

# PLATES WITH INCOMPATIBLE PRESTRAIN OF HIGHER ORDER

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ABSTRACT. We study the effective elastic behaviour of the incompatibly prestrained thin plates, characterized by a Riemann metric  $G$  on the reference configuration. We assume that the prestrain is “weak”, i.e. it induces scaling of the incompatible elastic energy  $E^h$  of order less than  $h^2$  in terms of the plate’s thickness  $h$ .

We essentially prove two results. First, we establish the  $\Gamma$ -limit of the scaled energies  $h^{-4}E^h$  and show that it consists of a von Kármán-like energy, given in terms of the first order infinitesimal isometries and of the admissible strains on the surface isometrically immersing  $G_{2 \times 2}$  (i.e. the prestrain metric on the midplate) in  $\mathbb{R}^3$ . Second, we prove that in the scaling regime  $E^h \sim h^\beta$  with  $\beta > 2$ , there is no other limiting theory: if  $\inf h^{-2}E^h \rightarrow 0$  then  $\inf E^h \leq Ch^4$ , and if  $\inf h^{-4}E^h \rightarrow 0$  then  $G$  is realizable and hence  $\min E^h = 0$  for every  $h$ .

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## 1. INTRODUCTION

The purpose of this paper is to study effective elastic behaviour of the incompatibly pre-stressed thin plates  $\Omega^h$ , characterized by a Riemann metric  $G$  given on their reference configuration. The incompatibility is measured through the energy  $E^h$  given below (sometimes called the “non-Euclidean” elastic energy).

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We will be concerned with the regime of curvatures of  $G$  which yields the incompatibility rate of order higher than  $h^2$ , in plate's thickness  $h$ . Indeed, in paper [6] we analyzed the scaling  $\inf E^h \sim h^2$  and proved that it only occurs when the metric  $G_{2 \times 2}$  on the mid-plate can be isometrically immersed in  $\mathbb{R}^3$  with the regularity  $W^{2,2}$  and when, at the same time, the three appropriate Riemann curvatures of  $G$  do not vanish identically (for details, see below). The relevant residual theory, obtained through  $\Gamma$ -convergence, yielded a bending Kirchhoff-like residual energy.

In the present paper we assume that:

$$(1.1) \quad h^{-2} \inf E^h \rightarrow 0$$

and prove that the only nontrivial residual theory in this regime is a von Kármán-like energy, valid when  $\inf E^h \sim h^4$ . It further turns out that this scaling is automatically implied by (1.1) and  $\inf E^h \neq 0$ . Indeed, we show that if (1.1) then  $h^{-4} \inf E^h \leq C$ , and that  $h^{-4} \inf E^h \rightarrow 0$  if and only if  $G$  is immersible whereas trivially  $\min E^h = 0$  for all  $h$ .

This scale separation is contrary to the findings of [29] valid in the Euclidean case of  $G = \text{Id}_3$ , where the residual energies are driven by presence of applied forces  $f^h \sim h^\alpha$ . In that context, three distinct limiting theories have been obtained for  $E^h \sim h^\beta$  with  $\beta > 2$  (equivalently  $\alpha > 2$ ). Namely:  $\beta \in (2, 4)$  corresponded to the linearized Kirchhoff (nonlinear bending) model subject to a nonlinear constraint on the displacements,  $\beta = 4$  to the classical von Kármán model, and  $\beta > 4$  to the linear elasticity. The present results are also contrary to the higher order hierarchy of scalings and of the resulting elastic theories of shells, as derived by an asymptotic calculus in [45]. The difference is due to the fact that while the magnitude of external forces is adjustable at will, it seems not to be the case for the interior mechanism of a given metric  $G$  which does not depend on  $h$ . In fact, it is the curvature tensor of  $G$  that induces the nontrivial stresses in the thin film. The Riemann tensor of a three-dimensional metric has only six independent components, namely the six sectional curvatures created out of the three principal directions, which further fall into two categories: including or excluding the thin direction variable. The simultaneous vanishing of curvatures in each of such categories correspond to the two scenarios at hand in terms of the scaling of the residual energy.

**1.1. Some background in dimension reduction for thin structures.** Early attempts for replacing the three-dimensional model of a thin elastic structure with planar mid-surface at rest, by a two-dimensional model, were based on *a priori* simplifying assumptions on the deformations and on the stresses. Later, the natural idea of using the thickness as a small parameter and of establishing a limit model was largely explored; we refer in particular to the works by Ciarlet and Destuynder who set the method in the appropriate framework of the weak formulation of the boundary value problems [12, 13], proved convergence to the linear plate model [23] in the context of linearized elasticity, and obtained formally the von Kármán plate model from finite elasticity [9]. See also [55, 56] for the time-dependent case and convergence results, and [10] for a comprehensive list of references.

The issue of deriving two-dimensional models valid for large deformations, by means of an asymptotic formalism, was subsequently tackled by Fox, Simo and the second author in [26]. They showed, in the context of the Saint Venant-Kirchhoff materials subject to appropriate

boundary conditions, how to recover a hierarchy of four models. This hierarchy, driven by the order of magnitude of the applied loads, consisted of: the nonlinear membrane model, the inextensional bending model, the von Kármán model and the linear plate model. The models thus obtained still required a justification through rigorous convergence results. In [34], Le Dret and the second author used the variational point of view and proved  $\Gamma$ -convergence of the three-dimensional elastic energies to a nonlinear membrane energy, valid for loads of magnitude of order 1. We remark that the expression of the limiting stored energy therein consisted of quasiconvexification of the three-dimensional energy, first minimized with respect to the normal stretches. This allowed to recover the degeneracy under compression; a feature that is otherwise missed by formal expansions. We further mention that for 3d  $\rightarrow$  1d reduction a similar point of view had been introduced by Acerbi, Buttazzo and Percivale in [1].

A key-point for deriving rigorously the above mentioned nonlinear bending model has been the geometric rigidity result due to Friesecke, James and Müller [28]. In the similar spirit, the same authors justified the von Kármán model, the linear model [29] and also they introduced novel intermediate models, in particular in the range of energies – or equivalently of loadings – between the scaling responsible for bending ( $\beta = 2$ ) and the von Kármán scaling ( $\beta = 4$ ). In this range of models, the three-dimensional stored energy appears in the limit stored energy through its second derivative at rest. Scaling the energy with exponents  $\beta$  other than integers had been explored for the membrane to bending range in [19] leading to convergence results for  $0 < \beta < 5/3$  while the regime  $5/3 \leq \beta < 2$  remains open and is conjectured to be relevant for crumpling of elastic sheets. Other significant extensions concern derivation of limit theories for incompressible materials [17, 18, 63, 47], for heterogeneous materials [61], through establishing convergence of equilibria rather than strict minimizers [52, 62, 36, 53, 37], and finally for shallow shells [39].

Extension of the above variational method valid in the framework of the large deformation model was conducted in parallel for slender structures whose midsurface at rest is non-planar. The first result by the second author and Le Dret [35] relates to scaling  $\beta = 0$  and models membrane shells: the limit stored energy depends then only on the stretching and shearing produced by the deformation on the midsurface. Another study is due to Friesecke, James, Mora and Müller [27] who analyzed the case  $\beta = 2$ . This scaling corresponds to a flexural shell model, where the only admissible deformations are those preserving the midsurface metric. The limit energy depends then on the change of curvature produced by the deformation. Further, the first author, Mora and Pakzad derived the relevant linear theories ( $\beta > 4$ ) and the von Kármán-like theories ( $\beta = 4$ ) in [41], and subsequently proceeded to finalize the analysis for elliptic shells in the full regime  $\beta > 2$  in [42]. A similar analysis has been performed in case of the developable shells in [32] leading to the proof of the collapse of all residual theories to the linear theory when  $\beta > 2$ . Following these findings, a conjecture was made in [45] about the infinite hierarchy of shell models and the various possible limiting scenarios differentiated by rigidity properties of shells. Let us recall that a comprehensive body of work had been previously devoted to the asymptotic derivation of shell models in the small displacement regime under clear hypotheses on the model taken for granted, three-dimensional or already two-dimensional and containing the thickness as a parameter. Several models were recovered by Ciarlet and coauthors [14, 15, 16], by Destuynder [22, 24] and by Sanchez-Palencia and coauthors [59, 60, 8, 7, 50]. Sanchez-Palencia, in particular, theorized

the role and interplay of the midsurface geometry and of the boundary conditions [58], as well as underlined the singular perturbation behavior. We refer to [11] for many additional references.

Most recently, there has been a sustained interest in studying similar problems where the shape formation is not driven by exterior forces but rather by the internal prestrain caused by *e.g.* growth, swelling, shrinkage or plasticity [33, 25, 62]. Variants of a thin plate theory can be used to study the self-similar structures which form due to variations in an intrinsic metric of a surface that is asymptotically flat at infinity [2], and also in the case of a circular disk with edge-localized growth [25], or in the shape of a long leaf [48]. Ben Amar and coauthors formally derived a variant of the Föppl-von Kármán equilibrium equations from finite incompressible elasticity [20, 21]: they use the multiplicative decomposition of the gradient proposed in [57] similar to ours. They take cockling of paper, grass blades and sympatulous flowers as examples [21, 5].

A systematic study of the possible limit problems when a target metric is prescribed was undertaken by the first author and collaborators: a residually strained version of the Kirchhoff theory for plates was, for the first time, rigorously derived in [44] under the assumption that the target metric is independent of thickness. This analysis was completed in [6], resulting in a necessary and sufficient condition that the elastic prestrained energy scales as  $h^2$ . The object of the present paper is to study higher order prestrains.

Let us also mention that in [39, 40, 43] similar derivations were carried out under a different assumption on the asymptotic behavior of the prescribed metric, which also implied energy scaling  $h^\beta$  in different regimes of  $\beta > 2$ . In [39] it was shown that the resulting equations are identical to those postulated to account for the effects of growth in elastic plates [48] and used to describe the shape of a long leaf. In [43] a model with a Monge-Ampère constraint was derived and analysed from various aspects. Other results concerning the energy scaling for the materials with residual strain are derived in [4], where by imposing suitable boundary data, conditions of [44, 6] are not satisfied and hence the residual energy scales larger than  $h^2$ , depending on the type of these boundary data (see also [62]).

**1.2. The set-up and notation.** Let  $\Omega$  be an open, bounded, smooth and simply connected subset of  $\mathbb{R}^2$ . For  $0 < h \ll 1$  we consider thin films  $\Omega^h$  around the mid-plate  $\Omega$ :

$$(1.2) \quad \Omega^h = \{x = (x', x_3); x' \in \Omega, x_3 \in (-h/2, h/2)\}.$$

Let  $G : \bar{\Omega}^h \rightarrow \mathbb{R}^{3 \times 3}$  be a given smooth Riemann metric on  $\Omega^h$ , uniform through the thickness:

$$G(x', x_3) = G(x') \quad \text{for every } (x', x_3) \in \Omega^h,$$

and let  $A = \sqrt{G}$  denote the unique positive definite symmetric square root of  $G$ . Consider the energy functional  $E^h : W^{1,2}(\Omega^h, \mathbb{R}^3) \rightarrow \bar{\mathbb{R}}_+$  defined as:

$$(1.3) \quad E^h(u^h) = \frac{1}{h} \int_{\Omega^h} W(\nabla u^h A^{-1}) dx.$$

The nonlinear elastic energy density  $W : \mathbb{R}^{3 \times 3} \rightarrow \bar{\mathbb{R}}_+$  is a Borel measurable function, assumed to be  $C^2$  in a neighborhood of  $SO(3)$  and to satisfy, for every  $F \in \mathbb{R}^{3 \times 3}$ , every  $R \in SO(3)$

and with a uniform constant  $c > 0$ , the conditions:

$$(1.4) \quad W(R) = 0, \quad W(RF) = W(F), \quad W(F) \geq c \operatorname{dist}^2(F, SO(3)).$$

The first condition states that the energy of a rigid motion is 0, while the second is the frame invariance. They imply that  $DW(\operatorname{Id}_3) = 0$  and that  $D^2W(\operatorname{Id}_3)(S, \cdot) = 0$  for all skew symmetric matrices  $S \in so(3)$ . The third assumption above reflects the quadratic growth of the density  $W$  away from the energy well  $SO(3)$ . Note that these assumptions are not contradictory with the physical condition  $W(F) = \infty$  for  $\det F \leq 0$ .

Throughout the paper, we use the following notation. Given a matrix  $F \in \mathbb{R}^{3 \times 3}$ , we denote its transpose by  $F^t$ , its symmetric part by  $\operatorname{sym}F = \frac{1}{2}(F + F^t)$ , and its skew part by  $\operatorname{skew}F = F - \operatorname{sym}F$ . By  $SO(n) = \{R \in \mathbb{R}^{n \times n}; R^t = R^{-1} \text{ and } \det R = 1\}$  we denote the group of special rotations, while  $so(n) = \{F \in \mathbb{R}^{n \times n}; \operatorname{sym}F = 0\}$  is the space of skew-symmetric matrices. We use the matrix norm  $|F| = (\operatorname{trace}(F^t F))^{1/2}$ , which is induced by the inner product  $\langle F_1 : F_2 \rangle = \operatorname{trace}(F_1^t F_2)$ . The  $2 \times 2$  principal minor of a matrix  $F \in \mathbb{R}^{3 \times 3}$  is denoted by  $F_{2 \times 2}$ . Conversely, for a given  $F_{2 \times 2} \in \mathbb{R}^{2 \times 2}$ , the  $3 \times 3$  matrix with principal minor equal  $F_{2 \times 2}$  and all other entries equal to 0, is denoted by  $(F_{2 \times 2})^*$ . All limits are taken as the thickness parameter  $h$  vanishes, i.e. when  $h \rightarrow 0$ . Finally, by  $C$  we denote any universal constant, independent of  $h$ .

**1.3. Some previous directly related results.** It has been proved in [44] that:

$$\inf_{u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)} E^h(u^h) = 0$$

if and only if the Riemann curvature tensor of  $G$  vanishes identically in  $\Omega^h$ , i.e.:  $\operatorname{Riem}(G) \equiv 0$ , and when (equivalently) the infimum above is achieved through a smooth isometric immersion  $u^h$  of the metric  $G$  on  $\Omega^h$ . Further, in [6] it is proved that:

$$\lim_{h \rightarrow 0} \frac{1}{h^2} \inf E^h = 0$$

if and only if the following Riemann curvatures of  $G$  vanish identically:

$$(1.5) \quad R_{1212} = R_{1213} = R_{1223} \equiv 0 \quad \text{in } \Omega^h.$$

More generally, the limit behavior of the rescaled energies  $h^{-2}E^h$  has been investigated in [6] and it has been proved that their  $\Gamma$ -limit is given by the functional:

$$\mathcal{I}_2(y) = \frac{1}{24} \int_{\Omega} \mathcal{Q}_{2,A}(x', (\nabla y)^t \nabla \vec{b}) \, dx',$$

effectively defined on the set of all  $y \in W^{2,2}(\Omega, \mathbb{R}^3)$  such that  $(\nabla y)^t \nabla y = G_{2 \times 2}$ . The quadratic forms  $\mathcal{Q}_{2,A}(x', \cdot)$  are given by means of the energy density  $W$  as in (3.4). The Cosserat vector  $\vec{b} \in W^{1,2} \cap L^\infty(\Omega, \mathbb{R}^3)$  is uniquely determined from the isometric immersion  $y$  by:

$$(1.6) \quad Q^t Q = G \quad \text{where} \quad Qe_1 = \partial_1 y, \quad Qe_2 = \partial_2 y, \quad Qe_3 = \vec{b}, \quad \text{with } \det Q > 0.$$

Observe that the functional  $\mathcal{I}_2$  is a Kirchhoff-like fully nonlinear bending energy, which in case of  $Ge_3 = e_3$  reduces to the classical bending content quantifying the second fundamental form  $(\nabla y)^t \nabla \vec{b} = (\nabla y)^t \nabla \vec{N}$  on the deformed surface  $y(\Omega)$  with the unit normal vector  $\vec{N}$ .

We recall that by Theorems 5.3, 5.5 and Corollary 5.4 in [6], the negation of condition (1.5) is equivalent to  $\min \mathcal{I}_2 > 0$ . For this reason, (1.5) is equivalent to the existence of a, necessarily unique and smooth, isometric immersion  $y_0 : \Omega \rightarrow \mathbb{R}^3$  of  $G_{2 \times 2}$ , such that:

$$(1.7) \quad \begin{cases} (\nabla y_0)^t \nabla y_0 = G_{2 \times 2} \\ \text{sym}((\nabla y_0)^t \nabla \vec{b}_0) = 0. \end{cases}$$

Above, the smooth vector field  $\vec{b}_0$  and the smooth matrix field  $Q_0$  are given as in (1.6):

$$(1.8) \quad Q_0^t Q_0 = G, \quad Q_0 e_1 = \partial_1 y_0, \quad Q_0 e_2 = \partial_2 y_0 \quad \text{and} \quad Q_0 e_3 = \vec{b}_0 \quad \text{with} \quad \det Q_0 > 0.$$

Equivalently, denoting the inverse matrix  $G^{-1} = [G^{ij}]_{i,j:1..3}$ , we have:

$$(1.9) \quad \vec{b}_0 = -\frac{1}{G^{33}}(G^{13}\partial_1 y_0 + G^{23}\partial_2 y_0) + \frac{1}{\sqrt{G^{33}}}\vec{N}.$$

Uniqueness of the immersion  $y_0$  in (1.7) follows from Theorem 5.3 in [6] which shows that the second fundamental form of the surface  $y_0(\Omega)$  is given in terms of  $G$ . Therefore, both fundamental forms are known. Also, the second equation in (1.7) comes from the fact that the kernel of each quadratic form  $\mathcal{Q}_{2,A}$  consists of  $so(2)$ .

**1.4. New results of this work.** In view of the above statements, in this paper we investigate smaller energy scalings and the limiting behaviour of the minimizing configurations to  $E^h$  under condition (1.5). We first prove (in Lemma 2.1) that (1.5), which as we recall is equivalent to (1.1), implies:

$$\inf E^h \leq Ch^4.$$

We then derive (in Theorem 3.1 and Theorem 4.1) the  $\Gamma$ -limit  $\mathcal{I}_4$  of the rescaled energies  $h^{-4}E^h$ , together with their compactness properties.

Namely, let  $y_0$  be the unique immersion satisfying (1.7), where  $\vec{b}_0$  is as in (1.8). Let  $\vec{d}_0 : \Omega \rightarrow \mathbb{R}^3$  be the smooth vector field given in terms of  $y_0$  by:

$$(1.10) \quad \langle Q_0^t \vec{d}_0, e_1 \rangle = -\langle \partial_1 \vec{b}_0, \vec{b}_0 \rangle, \quad \langle Q_0^t \vec{d}_0, e_2 \rangle = -\langle \partial_2 \vec{b}_0, \vec{b}_0 \rangle, \quad \langle Q_0^t \vec{d}_0, e_3 \rangle = 0.$$

The limit  $\mathcal{I}_4$  is then the following energy functional:

$$(1.11) \quad \begin{aligned} \mathcal{I}_4(V, \mathbb{S}) &= \frac{1}{2} \int_{\Omega} \mathcal{Q}_{2,A} \left( x', \mathbb{S} + \frac{1}{2}(\nabla V)^t \nabla V + \frac{1}{24}(\nabla \vec{b}_0)^t \nabla \vec{b}_0 \right) dx' \\ &+ \frac{1}{24} \int_{\Omega} \mathcal{Q}_{2,A} \left( x', (\nabla y_0)^t \nabla \vec{p} + (\nabla V)^t \nabla \vec{b}_0 \right) dx' \\ &+ \frac{1}{1440} \int_{\Omega} \mathcal{Q}_{2,A} \left( x', (\nabla y_0)^t \nabla \vec{d}_0 + (\nabla \vec{b}_0)^t \nabla \vec{b}_0 \right) dx', \end{aligned}$$

acting on the space of finite strains:

$$\mathbb{S} \in \text{cl}_{L^2} \{ \text{sym}((\nabla y_0)^t \nabla w); w \in W^{1,2}(\Omega, \mathbb{R}^3) \}$$

and the space of first order infinitesimal isometries:

$$V \in W^{2,2}(\Omega, \mathbb{R}^3) \quad \text{such that:} \quad \text{sym}((\nabla y_0)^t \nabla V)_{2 \times 2} = 0,$$

where the vector field  $\vec{p} \in W^{1,2}(\Omega, \mathbb{R}^3)$  is uniquely associated with each  $V$  by:  $(\nabla y_0)^t \vec{p} = -(\nabla V)^t \vec{b}_0$  and  $\langle \vec{b}_0, \vec{p} \rangle = 0$ .

The spaces consisting of  $\mathbb{S}$  and  $V$  contain the information about the admissible error displacements, relative to the leading order immersion  $y_0$ , under the energy scaling  $E^h \sim h^4$ . We discuss the geometrical significance of  $V$  and  $\mathbb{S}$  and of various bending and stretching tensors in the first two terms of  $\mathcal{I}_4(V, \mathbb{S})$  in section 5. We further prove in Theorem 6.2 that the last term in (1.11), which is obviously constant and as such does not play a role in the minimization process, is precisely given by the only potentially nonzero (in view of (1.5)) curvatures of  $G$ , namely:

$$\text{sym}((\nabla y_0)^t \nabla \vec{d}_0) + (\nabla \vec{b}_0)^t \nabla \vec{b}_0 = \begin{bmatrix} R_{1313} & R_{1323} \\ R_{1323} & R_{2323} \end{bmatrix}.$$

We may thus write, informally:

$$\begin{aligned} \mathcal{I}_4(V, \mathbb{S}) &= \frac{1}{2} \int_{\Omega} \mathcal{Q}_{2,A}(x', \text{stretching of order } h^2) dx' + \frac{1}{24} \int_{\Omega} \mathcal{Q}_{2,A}(x', \text{bending of order } h) dx' \\ &\quad + \frac{1}{1440} \int_{\Omega} \mathcal{Q}_{2,A}(x', \text{Riemann curvature of } G) dx'. \end{aligned}$$

In particular, since all three terms above are nonnegative, this directly implies that the condition  $\lim_{h \rightarrow 0} \frac{1}{h^4} \inf E^h = 0$ , which is equivalent to  $\min \mathcal{I}_4 = 0$ , is further equivalent to the immersability  $G$ , i.e. the vanishing of all its Riemann curvatures  $\text{Riem}(G) \equiv 0$  in  $\Omega^h$ .

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## 2. THE SCALING AND APPROXIMATION LEMMAS

We first introduce the following notation. Let  $B_0(x')$  be the matrix field satisfying:

$$(2.1) \quad B_0 e_1 = \partial_1 \vec{b}_0, \quad B_0 e_2 = \partial_2 \vec{b}_0 \quad \text{and} \quad B_0 e_3 = \vec{d}_0,$$

where  $\vec{d}_0$  is given by (1.10). Observe that in this way  $Q_0^t B_0$  is skew symmetric. Indeed, it has the following block form:

$$(2.2) \quad Q_0^t B_0 = \left[ \begin{array}{c|c} (\nabla y_0)^t \nabla \vec{b}_0 & (\nabla y_0)^t \vec{d}_0 \\ \hline (\vec{b}_0)^t \nabla \vec{b}_0 & \langle \vec{b}_0, \vec{d}_0 \rangle \end{array} \right]$$

and by (1.7) we see that  $(\nabla y_0)^t \nabla \vec{b}_0 \in so(2)$  is skew symmetric, while by (1.10):  $(\nabla y_0)^t \vec{d}_0 = -(\nabla \vec{b}_0)^t \vec{b}_0$  and  $\langle \vec{b}_0, \vec{d}_0 \rangle = 0$ .

**Lemma 2.1.** *Condition (1.5) implies:  $\inf_{W^{1,2}(\Omega^h, \mathbb{R}^3)} E^h \leq Ch^4$ .*

*Proof.* Let us construct a sequence  $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$  that has low energy. Let:

$$(2.3) \quad u^h(x', x_3) = y_0(x') + x_3 \vec{b}_0(x') + \frac{x_3^2}{2} \vec{d}_0(x'),$$

in fact each  $u^h$  is the restriction on its domain  $\Omega^h$  of the same deformation. We have:

$$\nabla u^h(x', x_3) = Q_0(x') + x_3 B_0(x') + \frac{x_3^2}{2} D_0(x'),$$

where the matrix field  $D_0(x') \in \mathbb{R}^{3 \times 3}$  is given through:

$$D_0(x')e_1 = \partial_1 \vec{d}_0, \quad D_0(x')e_2 = \partial_2 \vec{d}_0, \quad D_0(x')e_3 = 0,$$

so that:

$$\nabla u^h A^{-1} = Q_0 A^{-1} + x_3 B_0 A^{-1} + \frac{x_3^2}{2} D_0 A^{-1}.$$

For brevity, denote  $F^h = \nabla u^h A^{-1}$ . Obviously,  $F^h$  decomposes as:

$$(2.4) \quad F^h(x', x_3) = Q_0 A^{-1}(x')(\text{Id}_3 + x_3 S(x') + x_3^2 T(x')) = (Q_0 A^{-1}(x'))G^h(x', x_3)$$

with  $S = A^{-1}Q_0^t B_0 A^{-1}$ ,  $T = \frac{1}{2}A^{-1}Q_0^t D_0 A^{-1}$  and  $G^h = \text{Id}_3 + x_3 S + x_3^2 T$ . Since  $Q_0 A^{-1} \in SO(3)$  by construction, frame indifference implies that  $W(F^h) = W(((G^h)^t G^h)^{1/2})$ . Note that  $S$  is skew symmetric, because  $Q_0^t B_0$  is skew symmetric. Therefore,  $(G^h)^t G^h$  and the expansion of its square root do not contain terms linear in  $x_3$ . Indeed, letting  $K = T + T^t - S^2$ :

$$((G^h)^t G^h)(x', x_3) = \text{Id}_3 + x_3^2 K(x') + \mathcal{O}(x_3^3)$$

and:

$$(((G^h)^t G^h)^{1/2})(x', x_3) = \text{Id}_3 + \frac{x_3^2}{2} K(x') + \mathcal{O}(x_3^3).$$

As a consequence, using  $W(\text{Id}_3) = 0$  and  $DW(\text{Id}_3) = 0$ , we obtain:

$$W(F^h) = W(((G^h)^t G^h)^{1/2}) = \frac{x_3^4}{8} D^2 W(\text{Id}_3)(K, K) + \mathcal{O}(x_3^5).$$

Using (1.3), we get

$$E^h(u^h) = \frac{1}{h} \int_{\Omega^h} W(F^h) \, dx \leq Ch^4,$$

which accomplishes the proof of the lemma. ■

In Lemma 2.1 above, we constructed deformations whose gradient was sufficiently close to  $Q_0 + x_3 B_0$ , to provide the energy of the order  $h^4$ . Conversely, in Corollary 2.3 below, we establish that the gradients of deformations  $u^h$  whose energy scales like  $h^4$ , are close to  $Q_0 + x_3 B_0$  modulo local multiplications by  $R^h(x') \in SO(3)$ . Corollary 2.3 makes this statement precise and gives an estimation on  $\nabla R^h$  as well.

For any  $\mathcal{V}$  which is an open subset of  $\Omega$ , we let  $\mathcal{V}^h = \mathcal{V} \times (-h/2, h/2)$  and we define the local energy functional by:

$$E^h(u^h, \mathcal{V}^h) = \frac{1}{h} \int_{\mathcal{V}^h} W(\nabla u^h A^{-1}) \, dx.$$



**Lemma 2.2.** *Assume (1.5). There exists a constant  $C > 0$  with the following property. For any  $u^h \in W^{1,2}(\mathcal{V}^h, \mathbb{R}^3)$ , there exists  $\bar{R}^h \in SO(3)$  such that:*

$$(2.5) \quad \frac{1}{h} \int_{\mathcal{V}^h} \left| \nabla u^h(x) - \bar{R}^h(Q_0(x') + x_3 B_0(x')) \right|^2 dx \leq C \left( E^h(u^h, \mathcal{V}^h) + h^3 |\mathcal{V}^h| \right).$$

The constant  $C$  is uniform for all  $\mathcal{V}^h$  which are bi-Lipschitz equivalent with controlled Lipschitz constants.

*Proof.* By assumption (1.4), we have:

$$(2.6) \quad E^h(u^h, \mathcal{V}^h) \geq \frac{c}{h} \int_{\mathcal{V}^h} \text{dist}^2 \left( \nabla u^h A^{-1}, SO(3) \right) dx.$$

This suggests performing a change of variables in order to use the nonlinear geometric rigidity estimate [29]. For any  $u^h \in W^{1,2}(\mathcal{V}^h, \mathbb{R}^3)$ , we let  $v^h = u^h \circ Y^{-1}$  with  $Y : \mathcal{V}^h \rightarrow Y(\mathcal{V}^h) = \mathcal{U}^h \subset \mathbb{R}^3$  given as in (2.3), namely:

$$Y(x', x_3) = y_0(x') + x_3 \vec{b}_0(x') + \frac{x_3^2}{2} \vec{d}_0(x').$$

Obviously,  $v^h \in W^{1,2}(\mathcal{U}^h, \mathbb{R}^3)$  and:

$$(2.7) \quad \nabla u^h A^{-1}(x', x_3) = \nabla v^h(z) (\nabla Y A^{-1})(x', x_3), \quad z := Y(x', x_3).$$

Let  $S' = B_0 Q_0^{-1}$  and  $T' = \frac{1}{2} D_0 Q_0^{-1}$ . Note that  $S' = B_0 Q_0^{-1} = Q_0^{-1,t} (Q_0^t B_0 Q_0^{-1}) = -Q_0^{-1,t} B_0^t$  in view of  $Q_0^t B_0 \in so(3)$ . Therefore  $S' \in so(3)$ . Computations as in Lemma 2.1 now give:

$$(2.8) \quad \nabla Y(x', x_3) = Q_0(x') + x_3 B_0(x') + \frac{x_3^2}{2} D_0(x'),$$

and:

$$\nabla Y A^{-1} = (\text{Id}_3 + x_3 S'(x') + x_3^2 T'(x')) (Q_0 A^{-1}).$$

We see that for  $h$  small,  $\det(\nabla Y A^{-1}) > 0$ . Further, the left polar decomposition  $\nabla Y A^{-1} = (\nabla Y A^{-1} (\nabla Y A^{-1})^t)^{1/2} R$ , allows us to write:

$$\nabla Y A^{-1} = (\text{Id}_3 + x_3^2 M(x', x_3)) R(x', x_3),$$

where  $M = \mathcal{O}(1)$  is a symmetric matrix field and  $R \in SO(3)$ . Again, the symmetric term does not contain any term linear in  $x_3$ . Therefore:

$$\begin{aligned} \text{dist} \left( \nabla v^h \nabla Y A^{-1}, SO(3) \right) &= \text{dist} \left( \nabla v^h (\text{Id}_3 + x_3^2 M) R, SO(3) \right) \\ &= \text{dist} \left( \nabla v^h (\text{Id}_3 + x_3^2 M), SO(3) \right) \geq c \text{dist} \left( \nabla v^h, SO(3) (\text{Id}_3 + x_3^2 M)^{-1} \right) \\ &\geq c \text{dist} \left( \nabla v^h, SO(3) \right) + \mathcal{O}(x_3^2). \end{aligned}$$

Now, let  $J = |\det \nabla Y \circ Y^{-1}|^{-1}$ . By (2.7) and the above computation:

$$\int_{\mathcal{V}^h} \text{dist}^2 \left( \nabla u^h A^{-1}, SO(3) \right) dx \geq c \int_{\mathcal{U}^h} \text{dist}^2 \left( \nabla v^h, SO(3) \right) J dz - c \int_{\mathcal{V}^h} x_3^4 dx.$$

In other words, since  $J \geq c > 0$ :

$$\frac{1}{h} \int_{\mathcal{V}^h} \text{dist}^2 \left( \nabla u^h A^{-1}, SO(3) \right) dx + h^3 |\mathcal{V}^h| \geq \frac{c}{h} \int_{\mathcal{U}^h} \text{dist}^2 \left( \nabla v^h, SO(3) \right) dz.$$

By [29], there exists  $C > 0$  with the following property. For any  $v^h \in W^{1,2}(\mathcal{U}^h, \mathbb{R}^3)$ , there exists  $\bar{R}^h \in SO(3)$  such that:

$$C \int_{\mathcal{U}^h} \text{dist}^2 \left( \nabla v^h, SO(3) \right) dz \geq \int_{\mathcal{U}^h} \left| \nabla v^h - \bar{R}^h \right|^2 dz.$$

The constant  $C$  can be chosen uniformly for domains  $\mathcal{U}^h$  which are bi-Lipschitz equivalent with controlled Lipschitz constants. By (2.6) and the reverse change of variables which satisfies  $J^{-1} \geq c > 0$  and  $|\nabla Y| \leq C$ , we obtain:

$$C \left( E^h(u^h, \mathcal{V}^h) + h^3 |\mathcal{V}^h| \right) \geq \frac{1}{h} \int_{\mathcal{V}^h} \left| \nabla u^h - \bar{R}^h \nabla Y \right|^2 dx,$$

again with a constant  $C$  uniform for domains  $\mathcal{V}^h$  that are bi-Lipschitz equivalent with controlled Lipschitz constants. This accomplishes the proof in view of (2.8).  $\blacksquare$

**Corollary 2.3.** *Assume (1.5) and let  $u^h$  be a sequence of deformations such that:*

$$\lim_{h \rightarrow 0} h^{-2} E^h(u^h) = 0.$$

*Then, there exist matrix fields  $R^h \in W^{1,2}(\Omega, SO(3))$  such that:*

$$(2.9) \quad \frac{1}{h} \int_{\Omega^h} \left| \nabla u^h(x) - R^h(x') (Q_0(x') + x_3 B_0(x')) \right|^2 dx \leq C \left( E^h(u^h) + h^4 \right)$$

*and:*

$$(2.10) \quad \int_{\Omega} \left| \nabla R^h(x') \right|^2 dx' \leq \frac{C}{h^2} \left( E^h(u^h) + h^4 \right).$$

The proof follows the lines of [29, 44, 38], with necessary modifications in view of the expected error of the order  $h^4$ . For completeness, we will present the details in the Appendix.

### 3. THE LOWER BOUND

**Theorem 3.1.** *Let  $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$  be a sequence of deformations satisfying  $E^h(u^h) \leq Ch^4$ . Then there exists a sequence of translations  $c^h \in \mathbb{R}^3$  and rotations  $\bar{R}^h \in SO(3)$  such that the associated renormalizations:*

$$(3.1) \quad y^h(x', x_3) = (\bar{R}^h)^t u^h(x', hx_3) - c^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$$

*have the following properties, where  $y_0$  and  $\vec{b}_0$  are the unique solution to (1.7) (1.8). All convergences hold up to a subsequence:*

$$(i) \quad y^h \rightarrow y_0 \text{ in } W^{1,2}(\Omega^1, \mathbb{R}^3) \text{ and } \frac{1}{h} \partial_3 y^h \rightarrow \vec{b}_0 \text{ in } L^2(\Omega^1, \mathbb{R}^3);$$

(ii) *the scaled average displacements:*

$$(3.2) \quad V^h(x') = \frac{1}{h} \int_{-\frac{1}{2}}^{-\frac{1}{2}} \left( y^h(x', x_3) - (y_0(x') + hx_3 \vec{b}_0(x')) \right) dx_3$$

converge in  $W^{1,2}(\Omega, \mathbb{R}^3)$  to a limiting field  $V \in W^{2,2}(\Omega, \mathbb{R}^3)$ , satisfying the constraint:

$$(3.3) \quad \text{sym}((\nabla y_0)^t \nabla V) = 0;$$

(iii) *the scaled tangential strains:*

$$\frac{1}{h} \text{sym}((\nabla y_0)^t \nabla V^h)$$

converge weakly in  $L^2(\Omega, \mathbb{R}^{2 \times 2})$  to some  $\mathbb{S} \in L^2(\Omega, \mathbb{R}_{sym}^{2 \times 2})$ .

(iv) *Further, defining the quadratic forms  $\mathcal{Q}_3(F) = D^2W(\text{Id}_3)(F, F)$  and:*

$$(3.4) \quad \mathcal{Q}_{2,A}(x', F_{2 \times 2}) = \min \left\{ \mathcal{Q}_3(A(x')^{-1} \tilde{F} A(x')^{-1}); \tilde{F} \in \mathbb{R}^{3 \times 3} \text{ with } \tilde{F}_{2 \times 2} = F_{2 \times 2} \right\},$$

we have:

$$(3.5) \quad \begin{aligned} \liminf_{h \rightarrow 0} \frac{1}{h^4} E^h(u^h) &\geq \mathcal{I}_4(V, \mathbb{S}) = \frac{1}{2} \int_{\Omega} \mathcal{Q}_{2,A} \left( x', \mathbb{S} + \frac{1}{2} (\nabla V)^t \nabla V + \frac{1}{24} (\nabla \vec{b}_0)^t \nabla \vec{b}_0 \right) dx' \\ &+ \frac{1}{24} \int_{\Omega} \mathcal{Q}_{2,A} \left( x', (\nabla y_0)^t \nabla \vec{p} + (\nabla V)^t \nabla \vec{b}_0 \right) dx' \\ &+ \frac{1}{1440} \int_{\Omega} \mathcal{Q}_{2,A} \left( x', (\nabla y_0)^t \nabla \vec{d}_0 + (\nabla \vec{b}_0)^t \nabla \vec{b}_0 \right) dx', \end{aligned}$$

where the vector field  $\vec{p} \in W^{1,2}(\Omega, \mathbb{R}^3)$  is uniquely associated with  $V$  by:

$$(3.6) \quad \begin{cases} (\nabla y_0)^t \vec{p} = -(\nabla V)^t \vec{b}_0 \\ \langle \vec{b}_0, \vec{p} \rangle = 0. \end{cases}$$

*Proof. 1.* Corollary 2.3 yields existence of  $R^h \in W^{1,2}(\Omega, SO(3))$  such that (2.9) and (2.10) hold with  $Ch^4$  and  $Ch^2$  in their right hand sides, respectively. We rewrite these inequalities for the reader's convenience:

$$(3.7) \quad \frac{1}{h} \int_{\Omega^h} \left| \nabla u^h(x) - R^h(x') (Q_0(x') + x_3 B_0(x')) \right|^2 dx \leq Ch^4$$

and:

$$(3.8) \quad \int_{\Omega} \left| \nabla R^h(x') \right|^2 dx' \leq Ch^2.$$

To prove the claimed convergence properties for (3.1), it is natural in view of (3.7) to set:

$$\bar{R}^h = \mathbb{P}_{SO(3)} \int_{\Omega^h} \nabla u^h(x) Q_0(x')^{-1} dx.$$

This projection is well defined, because for every  $x' \in \Omega$ , in view of (3.7):

$$\begin{aligned}
\text{dist}^2 \left( \int_{\Omega^h} \nabla u^h Q_0^{-1} dx, SO(3) \right) &\leq \left| \int_{\Omega^h} \nabla u^h Q_0^{-1} dx - R^h(x') \right|^2 \\
&\leq C \left| \int_{\Omega^h} (\nabla u^h Q_0^{-1} - R^h) dx \right|^2 + C \left| \int_{\Omega^h} R^h dx - R^h(x') \right|^2 \\
&\leq C \left| \int_{\Omega^h} (\nabla u^h - R^h(Q_0 + x_3 B_0)) Q_0^{-1} dx \right|^2 + C \left| R^h(x') - \int_{\Omega} R^h \right|^2 \\
&\leq C \int_{\Omega^h} |\nabla u^h - R^h(Q_0 + x_3 B_0)|^2 dx + C |R^h(x') - \int_{\Omega} R^h|^2 \\
&\leq Ch^4 + C |R^h(x') - \int_{\Omega} R^h|^2
\end{aligned}$$

Now, taking the average on  $\Omega$ , by the Poincaré-Wirtinger inequality and (3.8), we get:

$$\text{dist}^2 \left( \int_{\Omega^h} \nabla u^h Q_0^{-1} dx, SO(3) \right) \leq Ch^4 + C \int_{\Omega} |\nabla R^h|^2 \leq Ch^2,$$

which proves that the average  $\int_{\Omega^h} \nabla u^h Q_0^{-1} dx$  is close to  $SO(3)$  and that:

$$(3.9) \quad \left| \int_{\Omega^h} \nabla u^h Q_0^{-1} dx - \bar{R}^h \right|^2 \leq Ch^2.$$

Moreover:

$$\begin{aligned}
&\int_{\Omega} |R^h - \bar{R}^h|^2 dx = \int_{\Omega^h} |R^h - \bar{R}^h|^2 dx \\
(3.10) \quad &\leq C \int_{\Omega^h} \left( |R^h - \int_{\Omega} R^h|^2 + \left| \left( \int_{\Omega} R^h \right) - \int_{\Omega^h} \nabla u^h Q_0^{-1} \right|^2 \right) + \int_{\Omega^h} |\bar{R}^h - \int_{\Omega^h} \nabla u^h Q_0^{-1}|^2 \\
&\leq C \int_{\Omega^h} |\nabla R^h|^2 dx + C \int_{\Omega^h} |\nabla u^h - R^h(Q_0 + x_3 B_0)|^2 dx + Ch^2 \leq Ch^2,
\end{aligned}$$

where the last estimate follows by (3.7), (3.8) and (3.9).

Let now  $c^h \in \mathbb{R}^3$  be such that  $\int_{\Omega} V^h = 0$  where  $V^h$  is defined as in (3.2). Denote by  $\nabla_h y^h$  the matrix whose columns are given by  $\partial_1 y^h$ ,  $\partial_2 y^h$  and  $\partial_3 y^h/h$ . Obviously:

$$(3.11) \quad \nabla_h y^h(x', x_3) = (\bar{R}^h)^t \nabla u^h(x', hx_3).$$

Observe that:

$$\begin{aligned}
\int_{\Omega^1} |\nabla_h y^h - Q_0|^2 dx &\leq C \int_{\Omega^h} |\nabla u^h - \bar{R}^h Q_0|^2 dx \\
&\leq C \left( \int_{\Omega^h} |\nabla u^h - R^h(Q_0 + x_3 B_0)|^2 dx + \int_{\Omega^h} |x_3 R^h B_0|^2 dx + \int_{\Omega^h} |R^h - \bar{R}^h|^2 dx \right) \leq Ch^2
\end{aligned}$$

by (3.7) and (3.10). Therefore,  $\nabla_h y^h$  converges in  $L^2(\Omega^1)$  to  $Q_0$ . Further, the sequence  $\{y^h\}$  is bounded in  $W^{1,2}(\Omega^1)$ , by the choice of  $c^h$ . Passing to a subsequence we get that  $y^h$

converges weakly in  $W^{1,2}(\Omega^1)$  and in view of the strong convergence of  $\nabla y^h$  we have:

$$y^h \rightarrow y_0 \quad \text{in } W^{1,2}(\Omega^1, \mathbb{R}^3) \quad \text{and} \quad \frac{1}{h} \partial_3 y^h \rightarrow \vec{b}_0 \quad \text{in } L^2(\Omega^1, \mathbb{R}^3).$$

**2.** Note that, for every  $x' \in \Omega$ :

$$\begin{aligned} \nabla V^h(x') &= \frac{1}{h} \left( \int_{-1/2}^{1/2} \nabla_h y^h(x) - Q_0(x') \, dx_3 \right)_{3 \times 2} \\ (3.12) \quad &= \frac{1}{h} \left( \int_{-1/2}^{1/2} \nabla_h y^h - (\bar{R}^h)^t R^h (Q_0 + hx_3 B_0) \, dx_3 \right)_{3 \times 2} + \frac{1}{h} \left( ((\bar{R}^h)^t R^h - \text{Id}_3) Q_0 \right)_{3 \times 2} \\ &= I_1^h + I_2^h. \end{aligned}$$

The first term above converges to 0. Indeed:

$$\begin{aligned} (3.13) \quad \left\| I_1^h \right\|_{L^2(\Omega)}^2 &\leq \frac{C}{h^2} \int_{\Omega^1} |(\bar{R}^h)^t \nabla u^h(x', hx_3) - (\bar{R}^h)^t R^h (Q_0(x') + hx_3 B_0)|^2 \, dx \\ &\leq \frac{C}{h^2} \int_{\Omega^h} |\nabla u^h(x', x_3) - R^h (Q_0 + x_3 B_0)|^2 \, dx \leq Ch^2. \end{aligned}$$

Towards estimating the second term in (3.12), denote:

$$S^h = \frac{1}{h} ((\bar{R}^h)^t R^h - \text{Id}_3).$$

By (3.10) and (3.8), it follows that:

$$\|S^h\|_{L^2(\Omega)}^2 \leq \frac{C}{h^2} \int_{\Omega} |R^h - \bar{R}^h|^2 \leq C \quad \text{and} \quad \|\nabla S^h\|_{L^2(\Omega)}^2 \leq \frac{C}{h^2} \int_{\Omega} |\nabla R^h|^2 \leq C.$$

Passing to a subsequence, we can assume that:

$$(3.14) \quad S^h \rightharpoonup S \quad \text{weakly in } W^{1,2}(\Omega),$$

which implies:

$$(3.15) \quad I_2^h \rightarrow (SQ_0)_{3 \times 2} \quad \text{in } L^2(\Omega, \mathbb{R}^{3 \times 2}).$$

Consequently, by (3.12):

$$(3.16) \quad \nabla V^h \rightarrow (SQ_0)_{3 \times 2} \quad \text{in } L^2(\Omega, \mathbb{R}^{3 \times 2}).$$

As before, we conclude that  $V^h$  converges in  $W^{1,2}(\Omega)$  and that its limit  $V$  belongs to  $W^{2,2}(\Omega, \mathbb{R}^3)$ , since  $\nabla V = (SQ_0)_{3 \times 2} \in W^{1,2}(\Omega)$ . We now prove (3.3). By definition of  $S^h$ :

$$(3.17) \quad \text{sym } S^h = -\frac{h}{2} (S^h)^t S^h,$$

so in view of the boundedness of  $\{S^h\}$  in  $W^{1,2}$ :

$$\|\text{sym } S^h\|_{L^2(\Omega)} \leq Ch \|S^h\|_{L^4(\Omega)}^2 \leq Ch \|S^h\|_{W^{1,2}(\Omega)}^2 \leq Ch.$$

Consequently,  $S$  is a skew symmetric field. But  $(\nabla y_0)^t \nabla V = (Q_0^t SQ_0)_{2 \times 2}$ , hence (3.3) follows.

For future use, let us define  $\vec{p} \in W^{1,2}(\Omega, \mathbb{R}^3)$  by:

$$(3.18) \quad [\nabla V \mid \vec{p}] = SQ_0.$$

Since  $Q_0^t[\nabla V \mid p] \in so(3)$ , it is easily checked that  $\vec{p}$  is given solely in terms of  $V$  by:

$$(3.19) \quad \begin{cases} (\nabla y_0)^t \vec{p} = -(\nabla V)^t \vec{b}_0 \\ \langle \vec{b}_0, \vec{p} \rangle = 0. \end{cases}$$

**3.** We now want to establish convergence in (iii). In view of (3.12) we write:

$$(3.20) \quad \begin{aligned} \frac{1}{h} \text{sym} (Q_0^t \nabla V^h)_{2 \times 2}(x') &= \frac{1}{h} \text{sym} (Q_0^t I_1^h)_{2 \times 2} + \frac{1}{h} \text{sym} (Q_0^t S^h Q_0)_{2 \times 2} \\ &= J_1^h + J_2^h. \end{aligned}$$

We first deal with the sequence  $J_2^h$ . By (3.14),  $S^h \rightarrow S$  in  $L^4(\Omega)$  and so (3.17) implies:

$$\frac{1}{h} \text{sym} S^h \rightarrow -\frac{1}{2} S^t S = \frac{1}{2} S^2 \quad \text{in } L^2(\Omega).$$

Therefore:

$$(3.21) \quad J_2^h \rightarrow -\frac{1}{2} (Q_0^t S^t S Q_0)_{2 \times 2} = -\frac{1}{2} (\nabla V)^t \nabla V \quad \text{in } L^2(\Omega).$$

We now prove that  $J_1^h$  converges. Recall that by (3.20), (3.12) and (3.11):

$$(3.22) \quad J_1^h = \frac{1}{h} \text{sym} (Q_0^t I_1^h)_{2 \times 2} = \text{sym} \left( Q_0^t (\bar{R}^h)^t \int_{-1/2}^{1/2} Z^h(x', x_3) dx_3 \right)_{2 \times 2}$$

where the rescaled strains  $Z^h$  are defined by:

$$(3.23) \quad Z^h(x', x_3) = \frac{1}{h^2} \left( \nabla u^h(x', hx_3) - R^h(x') (Q_0(x') + hx_3 B_0(x')) \right).$$

By (3.7), the sequence  $\{Z^h\}$  is bounded in  $L^2(\Omega^1, \mathbb{R}^3)$ . Therefore, up to a subsequence:

$$(3.24) \quad Z^h \rightharpoonup Z \quad \text{weakly in } L^2(\Omega^1, \mathbb{R}^3).$$

It follows that:

$$(3.25) \quad J_1^h \rightharpoonup J_1 := \text{sym} \left( Q_0^t (\bar{R})^t \int_{-1/2}^{1/2} Z(x', x_3) dx_3 \right)_{2 \times 2} \quad \text{weakly in } L^2(\Omega).$$

which yields (iii) by (3.20) and (3.21).

**4.** We now aim at giving the structure of the weak limit  $\mathbb{S}$  of  $\frac{1}{h} \text{sym} (Q_0^t \nabla V^h)_{2 \times 2}$  in terms of the limiting fields  $V$  and  $Z$ . We have just seen that:

$$(3.26) \quad \mathbb{S} = J_1 - \frac{1}{2} (\nabla V)^t \nabla V,$$

where  $J_1$  is given by (3.25). As a tool, consider the difference quotients  $f^{s,h}$ :

$$f^{s,h}(x', x_3) = \frac{1}{h^2 s} \left( y^h(x', x_3 + s) - y^h(x', x_3) - hs \left( \vec{b}_0 + h \left( x_3 + \frac{s}{2} \right) \vec{d}_0 \right) \right),$$

and let us study for any  $s$  the convergence of  $f^{s,h}$  as  $h \rightarrow 0$ . In fact, we will show that  $f^{s,h} \rightharpoonup \vec{p}$ , weakly in  $W^{1,2}(\Omega^1, \mathbb{R}^3)$ . Write:

$$f^{s,h}(x', x_3) = \frac{1}{h^2} \int_0^s \partial_3 y^h(x', x_3 + t) - h(\vec{b}_0 + h(x_3 + t)\vec{d}_0) dt,$$

and observe that:

$$\begin{aligned} \frac{1}{h^2} \left( \partial_3 y^h - h(\vec{b}_0 + hx_3\vec{d}_0) \right) &= \frac{1}{h} \left( (\bar{R}^h)^t \nabla u^h(x', hx_3) - (Q_0 + hx_3 B_0) \right) e_3 \\ &= \frac{1}{h} (\bar{R}^h)^t \left( \nabla u^h(x', hx_3) - R^h(Q_0 + hx_3 B_0) \right) e_3 + S^h(Q_0 + hx_3 B_0) e_3 \\ &= h(\bar{R}^h)^t Z^h(x', x_3) e_3 + S^h(Q_0 + hx_3 B_0) e_3. \end{aligned}$$

The first term in the right hand side above converges to 0 in  $L^2(\Omega^1)$  because  $\{Z^h\}$  is bounded in  $L^2(\Omega^1, \mathbb{R}^3)$ , while the second term converges to  $SQ_0 e_3 = S\vec{b}_0$  in  $L^2(\Omega^1)$  by (3.14). Note that  $SQ_0 e_3 = \vec{p}$  by (3.18). Therefore,  $f^{s,h} \rightarrow \vec{p}$  in  $L^2(\Omega^1)$ .

We now deal with the derivatives of the studied sequence. Firstly:

$$\begin{aligned} \partial_3 f^{s,h}(x', x_3) &= \frac{1}{s} \left( \frac{1}{h^2} \left( \partial_3 y^h(x', x_3 + s) - h(\vec{b}_0 + h(x_3 + s)\vec{d}_0) \right) \right. \\ &\quad \left. - \frac{1}{h^2} \left( \partial_3 y^h(x', x_3) - h(\vec{b}_0 + hx_3\vec{d}_0) \right) \right) \end{aligned}$$

converges to 0 in  $L^2(\Omega^1)$ . For  $i = 1, 2$ , the in-plane derivatives read as:

$$\begin{aligned} \partial_i f^{s,h}(x', x_3) &= \frac{1}{h^2 s} \left( (\bar{R}^h)^t \partial_i u^h(x', h(x_3 + s)) \right. \\ &\quad \left. - (\bar{R}^h)^t \partial_i u^h(x', hx_3) - hs \left( \partial_i \vec{b}_0 + h \left( x_3 + \frac{s}{2} \right) \partial_i \vec{d}_0 \right) \right) \\ &= \frac{1}{s} \left( (\bar{R}^h)^t Z^h(x', x_3 + s) - (\bar{R}^h)^t Z^h(x', x_3) \right) e_i \\ &\quad + \frac{1}{h^2 s} \left( (\bar{R}^h)^t R^h(Q_0 + h(x_3 + s)B_0) - (\bar{R}^h)^t R^h(Q_0 + hx_3 B_0) \right) e_i \\ &\quad - \frac{1}{h} \left( B_0 e_i + h \left( x_3 + \frac{s}{2} \right) \partial_i \vec{d}_0 \right). \end{aligned}$$

The last two terms above can be written as:  $S^h B_0 e_i - \left( x_3 + \frac{s}{2} \right) \partial_i \vec{d}_0$ , hence by (3.24):

$$\begin{aligned} \partial_i f^{s,h}(x', x_3) &\rightharpoonup \frac{1}{s} (\bar{R})^t \left( Z(x', x_3 + s) - Z(x', x_3) \right) e_i \\ &\quad + SB_0 e_i - \left( x_3 + \frac{s}{2} \right) \partial_i \vec{d}_0 \quad \text{weakly in } L^2(\Omega^1, \mathbb{R}^3), \end{aligned}$$

where  $\bar{R} \in SO(3)$  is an accumulation point of the rotations  $\bar{R}^h$ .

Consequently,  $f^{s,h} \rightharpoonup \vec{p}$  weakly in  $W^{1,2}(\Omega^1, \mathbb{R}^3)$  and, for  $i = 1, 2$ :

$$(3.27) \quad s \partial_i \vec{p} = (\bar{R})^t \left( Z(x', x_3 + s) - Z(x', x_3) \right) e_i + s B_0 e_i - s \left( x_3 + \frac{s}{2} \right) \partial_i \vec{d}_0,$$

which proves that  $Z(x', \cdot)e_i$  has polynomial form and that:

$$(3.28) \quad (\bar{R}^t Z(x', x_3))_{3 \times 2} = (\bar{R}^t Z(x', 0))_{3 \times 2} + x_3 (\nabla \bar{p} - (SB_0)_{3 \times 2}) + \frac{x_3^2}{2} \nabla \bar{d}_0.$$

By (3.24), it follows that:

$$J_1 = \text{sym} (Q_0^t (\bar{R})^t Z(x', 0))_{2 \times 2} + \frac{1}{24} \text{sym} (Q_0^t \nabla \bar{d}_0)_{2 \times 2}.$$

With (3.26), we finally arrive at the following identity that links  $\mathbb{S}$  and  $V$  and  $Z$ :

$$(3.29) \quad \mathbb{S}(x') = \text{sym} (Q_0^t (\bar{R})^t Z(x', 0))_{2 \times 2} + \frac{1}{24} \text{sym} (Q_0^t \nabla \bar{d}_0)_{2 \times 2} - \frac{1}{2} (\nabla V)^t \nabla V.$$

**5.** We now prove the lower bound in (iv). Recall that by (3.23):

$$\nabla u^h(x', hx_3) = R^h(x')(Q_0(x') + hx_3 B_0(x')) + h^2 Z^h(x', x_3).$$

Since  $Q_0 A^{-1} \in SO(3)$  we have:

$$W(\nabla u^h A^{-1}) = W((Q_0 A^{-1})^t (R^h)^t \nabla u^h A^{-1}) = W(\text{Id}_3 + h\mathcal{J} + h^2 \mathcal{G}^h),$$

where:

$$\mathcal{J}(x', x_3) = x_3 A^{-1} (Q_0^t B_0) A^{-1}(x') \in so(3), \quad \mathcal{G}^h(x', x_3) = A^{-1} Q_0^t (R^h)^t Z^h(x', x_3) A^{-1}.$$

Note that by (3.24):

$$\mathcal{G}^h(x', x_3) \rightharpoonup \mathcal{G} = A^{-1} Q_0^t (\bar{R}^t) Z(x', x_3) A^{-1} \quad \text{weakly in } L^2(\Omega^1, \mathbb{R}^{3 \times 3}).$$

Define the “good sets”:

$$\Omega_h = \{x \in \Omega^1; h|\mathcal{G}^h| < 1\}.$$

By the above, the characteristic functions  $\mathbb{1}_{\Omega_h}$  converge to  $\mathbb{1}$  in  $L^1(\Omega)$ . Further, by frame invariance and Taylor expanding  $W$  on  $\Omega_h$ :

$$\begin{aligned} W(\text{Id}_3 + h\mathcal{J} + h^2 \mathcal{G}^h) &= W(e^{-h\mathcal{J}} (\text{Id}_3 + h\mathcal{J} + h^2 \mathcal{G}^h)) \\ &= W(\text{Id}_3 + h^2 (\mathcal{G}^h - \frac{1}{2} \mathcal{J}^2) + o(h^2)) \\ &= \frac{1}{2} \mathcal{Q}_3 \left( h^2 (\mathcal{G}^h - \frac{1}{2} \mathcal{J}^2) \right) + o(h^4). \end{aligned}$$

Therefore:

$$(3.30) \quad \begin{aligned} \liminf_{h \rightarrow 0} \frac{1}{h^4} E^h(u^h) &\geq \liminf_{h \rightarrow 0} \frac{1}{h^4} \int_{\Omega^1} \mathbb{1}_{\Omega_h} W(\text{Id}_3 + h\mathcal{J} + h^2 \mathcal{G}^h) \, dx \\ &= \liminf_{h \rightarrow 0} \frac{1}{2} \int_{\Omega^1} \mathcal{Q}_3 \left( \mathbb{1}_{\Omega_h} \text{sym} \left( \mathcal{G}^h - \frac{1}{2} \mathcal{J}^2 \right) \right) \, dx \\ &\geq \frac{1}{2} \int_{\Omega^1} \mathcal{Q}_3 \left( \text{sym} \left( \mathcal{G} - \frac{1}{2} \mathcal{J}^2 \right) \right) \, dx, \end{aligned}$$

by the weak sequential lower semi-continuity of the quadratic form  $\mathcal{Q}_3$  in  $L^2$  and in view of:

$$\mathbb{1}_{\Omega_h} \text{sym} \left( \mathcal{G}^h - \frac{1}{2} \mathcal{J}^2 \right) \rightharpoonup \text{sym} \mathcal{G} - \frac{1}{2} \mathcal{J}^2 \quad \text{weakly in } L^2(\Omega^1).$$



Note that by (3.18) we have:  $(Q_0^t S B_0)_{2 \times 2} = -(\nabla V)^t \nabla \vec{b}_0$  and that:

$$\mathcal{J}^2 = -\mathcal{J}^t \mathcal{J} = -x_3^2 A^{-1} B_0^t B_0 A^{-1}.$$

Therefore, using (3.28), the right hand side of (3.30) is bounded below by:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega^1} \mathcal{Q}_{2,A} \left( x', \text{sym} \left( Q_0^t (\bar{R})^t Z(x', 0) + x_3 (Q_0^t \nabla \vec{p} + (\nabla V)^t \nabla \vec{b}_0) + \frac{x_3^2}{2} (Q_0^t \nabla \vec{d}_0 + (\nabla \vec{b}_0)^t \nabla \vec{b}_0) \right)_{2 \times 2} \right) dx \\ & = \frac{1}{2} \int_{\Omega^1} \mathcal{Q}_{2,A} \left( x', I(x') + x_3 III(x') + x_3^2 II(x') \right) dx. \end{aligned}$$

Above we used (3.29) and we denoted:

$$\begin{aligned} (3.31) \quad I(x') &= \mathbb{S} - \frac{1}{24} \text{sym} \left( (\nabla y_0)^t \nabla \vec{d}_0 \right) + \frac{1}{2} (\nabla V)^t \nabla V \\ II(x') &= \frac{1}{2} \text{sym} \left( (\nabla y_0)^t \nabla \vec{d}_0 \right) + \frac{1}{2} (\nabla \vec{b}_0)^t \nabla \vec{b}_0 \\ III(x') &= \text{sym} \left( (\nabla y_0)^t \nabla \vec{p} \right) + \text{sym} \left( (\nabla V)^t \nabla \vec{b}_0 \right). \end{aligned}$$

Let  $\mathcal{L}_{2,A}(x')$  be the symmetric bilinear form generating the quadratic form  $\mathcal{Q}_{2,A}(x')$ . Since the odd powers of  $x_3$  integrate to 0 on the symmetric interval  $(-1/2, 1/2)$ , we get:

$$\begin{aligned} & \int_{\Omega^1} \mathcal{Q}_{2,A} \left( x', I(x') + x_3 III(x') + x_3^2 II(x') \right) dx \\ & = \int_{\Omega} \mathcal{Q}_{2,A}(x', I(x')) dx' + \left( \int_{-1/2}^{1/2} x_3^2 dx_3 \right) \int_{\Omega} \mathcal{Q}_{2,A}(x', III(x')) dx' \\ & \quad + \left( \int_{-1/2}^{1/2} x_3^4 dx_3 \right) \int_{\Omega} \mathcal{Q}_{2,A}(x', II(x')) dx' + 2 \left( \int_{-1/2}^{1/2} x_3^2 dx_3 \right) \int_{\Omega} \mathcal{L}_{2,A}(x', I(x'), II(x')) dx' \\ & = \int_{\Omega} \mathcal{Q}_{2,A}(x', I) + \frac{1}{12} \int_{\Omega} \mathcal{Q}_{2,A}(x', III) + \frac{1}{80} \int_{\Omega} \mathcal{Q}_{2,A}(x', II) + \frac{2}{12} \int_{\Omega} \mathcal{L}_{2,A}(x', I, II) dx' \\ & = \int_{\Omega} \mathcal{Q}_{2,A} \left( x', I + \frac{1}{12} III \right) dx' + \frac{1}{12} \int_{\Omega} \mathcal{Q}_{2,A}(x', III) dx' + \frac{1}{180} \int_{\Omega} \mathcal{Q}_{2,A}(x', II) dx' \\ & = \mathcal{I}_4(V, \mathbb{S}), \end{aligned}$$

by a direct calculation. This completes the proof of Theorem 3.1 in view of (3.30).  $\blacksquare$

#### 4. THE UPPER BOUND

We now complete the proof of  $\mathcal{I}_4$  being the  $\Gamma$ -limit of  $h^{-4} E^h$ , by proving that the lower bound (3.5) is optimal.

**Theorem 4.1.** *Let  $V \in W^{2,2}(\Omega, \mathbb{R}^3)$  and  $\mathbb{S} \in L^2(\Omega, \mathbb{R}_{sym}^{2 \times 2})$  satisfy:*

$$(4.1) \quad \begin{aligned} & \text{sym} \left( (\nabla y_0)^t \nabla V \right) = 0, \\ & \mathbb{S} \in \mathcal{S} := \text{cl}_{L^2} \left\{ \text{sym} \left( (\nabla y_0)^t \nabla w \right); w \in W^{1,2}(\Omega, \mathbb{R}^3) \right\}. \end{aligned}$$

Then there exists a sequence  $w^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$  such that assertions (i), (ii) and (iii) of Theorem 3.1 are satisfied with  $R^h = \text{Id}$  and  $c^h = 0$ , and:

$$(4.2) \quad \limsup_{h \rightarrow 0} \frac{1}{h^4} E^h(w^h) \leq \mathcal{I}_4(V, \mathbb{S}).$$

*Proof.* In the construction below, we will use the following notation. In view of (3.4), for every  $F_{2 \times 2} \in \mathbb{R}^{2 \times 2}$  one can write:

$$(4.3) \quad \mathcal{Q}_{2,A}(x', F_{2 \times 2}) = \min_{c \in \mathbb{R}^3} \left\{ \mathcal{Q}_3(A^{-1}(F_{2 \times 2}^* + \text{sym}(c \otimes e_3))A^{-1}) \right\},$$

where  $F_{2 \times 2}^*$  denotes the  $\mathbb{R}^{3 \times 3}$  matrix whose principal  $2 \times 2$  minor equals  $F_{2 \times 2}$ . We will denote by  $c(x', F_{2 \times 2})$  the unique minimizer in (4.3). Note that  $c(x', \cdot)$  is a linear function of  $F_{2 \times 2}$  and it depends only on its symmetric part ( $\text{sym } F_{2 \times 2}$ ).

1. Since  $\mathbb{S} \in \mathcal{S}$ , there exists a sequence  $w^h \in W^{1,2}(\Omega, \mathbb{R}^3)$  such that:

$$(4.4) \quad \text{sym}((\nabla y_0)^t \nabla(w^h + \frac{1}{24} \vec{d}_0)) \rightarrow \mathbb{S} \quad \text{in } L^2(\Omega, \mathbb{R}^{2 \times 2})$$

and without loss of generality we can assume that each  $w^h$  is smooth up to the boundary of  $\Omega$ , together with:

$$(4.5) \quad \lim_{h \rightarrow 0} \sqrt{h} \|w^h\|_{W^{2,\infty}} = 0.$$

Fix a small  $\epsilon_0 \in (0, 1)$  and let  $v^h \in W^{2,\infty}(\Omega, \mathbb{R}^3)$  be a sequence of Lipschitz deformations with the properties:

$$(4.6) \quad \begin{aligned} v^h &\rightarrow V \quad \text{in } W^{2,2}(\Omega, \mathbb{R}^3), \\ h \|v^h\|_{W^{2,\infty}} &\leq \epsilon_0, \\ \lim_{h \rightarrow 0} \frac{1}{h^2} |\{x' \in \Omega; v^h(x') \neq V(x')\}| &= 0. \end{aligned}$$

We refer to [49] and [29] for the construction of such truncated sequence  $v^h$ . Define now  $\vec{p}^h \in W^{1,\infty}(\Omega, \mathbb{R}^3)$  by:

$$(4.7) \quad \vec{p}^h = (Q_0^t)^{-1} \begin{bmatrix} -(\nabla v^h)^t \vec{b}_0 \\ 0 \end{bmatrix},$$

and also define the fields  $\vec{q}^h \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ ,  $\vec{k}_0$  smooth and  $\vec{r}^h \in L^\infty(\Omega, \mathbb{R}^3)$  such that:

$$\begin{aligned} Q_0^t \vec{q}^h &= \frac{1}{2} c(x', 2(\nabla y_0)^t \nabla w^h + (\nabla v^h)^t \nabla v^h) - \begin{bmatrix} (\nabla w^h)^t \vec{b}_0 \\ 0 \end{bmatrix} - \begin{bmatrix} (\nabla v^h)^t \vec{p}^h \\ \frac{1}{2} |\vec{p}^h|^2 \end{bmatrix}, \\ Q_0^t \vec{k}_0 &= c(x', (\nabla y_0)^t \nabla \vec{d}_0 + (\nabla \vec{b}_0)^t \nabla \vec{b}_0) - \begin{bmatrix} (\nabla \vec{b}_0)^t \vec{d}_0 \\ |\vec{d}_0|^2 \end{bmatrix}, \\ Q_0^t \vec{r}^h &= c(x', (\nabla y_0)^t \nabla \vec{p}^h + (\nabla v^h)^t \nabla \vec{b}_0) - \begin{bmatrix} (\nabla v^h)^t \vec{d}_0 \\ \langle \vec{p}^h, \vec{d}_0 \rangle \end{bmatrix}. \end{aligned}$$

Finally, let  $\vec{r}^h \in W^{1,\infty}(\Omega, \mathbb{R}^3)$  be such that:

$$(4.8) \quad \lim_{h \rightarrow 0} \|\vec{r}^h - \tilde{r}^h\|_{L^2} = 0, \quad \lim_{h \rightarrow 0} \sqrt{h} \|\vec{r}^h\|_{W^{1,\infty}} = 0.$$

It follows from the definition of the minimizing map  $c$ , that:

$$(4.9) \quad \begin{aligned} & \mathcal{Q}_3 \left( A^{-1} (2Q_0^t [\nabla w^h \mid \vec{q}^h] + [\nabla v^h \mid \vec{p}^h]^t [\nabla v^h \mid \vec{p}^h]) A^{-1} \right) \\ & \quad = \mathcal{Q}_{2,A} \left( x', 2(\nabla y_0)^t \nabla w^h + (\nabla v^h)^t \nabla v^h \right), \\ & \mathcal{Q}_3 \left( A^{-1} (Q_0^t [\nabla \vec{d}_0 \mid \vec{k}_0] + [\nabla \vec{b}_0 \mid \vec{d}_0]^t [\nabla \vec{b}_0 \mid \vec{d}_0]) A^{-1} \right) \\ & \quad = \mathcal{Q}_{2,A} \left( x', (\nabla y_0)^t \nabla \vec{d}_0 + (\nabla \vec{b}_0)^t \nabla \vec{b}_0 \right), \\ & \mathcal{Q}_3 \left( A^{-1} (2Q_0^t [\nabla \vec{p}^h \mid \vec{r}^h] + 2[\nabla v^h \mid \vec{p}^h]^t [\nabla \vec{b}_0 \mid \vec{d}_0]^t) A^{-1} \right) \\ & \quad = \mathcal{Q}_{2,A} \left( x', (\nabla y_0)^t \nabla \vec{p}^h + (\nabla v^h)^t \nabla \vec{b}_0 \right). \end{aligned}$$

Moreover, we have the following pointwise bounds:

$$(4.10) \quad \begin{aligned} |\vec{p}^h| & \leq C |\nabla v^h|, \\ |\nabla \vec{p}^h| & \leq C (|\nabla v^h| + |\nabla^2 v^h|), \\ |\vec{q}^h| & \leq C (|\nabla w^h| + |\nabla v^h|^2 + |\nabla v^h| |\vec{p}^h| + |\vec{p}^h|^2) \leq C (|\nabla w^h| + |\nabla v^h|^2), \\ |\nabla \vec{q}^h| & \leq C (|\nabla w^h| + |\nabla^2 w^h| + |\nabla^2 v^h| |\nabla v^h| + |\nabla v^h|^2). \end{aligned}$$

**2.** Consider the sequence  $u^h \in W^{1,\infty}(\Omega^h, \mathbb{R}^3)$  defined as:

$$\begin{aligned} u^h(x', x_3) & = y_0(x') + h v^h(x') + h^2 w^h(x') + x_3 \vec{b}_0(x') + \frac{x_3^2}{2} \vec{d}_0(x') \\ & \quad + \frac{x_3^3}{6} \vec{k}_0(x') + h x_3 \vec{p}^h(x') + h^2 x_3 \vec{q}^h(x') + \frac{h x_3^2}{2} \vec{r}^h(x'). \end{aligned}$$

For every  $(x', x_3) \in \Omega^1$  we write:

$$\nabla u^h(x', h x_3) = Q_0(x') + Z_1^h(x', x_3) + Z_2^h(x', x_3),$$

where:

$$\begin{aligned} Z_1^h(x', x_3) & = h [\nabla v^h \mid \vec{p}^h] + h^2 [\nabla w^h \mid \vec{q}^h] + h x_3 [\nabla \vec{b}_0 \mid \vec{d}_0] + \frac{h^2 x_3^2}{2} [\nabla \vec{d}_0 \mid \vec{k}_0] + h^2 x_3 [\nabla \vec{p}^h \mid \vec{r}^h], \\ Z_2^h(x', x_3) & = \frac{h^3 x_3^3}{6} [\nabla \vec{k}_0 \mid 0] + h^3 x_3 [\nabla \vec{q}^h \mid 0] + \frac{h^3 x_3}{2} [\nabla \vec{r}^h \mid 0]. \end{aligned}$$

Since  $Q_0 A^{-1} \in SO(3)$ , we get:

$$\nabla u^h A^{-1}(x', h x_3) = Q_0 A^{-1} \left( \text{Id}_3 + A^{-1} Q_0^t Z_1^h A^{-1} + A^{-1} Q_0^t Z_2^h A^{-1} \right)$$

and, in view of (4.6), (4.8) and (4.10), there follows for  $h$  sufficiently small:

$$\begin{aligned} & \|A^{-1}Q_0^t Z_1^h A^{-1} + A^{-1}Q_0^t Z_2^h A^{-1}\|_{L^\infty} \\ & \leq C \left( h \|\nabla v^h\|_{L^\infty} + h \|\vec{p}^h\|_{L^\infty} + h^2 \|\nabla w^h\|_{L^\infty} + h^2 \|\vec{q}^h\|_{L^\infty} + h \|\nabla \vec{b}_0\|_{L^\infty} + h \|\vec{d}_0\|_{L^\infty} \right. \\ & \quad + h^2 \|\nabla \vec{d}_0\|_{L^\infty} + h^2 \|\vec{k}_0\|_{L^\infty} + h^2 \|\nabla \vec{p}^h\|_{L^\infty} + h^2 \|\vec{r}^h\|_{L^\infty} + h^3 \|\nabla \vec{k}_0\|_{L^\infty} \\ & \quad \left. + h^3 \|\nabla \vec{q}^h\|_{L^\infty} + h^3 \|\nabla \vec{r}^h\|_{L^\infty} \right) \leq C\epsilon_0. \end{aligned}$$

By the left polar decomposition, there exists a further rotation  $R \in SO(3)$  such that:

$$\begin{aligned} R \nabla u^h A^{-1} &= \left( (\text{Id}_3 + A^{-1}Q_0^t Z_1^h A^{-1} + A^{-1}Q_0^t Z_2^h A^{-1})^t (\text{Id}_3 + A^{-1}Q_0^t Z_1^h A^{-1} + A^{-1}Q_0^t Z_2^h A^{-1}) \right)^{1/2} \\ &= \left( \text{Id}_3 + 2A^{-1} \text{sym}(Q_0^t Z_1^h) A^{-1} + A^{-1} (Z_1^h)^t Z_1^h A^{-1} + \mathcal{O}(|Z_2^h|) \right)^{1/2} \\ &= \text{Id}_3 + A^{-1} \text{sym}(Q_0^t Z_1^h) A^{-1} + \frac{1}{2} A^{-1} (Z_1^h)^t Z_1^h A^{-1} \\ & \quad + \mathcal{O}(|\text{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h|) + \mathcal{O}(|Z_2^h|). \end{aligned}$$

**3.** Consider the set:

$$\Omega_h = \{(x', x_3) \in \Omega; v^h(x') = V(x')\}.$$

Note that on  $\Omega_h$  we have:  $\vec{p}^h = \vec{p}$  and  $Q_0^t [\nabla v^h \mid \vec{p}^h] \in so(3)$ . Using Taylor's expansion, it follows that:

$$\frac{1}{h^4} \int_{\Omega_h} W(\nabla u(x', hx_3) A^{-1}) dx = \frac{1}{2h^4} \int_{\Omega_h} \mathcal{Q}_3 \left( A^{-1} (Q_0^t Z_1^h + \frac{1}{2} (Z_1^h)^t Z_1^h) A^{-1} \right) dx + \mathcal{E}_1^h,$$

where the error term  $\mathcal{E}_1^h$  can be estimated by:

$$|\mathcal{E}_1^h| \leq \frac{C}{h^4} \int_{\Omega_h} |2 \text{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h|^3 + |Z_2^h|^2 + |2 \text{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h| |Z_2^h| dx.$$

Now on  $\Omega_h$  we also have, by (4.10):

$$\begin{aligned} |2 \text{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h| &\leq C \left( h^2 |\nabla w^h| + h^2 |\nabla v^h|^2 + h^2 + h^2 |\nabla v^h| + h^2 |\nabla^2 v^h| + h^2 |\vec{r}^h| \right), \\ |Z_2^h| &\leq Ch^3 (1 + |\nabla \vec{q}^h| + |\nabla \vec{r}^h|) \\ &\leq Ch^3 \left( 1 + |\nabla w^h| + |\nabla^2 w^h| + |\nabla^2 v^h| |\nabla v^h| + |\nabla v^h|^2 + |\nabla \vec{r}^h| \right), \end{aligned}$$

and therefore, in view of (4.5), (4.8), (4.6) and  $V \in W^{2,2}$ :

$$\begin{aligned}
& \frac{1}{h^4} \int_{\Omega_h} |2 \operatorname{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h|^3 \, dx \\
& \leq \frac{C}{h^4} \int_{\Omega_h} h^6 |\nabla w^h|^3 + h^6 |\nabla v^h|^6 + h^6 + h^6 |\nabla v^h|^3 + h^6 |\nabla^2 v^h|^3 + h^6 |\vec{r}^h|^3 \, dx \\
& \leq \frac{C}{h^4} \left( h^2 \|\nabla w^h\|_{L^\infty} (h^2 \|\nabla w^h\|_{L^2})^2 + h^6 \|\nabla V\|_{L^6}^6 + h^6 |\Omega| + h^6 \|\nabla V\|_{L^3}^3 \right. \\
& \quad \left. + h^6 \|\nabla^2 v^h\|_{L^\infty} \|\nabla^2 V\|_{L^2}^2 + (\sqrt{h} \|\vec{r}^h\|_{L^\infty})^3 h^{9/2} \right) \rightarrow 0 \quad \text{as } h \rightarrow 0.
\end{aligned}$$

Analogously:

$$\begin{aligned}
& \frac{1}{h^4} \int_{\Omega_h} |Z_2^h|^2 \, dx \leq \frac{C}{h^4} \int_{\Omega_h} h^5 + (h \|\nabla v^h\|_{L^\infty})^2 h^4 |\nabla^2 v^h|^2 + h^6 |\nabla v^h|^4 \, dx \rightarrow 0 \quad \text{as } h \rightarrow 0, \\
& \frac{1}{h^4} \int_{\Omega_h} |2 \operatorname{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h| |Z_2^h| \, dx \\
& \leq \frac{C}{h^4} \int_{\Omega_h} \left( h^5 |\nabla w^h|^2 + h^5 |\nabla^2 w^h|^2 + h^5 |\nabla v^h|^2 + h^5 + h^5 |\nabla V| + h^5 |\nabla^2 V| + h^5 |\vec{r}^h| \right. \\
& \quad \left. + h^5 |\nabla V|^2 |\nabla^2 V| + h^5 |\nabla V| |\nabla^2 V|^2 \right) \, dx \leq C \epsilon_0.
\end{aligned}$$

We therefore conclude that:

$$(4.11) \quad \limsup_{h \rightarrow 0} |\mathcal{E}_1^h| \leq C \epsilon_0.$$

4. Consider now the error due to integrating on the residual subdomain:

$$\mathcal{E}_2^h = \frac{1}{h^4} \int_{\Omega^1 \setminus \Omega_h} W \left( \nabla u^h A^{-1}(x', hx_3) \right) \, dx \leq \frac{C}{h^4} \int_{\Omega^1 \setminus \Omega_h} |2 \operatorname{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h|^2 + |Z_2^h|^2 \, dx.$$

Observe that, since the matrix field  $[\nabla v^h \mid \vec{p}^h]$  is Lipschitz, we have:

$$\begin{aligned}
& \left| \operatorname{sym}(Q_0^t [\nabla v^h \mid \vec{p}^h])(x') \right| \leq C \|\nabla v^h\|_{W^{1,\infty}} \operatorname{dist}(x', \{v^h = V\}) \\
& \leq \frac{C \epsilon_0}{h} \operatorname{dist}(x', \{v^h = V\}) \rightarrow 0 \quad \text{in } L^\infty(\Omega).
\end{aligned}$$

The last inequality above follows by a standard argument by contradiction. If there was a sequence  $x^h \in \Omega$  such that  $\operatorname{dist}(x^h, \{v^h = V\}) \geq ch$ , this would imply that:  $\{x'; v^h(x') \neq$

$V(x')\} \geq |\Omega \cap B(x^h, ch)| \geq ch^2$ , contradicting (4.6). Consequently, by (4.5), (4.8), (4.6):

$$\begin{aligned}
|\mathcal{E}_2^h| &\leq \frac{C}{h^4} \int_{\Omega^1 \setminus \Omega_h} h^2 |\text{sym}(Q_0^t[\nabla v^h | \vec{p}^h])| \, dx \\
&\quad + \frac{C}{h^4} \int_{\Omega^1 \setminus \Omega_h} h^4 |\nabla w^h|^2 + h^4 |\nabla v^h|^4 + h^4 |\nabla^2 v^h|^2 + h^4 |\vec{r}^h|^2 + h^4 + h^6 |\nabla v^h|^4 \, dx \\
&\leq \frac{C}{h^4} o(h^2) |\Omega^1 \setminus \Omega_h| + \frac{C}{h^4} \sqrt{h} \|\nabla w^h\|_{L^\infty} h^{7/2} |\Omega^1 \setminus \mathcal{U}^h|^{1/2} \|\nabla w^h\|_{L^2} \\
&\quad + C |\Omega^1 \setminus \mathcal{U}^h| \|\nabla v^h\|_{L^8}^4 + Ch \|\nabla^2 v^h\|_{L^\infty} \frac{1}{h} \|\nabla^2 v^h\|_{L^2} |\Omega^1 \setminus \mathcal{U}^h|^{1/2} + \frac{1}{h} (\sqrt{h} \|\vec{r}^h\|_{L^\infty})^2 |\Omega^1 \setminus \mathcal{U}^h| \\
&\quad + (h \|\nabla^2 v^h\|_{L^\infty})^2 \|\nabla v^h\|_{L^4}^2 |\Omega^1 \setminus \mathcal{U}^h|^{1/2} \quad \rightarrow 0 \quad \text{as } h \rightarrow 0.
\end{aligned}$$

Thus:

$$\limsup_{h \rightarrow 0} \frac{1}{h^4} E^h(u^h) \leq \limsup_{h \rightarrow 0} \frac{1}{h^4} \int_{\Omega_h} \frac{1}{2} \mathcal{Q}_3 \left( A^{-1} (\text{sym}(Q_0^t Z_1^h) + \frac{1}{2} (Z_1^h)^t Z_1^h) A^{-1} \right) \, dx + C\epsilon_0.$$

Now on  $\Omega_h$  we have:

$$\begin{aligned}
&2 \text{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h \\
&= 2h^2 \left( \text{sym}(Q_0^t[\nabla w^h | \vec{q}^h]) + \frac{x_3^2}{2} \text{sym}(Q_0^t[\nabla \vec{d}_0 | \vec{k}_0]) + x_3 \text{sym}(Q_0^t[\nabla \vec{p} | \vec{r}^h]) \right) \\
&\quad + h^2 \left( [\nabla V | \vec{p}]^t [\nabla V | \vec{p}] + x_3^2 [\nabla \vec{b}_0 | \vec{d}_0]^t [\nabla \vec{b}_0 | \vec{d}_0] + 2x_3 \text{sym}([\nabla V | \vec{p}]^t [\nabla \vec{b}_0 | \vec{d}_0]) \right) + \mathcal{E}^h,
\end{aligned}$$

where the present error  $\mathcal{E}^h$  is estimated by:

$$\begin{aligned}
|\mathcal{E}^h| &\leq C \left( h^3 |\nabla V| |\nabla w^h| + h^3 |\nabla V| + h^3 |\nabla V| |\nabla \vec{p}| + h^3 |\nabla V| |\vec{r}^h| \right. \\
&\quad \left. + h^4 |\nabla w^h|^2 + h^3 |\nabla w^h| + h^4 |\nabla w^h| |\nabla \vec{p}| + h^4 |\nabla w^h| |\vec{r}^h| + h^3 \right. \\
(4.12) \quad &\quad \left. + h^3 |\nabla \vec{p}| + h^3 |\vec{r}^h| + h^4 + h^4 |\nabla \vec{p}| + h^4 |\vec{r}^h| + h^4 |\nabla \vec{p}|^2 + h^4 |\vec{r}^h|^2 \right) \\
&\leq Ch^2 \left( o(1) \sqrt{h} |\nabla V| + \epsilon_0^2 |\nabla^2 V| + o(1) \sqrt{h} + o(1) \epsilon_0 \sqrt{h} \right).
\end{aligned}$$

Consequently:

$$\begin{aligned}
 & \limsup_{h \rightarrow 0} \frac{1}{h^4} E^h(u^h) \\
 & \leq \limsup_{h \rightarrow 0} \frac{1}{2} \int_{\Omega_h} \mathcal{Q}_3 \left( A^{-1}(\text{sym}(Q_0^t[\nabla w^h \mid \vec{q}^h]) + \frac{1}{2} x_3^2 \text{sym}(Q_0^t[\nabla \vec{d}_0 \mid \vec{k}_0]) \right. \\
 & \quad \left. + x_3 \text{sym}(Q_0^t[\nabla \vec{p} \mid \vec{r}^h]) + \frac{1}{2} [\nabla V \mid \vec{p}]^t [\nabla V \mid \vec{p}] \right. \\
 & \quad \left. + \frac{1}{2} x_3^2 [\nabla \vec{b}_0 \mid \vec{d}_0]^t [\nabla \vec{b}_0 \mid \vec{d}_0] + x_3 \text{sym}([\nabla V \mid \vec{p}]^t [\nabla \vec{b}_0 \mid \vec{d}_0]) A^{-1} \right) dx + C\epsilon_0 \\
 & = \limsup_{h \rightarrow 0} \frac{1}{2} \int_{\Omega_h} \mathcal{Q}_3 \left( A^{-1}(\text{sym}(Q_0^t[\nabla w^h \mid \vec{q}^h]) + \frac{1}{2} [\nabla V \mid \vec{p}]^t [\nabla V \mid \vec{p}] \right. \\
 & \quad \left. + \frac{1}{2} x_3^2 \text{sym}(Q_0^t[\nabla \vec{d}_0 \mid \vec{k}_0]) + \frac{1}{2} x_3^2 [\nabla \vec{b}_0 \mid \vec{d}_0]^t [\nabla \vec{b}_0 \mid \vec{d}_0]) A^{-1} \right) \\
 & \quad + \mathcal{Q}_3 \left( A^{-1}(x_3 \text{sym}(Q_0^t[\nabla \vec{p} \mid \vec{r}^h]) + x_3 \text{sym}([\nabla V \mid \vec{p}]^t [\nabla \vec{b}_0 \mid \vec{d}_0])) A^{-1} \right) dx + C\epsilon_0.
 \end{aligned}$$

Denoting by:

$$I_1(x') = \text{sym}((\nabla y_0)^t \nabla w^h) + \frac{1}{2} (\nabla v^h)^t \nabla v^h, \quad I_2(x') = \frac{1}{2} \text{sym}((\nabla y_0)^t \nabla \vec{d}_0) + (\nabla \vec{b}_0)^t \nabla \vec{b}_0,$$

we have:

$$\begin{aligned}
 & \mathcal{Q}_3 \left( A^{-1} (I_1^*(x') + \text{sym}(c(x', I_1(x')) \otimes e_3) + x_3^2 I_2^*(x') + x_3^2 \text{sym}(c(x', I_2(x')) \otimes e_3)) A^{-1} \right) \\
 & = \mathcal{Q}_3 \left( A^{-1} ((I_1(x') + x_3^2 I_2(x'))^* + \text{sym}(c(x', I_1(x') + x_3^2 I_2(x')) \otimes e_3)) A^{-1} \right) \\
 & = \mathcal{Q}_{2,A} \left( (I_1(x') + x_3^2 I_2(x')) \right),
 \end{aligned}$$

where we have used the definition and linearity of the minimizing map  $c$ . Recalling the definitions of the curvature forms  $I(x')$ ,  $II(x')$  and  $III(x')$  in (3.31), observe that  $I_2(x') = 2II(x')$  and that  $\frac{1}{2}I_1$  converges to  $I$  in  $L^2$  by (4.4). Hence:

$$\begin{aligned}
 \limsup_{h \rightarrow 0} \frac{1}{h^4} E^h(u^h) & \leq \frac{1}{2} \int_{\Omega^1} \mathcal{Q}_{2,A} \left( I(x') + x_3^2 II(x') \right) dx + \frac{1}{2} \int_{\Omega^1} \mathcal{Q}_{2,A} \left( x_3 III(x') \right) dx + C\epsilon_0 \\
 & = \mathcal{I}_4(V, \mathbb{S}) + C\epsilon_0.
 \end{aligned}$$

Since  $\epsilon_0 > 0$  was arbitrary, the proof is achieved by a diagonal argument.  $\blacksquare$

## 5. DISCUSSION OF THE VON KÁRMÁN-LIKE FUNCTIONAL (3.5)

Theorems 3.1 and 4.1 imply, as usual in the setting of  $\Gamma$ -convergence, convergence of almost-minimizers:

**Corollary 5.1.** *If  $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$  is a minimizing sequence to  $h^{-4}E^h$ , that is:*

$$\lim_{h \rightarrow 0} \left( \frac{1}{h^4} E^h(u^h) - \inf \frac{1}{h^4} E^h \right) = 0,$$

then the appropriate renormalizations  $y^h = (\bar{R}^h)^t u^h(x', hx_3) - c^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$  obey the convergence statements of Theorem 3.1 (i), (ii), (iii). The convergence of  $h^{-1} \text{sym}((\nabla y_0)^t \nabla V^h)$  to  $\mathbb{S}$  in (iii) is strong in  $L^2(\Omega)$ . Moreover, any limit  $(V, \mathbb{S})$  minimizes the functional  $\mathcal{I}_4$ .

*Proof.* The proof is standard. The only possibly nontrivial part is the strong convergence of the scaled tangential strains in (iii), which can be deduced as in Theorem 2.5 in [41].  $\blacksquare$

Let us now compare the functional (3.5) with the von-Kármán theory of thin shells that has been derived in [41]. Recall that when  $S$  is a smooth 2d surface in  $\mathbb{R}^3$ , the  $\Gamma$ -limit of the scaled elastic energies  $h^{-4} (\frac{1}{h} \int_{S^h} W(\nabla u^h))$  on thin shells  $S^h$  with mid-surface  $S$ , is:

$$(5.1) \quad \tilde{\mathcal{I}}_{4,S}(\tilde{V}, \tilde{\mathbb{S}}) = \frac{1}{2} \int_S \mathcal{Q}_2(\tilde{\mathbb{S}} - \frac{1}{2}(\tilde{A}^2)_{tan}) dy + \frac{1}{24} \int_S \mathcal{Q}_2((\nabla(\tilde{A}\tilde{N}) - \tilde{A}\Pi)_{tan}) dy.$$

Above,  $\Pi$  stands for the shape operator of  $S$  and  $\tilde{N}$  is the unit normal vector to  $S$ . The subscript *tan* means taking the restriction of a quadratic form (or an operator) to the tangent space  $T_y S$ . The arguments of  $\tilde{\mathcal{I}}_{4,S}$  are:

(i) First order infinitesimal isometries  $\tilde{V}$  on  $S$ . These are vector fields  $\tilde{V} \in W^{2,2}(S, \mathbb{R}^3)$  with skew symmetric covariant derivative, so that one may define:

$$(5.2) \quad \tilde{A} \in W^{1,2}(S, so(3)) \quad \text{with} \quad \tilde{A}(y)\tau = \partial_\tau \tilde{V}(y) \quad \forall y \in S \quad \forall \tau \in T_y S;$$

(ii) Finite strains  $\tilde{\mathbb{S}}$  on  $S$ . These are tensor fields  $\tilde{\mathbb{S}} \in L^2(S, \mathbb{R}_{sym}^{2 \times 2})$  such that:

$$(5.3) \quad \tilde{\mathbb{S}} = L^2 - \lim_{h \rightarrow 0} \text{sym}(\nabla \tilde{w}_h)_{tan} \quad \text{for some} \quad \tilde{w}_h \in W^{1,2}(S, \mathbb{R}^3).$$

In the present setting, denote  $S = y_0(\Omega)$  and observe that the 1-1 correspondence between  $\tilde{V}$  in (5.2) and  $V$  in (3.3) is given by the change of variables  $V = \tilde{V} \circ y_0$ . The skew-symmetric tensor field  $\tilde{A}$  on  $T_y S$  is then uniquely given by:

$$(5.4) \quad \tilde{A}(y_0(x')) \partial_e y_0 = \partial_e V(x') \quad \text{and} \quad \tilde{A} \tilde{b}_0 = \tilde{p} \quad \forall e \in \mathbb{R}^2,$$

and the finite strains in (5.3) are related to (4.1) by:

$$\langle \tilde{\mathbb{S}}(y_0(x')) \partial_e y_0, \partial_e y_0 \rangle = \langle \mathbb{S}(x') e, e \rangle \quad \forall e \in \mathbb{R}^2.$$

Recall that the first of the two terms in the functional (5.1) measures the difference of order  $h^2$ , between the (Euclidean) metric on  $S$  and the metric of the deformed surface. Indeed, the amount of stretching of  $S$  in the direction  $\tau \in T_y S$ , induced by the deformation  $u_h = id + h\tilde{V} + h^2\tilde{w}$ , has the expansion:

$$|\partial_\tau u_h|^2 - |\tau|^2 = h^2 \left( 2\langle \partial_\tau \tilde{w}, \tau \rangle + |\partial_\tau \tilde{V}|^2 \right) + \mathcal{O}(h^3) = 2h^2 \left( \langle (\text{sym} \nabla \tilde{w}) \tau, \tau \rangle - \frac{1}{2} \langle \tilde{A}^2 \tau, \tau \rangle \right) + \mathcal{O}(h^3).$$

The leading order quantity in the right hand side above coincides with:

$$\langle (\text{sym} \nabla w) e, e \rangle + \frac{1}{2} \langle \partial_e V, \partial_e V \rangle = \left\langle \left( \text{sym} \nabla w + \frac{1}{2} (\nabla V)^t \nabla V \right) e, e \right\rangle,$$

where we write  $\tau = \partial_e y_0$ , for any  $e \in \mathbb{R}^2$ . This is precisely the argument of the first term in  $\mathcal{I}_4(V, \mathbb{S})$ , modulo the correction  $(\nabla \tilde{b}_0)^t \nabla \tilde{b}_0$  (equal to the third fundamental form on  $S$  in case



$\vec{b}_0 = \vec{N}$ ), due to the incompatibility of the ambient Euclidean metric of  $S^h$  with the given prestrain  $G$  on  $\Omega^h$ .

The second term in (5.1) measures the difference of order  $h$ , between the shape operator  $\Pi$  on  $S$  and the shape operator  $\Pi^h$  on the deformed surface  $(id + h\tilde{V})(S)$  whose unit normal we denote by  $\vec{N}^h$ . The amount of bending of  $S$ , in the direction  $\tau \in T_y S$ , induced by the deformation  $u_h = id + h\tilde{V}$  can be estimated by [41]:

$$\begin{aligned} (\text{Id} + h\tilde{A})^{-1}\Pi^h(\text{Id} + h\tilde{A})\tau - \Pi\tau &= (\text{Id} + h\tilde{A})^{-1}(\partial_\tau\vec{N}^h + \mathcal{O}(h^2))\tau - \Pi\tau \\ &= (\text{Id} + h\tilde{A})^{-1}\left((\text{Id} + h\tilde{A})\Pi\tau + h(\partial_\tau A)\vec{N} + \mathcal{O}(h^2)\right) - \Pi\tau \\ &= (\text{Id} - h\tilde{A})h(\partial_\tau\tilde{A})\vec{N} + \mathcal{O}(h^2) \\ &= h(\partial_\tau\tilde{A})\vec{N} + \mathcal{O}(h^2) = h\left(\nabla(\tilde{A}\vec{N}) - \tilde{A}\Pi\right) + \mathcal{O}(h^2). \end{aligned}$$

The leading order term in this expansion coincides with the term  $(\nabla y_0)^t \nabla \vec{p} + (\nabla V)^t \nabla \vec{b}_0$  when  $\vec{b}_0 = \vec{N}$ , because in view of (5.4):

$$\begin{aligned} \langle (\partial_\tau \tilde{A})\vec{b}_0, \tau \rangle &= \langle (\partial_e(\tilde{A}\vec{b}_0), \partial_e y_0) - \langle (\tilde{A}\partial_e \vec{b}_0, \partial_e y_0) = \langle (\partial_e \vec{p}, \partial_e y_0) + \langle (\partial_e \vec{b}_0, \tilde{A}\partial_e y_0) \\ &= \langle (\nabla y_0)^t \nabla \vec{p}, e, e \rangle - \langle (\nabla V)^t \nabla \vec{b}_0, e, e \rangle, \end{aligned}$$

where we again wrote  $\tau = \partial_e y_0 \in T_{y_0(x')} S$ , for any  $e \in \mathbb{R}^2$ . This is precisely the argument in the second term in  $\mathcal{I}_4(V, \mathbb{S})$ .

In the next section we identify the geometric significance of the last term in (3.5).

## 6. THE SCALING OPTIMALITY

In this section, we prove the following crucial result:

**Theorem 6.1.** *Assume (1.5), together with:*

$$(6.1) \quad \text{sym}\left((\nabla y_0)^t \nabla \vec{d}_0\right) + (\nabla \vec{b}_0)^t \nabla \vec{b}_0 = 0,$$

where  $y_0$ ,  $\vec{b}_0$  and  $\vec{d}_0$  are defined in (1.7), (1.8), (1.10). Then the metric  $G$  is flat, i.e.  $\text{Riem}(G) \equiv 0$  in  $\Omega^h$ . Equivalently:  $\min E^h = 0$  for all  $h$ .

Observe that when  $\vec{b}_0 = \vec{N}$ , then by (1.10) there must be  $\vec{d}_0 = 0$ , and hence condition (6.1) becomes:  $\vec{N} \equiv \text{const}$ . This is consistent with our previous observation that when  $Ge_3 = e_3$ , then already condition (1.7) is enough to conclude immersability of  $G$  in  $\mathbb{R}^3$ . Equivalently,  $G_{2 \times 2}$  is immersible in  $\mathbb{R}^2$ , so that indeed  $y_0(\Omega)$  must be planar in this case.

Towards a proof of Theorem 6.1, recall that  $\text{Riem}(G)$  is the covariant Riemann curvature tensor, whose components  $R_{\dots}$  and their relation to the contravariant curvatures in  $R_{\dots}$  are:

$$\begin{aligned} R_{iklm} &= \frac{1}{2}(\partial_{kl}G_{im} + \partial_{im}G_{kl} - \partial_{km}G_{il} - \partial_{il}G_{km}) + G_{np}(\Gamma_{kl}^n \Gamma_{im}^p - \Gamma_{km}^n \Gamma_{il}^p) \\ R_{iklm} &= G_{is}R_{klm}^s, \end{aligned}$$

where we used the Einstein summation convention and the Christoffel symbols:

$$(6.2) \quad \Gamma_{kl}^n = \frac{1}{2} G^{ns} (\partial_k G_{sl} + \partial_l G_{sk} - \partial_s G_{kl}).$$

In view of the symmetries in  $Riem(G)$  of a 3-dimensional metric  $G$ , its flatness is equivalent to the vanishing of the following curvatures:

$$R_{1212}, R_{1213}, R_{1223}, R_{1313}, R_{1323}, R_{2323}.$$

The proof of Theorem 6.1 is a consequence of the following observation.

**Theorem 6.2.** *Assume (1.5) and let  $y_0$ ,  $\vec{b}_0$  and  $\vec{d}_0$  be defined as in (1.7), (1.10). Then:*

$$(6.3) \quad \text{sym}((\nabla y_0)^t \nabla \vec{d}_0) + (\nabla \vec{b}_0)^t \nabla \vec{b}_0 = \begin{bmatrix} R_{1313} & R_{1323} \\ R_{1323} & R_{2323} \end{bmatrix}.$$

*Proof. 1.* We have:

$$\begin{aligned} R_{1313} &= -\frac{1}{2} \partial_{11} G_{33} + G_{np} (\Gamma_{13}^n \Gamma_{13}^p - \Gamma_{11}^n \Gamma_{33}^p), \\ R_{2323} &= -\frac{1}{2} \partial_{22} G_{33} + G_{np} (\Gamma_{23}^n \Gamma_{23}^p - \Gamma_{22}^n \Gamma_{33}^p), \\ R_{1323} &= -\frac{1}{2} \partial_{12} G_{33} + G_{np} (\Gamma_{13}^n \Gamma_{23}^p - \Gamma_{12}^n \Gamma_{33}^p). \end{aligned}$$

On the other hand, in view of (1.10):

$$\begin{aligned} \forall i, j = 1, 2 \quad \frac{1}{2} (\langle \partial_i y_0, \partial_j \vec{d}_0 \rangle + \langle \partial_j y_0, \partial_i \vec{d}_0 \rangle) &= \frac{1}{2} (\partial_j \langle \partial_i y_0, \vec{d}_0 \rangle + \partial_i \langle \partial_j y_0, \vec{d}_0 \rangle) - \langle \partial_{ij} y_0, \vec{d}_0 \rangle \\ &= -\frac{1}{2} \partial_{ij} G_{33} - \langle \partial_{ij} y_0, \vec{d}_0 \rangle \end{aligned}$$

because:  $\partial_j \langle \partial_i y_0, \vec{d}_0 \rangle + \partial_i \langle \partial_j y_0, \vec{d}_0 \rangle = -\partial_{ij} |\vec{b}_0|^2 = -\partial_{ij} G_{33}$ . Consequently, the formula (6.3) will follow, if we establish:

$$(6.4) \quad \forall i, j = 1, 2 \quad \langle \partial_{ij} y_0, \vec{d}_0 \rangle = G_{np} \Gamma_{ij}^n \Gamma_{33}^p \quad \text{and} \quad \langle \partial_i \vec{b}_0, \partial_j \vec{b}_0 \rangle = G_{np} \Gamma_{i3}^n \Gamma_{j3}^p.$$

**2.** Before proving (6.4) we gather some useful formulas. Note that  $\partial_i G = 2 \text{sym}((\partial_i Q)^t Q)$  for  $i = 1, 2$ . Therefore, by direct inspection:

$$(6.5) \quad \forall i, j, k = 1, 2 \quad \langle \partial_{ij} y_0, \partial_k y_0 \rangle = \frac{1}{2} (\partial_i G_{kj} + \partial_j G_{ki} - \partial_k G_{ij}).$$

Also, recall that condition (1.7) is equivalent to (see [6], proof of Theorem 5.3, formula (5.8)):

$$(6.6) \quad \forall i, j = 1, 2 \quad \langle \partial_{ij} y_0, \vec{b}_0 \rangle = \frac{1}{2} (\partial_i G_{j3} + \partial_j G_{i3}).$$

Therefore, for all  $i, j = 1, 2$ :

$$(6.7) \quad \begin{aligned} \langle \partial_j y_0, \partial_i \vec{b}_0 \rangle &= \partial_i \langle \partial_j y_0, \vec{b}_0 \rangle - \langle \partial_{ij} y_0, \vec{b}_0 \rangle = \frac{1}{2} (\partial_i G_{j3} - \partial_j G_{i3}), \\ \langle \partial_i \vec{b}_0, \vec{b}_0 \rangle &= \frac{1}{2} \partial_i G_{33}. \end{aligned}$$

We now express  $\partial_{ij}y_0$ ,  $\partial_i\vec{b}_0$  and  $\vec{d}_0$  in the basis  $\{\partial_1y_0, \partial_2y_0, \vec{b}_0\}$ , writing:

$$(6.8) \quad \begin{aligned} \partial_{ij}y_0 &= \alpha_{ij}^1\partial_1y_0 + \alpha_{ij}^2\partial_2y_0 + \alpha_{ij}^3\vec{b}_0, \\ \partial_i\vec{b}_0 &= \beta_i^1\partial_1y_0 + \beta_i^2\partial_2y_0 + \beta_i^3\vec{b}_0, \\ \vec{d}_0 &= \gamma^1\partial_1y_0 + \gamma^2\partial_2y_0 + \gamma^3\vec{b}_0. \end{aligned}$$

By (6.5), (6.6), (6.7) and (1.10), it follows that:

$$\begin{aligned} G(\alpha_{ij}^1, \alpha_{ij}^2, \alpha_{ij}^3)^t &= GQ_0^{-1}\partial_{ij}y_0 = Q_0^t\partial_{ij}y_0 \\ &= \frac{1}{2}(\partial_iG_{1j} + \partial_jG_{1i} - \partial_1G_{ij}, \partial_iG_{2j} + \partial_jG_{2i} - \partial_2G_{ij}, \partial_iG_{3j} + \partial_jG_{3i}), \\ G(\beta_i^1, \beta_i^2, \beta_i^3)^t &= GQ_0^{-1}\partial_i\vec{b}_0 = Q_0^t\partial_i\vec{b}_0 = Q_0^t\partial_i\vec{b}_0 \\ &= \frac{1}{2}(\partial_iG_{13} - \partial_1G_{i3}, \partial_iG_{23} - \partial_2G_{i3}, \partial_iG_{33})^t, \\ G(\gamma^1, \gamma^2, \gamma^3)^t &= GQ_0^{-1}\vec{d}_0 = Q_0^t\vec{d}_0 = -\frac{1}{2}(\partial_1G_{33}, \partial_2G_{33}, 0)^t. \end{aligned}$$

In view of (6.2) we then obtain, for all  $i, j = 1, 2$ :

$$(\alpha_{ij}^1, \alpha_{ij}^2, \alpha_{ij}^3) = (\Gamma_{ij}^1, \Gamma_{ij}^2, \Gamma_{ij}^3), \quad (\beta_i^1, \beta_i^2, \beta_i^3) = (\Gamma_{i3}^1, \Gamma_{i3}^2, \Gamma_{i3}^3), \quad (\gamma^1, \gamma^2, \gamma^3)^t = (\Gamma_{33}^1, \Gamma_{33}^2, \Gamma_{33}^3),$$

so that (6.8) becomes:

$$(6.9) \quad \begin{aligned} \partial_{ij}y_0 &= \Gamma_{ij}^1\partial_1y_0 + \Gamma_{ij}^2\partial_2y_0 + \Gamma_{ij}^3\vec{b}_0, \\ \partial_i\vec{b}_0 &= \Gamma_{i3}^1\partial_1y_0 + \Gamma_{i3}^2\partial_2y_0 + \Gamma_{i3}^3\vec{b}_0, \\ \vec{d}_0 &= \Gamma_{33}^1\partial_1y_0 + \Gamma_{33}^2\partial_2y_0 + \Gamma_{33}^3\vec{b}_0. \end{aligned}$$

**3.** We now prove (6.4). Keeping in mind that  $Q_0^T Q_0 = G$ , the scalar products of expressions in (6.9) are:

$$\begin{aligned} \langle \partial_{ij}y_0, \vec{d}_0 \rangle &= \langle \Gamma_{ij}^n \partial_n y_0 + \Gamma_{ij}^3 \vec{b}_0, \Gamma_{33}^p \partial_p y_0 + \Gamma_{33}^3 \vec{b}_0 \rangle = G_{np} \Gamma_{ij}^n \Gamma_{33}^p, \\ \langle \partial_i \vec{b}_0, \partial_j \vec{b}_0 \rangle &= \langle \Gamma_{i3}^n \partial_n y_0 + \Gamma_{i3}^3 \vec{b}_0, \Gamma_{j3}^p \partial_p y_0 + \Gamma_{j3}^3 \vec{b}_0 \rangle = G_{np} \Gamma_{i3}^n \Gamma_{j3}^p. \end{aligned}$$

exactly as claimed in (6.4). This ends the proof of Theorem 6.2 and also of Theorem 6.1.  $\blacksquare$

## 7. TWO EXAMPLES

In this section we compute the energy  $\mathcal{I}_4(V, \mathbb{S})$  in the two particular cases of interest:

$$G(x', x_3) = \text{diag}(1, 1, \lambda(x')) \quad \text{and} \quad G(x', x_3) = \lambda(x')\text{Id}_3.$$

Let  $\vec{p}$  be as in the definition (3.6). Writing:  $\vec{p} = \alpha^1\partial_1y_0 + \alpha^2\partial_2y_0 + \alpha^3\vec{b}_0$ , we obtain:

$$G(\alpha^1, \alpha^2, \alpha^3)^t = -(\langle \partial_1 V, \vec{b}_0 \rangle, \langle \partial_2 V, \vec{b}_0 \rangle, 0)^t.$$

Consequently:

$$(7.1) \quad \vec{p} = -G^{1i} \langle \partial_i V, \vec{b}_0 \rangle \partial_1 y_0 - G^{2i} \langle \partial_i V, \vec{b}_0 \rangle \partial_2 y_0 - G^{3i} \langle \partial_i V, \vec{b}_0 \rangle \vec{b}_0.$$

**Lemma 7.1.** *Let  $\lambda : \bar{\Omega} \rightarrow \mathbb{R}$  be smooth and strictly positive. Consider the metric of the form:  $G(x', x_3) = \text{diag}(1, 1, \lambda(x'))$ . Then:*

(i)  *$G$  is immersible in  $\mathbb{R}^3$  if and only if:*

$$M_\lambda = \nabla^2 \lambda - \frac{1}{2\lambda} \nabla \lambda \otimes \nabla \lambda \equiv 0 \quad \text{in } \Omega,$$

*while the condition  $M_\lambda \neq 0$  is equivalent to:  $ch^4 \leq \inf E^h \leq Ch^4$ .*

(ii) *The  $\Gamma$ -limit energy functional  $\mathcal{I}_4$  in (3.5) becomes:*

$$\begin{aligned} \forall w \in W^{1,2}(\Omega, \mathbb{R}^2) \quad \forall v \in W^{2,2}(\Omega, \mathbb{R}) \\ \mathcal{I}_4(v, w) = \frac{1}{2} \int_{\Omega} \mathcal{Q}_2(\text{sym} \nabla w + \frac{1}{2} \nabla v \otimes \nabla v + \frac{1}{96\lambda} \nabla \lambda \otimes \nabla \lambda) \, dx' \\ + \frac{1}{24} \int_{\Omega} \mathcal{Q}_2(\sqrt{\lambda} \nabla^2 v) + \frac{1}{5760} \int_{\Omega} \mathcal{Q}_2(M_\lambda) \, dx', \end{aligned}$$

*where  $\mathcal{Q}_2$  is independent of  $x'$  and it is defined by  $\mathcal{Q}_{2,Id}$  in (3.4).*

*Proof.* Part (i) of the assertion has been shown in [6]. For (ii), note first that:

$$y_0(x') = x' \quad \text{and} \quad Q_0 = A = \text{diag}(1, 1, \sqrt{\lambda}).$$

Consequently, directly from (3.4) we see that  $\mathcal{Q}_{2,A} = \mathcal{Q}_{2,Id}$ , which we denote simply by  $\mathcal{Q}_2$ .

Further, in view of (4.1), every admissible limiting strain  $\mathbb{S} \in \mathbb{S}$  has the form  $\mathbb{S} = \text{sym} \nabla w$  for some  $w \in W^{1,2}(\Omega, \mathbb{R}^2)$ . Also, without loss of generality, every admissible limiting displacement  $V$  is of the form:  $V = (0, 0, v)$  for some  $v \in W^{2,2}(\Omega, \mathbb{R})$ . We now compute, using (1.9), (6.9) and (7.1):

$$\vec{b}_0 = \sqrt{\lambda} e_3, \quad \vec{d}_0 = -\frac{1}{2}(\partial_1 \lambda, \partial_2 \lambda, 0), \quad \vec{p} = -\sqrt{\lambda}(\partial_1 v, \partial_2 v, 0).$$

Therefore:

$$\begin{aligned} (\nabla \vec{b}_0)^t \nabla \vec{b}_0 &= \frac{1}{4\lambda} \nabla \lambda \otimes \nabla \lambda, & (\nabla y_0)^t \nabla \vec{d}_0 &= -\frac{1}{2} \nabla^2 \lambda, \\ (\nabla y_0)^t \nabla \vec{p} &= -\frac{1}{2\sqrt{\lambda}} \nabla v \otimes \nabla \lambda - \sqrt{\lambda} \nabla^2 v, & (\nabla V)^t \nabla \vec{b}_0 &= \frac{1}{2\sqrt{\lambda}} \nabla v \otimes \nabla \lambda. \end{aligned}$$

This ends the proof of Lemma 7.1 in view of (3.5). ■

**Lemma 7.2.** *Let  $\lambda : \bar{\Omega} \rightarrow \mathbb{R}$  be smooth and strictly positive. Consider the metric  $G(x', x_3) = \lambda(x') \text{Id}_3$ . Denote  $f = \frac{1}{2} \log \lambda$ . Then:*

- (i) *Condition (1.7) is equivalent to  $\Delta f = 0$ , which is also equivalent to the immersability of the metric  $G_{2 \times 2}$  in  $\mathbb{R}^2$ .*
- (ii) *Under condition (1.7), condition (6.1) can be directly seen as equivalent to  $\text{Ric}(G) = 0$  and therefore to the immersability of  $G$ .*

(iii) The  $\Gamma$ -limit energy functional in (3.5) has the following form:

$$\begin{aligned} \mathcal{I}_4(V, \mathbb{S}) &= \frac{1}{2} \int_{\Omega} e^{-2f} \mathcal{Q}_2(\mathbb{S} + \frac{1}{2}(\nabla V)^t \nabla V + \frac{1}{24} e^{2f} \nabla f \otimes \nabla f) \, dx' \\ &\quad + \frac{1}{24} \int_{\Omega} \mathcal{Q}_2(2\nabla V_3 \otimes \nabla f - \nabla^2 V_3 - \langle \nabla V_3, \nabla f \rangle \text{Id}_2) \, dx' \\ &\quad + \frac{1}{1440} \int_{\Omega} \mathcal{Q}_2(e^f \text{Ric}(G)_{2 \times 2}) \, dx', \end{aligned}$$

where  $\mathcal{Q}_2$  is as in Lemma 7.1, and where  $\text{Ric}(G)_{2 \times 2}$  denotes the tangential part of the Ricci curvature tensor of  $G$ , i.e.:

$$\text{Ric}(G)_{2 \times 2} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{bmatrix}.$$

*Proof.* The part (i) has been deduced in [6], together with the expression:

$$(7.2) \quad \text{Ric}(G) = -(\nabla^2 f - \nabla f \otimes \nabla f)^* - (\Delta f + |\nabla f|^2) \text{Id}_3.$$

We now consider the case when (1.7) holds. By (i) the metric  $G_{2 \times 2}$  is immersible in  $\mathbb{R}^2$  and in particular  $\vec{N} = e_3$ . Writing  $V = (V_1, V_2, V_3)$ , from (1.9), (6.9) and (7.1) we obtain:

$$\vec{b}_0 = \sqrt{\lambda} e_3, \quad \vec{d}_0 = -(\partial_1 f \partial_1 y_0 + \partial_2 f \partial_2 y_0), \quad \vec{p} = -\frac{1}{\sqrt{\lambda}} (\partial_1 V_3 \partial_1 y_0 + \partial_2 V_3 \partial_2 y_0).$$

$$(\nabla \vec{b}_0)^t \nabla \vec{b}_0 = e^{2f} \nabla f \otimes \nabla f, \quad (\nabla V)^t \nabla \vec{b}_0 = e^f \nabla V_3 \otimes \nabla f.$$

Further, observe that:  $\partial_i \vec{d}_0 = -(\partial_{1i} f \partial_1 y_0 + \partial_{2i} f \partial_2 y_0 + \partial_1 f \partial_{1i} y_0 + \partial_2 f \partial_{2i} y_0)$ , and so:

$$\frac{1}{\lambda} \langle \partial_1 y_0, \partial_1 \vec{d}_0 \rangle = -\frac{1}{\lambda} (\lambda \partial_{11} f + \frac{1}{2} \partial_1 \lambda \partial_1 f + \frac{1}{2} \partial_2 \lambda \partial_2 f) = -(\partial_{11} f + |\nabla f|^2).$$

In the same manner, we arrive at:

$$\frac{1}{\lambda} \langle \partial_2 y_0, \partial_2 \vec{d}_0 \rangle = -(\partial_{22} f + |\nabla f|^2), \quad \frac{1}{\lambda} \langle \partial_2 y_0, \partial_1 \vec{d}_0 \rangle = -\partial_{12} f, \quad \frac{1}{\lambda} \langle \partial_1 y_0, \partial_2 \vec{d}_0 \rangle = -\partial_{21} f.$$

Consequently,  $(\nabla y_0)^t \nabla \vec{d}_0$  is already a symmetric matrix, and:

$$(\nabla y_0)^t \nabla \vec{d}_0 = -e^{2f} (\nabla^2 f + |\nabla f|^2 \text{Id}_2).$$

In particular, under condition  $\Delta f = 0$ , the formula (7.2) yields:

$$\text{sym}(\nabla y_0)^t \nabla \vec{d}_0 + (\nabla \vec{b}_0)^t \nabla \vec{b}_0 = e^{2f} \text{Ric}(G)_{2 \times 2},$$

which we directly see to be equivalent with  $\nabla f = 0$  and hence with  $\text{Ric}(G) = 0$ . This establishes (ii).

We now compute the remaining quantities appearing in the expression of  $\mathcal{I}_4$ . Firstly:

$$\nabla \vec{p} = \frac{1}{2\lambda^{3/2}} \nabla y_0 (\nabla V_3 \otimes \nabla \lambda) - \frac{1}{\sqrt{\lambda}} \nabla y_0 \nabla^2 V_3 - \frac{1}{\sqrt{\lambda}} \left( \partial_1 V_3 (\partial_{11} y_0, \partial_{12} y_0) + \partial_2 V_3 (\partial_{12} y_0, \partial_{22} y_0) \right).$$

Using the relations between  $\langle \partial_{ij} y_0, \partial_k y_0 \rangle$  and  $\partial_l G$  in (6.5), we obtain:

$$(\nabla y_0)^t \nabla \vec{p} = \frac{1}{2\lambda^{3/2}} G_{2 \times 2} \nabla V_3 \otimes \nabla \lambda - \frac{1}{\sqrt{\lambda}} G_{2 \times 2} \nabla^2 V_3 - \frac{1}{2\sqrt{\lambda}} \left[ \begin{array}{c|c} \langle \nabla V_3, \nabla \lambda \rangle & \langle \nabla V_3, \nabla \lambda^\perp \rangle \\ \hline -\langle \nabla V_3, \nabla \lambda^\perp \rangle & \langle \nabla V_3, \nabla \lambda \rangle \end{array} \right],$$

and therefore:

$$\text{sym}(\nabla y_0)^t \nabla \vec{p} = \sqrt{\lambda} \text{sym}(\nabla V_3 \otimes \nabla f) - \sqrt{\lambda} \nabla^2 V_3 - \sqrt{\lambda} \langle \nabla V_3, \nabla \lambda \rangle \text{Id}_2.$$

In a similar manner, it follows that:

$$\text{sym}(\nabla y_0)^t \nabla \vec{d}_0 = -\lambda \left( \nabla^2 f + |\nabla f|^2 \text{Id}_2 \right).$$

Since  $\mathcal{Q}_{2,A}(x') = \lambda^{-1} \mathcal{Q}_2$ , the formula in (3.5) becomes:

$$(7.3) \quad \begin{aligned} \mathcal{I}_4(V, \mathbb{S}) &= \frac{1}{2} \int_{\Omega} e^{-2f} \mathcal{Q}_2 \left( \mathbb{S} + \frac{1}{2} (\nabla V)^t \nabla V + \frac{1}{24} e^{2f} \nabla f \otimes \nabla f \right) dx' \\ &+ \frac{1}{24} \int_{\Omega} e^{-2f} \mathcal{Q}_2 \left( 2e^f \nabla V_3 \otimes \nabla f - e^f \nabla^2 V_3 - e^f \langle \nabla V_3, \nabla f \rangle \text{Id}_2 \right) dx' \\ &+ \frac{1}{1440} \int_{\Omega} e^{-2f} \mathcal{Q}_2 \left( e^{2f} \text{Ric}(G)_{2 \times 2} \right) dx', \end{aligned}$$

which implies the result. ■

## 8. APPENDIX: A PROOF OF COROLLARY 2.3

**1.** For every  $x' \in \Omega$  denote  $D_{x',\delta} = B(x', \delta) \cap \Omega$  and  $B_{x',\delta,h} = D_{x',\delta} \times (-h/2, h/2)$ . For short, we write  $B_{x',2h} = B_{x',2h,h}$  and  $B_{x',h} = B_{x',h,h}$ . Apply Lemma 2.2 to the set  $\mathcal{V}^h = B_{x',2h}$  to get a rotation  $R_{x',2h} \in SO(3)$  such that, with a universal constant  $C$ :

$$(8.1) \quad \begin{aligned} \frac{1}{h} \int_{B_{x',2h}} \left| \nabla u^h(z) - R_{x',2h} (Q_0(z') + z_3 B_0(z')) \right|^2 dz \\ \leq C \left( E^h(u^h, B_{x',2h}) + h^3 |B_{x',2h}| \right). \end{aligned}$$

Consider a family of mollifiers  $\eta_{x'} \in C^\infty(\Omega, \mathbb{R})$ , parametrized by  $x' \in \Omega$ :

$$\int_{\Omega} \eta_{x'} = \frac{1}{h}, \quad \|\eta_{x'}\|_{L^\infty(\Omega)} \leq \frac{C}{h^3}, \quad \|\nabla_{x'} \eta_{x'}\|_{L^\infty(\Omega)} \leq \frac{C}{h^4} \quad \text{and} \quad (\text{supp } \eta_{x'}) \cap \Omega \subset D_{x',h}.$$

Define  $\tilde{R}^h \in W^{1,2}(\Omega, \mathbb{R}^{3 \times 3})$  as:

$$(8.2) \quad \tilde{R}^h(x') = \int_{\Omega^h} \eta_{x'}(z') \nabla u^h(z) (Q_0(z') + z_3 B_0(z'))^{-1} dz.$$

We then have:

$$(8.3) \quad \begin{aligned} \frac{1}{h} \int_{B_{x',h}} \left| \nabla u^h(z) - \tilde{R}^h(z') (Q_0(z') + z_3 B_0(z')) \right|^2 dz \\ \leq \frac{C}{h} \int_{B_{x',2h}} \left| \nabla u^h(z) - R_{x',2h} (Q_0(z') + z_3 B_0(z')) \right|^2 dz \\ + \frac{C}{h} \int_{B_{x',h}} \left| \tilde{R}^h(z') - R_{x',2h} \right|^2 |Q_0(z') + z_3 B_0(z')|^2 dz \\ \leq C \left( E^h(u^h, B_{x',2h}) + h^3 |B_{x',2h}| \right) + \frac{C}{h} \int_{B_{x',h}} \left| \tilde{R}^h(z') - R_{x',2h} \right|^2 dz, \end{aligned}$$

where we have used (8.1) and  $\|Q_0(z') + z_3 B_0(z')\|_{L^\infty} \leq C$ . Now, for every  $z' \in B_{x',h}$  we have:

$$\begin{aligned}
|\tilde{R}^h(z') - R_{x',2h}|^2 &= \left| \int_{\Omega^h} \eta_{z'}(y') \nabla u^h(y) (Q_0(y') + y_3 B_0(y'))^{-1} dy - R_{x',2h} \right|^2 \\
&= \left| \int_{\Omega^h} \eta_{z'}(y') \left( \nabla u^h(y) - R_{x',2h} (Q_0(y') + y_3 B_0(y')) \right) (Q_0(z') + y_3 B_0(z'))^{-1} dy \right|^2 \\
(8.4) \quad &\leq C \left( \int_{B_{z',h}} \eta_{z'}(y')^2 dy \right) \left( \int_{B_{z',h}} \left| \nabla u^h(y) - R_{x',2h} (Q_0(y') + y_3 B_0(y')) \right|^2 dy \right) \\
&\leq \frac{C}{h^2} \int_{B_{x',2h}} \left| \nabla u^h(y) - R_{x',2h} (Q_0(y') + y_3 B_0(y')) \right|^2 dy \\
&\leq \frac{C}{h^2} \left( E^h(u^h, B_{x',2h}) + h^3 |B_{x',2h}| \right).
\end{aligned}$$

In a similar way, in view of  $\int_{\Omega^h} \nabla_{z'} \eta_{z'}(y') dy = 0$ , it follows that:

$$\begin{aligned}
|\nabla \tilde{R}^h(z')|^2 &= \left( \int_{\Omega^h} \nabla_{z'} \eta_{z'}(y') \nabla u^h(y) (Q_0(y') + y_3 B_0(y'))^{-1} dy \right)^2 \\
&= \left( \int_{B_{x',2h}} \nabla_{z'} \eta_{z'}(y') \left( \nabla u^h(y) (Q_0(y') + y_3 B_0(y'))^{-1} - R_{x',2h} \right) dy \right)^2 \\
&\leq C \int_{\Omega^h} |\nabla_{z'} \eta_{z'}(y')|^2 dy \int_{B_{x',2h}} \left| \nabla u^h(y) - R_{x',2h} (Q_0(y') + y_3 B_0(y')) \right|^2 dy \\
&\leq \frac{C}{h^4} \left( E^h(u^h, B_{x',2h}) + h^3 |B_{x',2h}| \right).
\end{aligned}$$

From (8.4) we obtain:

$$\begin{aligned}
\int_{B_{x',h}} |\tilde{R}^h(z') - R_{x',2h}|^2 dz &\leq \frac{C}{h^2} \int_{B_{x',h}} \left( E^h(u^h, B_{x',2h}) + h^4 |B_{x',2h}| \right) dz \\
&\leq Ch \left( E^h(u^h, B_{x',2h}) + h^3 |B_{x',2h}| \right),
\end{aligned}$$

and therefore by (8.3) we further see that:

$$\begin{aligned}
(8.5) \quad \frac{1}{h} \int_{B_{x',h}} |\nabla u^h(z) - \tilde{R}^h(z') (Q_0(z') + z_3 B_0(z'))|^2 dz \\
\leq C \left( E^h(u^h, B_{x',2h}) + h^3 |B_{x',2h}| \right).
\end{aligned}$$

**2.** Covering  $\Omega^h$  by a finite family of sets  $\{B_{x',h}\}$ , such that the intersection number of the doubled covering  $\{B_{x',2h}\}$  is independent of  $h$ , applying (8.5) and summing over the covering, it follows that:

$$\frac{1}{h} \int_{\Omega^h} |\nabla u^h(z) - \tilde{R}^h(z') (Q_0(z') + z_3 B_0(z'))|^2 dz \leq C \left( E^h(u^h) + h^4 \right).$$

In a similar fashion we obtain:

$$\begin{aligned} \int_{D_{x',h}} |\nabla \tilde{R}^h(z')|^2 dz &\leq \frac{C}{h^4} \int_{D_{x',h}} \left( E^h(u^h, B_{x',2h}) + h^3 |B_{x',2h}| \right) dz \\ &\leq \frac{C}{h^2} \left( E^h(u^h, B_{x',2h}) + h^3 |B_{x',2h}| \right), \end{aligned}$$

and by the same covering argument:

$$\int_{\Omega^h} |\nabla \tilde{R}^h(z')|^2 dz \leq \frac{C}{h^2} \left( E^h(u^h) + h^4 \right).$$

**3.** Note that, in the above two estimates, we can replace  $\tilde{R}^h$  by  $R^h = \mathbb{P}_{SO(3)} \tilde{R}^h \in W^{1,2}(\Omega, SO(3))$ . Firstly, the projection in question is well defined in view of (8.4), since:

$$\text{dist}^2 \left( \tilde{R}^h, SO(3) \right) \leq |\tilde{R}^h - R_{x',2h}| \leq \frac{C}{h^2} \left( E^h(u^h) + h^4 \right),$$

which is small because of the hypothesis  $\alpha < 2$ . Moreover:

$$\begin{aligned} \frac{1}{h} \int_{B_{x',h}} |\nabla u^h(z) - R^h(z') (Q_0(z') + z_3 B_0(z'))|^2 dz \\ \leq \frac{C}{h} \int_{B_{x',h}} \left| \nabla u^h(z) - \tilde{R}^h(z') (Q_0(z') + z_3 B_0(z')) \right|^2 dz \\ + \frac{C}{h} \int_{B_{x',h}} |\tilde{R}^h(z') - R^h(z')|^2 |Q_0(z') + z_3 B_0(z')|^2 dz \\ \leq C \left( E^h(u^h, B_{x',2h}) + h^3 |B_{x',2h}| \right) \end{aligned}$$

because of (8.5) and (8.4). Finally, the previous covering argument clearly implies (2.9), and  $\int_{\Omega} |\nabla R^h|^2 dz \leq C \int_{\Omega} |\nabla \tilde{R}^h|^2 dz$  yields (2.10).

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