}^\infty\|D_i\|_{\mathcal{C}^0(\bar\Omega^i)} + (0.21)\delta_0\Big)\\
&\leq \delta_0(2.2)\big(1+\|\nabla v_0\|_{\mathcal{C}^0(\bar\Omega^0)}\big)\Big((12.6)\frac{16}{7}(\mathrm{diam}\,\Omega^0) + 0.21\Big)\\
&\leq \delta_0\big(1+\|\nabla v_0\|_{\mathcal{C}^0(\bar\Omega^0)}\big)\Big((63.4)(\mathrm{diam}\,\Omega^0)+0.5\Big).
\end{aligned}$$

The fact that $\{v_k\}_{k=0}^\infty$, $\{w_k\}_{k=0}^\infty$ are Cauchy in $\mathcal{C}^1(\bar\Omega)$ is obtained in:

$$\begin{aligned}
\|\nabla v_{k+1}-\nabla v_k\|_0 &\leq (1.1)10^8\Big(\frac{\delta_0}{\prod_{j=0}^{k-1}\sigma_j^s}\Big)^{\frac12} \leq \big(\frac34\big)^k(1.1)10^8\delta_0^{\frac12}\\
\|\nabla w_{k+1}-\nabla w_k\|_0 &\leq \big(\frac34\big)^k(1.1)10^8\delta_0^{\frac12}\cdot(2.2)\big(1+\|\nabla v_0\|_{\mathcal{C}^0(\bar\Omega^0)}\big).
\end{aligned}$$

Finally, we estimate the $\mathcal{C}^{1,\alpha}$ norms by interpolating from $\mathcal{C}^1$ to $\mathcal{C}^2$ norms in:

$$\begin{aligned}
[\nabla v_{k+1} - \nabla v_k]_\alpha &\leq C\|\nabla v_{k+1} - \nabla v_k\|^{1-\alpha}_{\mathcal{C}^0(\bar\Omega^{k+1})}\|\nabla^2 v_{k+1} - \nabla^2 v_k\|^{\alpha}_{\mathcal{C}^0(\bar\Omega^{k+1})} \\
&\leq C\frac{1}{\prod_{j=0}^{k-1}\sigma_j^{\frac{s}{2}(1-\alpha)}}\cdot\sigma_k^{3\alpha}\cdot\Big(\mathfrak{C}^k\prod_{j=0}^{k-1}\sigma_j^3\Big)^\alpha \leq C\sigma_{max}^{3\alpha}\cdot\mathfrak{C}^k\prod_{j=0}^{k-1}\sigma_j^{3\alpha-\frac{s}{2}(1-\alpha)} \\
&\leq C\big(\mathfrak{C}\cdot(\sigma_{max})^{3\alpha-\frac{s}{2}(1-\alpha)}\big)^k, \\
[\nabla w_{k+1} - \nabla w_k]_\alpha &\leq C\|\nabla w_{k+1} - \nabla w_k\|^{1-\alpha}_{\mathcal{C}^0(\bar\Omega^{k+1})}\|\nabla^2 w_{k+1} - \nabla^2 w_k\|^{\alpha}_{\mathcal{C}^0(\bar\Omega^{k+1})} \\
&\leq C(1+\|\nabla v_0\|_{\mathcal{C}^0(\bar\Omega^0)})^{1-\alpha}\big(\mathfrak{C}\cdot(\sigma_{max})^{3\alpha-\frac{s}{2}(1-\alpha)}\big)^k.
\end{aligned}$$

Above, the constant C depends, in its first appearance, only on the curvature of a smooth superset of $\bar\Omega$ contained in Ω_r. Eventually, C depends on various numerical values including M_0, α and σ_{max} but it is independent of k. We see that the last condition in (5.3) implies that the constant of the geometric progression in the right hand sides is less than 1 for large k, that is when σ_k is close to σ_{max}. This establishes that $\{v_k\}_{k=0}^\infty$, $\{w_k\}_{k=0}^\infty$ are Cauchy sequences in $\mathcal{C}^{1,\alpha}(\bar\Omega)$ and completes the proof of Proposition 5.1. ■

Proof of Theorem 1.2.

Let f, v_0 be as in Theorem 1.1. Fix a small parameter $\epsilon > 0$. We first extend v_0 to a continuous function on $\mathbb{R}^2$ and approximate this extension with $\bar v_0 \in \mathcal{C}^\infty(\mathbb{R}^2)$ satisfying:

$$\|\bar v_0 - v_0\|_0 \leq \epsilon.$$

Let $A = (\lambda + c)\mathrm{Id}_2 \in \mathcal{C}^{0,\beta}(\bar\Omega, \mathbb{R}^{2\times 2}_{sym})$ where $c > 0$ is a constant and $\lambda \in \mathcal{C}^{0,\beta}(\bar V)$ solves: $-\Delta\lambda = \chi_{\bar\Omega} f$ in an open superset V of $\bar\Omega$, with $\lambda = 0$ on ∂V. Then, as discussed in the introduction, $A = \frac{1}{2}\nabla v_0 \otimes \nabla v_0 + \mathrm{sym}\nabla w_0$ holds in $\bar\Omega_r$ for some $w_0 \in \mathcal{C}^1(\bar\Omega_r, \mathbb{R}^2)$ on $\Omega_r = \Omega + B_r(0)$ for some $r > 0$. Note that if c is large enough to guarantee:

$$\begin{aligned}
&\frac{3}{4}(\lambda + c) - \frac{1}{2}(\partial_1 \bar v_0)^2 + \frac{1}{8}(\partial_2 \bar v_0)^2 - \partial_1 w_0^1 + \frac{1}{4}\partial_2 w_0^2 > 0, \\
&(\lambda + c) - \frac{1}{2}(\partial_2 \bar v_0)^2 - (\partial_1 \bar v_0)(\partial_2 \bar v_0) - \partial_2 w_0^2 - \partial_1 w_0^2 - \partial_2 w_0^1 > 0, \qquad \text{in } \bar\Omega_r, \\
&(\lambda + c) - \frac{1}{2}(\partial_2 \bar v_0)^2 + (\partial_1 \bar v_0)(\partial_2 \bar v_0) - \partial_2 w_0^2 + \partial_1 w_0^2 + \partial_2 w_0^1 > 0,
\end{aligned}$$

then in view of Lemma 2.4 it follows that:

$$A - \big(\frac{1}{2}\nabla \bar v_0 \otimes \nabla \bar v_0 + \mathrm{sym}\nabla w_0\big) = \sum_{k=1}^3 \phi_k \eta_k \otimes \eta_k, \qquad \phi_k > 0 \ \text{ in } \bar\Omega_r.$$

Clearly, the field w_0 may be exchanged with a smooth field $\bar w_0 \in \mathcal{C}^\infty(\bar\Omega_r, \mathbb{R}^2)$ so that the positive coefficient decomposition above is still valid. Applying now Theorem 2.1, we get $v \in \mathcal{C}^\infty(\bar\Omega_r)$ and $w \in \mathcal{C}^\infty(\bar\Omega_r, \mathbb{R}^2)$ such that:

$$\|v - \bar v_0\|_0 \leq \epsilon \quad \text{ and } \quad \|A - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\nabla w\big)\|_0 \leq \min\big\{\epsilon, r, (5.4)10^{-16}\big\}.$$

In virtue of Proposition 5.1, we further find $\bar{v} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $\bar{w} \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ solving: $\frac{1}{2}\nabla\bar{v}\otimes\nabla\bar{v} + \mathrm{sym}\nabla\bar{w} = A$ in $\bar{\Omega}$, with the approximation error:

$$\|\bar{v} - v\|_0 \leq (0.21)\epsilon.$$

Concluding, the function $\bar{v}$ is a Hölder regular solution to (1.1), approximating the given v_0 with the arbitrarily small error:

$$\|\bar{v} - v_0\|_0 \leq \|\bar{v} - v\|_0 + \|v - \bar{v}_0\|_0 + \|\bar{v}_0 - v_0\|_0 \leq 3\epsilon,$$

as claimed in the result. ■

6. Numerical implementation of the $\mathcal{C}^{1,\alpha}$ convergence scheme

In section 3 we implemented the approximation construction of Theorem 2.1 and Proposition 2.5; we now implement the Hölder approximation specified in Proposition 4.3 and Proposition 5.1. Visualizations relevant to the $\mathcal{C}^{1,\alpha}$ case are much harder to obtain. The main difficulties come from the smallness of various parameter values and and the fact that the ratio σ between the frequencies of subsequent corrugations is by necessity very large. This first obstacle was overcome through the use of the Python package mpmath [15], allowing the user to define floating point arithmetics up to an arbitrary precision. The second obstacle of the necessity of large σ is insurmountable in most cases. Nonetheless, in the example of the degenerate Monge-Ampére equation $\det \nabla^2 v = 0$, small values of λ were found to produce a reduction in the deficit which justified the visualization obtained below.

The approach taken was significantly different from the case of $\mathcal{C}^1$ convergence. In each of the examples the problem was solved explicitly through symbolic calculations, defining symbolic variables and parameters in MATLAB (the only calculation which could not be done symbolically was the mollification step). The resulting text files, containing the outcomes were then modified through a script to make the syntax compatible with the mpmath package. We subsequently sampled these functions at 1000 randomly selected points on the square domain $\Omega = (-1,1) \times (-1,1)$, and the maximum value attained by each function was recorded. The main purpose of these calculations was to study how the results of convex integration varied by changing the value of σ. We argue that the maximum sample value, while not being the exact supremum of these functions on the domain, is a good approximation of the order of magnitude. In fact, when the process was repeated over different sets of 1000 sample points, the variation in the values obtained was negligible when considering the order of magnitude, as illustrated in the table below.

Test	$\|\tilde{D}\|_0$	$\|v_3\|_0$	$\|\nabla w_3\|_0$
1	$0.64 \cdot 10^{-11}$	$0.114 \cdot 10^{-23}$	$0.53 \cdot 10^{-8}$
2	$0.57 \cdot 10^{-11}$	$0.118 \cdot 10^{-23}$	$0.58 \cdot 10^{-8}$
3	$0.62 \cdot 10^{-11}$	$0.119 \cdot 10^{-23}$	$0.54 \cdot 10^{-8}$
4	$0.61 \cdot 10^{-11}$	$0.115 \cdot 10^{-23}$	$0.49 \cdot 10^{-8}$
5	$0.66 \cdot 10^{-11}$	$0.119 \cdot 10^{-23}$	$0.54 \cdot 10^{-8}$
6	$0.70 \cdot 10^{-11}$	$0.116 \cdot 10^{-23}$	$0.56 \cdot 10^{-8}$
7	$0.65 \cdot 10^{-11}$	$0.104 \cdot 10^{-23}$	$0.62 \cdot 10^{-8}$
8	$0.61 \cdot 10^{-11}$	$0.118 \cdot 10^{-23}$	$0.67 \cdot 10^{-8}$
9	$0.61 \cdot 10^{-11}$	$0.111 \cdot 10^{-23}$	$0.50 \cdot 10^{-8}$
10	$0.58 \cdot 10^{-11}$	$0.108 \cdot 10^{-23}$	$0.76 \cdot 10^{-8}$

In this table we show the results of running our code 10 times on the same example with the same parameters, where the only change was the choice of the sample points. In each test we evaluated the maximum defect recorded, the maximum value of $|v|$ and the maximum gradient of w. As can be seen, the order of magnitude of all these functions remains the same. Similar results were obtained with different examples, functions and parameters, but the details are omitted as they do not add particular insight.

Example 6.1. We approximate $v_0 = 0$ and $w_0 = 0$ with the solution to:

$$\mathcal{D}et\, \nabla^2 v = -10^{-18} < 0.$$

This results in $A = -10^{-18}(x^2 + y^2)\mathrm{Id}_2$, coinciding with the original defect $\|D\|_0 \sim 10^{-18}\sqrt{2}$. The defect evaluated for $\sigma = 35$ (1st stage of convex integration, third step and third corrugation added) becomes: $\|D_3\|_0 \sim 9.7 \cdot 10^{-19}$.

Example 6.2. We approximate $v_0 = 0$ and $w_0 = 0$ with the solution to:

$$\mathcal{D}et\, \nabla^2 v = 10^{-18} > 0.$$

This results in $A = 10^{-18}(x^2+y^2)\mathrm{Id}_2$, coinciding with the original defect, where $\|D\|_0 \sim 10^{-18}\sqrt{2}$. The defect evaluated for $\sigma = 35$ becomes: $\|D_3\|_0 \sim 9.1 \cdot 10^{-19}$. In Figures 6.1 and 6.2 we show visualizations of v on the subdomains $[1 - 10^{-19}, 1]^2$ and $[1 - 2 \cdot 10^{-21}, 1]^2$, respectively.

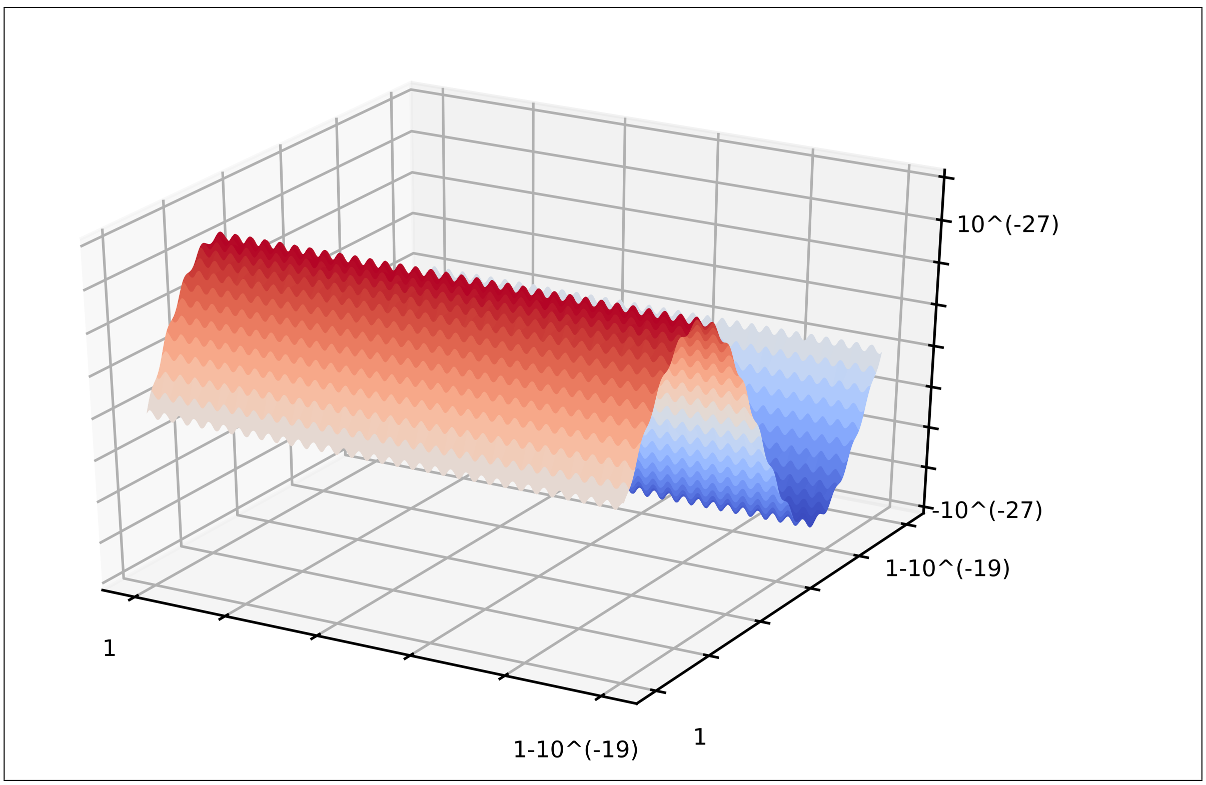


FIGURE 6.1. The three corrugations in Example 6.2.

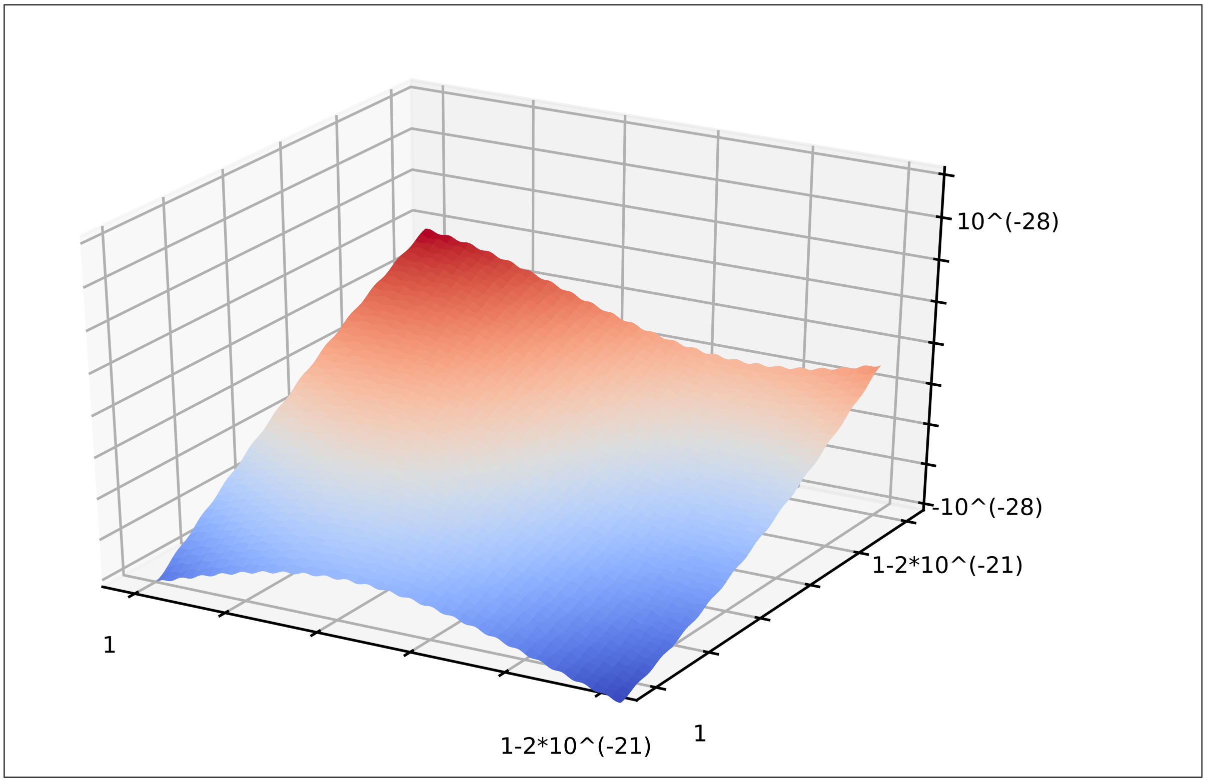


FIGURE 6.2. The detail of Figure 6.1.

Example 6.3. We approximate $v_0(x,y) = 10^{-9}(x^2+y^2)$ and $w_0 = 0$ with the solution to:

$$\mathcal{D}et\,\nabla^2 v = 0.$$

Thus $A = 0$ and the initial defect is given by: $D(x,y) = \frac{1}{2}\nabla v_0 \otimes \nabla v_0 = 10^{-18}\cdot 2\cdot (x,y)\otimes(x,y)$ with $\|D\|_0 \sim 4\cdot 10^{-18}$. The defect evaluated for $\sigma = 35$ becomes: $\|D_3\|_0 \sim 9.5\cdot 10^{-19}$. In Figure 6.1 we show a visualization of v on the subdomain $[1-10^{-19}, 1]^2$.

In each of the examples a value of $\lambda_1 = 10^{19}$ was chosen and σ was increased exponentially from 10^1 to 10^{18}, covering the orders of magnitude between the theoretical minimum of σ and σ_{max}. We then applied the modification of w and the three steps of convex integration in Proposition 4.3 and evaluated the maximum of the new defect, the norm of v together with is gradient and Hessian. The numerical results obtained are summarized in the tables in Appendix A.

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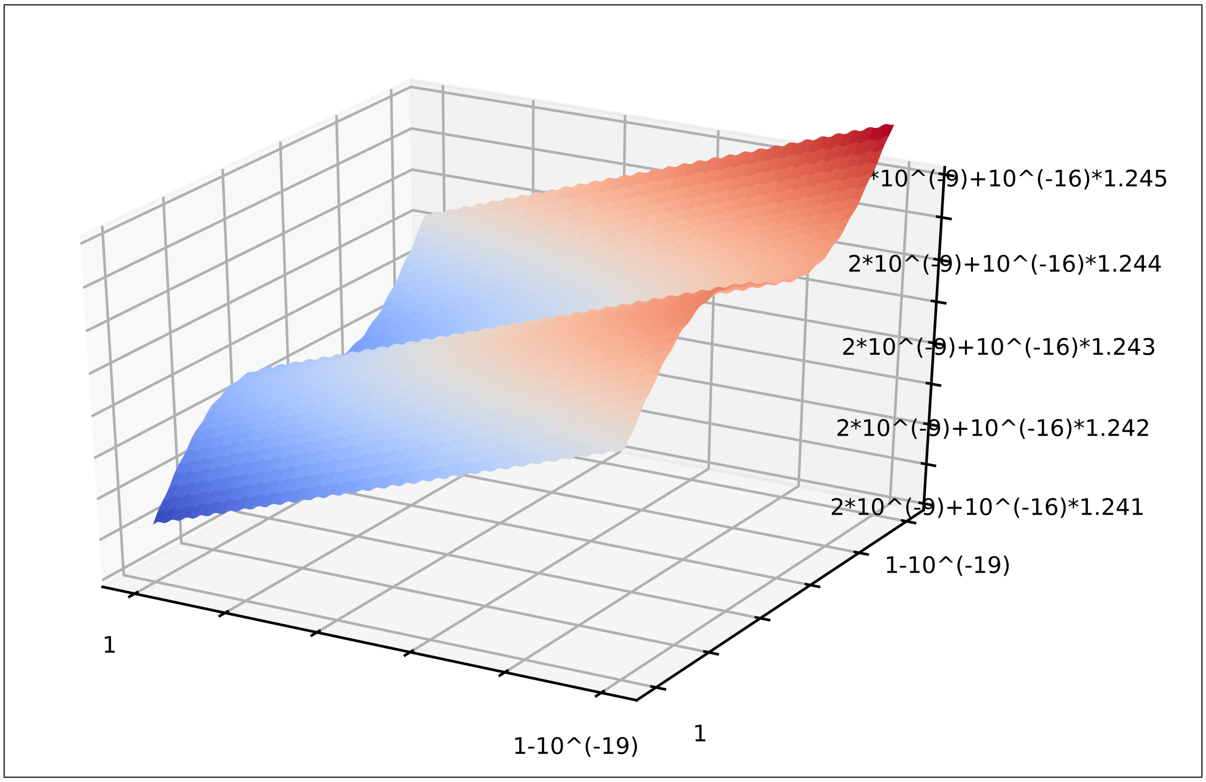

FIGURE 6.3. The corrugations in Example 6.3.

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Appendix A. Error tables for numerical calculations of the $\mathcal{C}^{1,\alpha}$ implementation

Table 1. Values of the defect $\|D_3\|_0$

σ	Example 6.1	Example 6.2	Example 6.3
10^1	$0.332 \cdot 10^{-17}$	$0.393 \cdot 10^{-17}$	$0.327 \cdot 10^{-17}$
10^2	$0.316 \cdot 10^{-18}$	$0.364 \cdot 10^{-18}$	$0.319 \cdot 10^{-18}$
10^3	$0.318 \cdot 10^{-19}$	$0.376 \cdot 10^{-19}$	$0.332 \cdot 10^{-19}$
10^4	$0.318 \cdot 10^{-20}$	$0.396 \cdot 10^{-20}$	$0.318 \cdot 10^{-20}$
10^5	$0.320 \cdot 10^{-21}$	$0.401 \cdot 10^{-21}$	$0.316 \cdot 10^{-21}$
10^6	$0.336 \cdot 10^{-22}$	$0.400 \cdot 10^{-22}$	$0.325 \cdot 10^{-22}$
10^7	$0.326 \cdot 10^{-23}$	$0.375 \cdot 10^{-23}$	$0.330 \cdot 10^{-23}$
10^8	$0.332 \cdot 10^{-24}$	$0.367 \cdot 10^{-24}$	$0.335 \cdot 10^{-24}$
10^9	$0.329 \cdot 10^{-25}$	$0.366 \cdot 10^{-25}$	$0.329 \cdot 10^{-25}$
10^{10}	$0.328 \cdot 10^{-26}$	$0.382 \cdot 10^{-26}$	$0.339 \cdot 10^{-26}$
10^{11}	$0.338 \cdot 10^{-27}$	$0.399 \cdot 10^{-27}$	$0.326 \cdot 10^{-27}$
10^{12}	$0.329 \cdot 10^{-28}$	$0.371 \cdot 10^{-28}$	$0.327 \cdot 10^{-28}$
10^{13}	$0.317 \cdot 10^{-29}$	$0.396 \cdot 10^{-29}$	$0.333 \cdot 10^{-29}$
10^{14}	$0.320 \cdot 10^{-30}$	$0.388 \cdot 10^{-30}$	$0.325 \cdot 10^{-30}$
10^{15}	$0.311 \cdot 10^{-31}$	$0.384 \cdot 10^{-31}$	$0.336 \cdot 10^{-31}$
10^{16}	$0.324 \cdot 10^{-32}$	$0.366 \cdot 10^{-32}$	$0.321 \cdot 10^{-32}$

TABLE 2. Values of $\|\nabla v_3\|_0$

σ	Example 6.1	Example 6.2	Example 6.3
10^1	$0.941 \cdot 10^{-8}$	$0.101 \cdot 10^{-7}$	$0.106 \cdot 10^{-7}$
10^2	$0.937 \cdot 10^{-8}$	$0.105 \cdot 10^{-7}$	$0.102 \cdot 10^{-7}$
10^3	$0.920 \cdot 10^{-8}$	$0.102 \cdot 10^{-7}$	$0.102 \cdot 10^{-7}$
10^4	$0.936 \cdot 10^{-8}$	$0.994 \cdot 10^{-8}$	$0.104 \cdot 10^{-7}$
10^5	$0.953 \cdot 10^{-8}$	$0.101 \cdot 10^{-7}$	$0.104 \cdot 10^{-7}$
10^6	$0.935 \cdot 10^{-8}$	$0.101 \cdot 10^{-7}$	$0.105 \cdot 10^{-7}$
10^7	$0.946 \cdot 10^{-8}$	$0.102 \cdot 10^{-7}$	$0.103 \cdot 10^{-7}$
10^8	$0.936 \cdot 10^{-8}$	$0.101 \cdot 10^{-7}$	$0.103 \cdot 10^{-7}$
10^9	$0.942 \cdot 10^{-8}$	$0.100 \cdot 10^{-7}$	$0.104 \cdot 10^{-7}$
10^{10}	$0.934 \cdot 10^{-8}$	$0.101 \cdot 10^{-7}$	$0.101 \cdot 10^{-7}$
10^{11}	$0.931 \cdot 10^{-8}$	$0.102 \cdot 10^{-7}$	$0.103 \cdot 10^{-7}$
10^{12}	$0.926 \cdot 10^{-8}$	$0.101 \cdot 10^{-7}$	$0.104 \cdot 10^{-7}$
10^{13}	$0.939 \cdot 10^{-8}$	$0.102 \cdot 10^{-7}$	$0.104 \cdot 10^{-7}$
10^{14}	$0.940 \cdot 10^{-8}$	$0.995 \cdot 10^{-8}$	$0.103 \cdot 10^{-7}$
10^{15}	$0.945 \cdot 10^{-8}$	$0.986 \cdot 10^{-8}$	$0.102 \cdot 10^{-7}$
10^{16}	$0.920 \cdot 10^{-8}$	$0.103 \cdot 10^{-7}$	$0.104 \cdot 10^{-7}$

TABLE 3. Values of $\|\nabla w_3\|_0$

σ	Example 6.1	Example 6.2	Example 6.3
10^1	$0.730 \cdot 10^{-16}$	$0.432 \cdot 10^{-16}$	$0.844 \cdot 10^{-16}$
10^2	$0.728 \cdot 10^{-16}$	$0.445 \cdot 10^{-16}$	$0.857 \cdot 10^{-16}$
10^3	$0.687 \cdot 10^{-16}$	$0.430 \cdot 10^{-16}$	$0.813 \cdot 10^{-16}$
10^4	$0.712 \cdot 10^{-16}$	$0.458 \cdot 10^{-16}$	$0.835 \cdot 10^{-16}$
10^5	$0.754 \cdot 10^{-16}$	$0.424 \cdot 10^{-16}$	$0.848 \cdot 10^{-16}$
10^6	$0.727 \cdot 10^{-16}$	$0.422 \cdot 10^{-16}$	$0.873 \cdot 10^{-16}$
10^7	$0.716 \cdot 10^{-16}$	$0.444 \cdot 10^{-16}$	$0.773 \cdot 10^{-16}$
10^8	$0.723 \cdot 10^{-16}$	$0.444 \cdot 10^{-16}$	$0.859 \cdot 10^{-16}$
10^9	$0.727 \cdot 10^{-16}$	$0.420 \cdot 10^{-16}$	$0.833 \cdot 10^{-16}$
10^{10}	$0.721 \cdot 10^{-16}$	$0.431 \cdot 10^{-16}$	$0.752 \cdot 10^{-16}$
10^{11}	$0.698 \cdot 10^{-16}$	$0.478 \cdot 10^{-16}$	$0.808 \cdot 10^{-16}$
10^{12}	$0.672 \cdot 10^{-16}$	$0.431 \cdot 10^{-16}$	$0.843 \cdot 10^{-16}$
10^{13}	$0.692 \cdot 10^{-16}$	$0.449 \cdot 10^{-16}$	$0.785 \cdot 10^{-16}$
10^{14}	$0.733 \cdot 10^{-16}$	$0.414 \cdot 10^{-16}$	$0.789 \cdot 10^{-16}$
10^{15}	$0.739 \cdot 10^{-16}$	$0.430 \cdot 10^{-16}$	$0.800 \cdot 10^{-16}$
10^{16}	$0.687 \cdot 10^{-16}$	$0.442 \cdot 10^{-16}$	$0.779 \cdot 10^{-16}$

Table 4. Values of $\|\nabla^2 v_3\|_0$

σ	Example 6.1	Example 6.2	Example 6.3
10^1	$3.37 \cdot 10^{13}$	$3.05 \cdot 10^{13}$	$3.26 \cdot 10^{13}$
10^2	$3.30 \cdot 10^{15}$	$3.00 \cdot 10^{15}$	$3.22 \cdot 10^{15}$
10^3	$3.27 \cdot 10^{17}$	$2.98 \cdot 10^{17}$	$3.28 \cdot 10^{17}$
10^4	$3.26 \cdot 10^{19}$	$2.98 \cdot 10^{19}$	$3.27 \cdot 10^{19}$
10^5	$3.27 \cdot 10^{21}$	$2.99 \cdot 10^{21}$	$3.29 \cdot 10^{21}$
10^6	$3.25 \cdot 10^{23}$	$2.99 \cdot 10^{23}$	$3.27 \cdot 10^{23}$
10^7	$3.28 \cdot 10^{25}$	$2.99 \cdot 10^{25}$	$3.23 \cdot 10^{25}$
10^8	$3.21 \cdot 10^{27}$	$2.99 \cdot 10^{27}$	$3.28 \cdot 10^{27}$
10^9	$3.26 \cdot 10^{29}$	$2.98 \cdot 10^{29}$	$3.28 \cdot 10^{29}$
10^{10}	$3.30 \cdot 10^{31}$	$2.99 \cdot 10^{31}$	$3.30 \cdot 10^{31}$
10^{11}	$3.25 \cdot 10^{33}$	$2.99 \cdot 10^{33}$	$3.23 \cdot 10^{33}$
10^{12}	$3.28 \cdot 10^{35}$	$2.99 \cdot 10^{35}$	$3.24 \cdot 10^{35}$
10^{13}	$3.23 \cdot 10^{37}$	$2.99 \cdot 10^{37}$	$3.25 \cdot 10^{37}$
10^{14}	$3.26 \cdot 10^{39}$	$2.98 \cdot 10^{39}$	$3.25 \cdot 10^{39}$
10^{15}	$3.25 \cdot 10^{41}$	$2.99 \cdot 10^{41}$	$3.27 \cdot 10^{41}$
10^{16}	$3.29 \cdot 10^{43}$	$2.98 \cdot 10^{43}$	$3.20 \cdot 10^{43}$

Table 5. Values of $\|\nabla^2 w_3\|_0$

σ	Example 6.1	Example 6.2	Example 6.3
10^1	$3.11 \cdot 10^{5}$	$2.80 \cdot 10^{5}$	$2.12 \cdot 10^{5}$
10^2	$3.12 \cdot 10^{7}$	$2.82 \cdot 10^{7}$	$2.05 \cdot 10^{7}$
10^3	$3.20 \cdot 10^{9}$	$2.87 \cdot 10^{9}$	$2.23 \cdot 10^{9}$
10^4	$3.17 \cdot 10^{11}$	$2.60 \cdot 10^{11}$	$2.18 \cdot 10^{11}$
10^5	$3.12 \cdot 10^{13}$	$2.85 \cdot 10^{13}$	$2.15 \cdot 10^{13}$
10^6	$3.18 \cdot 10^{15}$	$2.80 \cdot 10^{15}$	$2.08 \cdot 10^{15}$
10^7	$3.12 \cdot 10^{17}$	$2.79 \cdot 10^{17}$	$2.18 \cdot 10^{17}$
10^8	$3.17 \cdot 10^{19}$	$2.73 \cdot 10^{19}$	$2.18 \cdot 10^{19}$
10^9	$3.27 \cdot 10^{21}$	$2.74 \cdot 10^{21}$	$2.18 \cdot 10^{21}$
10^{10}	$3.31 \cdot 10^{23}$	$2.75 \cdot 10^{23}$	$2.10 \cdot 10^{23}$
10^{11}	$3.21 \cdot 10^{25}$	$2.65 \cdot 10^{25}$	$2.13 \cdot 10^{25}$
10^{12}	$3.25 \cdot 10^{27}$	$2.69 \cdot 10^{27}$	$2.19 \cdot 10^{27}$
10^{13}	$3.10 \cdot 10^{29}$	$2.79 \cdot 10^{29}$	$2.08 \cdot 10^{29}$
10^{14}	$3.08 \cdot 10^{31}$	$2.71 \cdot 10^{31}$	$2.21 \cdot 10^{31}$
10^{15}	$3.29 \cdot 10^{33}$	$2.79 \cdot 10^{33}$	$2.20 \cdot 10^{33}$
10^{16}	$3.39 \cdot 10^{35}$	$2.72 \cdot 10^{35}$	$2.22 \cdot 10^{35}$