ON ASYMPTOTIC EXPANSIONS FOR THE FRACTIONAL INFINITY LAPLACIAN

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Abstract. We propose two asymptotic expansions of the two interrelated integral-type averages, in the context of the fractional ∞-Laplacian Δ_∞^s for $s \in (\frac{1}{2}, 1)$. This operator has been introduced and first studied in [1]. Our expansions are parametrised by radius of the removed singularity ε, and allow for the identification of $\Delta_\infty^s \phi(x)$ as the ε^{2s}-order coefficient of the deviation of the ε-average from the value $\phi(x)$, in the limit $\varepsilon \to 0$. The averages are well posed for functions ϕ that are only Borel regular and bounded.

1. Introduction

This note concerns the fractional ∞-Laplace operator Δ_∞^s, as introduced in [1]. Given a function $\phi : \mathbb{R}^N \to \mathbb{R}$, our main result is the identification of $\Delta_\infty^s \phi(x)$ as the ε^{2s}-order coefficient in the asymptotic expansion of the deviation of an appropriate ε-average $\mathcal{A}_\varepsilon$ applied on ϕ, from the value $\phi(x)$. Such identification is of general interest in the analysis of partial differential operators, their related probabilistic interpretation via Tug-of-War games, a study of viscosity solutions and of numerical approximating schemes. The chief example of the said asymptotic expansions is given by the well known (local and linear) formula for the Laplace operator, where $\Delta\phi(x)$ emerges as the ε^2-order coefficient from the integral average $\mathcal{A}_\varepsilon = \fint_{B_\varepsilon}$:

$$\fint_{B_\varepsilon(x)} \phi(y)\,\mathrm{d}y = \phi(x) + \frac{\varepsilon^2}{2(N+2)}\Delta\phi(x) + o(\varepsilon^2) \qquad \text{as } \varepsilon \to 0+\,. \tag{1.1}$$

The parallel expansion for the ∞-Laplacian: $\Delta_\infty \phi(x) = \big\langle \nabla^2\phi(x) : \frac{\nabla\phi(x)}{|\nabla\phi(x)|} \otimes \frac{\nabla\phi(x)}{|\nabla\phi(x)|}\big\rangle$ utilizes the midpoint (local and nonlinear) average $\mathcal{A}_\varepsilon = \frac{1}{2}(\sup_{B_\varepsilon} + \inf_{B_\varepsilon})$ in:

$$\frac{1}{2}\big(\sup_{B_\varepsilon(x)} \phi + \inf_{B_\varepsilon(x)} \phi\big) = \phi(x) + \frac{\varepsilon^2}{2}\Delta_\infty\phi(x) + o(\varepsilon^2) \qquad \text{as } \varepsilon \to 0+\,. \tag{1.2}$$

1.1. **The asymptotic expansions and averaging operators in this paper.** In what follows, will prove that for every $s \in (\frac{1}{2}, 1)$ one counterpart formula of (1.1) for Δ_∞^s is:

$$\mathcal{A}_\varepsilon^o \phi(x) = \phi(x) + s\varepsilon^{2s}\Delta_\infty^s\phi(x) + o(\varepsilon^{2s}) \qquad \text{as } \varepsilon \to 0+, \tag{1.3}$$

based on the following (nonlocal and nonlinear) average:

$$\mathcal{A}_\varepsilon^o \phi(x) = \frac{1}{2}\Big(\sup_{|y|=1} \fint_\varepsilon^\infty \phi(x+ty)\,\mathrm{d}\mu_s(t) + \inf_{|y|=1} \fint_\varepsilon^\infty \phi(x+ty)\,\mathrm{d}\mu_s(t)\Big).$$

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The one-dimensional fractional measure μ_s and the structure of the error term $o(\varepsilon^{2s})$ will be explained below. When $\nabla\phi(x) \neq 0$, we also derive another identification through a local-nonlocal average, which is a convex combination of $\mathcal{A}_\varepsilon$-s in (1.3) and (1.2):

$$\mathcal{A}_\varepsilon\phi(x) = (1-s)\cdot\mathcal{A}^o_\varepsilon\phi(x) + s\cdot\frac{1}{2}\Big(\sup_{B_\varepsilon(x)}\phi + \inf_{B_\varepsilon(x)}\phi\Big).$$

We anticipate that the error quantity $o(\varepsilon^{2s})$ below is uniform in the whole considered range $s\in(\frac{1}{2},1)$, whereas the corresponding error in (1.3) blows up to ∞ as $s\to 1-$. Thus, the following asymptotic expansion can be seen as an improvement of (1.3):

$$\mathcal{A}_\varepsilon\phi(x) = \phi(x) + (1-s)s\varepsilon^{2s}\Delta^s_\infty\phi(x) + o(\varepsilon^{2s}) \qquad \text{as } \varepsilon\to 0+. \tag{1.4}$$

Precise statements of the formulas (1.3) and (1.4) will be given in Theorems 1.1 and 1.2.

1.2. **The fractional ∞-Laplacian.** Let $\phi:\mathbb{R}^N\to\mathbb{R}$ be a bounded, Borel function. We recall that $\phi\in C^{1,1}(x)$ at $x\in\mathbb{R}^N$, provided that there exists $p_x\in\mathbb{R}^N$ and $C_x,\eta_x>0$ such that:

$$\big|\phi(x+y)-\phi(x)-\langle p_x,y\rangle\big| \leq C_x|y|^2 \qquad \text{for all } |y|<\eta_x. \tag{1.5}$$

In [1][Definition 1.1], the fractional ∞-Laplacian $\Delta^s_\infty\phi(x)$, for $s\in(\frac{1}{2},1)$, has been introduced by means of two distinct formulas, distinguishing between cases $p_x=0$ and $p_x\neq 0$. In section 2, we observe that in both cases there holds:

$$\Delta^s_\infty\phi(x) = \frac{1}{C_s}\cdot\sup_{|y|=1}\inf_{|\tilde{y}|=1}\int_0^\infty L_\phi(x,ty,t\tilde{y})\,\mathrm{d}\mu_s(t). \tag{1.6}$$

To explain the notation in the right hand side above, for each $x,y,\tilde{y}\in\mathbb{R}^N$ we define:

$$L_\phi(x,y,\tilde{y}) \doteq \phi(x+y)+\phi(x-\tilde{y})-2\phi(x).$$

Further, μ_s is the measure[1] on the Borel subsets of $(0,\infty)$, given by:

$$\mathrm{d}\mu_s(t) \doteq \frac{C_s}{t^{1+2s}}\,\mathrm{d}t \quad \text{ where } C_s = \frac{4^s s\Gamma\big(\frac{1}{2}+s\big)}{\pi^{1/2}\Gamma\big(1-s\big)} = \Big(2\int_0^\infty\frac{1-\cos t}{t^{1+2s}}\,\mathrm{d}t\Big)^{-1}.$$

It is important to note [4] that one can express C_s by means of another constant[2] c_s, that is bounded and positive, uniformly in s. Namely, there holds:

$$C_s = s(1-s)c_s.$$

1.3. **Statements of main results.** We consider the operator in the right hand side of (1.6):

$$\mathcal{L}_s[\phi](x) \doteq \sup_{|y|=1}\inf_{|\tilde{y}|=1}\int_0^\infty L_\phi(x,ty,t\tilde{y})\,\mathrm{d}\mu_s(t). \tag{1.7}$$

[1]The role of the normalizing constant C_s is to ensure that the operator $-(-\Delta)^s u(x) \doteq \int_0^\infty L_u(x,t,t)\,\mathrm{d}\mu_s(t)$ defined for $u:\mathbb{R}\to\mathbb{R}$, is a pseudo-differential operator with symbol $|\xi|^{2s}$.

[2]A direct calculation shows that, for example, $c_s\in((\frac{12}{13})^2,(\frac{12}{5})^2)$ in the range $s\in(\frac{1}{2},1)$.

Given $\eta_x > 0$, we work with the following hypotheses on ϕ, relative to the ball $B_{\eta_x} \doteq B_{\eta_x}(x)$:

$$\left[\begin{array}{ll} \text{(i)} & \phi \in C^2(\bar{B}_{\eta_x}) \text{ where } p_x \doteq \nabla\phi(x) \text{ and } C_x \doteq \frac{1}{2}\|\nabla^2\phi\|_{L^\infty(B_{\eta_x})}. \\ \text{(ii)} & \phi \text{ is bounded and uniformly continuous on } \mathbb{R}^N \setminus \bar{B}_{\eta_x} \text{ with modulus of conti-} \\ & \text{nuity } \omega_\phi(a) \doteq \sup\big\{|\phi(x)-\phi(y)|;\ x,y \in \mathbb{R}^N \setminus \bar{B}_{\eta_x},\ |x-y| \leq a\big\}. \\ \text{(iii)} & \text{We further denote:} \\ & \qquad A_\varepsilon = \max\Big\{\dfrac{16C_x}{|p_x|}\cdot\dfrac{2s-1}{1-s}\cdot\dfrac{\eta_x^{2-2s}-\varepsilon^{2-2s}}{\varepsilon^{1-2s}-{\eta_x}^{1-2s}},\ \kappa_\varepsilon\Big\}, \\ & \qquad \kappa_\varepsilon = \sup\Big\{a;\ a\in[0,2] \text{ and } a^2 \leq \dfrac{8\omega_\phi(a)}{|p_x|}\cdot\dfrac{\frac{2s-1}{2s}\eta_x^{-2s}+\eta_x^{1-2s}}{\varepsilon^{1-2s}-\eta_x^{1-2s}}\Big\}. \end{array}\right] \qquad (\mathbf{H})$$

Our first main result regards the expansion (1.3):

Theorem 1.1. *Let $\phi : \mathbb{R}^N \to \mathbb{R}$ satisfy* ($\mathbf{H}$). *Then there holds, for every $\varepsilon < \eta_x$:*

$$\Big|\mathcal{A}^o_\varepsilon\phi(x) - \phi(x) - \frac{1}{c_s(1-s)}\varepsilon^{2s}\mathcal{L}_s[\phi](x)\Big| \leq \frac{s}{1-s}\cdot C_x\varepsilon^2 + \\ + \begin{cases} \varepsilon^{2s}\Big(4sC_x\frac{\eta_x^{2-2s}-\varepsilon^{2-2s}}{1-s}\cdot A_\varepsilon + \big(\eta_x^{-2s}+\frac{2s}{2s-1}\eta_x^{1-2s}\big)\cdot\omega_\phi(A_\varepsilon)\Big) & \textit{when } p_x \neq 0 \\ 0 & \textit{when } p_x = 0. \end{cases}$$

Our second main result regards the expansion (1.4):

Theorem 1.2. *Assume* ($\mathbf{H}$) *and $p_x \neq 0$. Then, for all $\varepsilon < \eta_x$ with $\varepsilon|\nabla^2\phi(x)| \leq |p_x|$, we have:*

$$\Big|\mathcal{A}_\varepsilon\phi(x) - \phi(x) - \frac{1}{c_s}\varepsilon^{2s}\mathcal{L}_s[\phi](x)\Big| \\ \leq 2\varepsilon^{2s}\Big(2sC_x\big(\eta_x^{2-2s}-\varepsilon^{2-2s}\big)A_\varepsilon + \Big(\frac{\eta_x^{-2s}}{2}+\frac{s\eta_x^{1-2s}}{2s-1}\Big)\cdot(1-s)\omega_\phi(A_\varepsilon)\Big) \\ + 2s\varepsilon^3\frac{|\nabla^2\phi(x)|^2}{|p_x|} + s\varepsilon^2\sup_{y\in B_\varepsilon(x)}\big|\nabla^2\phi(y)-\nabla^2\phi(x)\big|.$$

For a discussion of the error terms in the indicated estimates, we refer to Remarks 3.3 and 4.2. Finally, we note that another expansion for Δ^s_∞ has been recently proposed in [2,3]. The related averaging operator in [3, section 3.2] distinguishes, similarly to [1], between the cases $p_x \neq 0$ and $p_x = 0$. In comparison, the averages $\mathcal{A}_\varepsilon$ in the present paper neither rely on this distinction nor even necessitate the notion of the gradient being well posed. Thus, they can be applied on a larger class of functions ϕ that are only bounded Borel.

Our note is organized as follows: we prove (1.6) in section 2, Theorem 1.1 in section 3 and Theorem 1.2 in section 4. In the last section 5, we propose a version $\bar{\mathcal{A}}^o_\varepsilon$ of the average $\mathcal{A}^o_\varepsilon$ and its corresponding expansion, in which integration takes place on an open, bounded domain in $\mathbb{R}^N$ (a spherical-prism shaped), rather than an infinite line. We believe that this correction will be of importance in the implementation of the numerical schemes, approximating solutions to the nonlocal Dirichlet problem for Δ^s_∞. We also conjecture that the expected values of the stochastic process whose dynamic programming principle is modeled on $\bar{\mathcal{A}}^o_\varepsilon$ (or $\mathcal{A}^o_\varepsilon$), converge to these solutions in the limit $\varepsilon \to 0$, as in the pivotal study [6] of the classical operator Δ_∞.

2. The fractional ∞-Laplacian and a proof of (1.6)

Given a bounded Borel function $\phi : \mathbb{R}^N \to \mathbb{R}$, and two parameters $\varepsilon > 0$ and $s \in (\frac{1}{2}, 1)$, we will be concerned with values of the integral operators $\mathcal{L}_s^\varepsilon[\phi] : \mathbb{R}^N \to \mathbb{R}$, given in:

$$\begin{aligned}\mathcal{L}_s^\varepsilon[\phi](x) &\doteq \sup_{|y|=1} \inf_{|\tilde{y}|=1} \int_\varepsilon^\infty L_\phi(x, ty, t\tilde{y})\ \mathrm{d}\mu_s(t) \\ &= \sup_{|y|=1} \int_\varepsilon^\infty \phi(x+ty)\ \mathrm{d}\mu_s(t) + \inf_{|y|=1} \int_\varepsilon^\infty \phi(x+ty)\ \mathrm{d}\mu_s(t) - \frac{(1-s)c_s}{\varepsilon^{2s}}\phi(x),\end{aligned} \tag{2.1}$$

Note that, since the restriction of ϕ to any one-dimensional line is also Borel, the function $(0,\infty) \ni t \mapsto L_\phi(x, ty, t\tilde{y})$ is bounded and Borel for any $x, y, \tilde{y}$. Further, since $\mu_s(\varepsilon,\infty) = \frac{C_s}{2s\varepsilon^{2s}} < \infty$, each integral $\int_\varepsilon^\infty L_\phi(x, ty, t\tilde{y})\ \mathrm{d}\mu_s(t)$ and consequently also the quantities $\mathcal{L}_s^\varepsilon[\phi](x)$, are all well defined and finite. On the other hand, $\mu_s(0,\infty) = \infty$, so neither the definition of $\mathcal{L}_s[\phi]$ in (1.7) nor a version of its equivalent formulation as in $\mathcal{L}_s^\varepsilon[\phi]$ are necessarily valid, when ϕ is only bounded and Borel. However, one immediate consequence of (1.5) is that:

$$\big|L_\phi(x, ty, t\tilde{y}) - t\langle p_x, y - \tilde{y}\rangle\big| \leq 2C_x t^2 \qquad \text{for all } \ |y|, |\tilde{y}| = 1 \text{ and } |t| < \eta_x. \tag{2.2}$$

which yields (through an application of Taylor's expansion):

Lemma 2.1. *Let $\phi \in C^{1,1}(x)$ be a bounded, Borel function. Then $\{\mathcal{L}_s^\varepsilon[\phi](x)\}_{\varepsilon>0}$ are bounded independently of ε and $\mathcal{L}_s[\phi](x)$ is well defined. More precisely, for all $\varepsilon < \eta_x$ there holds:*

$$|\mathcal{L}_s^\varepsilon[\phi](x)|,\ |\mathcal{L}_s[\phi](x)| \leq 2c_s(1-s)\|\phi\|_{L^\infty}\eta_x^{-2s} + c_s s \cdot C_x \eta_x^{2-2s}. \tag{2.3}$$

Proof. We use (2.2) to obtain, for any $|y| = |\tilde{y}| = 1$:

$$\begin{aligned}\Big|\int_\varepsilon^\infty L_\phi(x, ty, t\tilde{y})\ \mathrm{d}\mu_s(t) - \int_\varepsilon^{\eta_x} t\langle p_x, y-\tilde{y}\rangle\ \mathrm{d}\mu_s(t)\Big| &\leq \int_{\eta_x}^\infty 4\|\phi\|_{L^\infty}\ \mathrm{d}\mu_s(t) + \int_\varepsilon^{\eta_x} 2C_x t^2\ \mathrm{d}\mu_s(t) \\ &= 2\frac{C_s}{s\eta_x^{2s}}\|\phi\|_{L^\infty} + C_s C_x \frac{\eta_x^{2-2s} - \varepsilon^{2-2s}}{1-s}.\end{aligned}$$

On the other hand:

$$\sup_{|y|=1} \inf_{|\tilde{y}|=1} \int_\varepsilon^{\eta_x} t\langle p_x, y-\tilde{y}\rangle\ \mathrm{d}\mu_s(t) = \int_\varepsilon^{\eta_x} t\ \mathrm{d}\mu_s(t) \cdot \sup_{|y|=1} \inf_{|\tilde{y}|=1} \langle p_x, y-\tilde{y}\rangle = 0.$$

This results in:

$$\begin{aligned}|\mathcal{L}_s^\varepsilon[\phi](x)| &= \Big|\mathcal{L}_s^\varepsilon[\phi](x) - \sup_{|y|=1} \inf_{|\tilde{y}|=1} \int_\varepsilon^{\eta_x} t\langle p_x, y-\tilde{y}\rangle\ \mathrm{d}\mu_s(t)\Big| \\ &\leq \sup_{|y|=|\tilde{y}|=1} \Big|\int_\varepsilon^\infty L_\phi(x, ty, t\tilde{y})\ \mathrm{d}\mu_s(t) - \int_\varepsilon^{\eta_x} t\langle p_x, y-\tilde{y}\rangle\ \mathrm{d}\mu_s(t)\Big|,\end{aligned}$$

which proves the bound for $|\mathcal{L}_s^\varepsilon[\phi](x)|$. The bound for $|\mathcal{L}_s[\phi](x)|$ follows similarly. ■

Our proofs throughout the paper largely depend on analyzing the behaviour of approximate extremizers $y, \tilde{y}$ in the definition (2.1) We now observe that for the operator $\mathcal{L}_s[\phi]$ these extremizers are explicit, for a generic function ϕ.

Proposition 2.2. *Let $\phi \in C^{1,1}(x)$ be a bounded, Borel function such that $p_x \neq 0$. Then:*

$$\mathcal{L}_s[\phi](x) = \int_0^\infty L_\phi\big(x, t\frac{p_x}{|p_x|}, t\frac{p_x}{|p_x|}\big)\ \mathrm{d}\mu_s(t)$$

Proof. For $\delta > 0$, let y_δ be such that $\mathcal{L}_s[\phi](x) \le \inf_{|\tilde y|=1} \int_0^\infty L_\phi(x, ty_\delta, t\tilde y)\,\mathrm{d}\mu_s(t) + \delta$. Splitting the integral and applying (2.2) on the interval $(0, \eta_x)$ yields:

$$\mathcal{L}_s[\phi](x) \le \int_0^\infty L_\phi\big(x, ty_\delta, t\frac{p_x}{|p_x|}\big)\,\mathrm{d}\mu_s(t) + \delta \le \int_0^{\eta_x} t\big\langle p_x, y_\delta - \frac{p_x}{|p_x|}\big\rangle\,\mathrm{d}\mu_s(t) + C + \delta,$$

where the constant C depends on s and ϕ. Using the lower bound in (2.3), we conclude that:

$$\big\langle p_x, \frac{p_x}{|p_x|} - y_\delta\big\rangle \cdot \int_0^{\eta_x} t\,\mathrm{d}\mu_s(t) < \infty.$$

Since the product above is nonnegative while the integral diverges to ∞, we get $y_\delta = \frac{p_x}{|p_x|}$ and:

$$\mathcal{L}_s[\phi](x) = \inf_{|\tilde y|=1} \int_0^\infty L_\phi\big(x, t\frac{p_x}{|p_x|}, t\tilde y\big)\,\mathrm{d}\mu_s(t). \tag{2.4}$$

Let now $\tilde y_\delta$ be such that $\mathcal{L}_s[\phi](x) \ge \int_0^\infty L_\phi\big(x, t\frac{p_x}{|p_x|}, t\tilde y_\delta\big)\,\mathrm{d}\mu_s(t) - \delta$. As before, in virtue of (2.2) we get: $\mathcal{L}_s[\phi](x) \ge \int_0^{\eta_x} t\big\langle p_x, \frac{p_x}{|p_x|} - \tilde y_\delta\big\rangle\,\mathrm{d}\mu_s(t) - C - \delta$, so the upper bound in (2.3) gives:

$$\big\langle p_x, \frac{p_x}{|p_x|} - \tilde y_\delta\big\rangle \cdot \int_0^{\eta_x} t\,\mathrm{d}\mu_s(t) < \infty.$$

Consequently $\tilde y_\delta = \frac{p_x}{|p_x|}$, so that: $\mathcal{L}_s[\phi](x) \ge \int_0^\infty L_\phi\big(x, t\frac{p_x}{|p_x|}, t\frac{p_x}{|p_x|}\big)\,\mathrm{d}\mu_s(t) - \delta$ for all $\delta > 0$. The proof is done, in view of (2.4). ∎

3. A proof of Theorem 1.1

We first observe that taking $p_x = 0$ in (2.2) implies the following bound, for all $\varepsilon < \eta_x$:

$$\begin{aligned} |\mathcal{L}_s^\varepsilon[\phi](x) - \mathcal{L}_s[\phi](x)| &\le \sup_{|y|=|\tilde y|=1} \Big| \int_\varepsilon^\infty L_\phi(x, ty, t\tilde y)\,\mathrm{d}\mu_s(t) - \int_0^\infty L_\phi(x, ty, t\tilde y)\,\mathrm{d}\mu_s(t)\Big| \\ &\le \int_0^\varepsilon 2C_x t^2\,\mathrm{d}\mu_s(t) = c_s s \cdot C_x \varepsilon^{2-2s}. \end{aligned} \tag{3.1}$$

In order to estimate the same difference when $p_x \ne 0$, we will quantify estimates in the proof of Proposition 2.2 for higher regular functions, as specified below.

Proposition 3.1. *Assume (**H**) with $p_x \ne 0$. Then, for every $\varepsilon < \eta_x$ there holds:*

$$\begin{aligned} \Big|\mathcal{L}_s^\varepsilon[\phi](x) - &\int_\varepsilon^\infty L_\phi\big(x, t\frac{p_x}{|p_x|}, t\frac{p_x}{|p_x|}\big)\,\mathrm{d}\mu_s(t)\Big| \\ &\le 4c_s s \cdot C_x\big(\eta_x^{2-2s} - \varepsilon^{2-2s}\big) \cdot A_\varepsilon + c_s(1-s) \cdot \Big(\eta_x^{-2s} + \frac{2s}{2s-1}\eta_x^{1-2s}\Big) \cdot \omega_\phi(A_\varepsilon), \end{aligned} \tag{3.2}$$

Proof. **1.** For every $\varepsilon < \eta_x$ and every small $\delta > 0$ let $|y_\delta^\varepsilon| = 1$ satisfy: $\sup_{|y|=1} \int_\varepsilon^\infty \phi(x + ty)\,\mathrm{d}\mu_s(t) \le \int_\varepsilon^\infty \phi(x + ty_\delta^\varepsilon)\,\mathrm{d}\mu_s(t) + \delta$. In particular, this implies:

$$\int_\varepsilon^\infty \phi\big(x + t\frac{p_x}{|p_x|}\big)\,\mathrm{d}\mu_s(t) \le \int_\varepsilon^\infty \phi(x + ty_\delta^\varepsilon)\,\mathrm{d}\mu_s(t) + \delta,$$

Denote $A = \left|\frac{p_x}{|p_x|} - y_\delta^\varepsilon\right|$. Together with (1.5), the above bound results in:

$$\begin{aligned}\delta &\geq \int_\varepsilon^{\eta_x} \left(\phi\big(x + t\frac{p_x}{|p_x|}\big) - \phi(x + ty_\delta^\varepsilon)\right) \mathrm{d}\mu_s(t) - \int_{\eta_x}^\infty \left|\phi\big(x + t\frac{p_x}{|p_x|}\big) - \phi(x + ty_\delta^\varepsilon)\right| \mathrm{d}\mu_s(t) \\ &\geq \int_\varepsilon^{\eta_x} t\Big\langle \nabla\phi\big(x + t\frac{p_x}{|p_x|}\big), \frac{p_x}{|p_x|} - y_\delta^\varepsilon\Big\rangle \mathrm{d}\mu_s(t) - \int_\varepsilon^{\eta_x} C_x t^2 A^2 \, \mathrm{d}\mu_s(t) - \int_{\eta_x}^\infty (1+t)\cdot\omega_\phi(A)\,\mathrm{d}\mu_s(t) \\ &\geq \Big\langle p_x, \frac{p_x}{|p_x|} - y_\delta^\varepsilon\Big\rangle \int_\varepsilon^{\eta_x} t\,\mathrm{d}\mu_s(t) - 4C_x A \int_\varepsilon^{\eta_x} t^2 \,\mathrm{d}\mu_s(t) - \omega_\phi(A)\int_{\eta_x}^\infty (1+t)\,\mathrm{d}\mu_s(t).\end{aligned}$$

The last bound above follows by observing that $\left|\nabla\phi\big(x + t\frac{p_x}{|p_x|}\big) - \nabla\phi(x)\right| \leq 2C_x t$ for all $|t| \leq \eta_x$ and that $A \leq 2$. Consequently, we get:

$$\Big\langle p_x, \frac{p_x}{|p_x|} - y_\delta^\varepsilon\Big\rangle \leq \frac{1}{\int_\varepsilon^{\eta_x} t\,\mathrm{d}\mu_s(t)}\Big(\delta + 4C_x A\int_\varepsilon^{\eta_x} t^2\,\mathrm{d}\mu_s(t) + \omega_\phi(A)\int_{\eta_x}^\infty (1+t)\,\mathrm{d}\mu_s(t)\Big).$$

On the other hand, by a straightforward calculation:

$$A^2 = \left|\frac{p_x}{|p_x|} - y_\delta^\varepsilon\right|^2 = 2 - 2\Big\langle\frac{p_x}{|p_x|}, y_\delta^\varepsilon\Big\rangle = \frac{2}{|p_x|}\Big\langle p_x, \frac{p_x}{|p_x|} - y_\delta^\varepsilon\Big\rangle,$$

the last two displayed formulas yields that:

$$A^2 \leq \frac{2}{|p_x|\int_\varepsilon^{\eta_x} t\,\mathrm{d}\mu_s(t)}\Big(\delta + 4C_x A\int_\varepsilon^{\eta_x} t^2\,\mathrm{d}\mu_s(t) + \omega_\phi(A)\int_{\eta_x}^\infty (1+t)\,\mathrm{d}\mu_s(t)\Big). \tag{3.3}$$

We now simplify (3.3) as follows. Without loss of generality, we may assume that $\delta > 0$ satisfies: $\delta \cdot |p_x|\int_\varepsilon^{\eta_x} t\,\mathrm{d}\mu_s(t) \leq \big(16C_x\int_\varepsilon^{\eta_x} t^2\,\mathrm{d}\mu_s(t)\big)^2$, In case when δ is larger than the two other terms in the right hand side of (3.3), we get:

$$A^2 \leq \frac{4\delta}{|p_x|\int_\varepsilon^{\eta_x} t\,\mathrm{d}\mu_s(t)} \leq \Big(\frac{32C_x\int_\varepsilon^{\eta_x} t^2\,\mathrm{d}\mu_s(t)}{|p_x|\int_\varepsilon^{\eta_x} t\,\mathrm{d}\mu_s(t)}\Big)^2, \tag{3.4}$$

In the opposite case, there holds:

$$A^2 \leq \frac{4}{|p_x|\int_\varepsilon^{\eta_x} t\,\mathrm{d}\mu_s(t)}\Big(4C_x A\int_\varepsilon^{\eta_x} t^2\,\mathrm{d}\mu_s(t) + \omega_\phi(A)\int_{\eta_x}^\infty (1+t)\,\mathrm{d}\mu_s(t)\Big) \doteq I_1 + I_2.$$

Further, when $I_2 \leq I_1$, then we obtain the same bound as in (3.4), namely:

$$A \leq \frac{32C_x\int_\varepsilon^{\eta_x} t^2\,\mathrm{d}\mu_s(t)}{|p_x|\int_\varepsilon^{\eta_x} t\,\mathrm{d}\mu_s(t)} = \frac{16C_x}{|p_x|}\cdot\frac{2s-1}{1-s}\cdot\frac{\eta_x^{2-2s} - \varepsilon^{2-2s}}{\varepsilon^{1-2s} - {\eta_x}^{1-2s}}.$$

On the other hand, $I_1 < I_2$ implies:

$$A^2 \leq \frac{8\omega_\phi(A)\int_{\eta_x}^\infty (1+t)\,\mathrm{d}\mu_s(t)}{|p_x|\int_\varepsilon^{\eta_x} t\,\mathrm{d}\mu_s(t)} = \frac{8\omega_\phi(A)}{|p_x|}\cdot\frac{\frac{2s-1}{2s}\eta_x^{-2s} + \eta_x^{1-2s}}{\varepsilon^{1-2s} - \eta_x^{1-2s}}.$$

We hence conclude that $A \leq A_\varepsilon$ in either of the above cases.

2. Similarly as in step 1, we see that the unit vector $\tilde{y}_\delta^\varepsilon$ with the property: $\inf_{|y|=1}\int_\varepsilon^\infty \phi(x - ty)\,\mathrm{d}\mu_s(t) \geq \int_\varepsilon^\infty \phi(x - t\tilde{y}_\delta^\varepsilon)\,\mathrm{d}\mu_s(t) - \delta$, satisfies: $\left|\frac{p_x}{|p_x|} - \tilde{y}_\delta^\varepsilon\right| \leq A_\varepsilon$. We now write:

$$\int_\varepsilon^\infty L_\phi(x, t\tilde{y}_\delta^\varepsilon, t\tilde{y}_\delta^\varepsilon)\,\mathrm{d}\mu_s(t) - \delta \leq \mathcal{L}_s^\varepsilon[\phi](x) \leq \int_\varepsilon^\infty L_\phi(x, ty_\delta^\varepsilon, ty_\delta^\varepsilon)\,\mathrm{d}\mu_s(t) + \delta,$$

which implies:

$$\begin{aligned}&\Big|\mathcal{L}_s^\varepsilon[\phi](x)-\int_\varepsilon^\infty L_\phi\big(x,t\frac{p_x}{|p_x|},t\frac{p_x}{|p_x|}\big)\ \mathrm{d}\mu_s(t)\Big|\leq\delta+\max\big\{|I(y_\delta^\varepsilon)|,|I(\tilde y_\delta^\varepsilon|)\big\},\\&\text{where: } J(y)\doteq\int_\varepsilon^\infty\phi(x+ty)-\phi\big(x+t\frac{p_x}{|p_x|}\big)+\phi(x-ty)-\phi\big(x-t\frac{p_x}{|p_x|}\big)\ \mathrm{d}\mu_s(t).\end{aligned}\tag{3.5}$$

Observe that:

$$\begin{aligned}|I(y_\delta^\varepsilon)|\leq&\Big|\int_\varepsilon^{\eta_x}\phi(x+ty_\delta^\varepsilon)-\phi\big(x+t\frac{p_x}{|p_x|}\big)+\phi(x-ty_\delta^\varepsilon)-\phi\big(x-t\frac{p_x}{|p_x|}\big)\ \mathrm{d}\mu_s(t)\Big|\\&+\int_{\eta_x}^\infty\big|\phi(x+ty_\delta^\varepsilon)-\phi\big(x+t\frac{p_x}{|p_x|}\big)\big|+\big|\phi(x-ty_\delta^\varepsilon)-\phi\big(x-t\frac{p_x}{|p_x|}\big)\big|\ \mathrm{d}\mu_s(t)\doteq\bar I_1+\bar I_2.\end{aligned}$$

In order to deal with $\bar I_1$, we use the Taylor expansion:

$$\Big|\phi(x\pm ty_\delta^\varepsilon)-\phi\big(x\pm t\frac{p_x}{|p_x|}\big)-\Big\langle\nabla\phi\big(x\pm t\frac{p_x}{|p_x|}\big),\pm t\big(y_\delta^\varepsilon-\frac{p_x}{|p_x|}\big)\Big\rangle\Big|\leq C_xt^2\big|y_\delta^\varepsilon-\frac{p_x}{|p_x|}\big|^2,$$

which upon integration implies:

$$\bar I_1\leq\int_\varepsilon^{\eta_x}C_xt^2\big(4A+2A^2\big)\ \mathrm{d}\mu_s(t)\leq 8C_xA\int_\varepsilon^{\eta_x}t^2\ \mathrm{d}\mu_s(t)=4C_xC_s\frac{\eta_x^{2-2s}-\varepsilon^{2-2s}}{1-s}\cdot A.$$

For the term J_2, we get:

$$\bar I_2\leq 2\int_{\eta_x}^\infty(1+t)\cdot\omega_\phi(A)\ \mathrm{d}\mu_s(t)=2C_s\Big(\frac{\eta_x^{-2s}}{2s}+\frac{\eta_x^{1-2s}}{2s-1}\Big)\cdot\omega_\phi(A).$$

In conclusion, we obtain the following bounds:

$$|I(y_\delta^\varepsilon)|,\quad|I(\tilde y_\delta^\varepsilon)|\leq 4C_xC_s\frac{\eta_x^{2-2s}-\varepsilon^{2-2s}}{1-s}A_\varepsilon+2C_s\Big(\frac{\eta_x^{-2s}}{2s}+\frac{\eta_x^{1-2s}}{2s-1}\Big)\cdot\omega_\phi(A_\varepsilon).$$

This ends the proof in virtue of (3.5). ■

Corollary 3.2. *Under the same assumptions and notation as in Proposition 3.1, we have:*

$$\begin{aligned}&\Big|\mathcal{L}_s^\varepsilon[\phi](x)-\mathcal{L}_s[\phi](x)\Big|\\&\quad\leq 4c_ss\cdot C_x\big(\eta_x^{2-2s}-\varepsilon^{2-2s}\big)\cdot A_\varepsilon+c_s(1-s)\cdot\Big(\eta_x^{-2s}+\frac{2s}{2s-1}\eta_x^{1-2s}\Big)\cdot\omega_\phi(A_\varepsilon)+c_ss\cdot C_x\varepsilon^{2-2s}.\end{aligned}$$

Proof. Observe that for all $t<\eta_x$ there holds:

$$\big|L_\phi\big(x,t\frac{p_x}{|p_x|},t\frac{p_x}{|p_x|}\big)\big|\leq t^2\|\nabla^2\phi\|_{L^\infty(B_t)}.$$

Consequently and in view of Proposition 2.2 we get:

$$\Big|\mathcal{L}_s[\phi](x)-\int_\varepsilon^\infty L_\phi\big(x,t\frac{p_x}{|p_x|},t\frac{p_x}{|p_x|}\big)\ \mathrm{d}\mu_s(t)\Big|\leq\int_0^\varepsilon\Big|L_\phi\big(x,t\frac{p_x}{|p_x|},t\frac{p_x}{|p_x|}\big)\Big|\ \mathrm{d}\mu_s(t)\leq\frac{C_s}{1-s}\cdot C_x\varepsilon^{2-2s}.$$

This achieves the proof by Proposition 3.1. ■

Note that the bound in Corollary 3.2 is essentially valid in both cases $p_x\neq0$ and $p_x=0$, because of (3.1). Scaling the said bound by the factor $\frac{s\varepsilon^{2s}}{C_s}$, we directly deduce Theorem 1.1.

Remark 3.3. (i) Observing that: $\omega_\phi(a) \leq 2\|\phi\|_{L^\infty}$, we get for all $\varepsilon < \frac{\eta_x}{2}$:

$$\kappa_\varepsilon \leq \Big(\frac{16\|\phi\|_{L^\infty}}{|p_x|} \cdot \frac{\frac{2s-1}{2s}\eta_x^{-2s} + \eta_x^{1-2s}}{\varepsilon^{1-2s} - \eta_x^{1-2s}}\Big)^{1/2} \leq 8\Big(\frac{\|\phi\|_{L^\infty}}{|p_x|} \cdot \frac{\eta_x^{-2s} + \eta_x^{1-2s}}{2s-1}\Big)^{1/2} \varepsilon^{s-1/2}.$$

In the second inequality we used that for all $\varepsilon < \frac{\eta_x}{2}$ and all $s \in (\frac{1}{2}, 1)$ there holds:

$$\varepsilon^{1-2s} - \eta_x^{1-2s} > \varepsilon^{1-2s}(1 - 2^{1-2s}), \qquad 1 - 2^{1-2s} \geq (2s-1)\ln\sqrt{2} > \frac{2s-1}{4}.$$

The first quantity in A_ε is of order ε^{2s-1}, so the right hand side in (3.2) is:

$$C(s) \cdot C\big(C_x, \frac{1}{|p_x|}, \|\phi\|_{L^\infty}\big) \cdot C(\eta_x)\varepsilon^{s-1/2} + C(s) \cdot C\big(\frac{1}{|p_x|}, \|\phi\|_{L^\infty}\big) \cdot C(\eta_x)\omega_\phi(\varepsilon^{s-1/2}),$$

where $C(s)$ depends only on s and $C(\eta_x)$ only on η_x and the remaining constants depend on the other displayed terms, in a nondecreasing manner.

(ii) When $\phi \in C^{0,\alpha}(\mathbb{R}^N \setminus \bar{B}_{\eta_x})$ with $\alpha \in (0,1)$, then $\omega_\phi(a) = [\phi]_\alpha a^\alpha$. Therefore:

$$\kappa_\varepsilon \leq \Big(\frac{32\ [\phi]_\alpha}{|p_x|} \cdot \frac{\eta_x^{-2s} + \eta_x^{1-2s}}{2s-1}\Big)^{\frac{1}{2-\alpha}} \varepsilon^{\frac{2s-1}{2-\alpha}},$$

whereas (3.2) can be replaced with: $C(s) \cdot C\big(\frac{1}{|p_x|}, [\phi]_\alpha\big) \cdot C(\eta_x)\varepsilon^{\alpha\cdot\frac{2s-1}{2-\alpha}}$.

(iii) Finally, for ϕ Lipschitz on $\mathbb{R}^N \setminus \bar{B}_{\eta_x}$ with the Lipschitz constant Lip_ϕ, we get:

$$\kappa_\varepsilon \leq \frac{32\ \mathrm{Lip}_\phi}{|p_x|} \cdot \frac{\eta_x^{-2s} + \eta_x^{1-2s}}{2s-1}\varepsilon^{2s-1}, \quad A_\varepsilon \leq \frac{32}{|p_x|} \cdot \max\Big\{\frac{2C_x\eta_x^{2-2s}}{1-s}, \frac{\mathrm{Lip}_\phi(\eta_x^{-2s} + \eta_x^{1-2s})}{2s-1}\Big\}\varepsilon^{2s-1}.$$

Indeed, both quantities in A_ε have ε^{2s-1}-order. The expression in (3.2) is then: $C(s) \cdot C\big(C_x, \frac{1}{|p_x|}, \mathrm{Lip}_\phi\big) \cdot C(\eta_x)\varepsilon^{2s-1}$, whereas the order of the error bounding quantity in Theorem 1.1 is $C(s) \cdot (\varepsilon^{4s-1} + \varepsilon^2)$ as $\varepsilon \to 0+$, and $C(s) \to \infty$ as $s \to 1-$.

4. A proof of Theorem 1.2

We note the following refinement of the argument in the proof of Corollary 3.2:

Proposition 4.1. *Let $\phi \in C^2(\bar{B}_{\eta_x})$ satisfy: $p_x \doteq \nabla\phi(x) \neq 0$. Then, for every $\varepsilon < \eta_x$ such that $\varepsilon|\nabla^2\phi(x)| \leq |p_x|$, there holds:*

$$\begin{aligned}\Big|c_s s \cdot \varepsilon^{-2s} \cdot \frac{1}{2}\Big(\sup_{B_\varepsilon(x)} \phi + \inf_{B_\varepsilon(x)} \phi - 2\phi(x)\Big) - \int_0^\varepsilon L_\phi\big(x, t\frac{p_x}{|p_x|}, t\frac{p_x}{|p_x|}\big)\ \mathrm{d}\mu_s(t)\Big| \\ \leq c_s s\Big(2\varepsilon^{3-2s}\frac{|\nabla^2\phi(x)|^2}{|p_x|} + \varepsilon^{2-2s}\sup_{y\in B_\varepsilon(x)}|\nabla^2\phi(y) - \nabla^2\phi(x)|\Big).\end{aligned} \tag{4.1}$$

Proof. A simple application of Taylor's expansion yields:

$$\big|L_\phi\big(x, t\frac{p_x}{|p_x|}, t\frac{p_x}{|p_x|}\big) - t^2\Delta_\infty\phi(x)\big| \leq t^2 \sup_{y\in B_t}|\nabla^2\phi(y) - \nabla^2\phi(x)|,$$

where we recall that $\Delta_\infty\phi(x) = \big\langle\nabla^2\phi(x) : \frac{p_x}{|p_x|} \otimes \frac{p_x}{|p_x|}\big\rangle$. Integrating the above $\int_0^\varepsilon\ \mathrm{d}\mu_s(t)$, we get:

$$\Big|\int_0^\varepsilon L_\phi\big(x, t\frac{p_x}{|p_x|}, t\frac{p_x}{|p_x|}\big)\ \mathrm{d}\mu_s(t) - \frac{C_s}{2(1-s)}\varepsilon^{2-2s}\Delta_\infty\phi(x)\Big| \leq \frac{C_s}{2(1-s)}\varepsilon^{2-2s}\sup_{y\in B_\varepsilon}|\nabla^2\phi(y) - \nabla^2\phi(x)|.$$

Recalling that (see for example [5]):

$$\Big|\big(\sup_{B_\varepsilon}\phi+\inf_{B_\varepsilon}\phi-2\phi(x)\big)-\varepsilon^2\Delta_\infty\phi(x)\Big|\leq 4\varepsilon^3\frac{|\nabla^2\phi(x)|^2}{|p_x|}+\varepsilon^2\sup_{y\in B_\varepsilon}|\nabla^2\phi(y)-\nabla^2\phi(x)|, \tag{4.2}$$

and taking the linear combination of the two above formulas, the proof is done. ■

The proof of Theorem 1.2 follows directly by summing up formulas (3.2), (4.1), and multiplying the result by the factor $\frac{(1-s)s}{C_s}\varepsilon^{2s}$. Since:

$$\Big(\frac{(1-s)s}{C_s}\varepsilon^{2s}\Big)\mathcal{L}_s^\varepsilon[\phi](x)+\Big(\frac{(1-s)s}{C_s}\varepsilon^{2s}\Big)\cdot\frac{C_s}{2(1-s)}\varepsilon^{-2s}\big(\sup_{B_\varepsilon}\phi+\inf_{B_\varepsilon}\phi-2\phi(x)\big)=\mathcal{A}_\varepsilon\phi(x)-\phi(x),$$

the error in the claimed expansion is the sum of errors in (3.2) and (4.1), multiplied by $\frac{(1-s)s}{C_s}\varepsilon^{2s}$.

Remark 4.2. (i) Analysis similar to Remark 3.3 allows for computing the order of the error term in Theorem 1.2 when ϕ is Lipschitz:

$$C(s)\cdot C\big(C_x,\frac{1}{|p_x|},\mathrm{Lip}_\phi\big)\cdot C(\eta_x)\varepsilon^{4s-1}+C(s)\cdot C\big(|\nabla^2\phi(x)|,\frac{1}{|p_x|}\big)\varepsilon^3+C(s)\cdot o(\varepsilon^2).$$

As before, $C(s)$ depends only on s, and $C(\eta_x)$ only on η_x, while the remaining constants depend on the displayed terms in a nondecreasing manner. For $\phi\in C^{2,1}(B_{\eta_x})$, the above quantity has order $\varepsilon^{4s-1}+\varepsilon^3$, which equals ε^3 at $s=1$.

(ii) For a more precise analysis of the asymptotic expansion when $s\to 1-$, note that:

$$\kappa_\varepsilon\leq\sup\Big\{a;\ a\in[0,2]\ \text{ and }\ a^2\leq\frac{32\,\omega_\phi(a)}{|p_x|}\cdot\frac{\eta_x^{-2s}+\eta_x^{1-2s}}{2s-1}\varepsilon^{2s-1}\Big\},$$
$$\frac{16C_x}{|p_x|}\cdot\frac{2s-1}{1-s}\cdot\frac{\eta_x^{2-2s}-\varepsilon^{2-2s}}{\varepsilon^{1-2s}-{\eta_x}^{1-2s}}\leq\frac{16C_x}{|p_x|}\cdot 16\eta_x^{2-2s}|\ln\varepsilon|\varepsilon^{2s-1}.$$

The first bound above is valid when $\varepsilon<\frac{\eta_x}{2}$, while for the second bound we used: $\eta_x^{2-2s}-\varepsilon^{2-2s}\leq(2-2s)(\ln\eta_x-\ln\varepsilon)\eta_x^{2-2s}\leq 4(1-s)|\ln\varepsilon|\eta_x^{2-2s}$, when $\varepsilon<e^{-|\ln\eta_x|}$. Consequently, $A_\varepsilon\leq o(1)$ as $\varepsilon\to 0+$, uniformly in $s\in(\frac12+\delta,1)$. For each fixed ε, the bound in Theorem 1.2 converges to (consistently with (4.2) as $s\to 1-$):

$$2\varepsilon^3\frac{|\nabla^2\phi(x)|^2}{|p_x|}+\varepsilon^2\sup_{y\in B_\varepsilon}\big|\nabla^2\phi(y)-\nabla^2\phi(x)\big|.$$

We also observe that when ϕ is Lipschitz on $\mathbb{R}^N\setminus\bar{B}_{\eta_x}$, the said bound becomes:

$$(1-s)\varepsilon^{4s-1}|\ln\varepsilon|^2\cdot\frac{2^9C_x}{|p_x|}\ \cdot\cdot\Big(16sC_x\eta_x^{4-4s}+\mathrm{Lip}_\phi\cdot\big(\frac{\eta_x^{2-4s}}{2}+\frac{s\eta_x^{3-4s}}{2s-1}\big)|\ln\varepsilon|^{-1}\Big)$$
$$+\,s\Big(2\varepsilon^3\frac{|\nabla^2\phi(x)|^2}{|p_x|}+\varepsilon^2\sup_{y\in B_\varepsilon}\big|\nabla^2\phi(y)-\nabla^2\phi(x)\big|\Big).$$

5. Further remarks

With an eye towards future applications, we now consider another averaging operator:

$$\bar{\mathcal{A}}_\varepsilon^o\phi(x)=\frac12\Big(\sup_{|y|=1}\fint_{T^{\varepsilon,R,\alpha}(y)}\phi(x+z)\,\mathrm{d}\mu_s^N(z)+\inf_{|y|=1}\fint_{T^{\varepsilon,R,\alpha}(y)}\phi(x+z)\,\mathrm{d}\mu_s^N(z)\Big).$$

Above, the integration is taken with respect to the measure μ_s^N on the Borel subsets of $\mathbb{R}^N$:

$$\mathrm{d}\mu_s^N(z) \doteq \frac{C(N,s)}{|z|^{N+2s}}\,\mathrm{d}z \quad \text{ where } C(N,s) = \frac{4^s s\Gamma\big(\frac{N}{2}+s\big)}{\pi^{N/2}\Gamma\big(1-s\big)} = \Big(\int_{\mathbb{R}^N} \frac{1-\cos\langle z, e_1\rangle}{|z|^{N+2s}}\,\mathrm{d}z\Big)^{-1}.$$

Clearly, $C(1,s) = C_s$ and $\mu_s^1 = \mu_s$. The integration domain $T^{\varepsilon,R,\alpha}(y)$ is the following regular spherical prism in $\mathbb{R}^N$, oriented in the direction $y \in \mathbb{R}^N \setminus \{0\}$:

$$T^{\varepsilon,R,\alpha}(y) = \Big\{z \in \mathbb{R}^N;\ \sin\frac{\angle(y,z)}{2} < \alpha,\ \langle y, z\rangle > 0 \ \text{ and } \ \varepsilon < |z| < R\Big\}.$$

Lemma 5.1. *Assume (**H**). Then, for every $\varepsilon < \eta_x$, $R > \max\{\eta_x, 1\}$ and $\alpha < \frac{1}{2}$, there holds:*

$$\begin{aligned}\sup_{|y|=1}\Big| &\fint_{T^{\varepsilon,R,\alpha}(y)} \phi(x+z)\,\mathrm{d}\mu_s^N(z) - \fint_\varepsilon^\infty \phi(x+ty)\,\mathrm{d}\mu_s(t)\Big| \\ &\leq 2\Big|\frac{\varepsilon}{R}\Big|^{2s}\cdot\|\phi\|_{L^\infty} + \max\Big\{2\big(|p_x| + 2C_x\eta_x\big)\eta_x\cdot\alpha,\ 3R\cdot\omega_\phi(\alpha)\Big\}.\end{aligned}$$

Proof. We first estimate the difference:

$$\begin{aligned}\Big|&\fint_\varepsilon^\infty \phi(x+ty)\,\mathrm{d}\mu_s(t) - \fint_\varepsilon^R \phi(x+ty)\,\mathrm{d}\mu_s(t)\Big| \\ &\leq \frac{1}{\mu_s(\varepsilon,\infty)}\int_R^\infty |\phi(x+ty)|\,\mathrm{d}\mu_s(t) + \Big|\frac{1}{\mu_s(\varepsilon,\infty)} - \frac{1}{\mu_s(\varepsilon,R)}\Big|\int_\varepsilon^R |\phi(x+ty)|\,\mathrm{d}\mu_s(t) \\ &\leq 2\Big|\frac{\varepsilon}{R}\Big|^{2s}\cdot\|\phi\|_{L^\infty}.\end{aligned}$$

Next, observe that:

$$\begin{aligned}\int_{T^{\varepsilon,R,\alpha}(y)} \phi(x+|z|y)\,\mathrm{d}\mu_s^N(z) &= \int_\varepsilon^R \phi(x+ty)t^{N-1}\cdot\mathrm{area}\big(\{|z|=1,\ z\in T^{\varepsilon,R,\alpha}\}\big)\frac{\mathrm{d}t}{t^{N+2s}} \\ &= \mathrm{area}\big(\{|z|=1,\ z\in T^{\varepsilon,R,\alpha}\}\big)\cdot\int_\varepsilon^R \phi(x+ty)\,\mathrm{d}\mu_s(t)\end{aligned}$$

which implies: $\fint_{T^{\varepsilon,R,\alpha}(y)} \phi(x+|z|y)\,\mathrm{d}\mu_s^N(z) = \fint_\varepsilon^R \phi(x+ty)\,\mathrm{d}\mu_s(t)$. It remains to bound:

$$\begin{aligned}\fint_{T^{\varepsilon,R,\alpha}(y)} |\phi(x+z) - \phi(x+|z|y)|\,\mathrm{d}\mu_s^N(z) &\leq \sup_{z\in T^{\varepsilon,R,\alpha}(y)} |\phi(x+z) - \phi(x+|z|y)| \\ &\leq \max\Big\{\|\nabla\phi\|_{L^\infty(B_{\eta_x})}\cdot 2\eta_x\cdot\alpha,\ \omega_\phi(2R\alpha)\Big\} \\ &\leq \max\Big\{2\big(|p_x| + 2C_x\eta_x\big)\eta_x\cdot\alpha,\ (1+2R)\cdot\omega_\phi(\alpha)\Big\}.\end{aligned}$$

This yields the desired estimate and ends the proof. ■

From Lemma 5.1, Remark 3.3 and Theorem 1.1, we directly deduce:

Corollary 5.2. *Assume* (**H**) *with* $\eta_x \leq 1$, *and that* ϕ *is Lipschitz on* $\mathbb{R}^N \setminus \bar{B}_{\eta_x}$ *with Lipschitz constant* Lip_ϕ. *For every* $\varepsilon \ll \eta_x$, *we set* $R = \varepsilon^{\frac{1}{2s}-1}$ *and* $\alpha = \varepsilon^{4s-\frac{1}{2s}}$. *Then there holds:*

$$\Big|\bar{\mathcal{A}}^o_\varepsilon\phi(x) - \phi(x) - \frac{1}{c_s(1-s)}\varepsilon^{2s}\mathcal{L}_s[\phi](x)\Big| \leq \varepsilon^{4s-1}\big(2\|\phi\|_{L^\infty} + 3Lip_\phi\big) + \frac{s}{1-s}\cdot 2C_x\varepsilon^2$$

$$+\begin{cases} \dfrac{32}{|p_x|}\,\varepsilon^{4s-1}\Big(\dfrac{8s}{1-s} + \big(\eta_x^{-2s} + \dfrac{2s}{2s-1}\eta_x^{1-2s}\big)Lip_\phi\Big)\cdot \\ \qquad\qquad \cdot\max\Big\{\dfrac{2C_x}{1-s}, \dfrac{(\eta_x^{-2s}+\eta_x^{1-2s})}{2s-1}Lip_\phi\Big\} & \text{when } p_x \neq 0 \\ 0 & \text{when } p_x = 0. \end{cases}$$

Remark 5.3. Towards the applications in the numerical approximating of solutions to the nonlocal Dirichlet problem for the operator Δ^s_∞, one has to consider a discrete version of the result in Theorem 1.1. To this end, let $\{\theta_i\}_{i=1}^n$ be an equidistributed spherical grid on $\{|z|=1\} \subset \mathbb{R}^N$; when $N=2$ then $\theta_i = e^{2\pi i/n}$. Next, for all x_k in the cubical grid $h\mathbb{Z}^N$ define:

$$\bar{\mathcal{A}}^d_\varepsilon\phi(x_k) \doteq \frac{sh^N}{|S_\alpha|\big(\varepsilon^{-2s} - R^{-2s}\big)}\cdot\big(\max_{i=1\ldots n} + \min_{i=1\ldots n}\big)\sum_{x_j\in T^{\varepsilon,R,\alpha}(\theta_i)\cap h\mathbb{Z}^N}\frac{\phi(x_k+x_j)}{|x_j|^{N+2s}},$$

where we used that $\mu_s^N(T^{\varepsilon,R,\alpha}) = C(N,s)|S_\alpha|\cdot\frac{\varepsilon^{-2s}-R^{-2s}}{2s}$, with $S_\alpha \doteq T^{0,\infty,\alpha}\cap\{|z|=1\}$.

It is clear that for h and n scaling in ε with sufficiently high positive and negative powers, respectively, the averaging operator $\bar{\mathcal{A}}^d_\varepsilon$ is a discrete approximation of $\mathcal{A}^o_\varepsilon$ at the same rate of the error proved in Corollary 5.2. The details of this construction as well as its implementation for a numerical scheme, are left for the future work.

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