CONVERGENCE OF EQUILIBRIA FOR INCOMPRESSIBLE ELASTIC PLATES IN THE VON KÁRMÁN REGIME

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ABSTRACT. We prove convergence of critical points to the nonlinear elastic energies J^h of 3d thin incompressible plates, to critical points of the 2d energy obtained as the Γ-limit of J^h in the von Kármán scaling regime. The presence of incompressibility constraint requires to restrict the class of admissible test functions to bounded divergence-free variations on the 3d deformations. This poses new technical obstacles, which we resolve by means of introducing 3d extensions and truncations of the 2d limiting deformations, specific to the problem at hand.

1. Introduction and the main result

The derivation of asymptotic theories for thin elastic films has been a longstanding problem in the mathematical theory of elasticity [3]. Recently, various lower dimensional theories have been rigorously derived from the nonlinear three-dimensional model, through Γ-convergence methods. Consequently, what seemed to be competing and contradictory theories for elastica (rods, plates, shells, etc) are now revealed to be each valid in their own specific range of parameters such as material elastic constants, boundary conditions and force magnitudes. In this line, Friesecke, James and Müller gave a detailed description of the so called hierarchy of plate theories in [8], corresponding to distinct energy scaling laws in terms of the plate thickness. Similar results have been obtained for elastic shells [6, 12, 13, 14], elastic incompressible plates [4, 5, 17] and in presence of residual stress [15, 10, 11].

The variational approach, described in detail in the below subsections, provides, among its other features, a rigorous justification of convergence of three-dimensional energy minimizing deformations, to minimizers of suitable lower dimensional limit energies. As shown in [20] (see also the first result [18] where the converge of equilibria is considered, although in the setting of beams, not plates), one can also expect convergence of stationary points, in the same regimes. The results in [20] relied on the crucial assumption that the elastic energy density is differentiable everywhere and its derivative satisfies a linear growth condition. This assumption is contradictory with the physically expected non-interpenetration condition, and subsequently it has been removed in [19] and exchanged with Ball's related notion of the outer variations.

In the present paper, we prove convergence of critical points of the nonlinear elastic energies of thin incompressible plates, to critical points of the limiting energy in the von Kármán scaling regime, derived in [17]. We follow the same approach as in [19]; indeed the concept of outer variations comes up naturally in the context of incompressible elasticity. The presence of incompressibility constraint requires to modify the class of admissible test functions in the stationarity conditions, and subsequently one is allowed to work only with bounded divergence-free variations on the three-dimensional deformations. On the other hand, the limiting two-dimensional displacements may be arbitrary, while the incompressibility constraint is seen in the limiting equations only by tracelessness of the limiting stress. This poses new technical obstacles, which we resolve by means of introducing 3d extensions and truncations of the 2d limiting deformations, specific to the problem at hand.

We now turn to describing the framework of the problem and our results in detail.

1.1. **Elastic energy of thin incompressible plates.** Let $\Omega\subset\mathbb{R}^2$ be an open, bounded, simply connected domain. For $h>0$, define Ω^h to be the 3d plate with the midplate Ω and thickness h:

$$\Omega^h=\Big\{x=(x',x_3);\ x'\in\Omega,\ x_3\in\Big(-\frac{h}{2},\frac{h}{2}\Big)\Big\}.$$

The elastic energy of a deformation $u^h\in W^{1,2}(\Omega^h,\mathbb{R}^3)$ of the homogeneous plate Ω^h, scaled by its unit thickness, is given by:

$$I^h(u^h)=\frac{1}{h}\int_{\Omega^h}W_{in}(\nabla u^h)\ \mathrm{d}x, \tag{1.1}$$

while the total energy, relative to the external force with the density $f^h\in L^2(\Omega^h,\mathbb{R}^3)$, is:

$$J^h(u^h)=\frac{1}{h}\int_{\Omega^h}W_{in}(\nabla u^h)\ \mathrm{d}x-\frac{1}{h}\int_{\Omega^h}f^h\cdot u^h\ \mathrm{d}x. \tag{1.2}$$

The elastic energy density $W_{in}:\mathbb{R}^{3\times3}\to[0,\infty]$ in (1.1) is assumed to be infinite at compressible deformations:

$$W_{in}(F)=\begin{cases}W(F) & \text{if } \det F=1,\\ +\infty & \text{otherwise.}\end{cases}$$

The effective density $W:\mathbb{R}^{3\times3}\to[0,\infty)$ above, which acts when $\det F=1$, is required to satisfy the following conditions:

(i) *(frame invariance)* $W(RF)=W(F)$, for each proper rotation $R\in SO(3)$, and each $F\in\mathbb{R}^{3\times3}$.
(ii) *(normalisation)* $W(F)=0$ for all $F\in SO(3)$.
(iii) *(non-interpenetration)* $W(F)=+\infty$ if $\det F\leq0$, and $W(F)\to+\infty$ as $\det F\to0+$.
(iv) *(bound from below)* $W(F)\geq c\ \mathrm{dist}^2(F,SO(3))$ with a constant $c>0$ independent of F.
(v) *(bound from above)* There exists a constant $C>0$ such that for each F with $\det F>0$, i.e. for each $F\in\mathbb{R}^{3\times3}_+$ there holds:

$$|DW(F)F^T|\leq C(W(F)+1). \tag{1.3}$$

(vi) *(regularity)* W is of class $\mathcal{C}^1$ on $\mathbb{R}^{3\times3}_+$.
(vii) *(local regularity)* W is of class $\mathcal{C}^2$ in a small neighborhood of $SO(3)$.

The growth conditions in (iv) and (v) will be crucial in the present analysis. Condition (iv) has been introduced in the context of [8] and it allows to use the nonlinear version of Korn's inequality [7], ultimately serving to control the local deviations of the deformation u^h from rigid motions, by the elastic energy $I^h(u^h)$. Condition (v) has been introduced in [1] (see also [2]) in the context of outer variations, in order to control the related strain in terms of the energy. Both conditions are compatible with other requirements above. Indeed, examples of W satisfying (i) – (vii) are:

$$W_1(F)=|(F^TF)^{1/2}-\mathrm{Id}|^2+|\log\det F|^q,$$
$$W_2(F)=|(F^TF)^{1/2}-\mathrm{Id}|^2+\left|\frac{1}{\det F}-1\right|^q\ \text{ for }\ \det F>0,$$

where $q>1$ and W equals $+\infty$ if $\det F\leq0$ [19].

1.2. **Notation.** Given a matrix $F \in \mathbb{R}^{n\times n}$, we denote its trace by Tr F and its transpose by F^T. The symmetric part of F is given by sym $F = \frac{1}{2}(F+F^T)$. The cofactor of F is the matrix: cof F, where $[\text{cof } F]_{ij} = (-1)^{i+j}\det \hat{F}_{ij}$ and each $\hat{F}_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ is obtained from F by deleting its ith row and jth column. The identity matrix is denoted by Id_n.

In what follows, we shall use the matrix norm $|F| = (\text{Tr}(F^TF))^{1/2}$, which is induced by the inner product: $F_1 : F_2 = \text{Tr}(F_1^TF_2)$. To avoid notational confusion, we will often write $\langle F_1 : F_2\rangle$ instead of $F_1 : F_2$. In general, 3×3 matrices will be denoted by F and 2×2 matrices will be denoted by F''. Unless noted otherwise, F'' is the principal 2×2 minor of F.

Finally, by $\mathcal{C}_b^k(\mathbb{R}^n, \mathbb{R}^s)$ we denote the space of continuous functions whose derivatives up to the order k are continuous and bounded in $\mathbb{R}^n$.

1.3. **The limiting energy.** The following 2d energy functional has been rigorously derived in [17] as the Γ-limit of the scaled incompressible energies $h^{-4}I^h$ in (1.1), when $h \to 0$:

$$\mathcal{I}(w,v) = \frac{1}{2}\int_\Omega \mathcal{Q}_2^{in}\left(\text{sym}\nabla u + \frac{1}{2}\nabla v\otimes\nabla v\right)\,\mathrm{d}x + \frac{1}{24}\int_\Omega \mathcal{Q}_2^{in}\left(\nabla^2 v\right)\,\mathrm{d}x, \tag{1.4}$$

acting on couples $w \in W^{1,2}(\Omega,\mathbb{R}^2), v\in W^{2,2}(\Omega,\mathbb{R})$. The fields (w,v) may be identified as the in-plane and the out-of-plane displacements, respectively. Roughly speaking, any minimizing sequence of $h^{-4}J^h$, where $f^h(x) \approx h^3 f(x')e_3$ and $\int_\Omega f = 0$, will have the structure:

$$u^h_{|\Omega} \approx (\bar{R})^T\left(\text{id} + hve_3 + h^2 w\right) - c^h$$

asymptotically as $h\to 0$, with (w,v) as above and $\bar{R}\in SO(3)$ maximizing $\int_\Omega f(x')e_3\cdot Rx'\,\mathrm{d}x'$ among all rotations R, while $c^h \in \mathbb{R}^3$ are constant translation vectors. Moreover, $(w,v,\bar{R})$ minimize the following total limiting energy:

$$\mathcal{J}(w,v,\bar{R}) = \mathcal{I}(w,v) - \bar{R}_{33}\int_\Omega fv.$$

A precise formulation of the statements above can be found in [12].

The energy in (1.4) is the incompressible version of the von Kármán functional, which has been derived (for compressible case, i.e. without the assumption that $\det\nabla u^h = 1$) by means of Γ-convergence in [8]. The quadratic forms $\mathcal{Q}_2^{in}$ differ from the standard $\mathcal{Q}_2$ in [8] in as much as minimization in (1.5) below is taken over the out-of-plane stretches which preserve the incompressibility constraint. Namely, $\mathcal{Q}_2^{in}$ in (1.4) are given as:

$$\begin{aligned}&\forall F''\in\mathbb{R}^{2\times 2}\quad \mathcal{Q}_2^{in}(F'') = \min_{d\in\mathbb{R}^3}\Big\{Q_3(F''+d\otimes e_3 + e_3\otimes d);\ \text{Tr}(F''+d\otimes e_3+e_3\otimes d) = 0\Big\},\\ &\forall F\in\mathbb{R}^{3\times 3}\quad \mathcal{Q}_3(F) = D^2W(\text{Id})(F,F).\end{aligned} \tag{1.5}$$

Both forms $\mathcal{Q}$ above are positive semidefinite, and strictly positive definite on symmetric matrices. We also introduce the linear operators $\mathcal{L}_2^{in}:\mathbb{R}^{2\times 2}\to\mathbb{R}^{2\times 2}$ and $\mathcal{L}_3:\mathbb{R}^{3\times 3}\to\mathbb{R}^{3\times 3}$ such that:

$$\begin{aligned}&\forall F''\in\mathbb{R}^{2\times 2}\quad \langle\mathcal{L}_2^{in}(F''):F''\rangle = \mathcal{Q}_2^{in}(F''),\\ &\forall F\in\mathbb{R}^{3\times 3}\quad \langle\mathcal{L}_3(F):F\rangle = \mathcal{Q}_3(F).\end{aligned} \tag{1.6}$$

Note that symmetric operators $\mathcal{L}$ are uniquely given by: $\langle\mathcal{L}(F_1):F_2\rangle = \frac{1}{4}\left(\mathcal{Q}(F_1+F_2) - \mathcal{Q}(F_1-F_2)\right)$.

1.4. **Critical points and the incompressible outer variations.** Following [2], we now define the critical points u^h of the functionals J^h in (1.2) with respect to outer variations, that is requesting that the derivative of J^h at an incompressible equilibrium u^h be zero:

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}_{|\epsilon=0} J^h(u^h_\epsilon) = 0,$$

along all curves $\epsilon \mapsto u^h_\epsilon$ of incompressible deformations of Ω^h having the form: $u^h_\epsilon(x) = \Phi(\epsilon, u^h(x))$, with $u^h_0 = u^h$ at $\epsilon = 0$. This requirement is translated into the following condition:

(1.7)
$$\int_{\Omega^h} \Big\langle DW(\nabla u^h)(\nabla u^h)^T : \nabla\phi(u^h(x))\Big\rangle ~\mathrm{d}x = \int_{\Omega^h} f^h \cdot \phi(u^h) ~\mathrm{d}x, \quad \forall \phi \in \mathcal{C}^1_b(\mathbb{R}^3, \mathbb{R}^3) \text{ with div } \phi = 0.$$

We refer to section 2 for the derivation and discussion of (1.7). Let us only note now that the incompressible outer variations:

$$u^h_\epsilon(x) = \Phi(\epsilon, u^h(x)) = u^h(x) + \epsilon\phi(u^h(x)) + \mathcal{O}(\epsilon^2).$$

replace the classical variations $u^h_\epsilon(x) = u^h(x) + \epsilon w^h(x)$ used in definition of minimizers of J^h, and also they replace the outer variations $u^h_\epsilon(x) = u^h(x) + \epsilon\phi(u^h(x))$ considered in [2] and [19] in the compressible case.

1.5. **The main result.** The following is our main result:

Theorem 1.1. *For each $h << 1$, let $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ be a critical point of J^h, i.e. it satisfies (1.7) subject to the external forces $f^h(x) = h^3 f(x')e_3$. Assume that:*

$$I^h(u^h) \leq Ch^4, \tag{1.8}$$

for a constant $C > 0$ independent of h. Then there exists a sequence of proper rotations $\bar{R}^h \in SO(3)$, and translations $c^h \in \mathbb{R}^3$, such that for the renormalized deformations:

$$y^h(x', x_3) = (\bar{R}^h)^T u^h(x', hx_3) - c^h \in W^{1,2}(\Omega^1, \mathbb{R}^3), \tag{1.9}$$

the following convergences hold, up to a subsequence in h, as $h \to 0$:

(i) *$\bar{R}^h \to \bar{R} = [\bar{R}_{ij}]_{i,j:1..3} \in SO(3)$.*
(ii) *$y^h \to x'$ in $W^{1,2}(\Omega^1)$.*
(iii) *For the scaled out-of-plane displacements:*

$$v^h(x') = \frac{1}{h}\int_{-1/2}^{1/2} y^h_3(x', x_3) ~\mathrm{d}x_3, \tag{1.10}$$

there exists $v \in W^{2,2}(\Omega, \mathbb{R})$ such that $v^h \to v$ strongly in $W^{1,2}(\Omega)$.
(iv) *For the scaled in-plane displacements:*

$$w^h(x') = \frac{1}{h^2}\int_{-1/2}^{1/2} \Big((y^h)'(x', x_3) - x'\Big) ~\mathrm{d}x_3 \tag{1.11}$$

there exists $w \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that $w^h \rightharpoonup w$ weakly in $W^{1,2}(\Omega, \mathbb{R}^2)$.
(v) *The limiting displacements (w, v) solve the following Euler-Lagrange equations of the functional (1.4), expressed in the variational form:*

$$\int_\Omega \Big\langle \mathcal{L}^{in}_2 \Big(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v \otimes \nabla v\Big) : \nabla\tilde{w}\Big\rangle ~\mathrm{d}x' = 0 \tag{1.12}$$

$$\int_\Omega \left\langle \mathcal{L}_2^{in}\left(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v \otimes \nabla v\right) : (\nabla v \otimes \nabla \tilde{v})\right\rangle \, \mathrm{d}x' + \frac{1}{12}\int_\Omega \left\langle \mathcal{L}_2^{in}(\nabla^2 v) : \nabla^2 \tilde{v}\right\rangle \, \mathrm{d}x' = \bar{R}_{33}\int_\Omega f\tilde{v}\, \mathrm{d}x', \tag{1.13}$$

for every $\tilde{w} \in W^{1,2}(\Omega, \mathbb{R}^2)$ *and every* $\tilde{v} \in W^{2,2}(\Omega, \mathbb{R})$.

We note that (1.8) are automatically satisfied by any minimizing sequence of u^h of the total energy J^h, under the assumption that $f^h(x) = h^3 f(x')e_3$ [8]. Also, (1.7) holds for every minimum of J^h (see Theorem 2.3), and the assertions (i) - (v) are then a direct consequence [17] of the fact that $\frac{1}{h^4}J^h$ Γ-converges to $\mathcal{J}$. In general, Γ-convergence does not assure that a limit of a sequence of equlibria is an equilibrium of the Γ-limit. In the present situation, this turns out to be the case.

1.6. **The isotropic case.** For an isotropic energy density W with the Lamé constants λ and μ, the Euler-Lagrange equations (1.12) – (1.13) of (1.4) are:

$$\frac{\mu}{3}\Delta^2 v = [v, \Phi], \qquad \Delta^2\Phi = -\frac{3\mu}{2}[v, v], \tag{1.14}$$

where v is the out-of-plane displacement, while the in-plane displacement w can be recovered through the Airy stress potential Φ, by means of:

$$\mathrm{cof}\nabla^2\Phi = 2\mu\Big[\mathrm{sym}\nabla w + \frac{1}{2}\nabla v \otimes \nabla v + \Big(\mathrm{div} w + \frac{1}{2}|\nabla v|^2\Big)\mathrm{Id}\Big].$$

The Airy's bracket $[\cdot, \cdot]$ is defined as: $[v, \Phi] = \nabla^2 v : (\mathrm{cof}\nabla^2\Phi)$. As expected, the system (1.14) can be now obtained as the incompressible limit, i.e. when passing with the Poisson ratio $\nu \to \frac{1}{2}$, of the classical (compressible) von Kármán system:

$$B\Delta^2 v = [v, \Phi], \qquad \Delta^2\Phi = -\frac{S}{2}[v, v],$$

where $S = 2\mu(1 + \nu)$ is Young's modulus, $\nu = \frac{\lambda}{2(\mu+\lambda)}$ is the Poisson ratio, and $B = \frac{S}{12(1-\nu^2)}$ is bending stiffness. By the change of variable $\Phi = 2\mu\Phi_1$ one can eliminate the parameter μ entirely and write (1.14) in its equivalent form:

$$\Delta^2 v = 6[v, \Phi_1], \qquad \Delta^2\Phi_1 = -\frac{3}{4}[v, v].$$

1.7. **Relation to other works and the layout of the paper.** To put our work in a larger perspective, recall that one of the fundamental questions in the mathematical theory of elasticity has been to rigorously justify various 2d plate models present in the engineering literature, in relation to the three-dimensional theory. This goal has been largely accomplished in [8], where a hierarchy of limiting 2d energies has been derived; the distinct theories are differentiated by their validity in the corresponding scaling regimes h^β, $\beta \geq 2$, i.e. in presence of assumption (1.8) where h^4 is replaced by h^β.

Under the additional incompressibility constraint, the works [4, 5] proved compactness properties and the Γ-convergence of the functionals $\frac{1}{h^\beta}I^h$ as in (1.1), for the so-called Kirchhoff scaling $\beta = 2$, while [17] treated the case $\beta = 4$ including as well a more complex case of shells when the midsurface Ω is a generic 2d hypersurface in $\mathbb{R}^3$. In view of the fundamental property of Γ-convergence, it follows that the global almost-minimizers of the energies (1.2) converge to the minimizers of the limiting energy (given by (1.4) in the von Kármán regime).

Regarding convergence of stationary points for thin plates, the first result has been obtained in [20] under the von Kármán scaling $\beta = 4$ (see also [9] for an extension to thin shells). The first result on converge of equilibria for beams has been obtained in [18]. These results relied on the crucial assumption that the elastic energy density W is differentiable everywhere and that:

$|DW(F)| \leq C(|F|+1)$. This linear growth condition is, however, contradictory with the physically expected non-interpenetration condition. In [19], it has been exchanged with Ball's condition (1.3), while the equilibrium equations have been rephrased in terms of the outer variations. In the present paper we follow the same approach; indeed the concept of outer variations comes up naturally in the context of incompressible elasticity.

Our analysis is inspired by [20] and [19], with necessary improvements to overcome the difficulties imposed by the incompressibility constraint, which in particular requires to modify the class of admissible test functions in the stationarity conditions. These conditions are derived in section 2, along with introducing the incompressible outer variations and the corresponding weak form of the Euler-Lagrange equations, relying on the flow (2.2) with divergence-free velocity field.

In section 3 we obtain the equilibrium equations (1.7), and in section 4 we identify the operators $\mathcal{L}_2^{in}$ in (1.12) and (1.13). In section 5, we show that in the present incompressible case, the limiting strain is traceless (Lemma 5.1); we also construct a sequence of appropriate truncation functions (Lemma 5.2) which are further used (Lemma 5.3) to show that the limiting stress satisfies the required condition of section 4. One major difficulty in the proof of Lemma 5.3 and the subsequent results is that one is allowed to work only with bounded divergence-free variations on the three-dimensional deformations, while the limiting two-dimensional displacements may be arbitrary. This poses new technical obstacles, which we resolve by introducing 3d extensions and truncations of the 2d limiting deformations, specific to the problems at hand. In section 6 and 7, we then prove the first and second Euler-Lagrange equations (1.12) and (1.13), which concludes the proof of our main result Theorem 1.1.

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2. Incompressible outer variations and critical points

Following [2], we want to define the critical points u^h of the functionals J^h in (1.2) by taking outer variations. That is, we request that the derivative of J^h at an incompressible equilibrium u^h be zero along all curves $\epsilon \mapsto u^h_\epsilon$ of incompressible deformations of Ω^h having the form: $u^h_\epsilon(x) = \Phi(\epsilon, u^h(x))$, with $u^h_0 = u^h$ at $\epsilon = 0$. This requirement imposes the following conditions on the flow $\Phi : [0, \epsilon_0) \times \mathbb{R}^3 \to \mathbb{R}^3$:

$$\begin{aligned} &\forall \epsilon \quad \Phi(\epsilon, \cdot) \text{ is incompressible, i.e} \quad \forall y \in \mathbb{R}^3 \quad \det \nabla\Phi(\epsilon, y) = 1, \\ &\forall y \in \mathbb{R}^3 \quad \Phi(0, y) = y. \end{aligned} \tag{2.1}$$

Assuming sufficient smoothness of Φ, the above immediately implies:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} \det \nabla\Phi(0, y) = \left\langle \mathrm{cof}\nabla\Phi(0, y) : \frac{\mathrm{d}}{\mathrm{d}\epsilon}\nabla\Phi(0, y)\right\rangle = \left\langle \mathrm{Id} : \frac{\mathrm{d}}{\mathrm{d}\epsilon}\nabla\Phi(0, y)\right\rangle \\ &= \mathrm{Tr}\left(\frac{\mathrm{d}}{\mathrm{d}\epsilon}\nabla\Phi(0, y)\right) = \mathrm{div}\left(\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi(0, y)\right) =: \mathrm{div}\ \phi(y). \end{aligned}$$

On the other hand, any divergence-free vector field ϕ generates a path of incompressible deformations. We recall this standard fact below, for the sake of completeness.

Lemma 2.1. *Let $\phi \in \mathcal{C}^1_b(\mathbb{R}^n, \mathbb{R}^n)$ such that* div $\phi = 0$*. Consider the ODE:*

$$\left\{ \begin{array}{l} u'(\epsilon) = \phi(u(\epsilon)), \\ u(0) = y. \end{array} \right. \tag{2.2}$$

and denote its flow by $\Phi(\epsilon, y) = u(\epsilon)$ solving (2.2). Then Φ satisfies (2.1).

Proof. Let $\epsilon, \delta > 0$ and note that: $\Phi(\epsilon+\delta, y) = \Phi(\delta, \Phi(\epsilon, y)) = \Phi(\delta, y_1)$ where we put $y_1 = \Phi(\epsilon, y)$. Hence, denoting the spacial gradient by ∇, we obtain:

$$\det \nabla\Phi(\epsilon+\delta, y) = \det \nabla\Phi(\delta, y_1) \det \nabla\Phi(\epsilon, y),$$

Consequently:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\epsilon} \left(\det \nabla\Phi(\epsilon+\delta, y)\right) &= \frac{\mathrm{d}}{\mathrm{d}\delta} \left(\det \nabla\Phi(\epsilon+\delta, y)\right) = \frac{\mathrm{d}}{\mathrm{d}\delta} \left(\det \nabla\Phi(\delta, y_1)\right) \left(\det \nabla\Phi(\epsilon, y)\right) \\ &= \left\langle \mathrm{cof}\ \nabla\Phi(\delta, y_1) : \frac{\mathrm{d}}{\mathrm{d}\delta}\nabla\Phi(\delta, y_1) \right\rangle \det \nabla\Phi(\epsilon, y). \end{aligned} \tag{2.3}$$

Above, we used the formula for the derivative of the determinant of a matrix function $A(t)$, namely: $(\det A(t))' = \mathrm{cof}A(t) : A(t)'$. For $\delta = 0$, (2.3) implies:

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \left(\det \nabla\Phi(\epsilon, y)\right) = \langle \mathrm{cof}\nabla\Phi(0, y_1) : \nabla\phi(y_1)\rangle = \langle \mathrm{Id} : \nabla\phi(y_1)\rangle = \mathrm{Tr}\nabla\phi = \mathrm{div}\ \phi = 0.$$

But $\det \nabla\Phi(0, y) = \det \mathrm{Id}_n = 1$, which achieves the claim. ∎

We are now ready to derive the equilibrium equations (1.7). The result is essentially similar to Theorem 2.4 [2], which dealt with the compressible outer variations $u^h_\epsilon = u^h(x) + \epsilon\phi \circ u^h$ of a deformation u^h with clamped boundary conditions. The growth condition (1.3) will be crucial in passing to the limit in the nonlinear term in J^h, to which end we are going to use the following Lemma from [2]:

Lemma 2.2. *(Lemma 2.5 (i) [2]) Assume that W satisfies (1.3). Then there exists $\gamma > 0$ such that if $A \in \mathbb{R}^{3\times3}_+$ and $|A - \mathrm{Id}| < \gamma$, then:*

$$|DW(AF)F^T| \leq 3C(W(F) + 1) \qquad \forall F \in \mathbb{R}^{3\times3}_+,$$

where C is the constant in condition (1.3).

Theorem 2.3. *Let $\phi \in \mathcal{C}^1_b(\mathbb{R}^3, \mathbb{R}^3)$ be such that* $\mathrm{div}\ \phi = 0$*. Given a deformation $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ with* $\det \nabla u^h = 1$*, and such that $\int_{\Omega^h} W(\nabla u^h)\ \mathrm{d}x < +\infty$, define $u^h_\epsilon(x) = \Phi(\epsilon, u^h(x))$. Then:*

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}_{|\epsilon=0} J^h(u^h_\epsilon) = 0$$

is equivalent to:

$$\int_{\Omega^h} \left\langle DW(\nabla u^h)(\nabla u^h)^T : \nabla\phi(u^h(x))\right\rangle\ \mathrm{d}x = \int_{\Omega^h} f^h \cdot \phi(u^h)\ \mathrm{d}x.$$

Proof. For the notational convenience, in what follows we drop the index h and write U instead of Ω^h, which stands now for a fixed open bounded domain in $\mathbb{R}^3$. It is easy to notice that:

$$\lim_{\epsilon\to0} \frac{1}{\epsilon}(\Phi(\epsilon, y) - y) = \phi(y) \qquad \text{uniformly in } \mathbb{R}^3. \tag{2.4}$$

It directly implies that:

$$\lim_{\epsilon\to0} \frac{1}{\epsilon} \int_U f \cdot (\Phi(\epsilon, u(x)) - u(x))\ \mathrm{d}x = \int_U f \cdot \phi(u(x))\ \mathrm{d}x.$$

To treat the nonlinear term, consider:

$$\begin{aligned} &\frac{1}{\epsilon} \int_U \big(W(\nabla u_\epsilon) - W(\nabla u)\big)\ \mathrm{d}x - \int_U \left\langle DW(\nabla u)(\nabla u)^T : \nabla\phi(u)\right\rangle\ \mathrm{d}x \\ &= \int_U \fint_0^\epsilon \left\langle DW\big(\nabla\Phi(s, u)\nabla u\big)(\nabla u)^T : \nabla\phi(\Phi(s, u))\right\rangle - \left\langle DW(\nabla u)(\nabla u)^T : \nabla\phi(u)\right\rangle\ \mathrm{d}s\ \mathrm{d}x. \end{aligned} \tag{2.5}$$

Since the integrand below converges to 0 pointwise by (2.4), and it is bounded by the function $2\|\nabla\phi\|_{L^\infty}|DW(\nabla u)(\nabla u)^T|$ which is integrable in view of (1.3), we obtain:

$$\lim_{\epsilon\to 0}\int_U \Big\langle DW(\nabla u)(\nabla u)^T : \fint_0^\epsilon \nabla\phi(\Phi(s,u)) - \nabla\phi(u)~\mathrm{d}s\Big\rangle~\mathrm{d}x = 0,$$

by the dominated convergence theorem. Similarly:

$$\lim_{\epsilon\to 0}\int_U\fint_0^\epsilon \Big\langle\Big(DW(\nabla\Phi(s,u)\nabla u) - DW(\nabla u)\Big)(\nabla u)^T : \nabla\phi(\Phi(s,u))\Big\rangle~\mathrm{d}s~\mathrm{d}x = 0,$$

where the pointwise convergence follows by the formula (2.4), its counterpart for $\nabla\Phi$, and the continuity of DW on $\mathbb{R}^{3\times 3}_+$. The integrands, for small ϵ, are dominated by the $L^1(U)$ function $4C\|\nabla\phi\|_{L^\infty}(W(\nabla u)+1)$ in view of Lemma 2.2 and the growth condition (1.3).

Therefore, the left hand side in (2.5) converges to 0 as well. This completes the proof. ■

3. The equilibrium equation (1.7)

In this section, we review several facts from [8] and [19], to set the stage for a proof of Theorem 1.1 and to rewrite the equation (1.7) using the change of variables (1.9).

The first crucial step in the dimension reduction argument of [8] is finding the appropriate approximations of the deformations gradients u^h. Under the sole assumption:

$$\frac{1}{h}\int_{\Omega^h} W(\nabla u^h)~\mathrm{d}x \leq Ch^4, \tag{3.1}$$

an application of a nonlinear verion of Korn's inequality [7], yields existence of rotation fields $R^h \in W^{1,2}(\Omega,\mathbb{R}^{3\times 3})$ with $R^h(x)\in SO(3)$ a.e. in Ω, so that:

$$\|\nabla u^h(x',hx_3) - R^h\|_{L^2(\Omega^1)} \leq Ch^2 \qquad\text{and}\qquad \|\nabla R^h\|_{L^2(\Omega)} \leq Ch. \tag{3.2}$$

Recall that $\Omega^1 = \Omega\times(-\frac{1}{2},\frac{1}{2})$ is the common domain of the rescaled deformations $y^h(x',x_3) = (\bar R^h)^T u^h(x',hx_3) - c^h$, and the typical point in Ω^1 is denoted by $x = (x',x_3)$. Then, the detailed analysis in [8] shows that convergences in (i) – (iv) of Theorem 1.1 hold, as a consequence of (1.8) implying (3.1). The constant rotations $\bar R^h \in SO(3)$ are given by:

$$\bar R^h = \mathbb{P}_{SO(3)}\left(\fint_{\Omega^h}\nabla u^h~\mathrm{d}x\right),$$

where the orthogonal projection $\mathbb{P}_{SO(3)}$ onto $SO(3)$ above is well defined; see also [10] for detailed calculations. Further, there holds:

$$\|R^h - \bar R^h\|_{L^2(\Omega)} \leq Ch \qquad\text{and}\qquad \lim_{h\to 0}(\bar R^h)^T R^h = \mathrm{Id}\quad\text{in } W^{1,2}(\Omega,\mathbb{R}^{3\times 3}), \tag{3.3}$$

and upon defining the matrix fields $A^h \in W^{1,2}(\Omega,\mathbb{R}^{3\times 3})$:

$$A^h(x') = \frac{1}{h}\Big((\bar R^h)^T R^h(x') - \mathrm{Id}\Big), \tag{3.4}$$

it also follows that:

$$A^h \rightharpoonup A = \left[\begin{array}{c|c} 0 & -\nabla v \\ \hline \nabla v & 0\end{array}\right] \qquad\text{weakly in } W^{1,2}(\Omega,\mathbb{R}^{3\times 3}). \tag{3.5}$$

The same convergence holds strongly in $L^q(\Omega,\mathbb{R}^{3\times 3})$ for each $q\geq 1$.

Lemma 3.1. *We have:*

$$\lim_{h\to 0} y^h = (x',0) \quad \text{and} \quad \lim_{h\to 0} \frac{y_3^h}{h} = x_3 + v(x') \quad \text{in } W^{1,2}(\Omega^1). \tag{3.6}$$

Consequently, for every $\omega_h > 0$ *and* $p \in [1,5]$*:*

$$\left|\left\{x \in \Omega^1;\ \frac{|y_3^h(x)|}{h} \geq \omega_h\right\}\right| \leq \frac{C}{\omega_h^2} \quad \text{and} \quad \int_{\left\{x\in\Omega^1;\ \frac{|y_3^h(x)|}{h}\geq\omega_h\right\}} \left|\frac{y_3^h(x)}{h}\right|^p \,\mathrm{d}x \leq \frac{C}{\omega_h^{\frac{2}{p+1}}}. \tag{3.7}$$

Proof. By (3.2), (3.3), and applying the Poincaré-Wirtinger inequality on segments $\{x'\}\times(-\frac{1}{2},\frac{1}{2})$, we see that:

$$\begin{aligned}\left\|\frac{y_3^h}{h} - x_3 - v^h(x')\right\|_{L^2(\Omega^1)} &\leq C\left\|\frac{\partial_3 y_3^h}{h} - 1\right\|_{L^2(\Omega^1)} = C\left\|[(\bar{R}^h)^T\nabla u^h(x',hx_3)]_{33} - 1\right\|_{L^2(\Omega^1)} \\ &\leq C\|(\bar{R}^h)^T\nabla u^h(x',hx_3) - \mathrm{Id}\|_{L^2(\Omega^1)} \\ &\leq C\|\nabla u^h(x',hx_3) - R^h\|_{L^2(\Omega^1)} + C\|R^h - \bar{R}^h\|_{L^2(\Omega^1)} \leq Ch.\end{aligned}$$

Together with (1.10), the above inequality implies the second assertion in (3.6). The first assertion follows then directly in view of (1.11).

To prove (3.7), note that for every $p \in [1,5]$:

$$\begin{aligned}\int_{\left\{x\in\Omega^1;\ \frac{|y_3^h(x)|}{h}\geq\omega_h\right\}} \left|\frac{y_3^h(x)}{h}\right|^p \,\mathrm{d}x &\leq \left\|\frac{y_3^h}{h}\right\|_{L^{p+1}}^p \left|\left\{x \in \Omega^1;\ \frac{|y_3^h(x)|}{h} \geq \omega_h\right\}\right|^{\frac{1}{p+1}} \\ &\leq C\left|\left\{x \in \Omega^1;\ \frac{|y_3^h(x)|}{h} \geq \omega_h\right\}\right|^{\frac{1}{p+1}},\end{aligned} \tag{3.8}$$

by the Hölder inequality and the Sobolev embedding $W^{1,2}(\Omega^1) \hookrightarrow L^6(\Omega^1)$ combined with (3.6). When $p = 1$, it implies:

$$\left|\left\{x \in \Omega;\ \frac{|y_3^h(x)|}{h} \geq \omega_h\right\}\right| \leq \frac{1}{\omega_h}\int_{\left\{x\in\Omega;\ \frac{|y_3^h(x)|}{h}\geq\omega_h\right\}} \frac{|y_3^h(x)|}{h}\,\mathrm{d}x \leq \frac{C}{\omega_h}\left|\left\{x \in \Omega;\ \frac{|y_3^h(x)|}{h} \geq \omega_h\right\}\right|^{1/2}$$

Hence, the first assertion in (3.7) follows, as well as the second one, in view of (3.8). ■

Define the strain $G^h \in L^2(\Omega^1,\mathbb{R}^{3\times 3})$ and the scaled stress $E^h \in L^1(\Omega^1,\mathbb{R}^{3\times 3})$ as:

$$\begin{aligned}G^h(x',x_3) &= \frac{1}{h^2}\Big((R^h)^T\nabla u^h(x',hx_3) - \mathrm{Id}\Big), \\ E^h(x',x_3) &= \frac{1}{h^2}DW(\mathrm{Id} + h^2G^h(x',x_3))(\mathrm{Id} + h^2G^h(x',x_3))^T.\end{aligned}$$

We now gather the fundamental properties of E^h and G^h from [19], that will be used in the sequel.

Lemma 3.2. *(Section 4,* [19]*)*

(i) *Up to a subsequence,* $G^h \rightharpoonup G$ *weakly in* $L^2(\Omega^1,\mathbb{R}^{3\times 3})$*, where* G *is the limiting strain whose principal* 2×2 *minor* G'' *satisfies:*

$$\begin{aligned}&G''(x',x_3) = G_0(x') - x_3G_1(x'), \qquad \text{with:} \\ &\mathrm{sym}\ G_0 = \mathrm{sym}\nabla w + \frac{1}{2}\nabla v\otimes\nabla v, \qquad G_1 = \nabla^2 v.\end{aligned} \tag{3.9}$$

(ii) *Each $E^h(x)$ is symmetric, and there holds:*

$$|E^h| \leq C\left(\frac{1}{h^2}W(\mathrm{Id} + h^2G^h) + |G^h|\right). \tag{3.10}$$

(iii) *Up to a subsequence, $E^h \rightharpoonup E$ weakly in $L^1(\Omega^1, \mathbb{R}^{3\times 3})$, and $E = \mathcal{L}_3(G) \in L^2(\Omega^1, \mathbb{R}^{3\times 3})$.*
(iv) *For a given, fixed $\gamma \in (0,2)$, define $B_h = \{x \in \Omega^1;\ h^{2-\gamma}|G^h(x)| \leq 1\}$. Then:*

$$|\Omega^1 \setminus B_h| \leq Ch^{2(2-\gamma)} \qquad \textit{and} \qquad \int_{\Omega^1\setminus B_h} |E^h|\ \mathrm{d}x \leq Ch^{2-\gamma}. \tag{3.11}$$

Moreover, calling χ_h the characteristic function of B_h, we have:

$$\chi_h E^h \rightharpoonup E \quad \textit{weakly in } L^2(\Omega^1, \mathbb{R}^{3\times 3}). \tag{3.12}$$

The below more convenient form of the equilibrium condition will be repeatedly used in the proof of Theorem 1.1.

Lemma 3.3. *Condition (1.7) is equivalent to:*

$$\begin{aligned} &\int_{\Omega^1} \Big\langle (\bar{R}^h)^T R^h E^h(x', x_3)(R^h)^T \bar{R}^h : \nabla\phi(y^h(x', x_3))\Big\rangle\ \mathrm{d}x_3\ \mathrm{d}x' \\ &\qquad\qquad = h\int_{\Omega^1} \Big\langle f(x')e_3, \bar{R}^h\phi(y^h(x', x_3))\Big\rangle\ \mathrm{d}x_3\ \mathrm{d}x', \end{aligned} \tag{3.13}$$

for each $\phi \in \mathcal{C}^1_b(\mathbb{R}^3, \mathbb{R}^3)$ with $\mathrm{div}\phi = 0$.

Proof. For a given divergence free $\phi \in \mathcal{C}^1_b(\mathbb{R}^3, \mathbb{R}^3)$, define $\psi(y) = \bar{R}^h\phi\left((\bar{R}^h)^T y - c^h\right)$, which satisfies $\psi \in \mathcal{C}^1_b$ and $\mathrm{div}\psi = 0$, and moreover $\nabla\psi\left(u^h(x', hx_3)\right) = \bar{R}^h\nabla\phi\left(y^h(x', x_3)\right)(\bar{R}^h)^T$. Use now (1.7) with the divergence-free test function ψ:

$$\begin{aligned} &\int_\Omega\int_{-1/2}^{1/2} \Big\langle DW\Big(\nabla u^h(x', hx_3)\Big)\Big(\nabla u^h(x', hx_3)\Big)^T : \bar{R}^h\nabla\phi(y^h(x', x_3))(\bar{R}^h)^T\Big\rangle\ \mathrm{d}x_3\ \mathrm{d}x' \\ &\qquad\qquad = h^3\int_\Omega\int_{-1/2}^{1/2} f(x')e_3\cdot\bar{R}^h\phi(y^h(x', x_3))\ \mathrm{d}x_3\ \mathrm{d}x'. \end{aligned}$$

The formula (3.13) follows directly, in view of:

$$\begin{aligned} DW(\nabla u^h(x', hx_3))(\nabla u^h(x', hx_3))^T &= R^h DW(\mathrm{Id} + h^2G^h(x))(\mathrm{Id} + h^2G^h(x))^T(R^h)^T \\ &= h^2R^hE^h(x', x_3)(R^h)^T. \end{aligned}$$

∎

4. Identification of the operators in (1.12) – (1.13)

Lemma 4.1. *Let $G \in \mathbb{R}^{3\times 3}$ and a symmetric matrix $E \in \mathbb{R}^{3\times 3}$ satisfy:*

$$\mathcal{L}_3(G) = E, \quad \mathrm{Tr}\ G = 0 \quad \textit{and} \quad E_{13} = E_{23} = 0.$$

Then:

$$\mathcal{L}_2^{in}(G'') = E'' - E_{33}\mathrm{Id}_2. \tag{4.1}$$

Proof. Since $\mathcal{L}$ and $\mathcal{Q}$ depend only on the symmetric parts of their arguments, we may without loss of generality assume that G is symmetric.

Firstly, by definitions in (1.5), (1.6), it follows that for every $F'' \in \mathbb{R}^{2\times 2}$ there is a unique tangential minimizer $d = d(F'') \in \mathbb{R}^2$, in the sense that:

(4.2)
$$\mathcal{Q}_2^{in}(F'') = \mathcal{Q}_3(\begin{bmatrix} F'' & d \\ d & -\mathrm{Tr}\ F'' \end{bmatrix}) \quad \text{and} \quad \Big\langle \mathcal{L}_3(\begin{bmatrix} F'' & d \\ d & -\mathrm{Tr}\ F'' \end{bmatrix}) : \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \Big\rangle = 0 \quad \forall c \in \mathbb{R}^2.$$

The second identity above is just the Euler-Lagrange equation for the minimization in (1.5). By convexity of this minimization problem, it also follows that d is linear:

$$d(F'' + G'') = d(F'') + d(G'') \tag{4.3}$$

Observe now that:

$$\begin{aligned}
\mathcal{Q}_2(G'') &= \mathcal{Q}_3(\begin{bmatrix} G'' & d(G'') \\ d(G'') & G_{33} \end{bmatrix}) = \Big\langle \mathcal{L}_3(\begin{bmatrix} G'' & d(G'') \\ d(G'') & G_{33} \end{bmatrix}) : \begin{bmatrix} G'' & d(G'') \\ d(G'') & G_{33} \end{bmatrix} \Big\rangle \\
&= \Big\langle \Big(E + \mathcal{L}_3(\begin{bmatrix} 0 & d(G'') - G_{13,23} \\ d(G'') - G_{13,23} & 0 \end{bmatrix})\Big) : \begin{bmatrix} G'' & d(G'') \\ d(G'') & G_{33} \end{bmatrix} \Big\rangle \\
&= \langle E'' : G'' \rangle + E_{33} G_{33} \\
&\qquad + \Big\langle \mathcal{L}_3(\begin{bmatrix} G'' & d(G'') \\ d(G'') & G_{33} \end{bmatrix}) : \begin{bmatrix} 0 & d(G'') - G_{13,23} \\ d(G'') - G_{13,23} & 0 \end{bmatrix} \Big\rangle \\
&= \langle E'' : G'' \rangle + E_{33} G_{33} = \langle E : G \rangle = \mathcal{Q}_3(G),
\end{aligned}$$

where we repeatedly used the assumptions on G and E, and (4.2). Consequently, by uniqueness of the minimizer d, it follows that:

$$d(G'') = G_{13,23}. \tag{4.4}$$

Take any $F'' \in \mathbb{R}^{2\times 2}$. By (4.2) and (4.3), we see that:

$$\mathcal{Q}_2(G'' + F'') = \mathcal{Q}_3(\begin{bmatrix} G'' + F'' & d(G'') + d(F'') \\ d(G'') + d(F'') & G_{33} - \mathrm{Tr}\ F'' \end{bmatrix}).$$

Expanding the above and removing $\mathcal{Q}_2(G'')$ and $\mathcal{Q}_2(F'')$ from both sides, we obtain:

$$\begin{aligned}
\langle \mathcal{L}_2(G'') : F'' \rangle &= \Big\langle \mathcal{L}_3(\begin{bmatrix} G'' & d(G'') \\ d(G'') & -\mathrm{Tr}\ G'' \end{bmatrix}) : \begin{bmatrix} F'' & d(F'') \\ d(F'') & -\mathrm{Tr}\ F'' \end{bmatrix} \Big\rangle \\
&= \Big\langle \mathcal{L}_3(\begin{bmatrix} G'' & d(G'') \\ d(G'') & -\mathrm{Tr}\ G'' \end{bmatrix}) : \begin{bmatrix} F'' & 0 \\ 0 & -\mathrm{Tr}\ F'' \end{bmatrix} \Big\rangle \\
&= \Big\langle \mathcal{L}_3(G) : \begin{bmatrix} F'' & d(F'') \\ d(F'') & -\mathrm{Tr}\ F'' \end{bmatrix} \Big\rangle = \Big\langle E : \begin{bmatrix} F'' & d(F'') \\ d(F'') & -\mathrm{Tr}\ F'' \end{bmatrix} \Big\rangle \\
&= \langle E'' - E_{33}\mathrm{Id}_2 : F'' \rangle,
\end{aligned}$$

by (4.4) and assumptions on E and G. The expression (4.1) follows now directly. ■

In section 5 below we shall prove that for almost every $x \in \Omega^1$ there holds:

$$\mathrm{Tr}\ G(x) = 0 \quad \text{and} \quad E_{13}(x) = E_{23}(x) = 0. \tag{4.5}$$

Therefore, recalling Lemma 3.2 (iii), we observe that the limiting stress and strain satisfy the assumptions of Lemma 4.1 pointwise almost everywhere. We now record the following simple conclusion which will be used in deriving the Euler-Lagrange equations (1.12), (1.13).

Lemma 4.2. *Let $E, G \in L^2(\Omega^1, \mathbb{R}^{3\times 3})$ be the limiting strain and stress as in Lemma 3.2, which are related to (w,u) by (3.9). Then, for almost every $x' \in \Omega$, there holds:*

$$\begin{aligned}\int_{-1/2}^{1/2} (E'' - E_{33}\mathrm{Id}_2)\ \mathrm{d}x_3 &= \mathcal{L}_2^{in}\left(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v \otimes \nabla v\right), \\ \int_{-1/2}^{1/2} x_3(E'' - E_{33}\mathrm{Id}_2)\ \mathrm{d}x_3 &= -\frac{1}{12}\mathcal{L}_2^{in}\left(\nabla^2 v\right).\end{aligned} \tag{4.6}$$

Proof. By Lemma 5.1, Lemma 5.3, Lemma 4.1 and (3.9) we see that:

$$\begin{aligned}\int_{-1/2}^{1/2} (E'' - E_{33}\mathrm{Id}_2)\ \mathrm{d}x_3 &= \int_{-1/2}^{1/2} \mathcal{L}_2^{in}(G'')\ \mathrm{d}x_3 \\ &= \mathcal{L}_2^{in}\left(\int_{-1/2}^{1/2} G''(x', x_3)\ \mathrm{d}x_3\right) = \mathcal{L}_2^{in}(G_0(x')) = \mathcal{L}_2^{in}(\mathrm{sym}\ G_0(x'))\end{aligned}$$

$$\begin{aligned}\int_{-1/2}^{1/2} x_3(E'' - E_{33}\mathrm{Id}_2)\ \mathrm{d}x_3 &= \int_{-1/2}^{1/2} x_3\mathcal{L}_2^{in}(G'')\ \mathrm{d}x_3 \\ &= \mathcal{L}_2^{in}\left(\int_{-1/2}^{1/2} x_3 G''(x', x_3)\ \mathrm{d}x_3\right) = -\mathcal{L}_2^{in}\left(\int_{-1/2}^{1/2} x_3^2 G_1(x')\ \mathrm{d}x_3\right) = -\frac{1}{12}\mathcal{L}_2^{in}(G_1(x')).\end{aligned}$$

This concludes the proof, in view of (3.9). ■

5. Two further properties of G and E

In this section we derive the two fundamental properties of the incompressible stress and strain, allowing for pointwise application of Lemma 4.1, and ultimately leading to formulas in (4.6).

Lemma 5.1. *The limiting strain $G(x)$ is traceless, for almost every $x \in \Omega^1$.*

Proof. Recall that $\nabla u^h(x', hx_3) = R^h(x')\big(\mathrm{Id} + h^2 G^h(x', x_3)\big)$. Therefore:

$$1 = \det \nabla u^h = \det(\mathrm{Id} + h^2 G^h) = 1 + h^2 \mathrm{Tr}\ G^h + h^4 \mathrm{Tr}\ \mathrm{cof}\ G^h + h^6 \det G^h,$$

and consequently:

$$\mathrm{Tr}\ G^h + h^2 \mathrm{Tr}\ \mathrm{cof}\ G^h + h^4 \det G^h = 0. \tag{5.1}$$

Fix an exponent $\gamma \in (\frac{2}{3}, 2)$ and define $B_h = \{x \in \Omega^1;\ h^{2-\gamma}|G^h(x) \leq 1\}$ as in Lemma 3.2 (iv). Then:

$$\begin{aligned}\int_{\Omega^1\setminus B_h} |h^4 \det G^h| &= \int_{\Omega^1\setminus B_h} |\mathrm{Tr}\ G^h + h^2 \mathrm{Tr}\ \mathrm{cof}\ G^h| \\ &\leq |\Omega^1 \setminus B_h|^{1/2}\left(\int_{\Omega^1\setminus B_h} |\mathrm{Tr}\ G^h|^2\right)^{1/2} + h^2\int_{\Omega^1} |\mathrm{Tr}\ \mathrm{cof}\ G^h| \leq C(h^{2-\gamma} + h^2),\end{aligned}$$

where we used (3.11) and the boundedness of G^h in $L^2(\Omega^1)$. On the other hand, we have:

$$\int_{B_h} |h^4 \det G^h| = \frac{h^4}{h^{6-3\gamma}}\int_{B_h} |\det(h^{2-\gamma}G^h)| \leq Ch^{3\gamma-2}.$$

Hence, by (5.1) and, again the boundedness of Tr cof G^h in $L^1(\Omega^1)$, it follows that:

$$\int_{\Omega^1} |\mathrm{Tr}\ G^h| \leq \int_{\Omega^1} |h^2 \mathrm{Tr}\ \mathrm{cof}\ G^h| + \int_{\Omega^1} |h^4 \det G^h| \to 0, \quad \text{as } h \to 0.$$

Observing that Tr $G^h \rightharpoonup$ Tr G weakly in $L^2(\Omega^1)$, we conclude that Tr $G = 0$. ■

We now prove the remaining property of the strain E in (4.5). The strategy of proof is the same as in the later proofs of the Euler-Lagrange equations; we will apply the equilibrium equation (3.13) to appropriate test functions ϕ^h, such that after passing to the limit with $h \to 0$ only some chosen terms will survive, yielding the week formulation of (4.5). One difficulty with (3.13) is that it only allows for globaly bounded ϕ^h. For this reason, following [19], we introduce a family of truncation functions θ^h which coincide with the identity on intervals $(-\omega_h, \omega_h)$ with a suitable rate of convergence of $\omega_h \to \infty$.

Lemma 5.2. *Let $\{\omega_h\}$ be a sequence of positive numbers, increasing to $+\infty$ as $h \to 0$. There exists a sequence of nondecreasing functions $\theta^h \in \mathcal{C}^2_b(\mathbb{R}, \mathbb{R})$ with the following properties:*

$$\begin{aligned} &\theta^h(t) = t \quad \forall |t| \leq \omega_h \qquad \textit{and} \qquad \theta^h(t) = (\mathrm{sgn}\ t)\frac{3}{2}\omega_h \quad \forall |t| \geq 2\omega_h \\ &|\theta^h(t)| \leq t \quad \forall t \qquad \textit{and} \qquad \|\theta^h\|_{L^\infty} \leq \frac{3}{2}\omega_h \\ &\|\frac{\mathrm{d}}{\mathrm{d}t}\theta^h\|_{L^\infty} \leq 1 \qquad \textit{and} \qquad \|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\theta^h\|_{L^\infty} \leq \frac{C}{\omega_h}. \end{aligned} \tag{5.2}$$

Proof. One may take:

$$\theta^h(t) = \begin{cases} t & \text{for } |t| \leq \omega_h \\ (\mathrm{sgn}\ t)\frac{1}{2}\left(|t| + \omega_h + \frac{\omega_h}{\pi}\sin\left(\frac{\pi|t| - \omega_h}{\omega_h}\right)\right) & \text{for } |t| \in [\omega_h, 2\omega_h] \\ (\mathrm{sgn}\ t)\frac{3}{2}\omega_h & \text{for } |t| \geq \omega_h \end{cases}$$

■

Lemma 5.3. *The limiting stress $E(x)$ satisfies: $E_{13}(x) = E_{23}(x) = 0$ for almost every $x \in \Omega^1$.*

Proof. **1.** Let $\eta = (\eta_1, \eta_2) \in \mathcal{C}^2_b(\mathbb{R}^3, \mathbb{R}^2)$ be a given test function, and define:

$$\eta_3(x', x_3) = -\int_0^{x_3} \mathrm{div}\ \eta(x', s)\ \mathrm{d}s. \tag{5.3}$$

Since $\partial_3\eta_3 = -\mathrm{div}\ \eta$, the following test functions $\phi^h \in \mathcal{C}^1_b(\mathbb{R}^3, \mathbb{R}^3)$ are divergence-free:

$$\phi^h(x', x_3) = \begin{bmatrix} h\theta^{h\prime}\left(\frac{x_3}{h}\right)\eta\left(x', \theta^h\left(\frac{x_3}{h}\right)\right) \\ h^2\eta_3\left(x', \theta^h\left(\frac{x_3}{h}\right)\right) \end{bmatrix},$$

and denoting ∇_{tan} the gradient in the tangential directions e_1, e_2, we have:

$$\nabla\phi^h(x', x_3) = \left[\begin{array}{c|c} h\theta^{h\prime}\left(\frac{x_3}{h}\right)\nabla_{tan}\eta\left(x', \theta^h\left(\frac{x_3}{h}\right)\right) & \begin{array}{r}\left(\theta^{h\prime}\left(\frac{x_3}{h}\right)\right)^2\partial_3\eta\left(x', \theta^h\left(\frac{x_3}{h}\right)\right) \\ + \theta^{h\prime\prime}\left(\frac{x_3}{h}\right)\eta\left(x', \theta^h\left(\frac{x_3}{h}\right)\right)\end{array} \\ \hline h^2\nabla_{tan}\eta_3\left(x', \theta^h\left(\frac{x_3}{h}\right)\right) & h\theta^{h\prime}\left(\frac{x_3}{h}\right)\partial_3\eta_3\left(x', \theta^h\left(\frac{x_3}{h}\right)\right) \end{array}\right].$$

The truncations θ^h are chosen as in Lemma 5.2 and such that:

$$\lim_{h\to 0}\omega_h = +\infty \quad \text{and} \quad h^2\omega_h \leq C. \tag{5.4}$$

2. Applying the equilibrium equation (3.13) with $\phi = \phi^h$, we obtain:

(5.5)

$$\begin{aligned} h\int_{\Omega^1} \Big\langle \Big((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\Big)'' - \Big((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\Big)_{33}\mathrm{Id}_2 : \theta^{h\prime}\left(\frac{y_3^h}{h}\right)\nabla_{tan}\eta(y^{h\prime}, \theta^h\left(\frac{y_3^h}{h}\right))\Big\rangle \\ + \int_{\Omega^1} \Big\langle \Big((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\Big)_{13,23}, (\theta^{h\prime}\left(\frac{y_3^h}{h}\right))^2\partial_3\eta(y^{h\prime}, \theta^h\left(\frac{y_3^h}{h}\right))\Big\rangle \\ + \int_{\Omega^1} \Big\langle \Big((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\Big)_{13,23}, \theta^{h\prime\prime}\left(\frac{y_3^h}{h}\right)\eta(y^{h\prime}, \theta^h\left(\frac{y_3^h}{h}\right))\Big\rangle \\ + h^2\int_{\Omega^1} \Big\langle \Big((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\Big)_{31,32}, \nabla_{tan}\eta_3(y^{h\prime}, \theta^h\left(\frac{y_3^h}{h}\right))\Big\rangle \\ = h^2\int_{\Omega^1} \Big\langle f(x')(\bar R^h)_{31,32}, \eta(y^{h\prime}, \theta^h\left(\frac{y_3^h}{h}\right))\Big\rangle + h^3\int_{\Omega^1} f(x')(\bar R^h)_{33}\eta_3(y^{h\prime}, \theta^h\left(\frac{y_3^h}{h}\right)). \end{aligned}$$

Now, we will discuss the convergence as $h \to 0$ of each term in (5.5). The first term converges to 0, because $\Big((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\Big)'' - \Big((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\Big)_{33}\mathrm{Id}_2$ is bounded in $L^1(\Omega^1)$ in view of Lemma 3.2 (iii), while $\theta^{h\prime}\left(\frac{y_3^h}{h}\right)\nabla_{tan}\eta(y^{h\prime}, \theta^h\left(\frac{y_3^h}{h}\right))$ is pointwise bounded by (5.2).

3. The second term in (5.5) when integrated over $\Omega^1 \setminus B_h$, goes to 0 in view of (3.11) and of the pointwise boundedness of $(\theta^{h\prime}\left(\frac{y_3^h}{h}\right))^2\partial_3\eta(y^{h\prime}, \theta^h\left(\frac{y_3^h}{h}\right))$ by (5.2). On the other hand, the limit of this integral over B_h is the same as the limit of:

$$\int_{\Omega^1} \Big\langle \chi_h E^h_{13,23}, (\theta^{h\prime}\left(\frac{y_3^h}{h}\right))^2\partial_3\eta(y^{h\prime}, \theta^h\left(\frac{y_3^h}{h}\right))\Big\rangle \,\mathrm{d}x \tag{5.6}$$

because of (3.3). We now conclude that the integrals in (5.6) converge to:

$$\int_{\Omega^1} \Big\langle E_{13,23}, \partial_3\eta(x', x_3 + v(x'))\Big\rangle \,\mathrm{d}x.$$

This follows by recalling (3.12) and observing that:

$$(\theta^{h\prime}\left(\frac{y_3^h}{h}\right))^2\partial_3\eta(y^{h\prime}, \theta^h\left(\frac{y_3^h}{h}\right)) \to \partial_3\eta(x', x_3 + v(x')) \quad \text{in } L^2(\Omega^1) \tag{5.7}$$

Indeed:

$$\begin{aligned} &\int_{\Omega^1} \left|(\theta^{h\prime}\left(\frac{y_3^h}{h}\right))^2\partial_3\eta(y^{h\prime}, \theta^h\left(\frac{y_3^h}{h}\right)) - \partial_3\eta(x', x_3 + v(x'))\right|^2 \,\mathrm{d}x \\ &\quad \leq C\int_{\Omega^1} \left|\theta^{h\prime}\left(\frac{y_3^h}{h}\right)\right|^4 \left(|y^{h\prime} - x'|^2 + \left|\theta^h\left(\frac{y_3^h}{h}\right) - (x_3 + v(x'))\right|^2\right) \,\mathrm{d}x \\ &\quad\quad + C\int_{\Omega^1} \left|\theta^{h\prime}\left(\frac{y_3^h}{h}\right) - 1\right|^2 \,\mathrm{d}x \\ &\quad \leq C\int_{\Omega^1} |y^{h\prime} - x'|^2 + \left|\frac{y_3^h}{h} - (x_3 + v(x'))\right|^2 \,\mathrm{d}x + C\int_{\left\{x\in\Omega^1;\ \frac{|y_3^h|}{h}\geq\omega_h\right\}} 1 + \left|\frac{y_3^h}{h}\right|^2 \,\mathrm{d}x \end{aligned}$$

converges to 0 as $h \to 0$, by (3.6), (3.7) and (5.4), proving hence (5.7).

4. The third term in (5.5) is bounded by: $\frac{C}{\omega_h}\int_{\Omega^1}|E^h|$ by (5.2). It therefore converges to 0 in view of the boundedness of E^h in $L^1(\Omega^1)$ and (5.4).

The fourth term in (5.5) is bounded by:

$$Ch^2\int_{\Omega^1}|E^h|\left|\theta^h\left(\frac{y_3^h}{h}\right)\right|\ \mathrm{d}x \le Ch^2\omega_h\int_{\Omega^1\setminus B_h}|E^h| + Ch^2\int_{\Omega^1}\chi_h|E^h|\frac{|y_3^h|}{h}$$
$$\le Ch^2\omega_h\ o(1) + Ch^2\|\chi_h E^h\|_{L^2(\Omega^1)}\left\|\frac{y_3^h}{h}\right\|_{L^2(\Omega^1)},$$

and it converges to 0 by (3.11), (3.12), (5.4) and the boundedness of $\frac{y_3^h}{h}$ in $L^2(\Omega^1)$.

Finally, both terms in the right hand side of (5.5) are bounded by:

$$Ch^2\int_{\Omega^1}|f(x')|\left(\left|\theta^{h\prime}\left(\frac{y_3^h}{h}\right)\right| + h\left|\theta^h\left(\frac{y_3^h}{h}\right)\right|\right)\ \mathrm{d}x \le Ch^2\int_{\Omega^1}|f(x')|(1+h\omega_h)\ \mathrm{d}x \le Ch\|f\|_{L^2(\Omega)},$$

which clearly converges to 0. Above, we used (5.2) and (5.4).

5. In conclusion, passing to the limit with $h \to 0$ in (5.5), results in:

$$\int_{\Omega^1}\Big\langle E_{13,23}, \partial_3\eta(x', x_3 + v(x'))\Big\rangle\ \mathrm{d}x = 0 \qquad \forall \eta \in \mathcal{C}_b^2(\mathbb{R}^3, \mathbb{R}^2). \tag{5.8}$$

We now reproduce an argument from [19], in order to deduce that $E_{13,23} = 0$. Take an arbitrary $\phi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^2)$. Let $\mathcal{C}_c^\infty(\Omega, \mathbb{R}) \ni v_k \to v$ in $L^2(\Omega)$, and define:

$$\phi_k(x', x_3) = \phi(x', x_3 - v_k(x')), \qquad \eta(x', x_3) = \int_0^{x_3}\phi_k(x', s)\ \mathrm{d}s$$

Clearly $\phi_k \in \mathcal{C}_c^2(\mathbb{R}^3, \mathbb{R}^2)$, $\eta \in \mathcal{C}_b^2(\mathbb{R}^3, \mathbb{R}^2)$, and thus by (5.8) we obtain:

$$0 = \int_{\Omega^1}\Big\langle E_{13,23}, \phi_k(x', x_3 + v(x'))\Big\rangle\ \mathrm{d}x = \int_{\Omega^1}\Big\langle E_{13,23}, \phi(x', x_3 + v(x') - v_k(x'))\Big\rangle\ \mathrm{d}x$$

Passing to the limit with $k \to \infty$, it follows that:

$$\int_{\Omega^1} E_{13,23}\phi(x', x_3)\ \mathrm{d}x = 0 \qquad \forall \phi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^2)$$

which concludes the proof. ■

6. Derivation of the first Euler-Lagrange equation (1.12)

1. Let $\eta = (\eta_1, \eta_2) \in \mathcal{C}_b^2(\mathbb{R}^2, \mathbb{R}^2)$ be a given test function, and let $\eta_3(x') = -\mathrm{div}\ \eta(x')$. Given θ^h as in Lemma 5.2, with:

$$\lim_{h\to 0}\omega_h = \lim_{h\to 0}h\omega_h^2 = +\infty \quad \text{and} \quad h\omega_h \le C, \tag{6.1}$$

consider the following divergence-free test functions $\phi^h \in \mathcal{C}_b^1(\mathbb{R}^3, \mathbb{R}^3)$:

$$\phi^h(x', x_3) = \begin{bmatrix} \theta^{h\prime}\left(\frac{x_3}{h}\right)\eta(x') \\ h\theta^h\left(\frac{x_3}{h}\right)\eta_3(x') \end{bmatrix},$$

Denoting ∇_{tan} the gradient in the tangential directions e_1, e_2, we have:

$$\nabla\phi^h(x', x_3) = \left[\begin{array}{c|c} \theta^{h'}\left(\frac{x_3}{h}\right)\nabla_{tan}\eta(x') & \frac{1}{h}\theta^{h''}\left(\frac{x_3}{h}\right)\eta(x') \\ \hline h\theta^h\left(\frac{x_3}{h}\right)\nabla_{tan}\eta_3(x') & \theta^{h'}\left(\frac{x_3}{h}\right)\eta_3(x') \end{array}\right].$$

2. Applying the equilibrium equation (3.13) with $\phi = \phi^h$, we obtain:

(6.2)
$$\begin{aligned}
&\int_{\Omega^1}\Big\langle\Big((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\Big)'' - \Big((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\Big)_{33}\mathrm{Id}_2 : \theta^{h'}\left(\frac{y_3^h}{h}\right)\nabla_{tan}\eta(y^{h'})\Big\rangle \\
&\qquad + h\int_{\Omega^1}\Big\langle\Big((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\Big)_{31,32}, \theta^h\left(\frac{y_3^h}{h}\right)\nabla_{tan}\eta_3(y^{h'})\Big\rangle \\
&\qquad + \frac{1}{h}\int_{\Omega^1}\Big\langle\Big((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\Big)_{13,23}, \theta^{h''}\left(\frac{y_3^h}{h}\right)\eta(y^{h'})\Big\rangle \\
&= h\int_{\Omega^1}\Big\langle f(x')(\bar R^h)_{31,32}, \theta^{h'}\left(\frac{y_3^h}{h}\right)\eta(y^{h'})\Big\rangle\ \mathrm{d}x + h^2\int_{\Omega^1} f(x')(\bar R^h)_{33}\theta^h\left(\frac{y_3^h}{h}\right)\eta_3(y^{h'})\ \mathrm{d}x.
\end{aligned}$$

Now, we will check convergence as $h \to 0$ of each of the four terms in the identity (6.2). Regarding the first term, it converges to 0 when integrated over $\Omega^1 \setminus B_h$, by (3.11) and by the pointwise boundedness of $\theta^{h'}\left(\frac{y_3^h}{h}\right)\nabla_{tan}\eta(y^{h'})$ in view of (5.2). On the other hand, the limit of this integral over B_h is the same as the limit of:

$$\int_{\Omega^1}\Big\langle\chi_h\big(E^{h''} - E^h_{33}\mathrm{Id}_2\big) : \theta^{h'}\left(\frac{y_3^h}{h}\right)\nabla_{tan}\eta(y^{h'})\Big\rangle\ \mathrm{d}x, \tag{6.3}$$

because of the convergence in (3.3). Now, the limit of integrals in (6.3) equals:

$$\int_{\Omega^1}\Big\langle E'' - E_{33}\mathrm{Id}_2 : \nabla\eta(x')\Big\rangle\ \mathrm{d}x,$$

in view of (3.12) and:

$$\begin{aligned}
&\int_{\Omega^1}\left|\theta^{h'}\left(\frac{y_3^h}{h}\right)\nabla_{tan}\eta(y^{h'}) - \nabla\eta(x')\right|^2\ \mathrm{d}x \\
&\qquad \le C\int_{\Omega^1}\left|\nabla_{tan}\eta(y^{h'}) - \nabla\eta(x')\right|^2 + C\int_{\Omega^1}\left|\theta^{h'}\left(\frac{y_3^h}{h}\right) - 1\right|^2 \\
&\qquad \le C\int_{\Omega^1}\left|y^{h'} - x'\right|^2\ \mathrm{d}x + C\left|\left\{x \in \Omega^1;\ \frac{|y_3^h(x)|}{h} \ge \omega_h\right\}\right| \\
&\qquad \le C\int_{\Omega^1}\left|y^{h'} - x'\right|^2\ \mathrm{d}x + \frac{C}{\omega_h^2},
\end{aligned}$$

where we apply (3.7), and then (3.7) and (6.1) to conclude the convergence of both terms in the right hand side of the above displayed expression to 0.

3. The second term in (6.2) is bounded by:

$$\begin{aligned} Ch\int_{\Omega^1\setminus B_h} \theta^h\left(\frac{|y_3^h|}{h}\right)|E^h|\ \mathrm{d}x + Ch\int_{\Omega^1}|\chi_h E^h|\frac{|y_3^h|}{h}\ \mathrm{d}x \\ \leq Ch\omega_h\int_{\Omega^1\setminus B_h}|E^h|\ \mathrm{d}x + C\|y_3^h\|_{L^2(\Omega^1)}\|\chi_h E^h\|_{L^2(\Omega^1)} \end{aligned}$$

and it clearly converges to 0 by (3.11), (3.12), (3.6) and (6.1).

The third term in (6.2) is bounded by:

$$\begin{aligned} \frac{C}{h\omega_h}\int_{\left\{x\in\Omega^1;\ \frac{|y_3^h(x)|}{h}\geq\omega_h\right\}}|E^h|\ \mathrm{d}x &\leq \frac{C}{h\omega_h}\int_{\left\{x\in\Omega^1;\ \frac{|y_3^h(x)|}{h}\geq\omega_h\right\}}\frac{1}{h^2}W(\mathrm{Id}+h^2G^h)+|G^h|\ \mathrm{d}x \\ &\leq \frac{C}{h^3\omega_h}\int_{\Omega^1}W\big(\nabla u^h(x',hx_3)\big)\ \mathrm{d}x + \frac{C}{h\omega_h}\|G^h\|_{L^2(\Omega^1)}\left|\left\{x\in\Omega^1;\ \frac{|y_3^h(x)|}{h}\geq\omega_h\right\}\right|^{1/2} \\ &\leq C\left(\frac{h}{\omega_h}+\frac{1}{h\omega_h^2}\right), \end{aligned}$$

by (3.10), (3.7), the boundedness of G^h in $L^2(\Omega^1)$ and (1.8). Then, the right hand side above converges to 0 by (6.1).

Finally, the right hand side of (6.2) converges to 0 as well, as it is bounded by:

$$Ch\int_{\Omega^1}|f(x')|(1+h\omega_h)\ \mathrm{d}x \leq Ch\|f\|_{L^2(\Omega)}.$$

In conclusion, passing to the limit with $h\to 0$ in (6.2) we obtain:

$$\int_{\Omega^1}\Big\langle E''-E_{33}\mathrm{Id}_2 : \nabla\eta(x')\Big\rangle\ \mathrm{d}x = 0 \quad \forall\eta\in\mathcal{C}_b^2(\mathbb{R}^2,\mathbb{R}^2). \tag{6.4}$$

and thus the Euler-Lagrange equation (1.12) follows directly, in view of (4.6) and the density of test functions η as above in $W^{1,2}(\Omega,\mathbb{R}^2)$. ■

7. Derivation of the second Euler-Lagrange equation (1.13)

Lemma 7.1. *For every $\eta_3\in\mathcal{C}_b^3(\mathbb{R}^2,\mathbb{R})$, it follows that:*

$$\begin{aligned} \int_{\Omega^1}\Big\langle(E''-E_{33}\mathrm{Id}_2):\nabla v\otimes\nabla\eta_3\Big\rangle\mathrm{d}x + \lim_{h\to 0}\frac{1}{h}\int_{\Omega^1}\Big\langle E^h_{31,32},\nabla\eta_3(y^{h'})\Big\rangle\ \mathrm{d}x \\ = \bar{R}_{33}\int_{\Omega}f(x')\eta_3(x')\ \mathrm{d}x'. \end{aligned} \tag{7.1}$$

Proof. **1.** Given $\eta_3\in\mathcal{C}_b^3(\mathbb{R}^2,\mathbb{R})$ consider the divergence-free test functions $\phi^h\in\mathcal{C}_b^!(\mathbb{R}^3,\mathbb{R}^3)$:

$$\phi^h(x',x_3) = \begin{bmatrix} 0 \\ \frac{1}{h}\eta_3(x') \end{bmatrix}, \quad \text{so that} \quad \nabla\phi^h(x',x_3) = \left[\begin{array}{c|c} 0 & 0 \\ \hline \frac{1}{h}\nabla_{tan}\eta_3(x') & 0 \end{array}\right].$$

Applying the equilibrium equation (3.13) with $\phi=\phi^h$, we obtain:

$$\frac{1}{h}\int_{\Omega^1}\Big\langle\Big((\bar{R}^h)^T R^h E^h (R^h)^T\bar{R}^h\Big)_{31,32},\nabla_{tan}\eta_3(y^{h'})\Big\rangle\ \mathrm{d}x = \bar{R}^h_{33}\int_{\Omega^1}f(x')\eta_3(y^{h'})\ \mathrm{d}x. \tag{7.2}$$

Recall that the tensor field A^h in (3.4) is defined as: $A^h(x') = \frac{1}{h}\big((\bar{R}^h)^T R^h(x') - \mathrm{Id}\big)$. Hence:

$$\frac{1}{h}(\bar{R}^h)^T R^h E^h (R^h)^T \bar{R}^h = A^h E^h (R^h)^T \bar{R}^h + E^h (A^h)^T + \frac{1}{h}E^h, \tag{7.3}$$

and therefore the left hand side of (7.2) can be written as:

$$\begin{aligned}&\int_{\Omega^1} \Big\langle (A^h E^h (R^h)^T \bar{R}^h)_{31,32}, \nabla\eta_3(y^{h'})\Big\rangle ~\mathrm{d}x \\ &\qquad + \int_{\Omega^1} \Big\langle (E^h (A^h)^T)_{31,32}, \nabla\eta_3(y^{h'})\Big\rangle ~\mathrm{d}x + \frac{1}{h}\int_{\Omega^1} \Big\langle E^h_{31,32}, \nabla\eta_3(y^{h'})\Big\rangle ~\mathrm{d}x.\end{aligned} \tag{7.4}$$

2. Let the sets B_h be defined as in Lemma 3.2 (iv), for some exponent $\gamma \in (0,1)$. The first two terms in (7.4), when considered on $\Omega^1 \setminus B_h$, converge to 0 because they are bounded by:

$$C\int_{\Omega^1\setminus B_h} |A^h||E^h| ~\mathrm{d}x \leq \frac{C}{h}\int_{\Omega^1\setminus B_h} |E^h| ~\mathrm{d}x \leq \frac{C}{h} h^{2-\gamma},$$

in view of (3.11) and $|A^h| \leq \frac{C}{h}$. On the other hand, the same two terms while on B_h, converge to:

$$\int_{\Omega^1} \big\langle (AE)_{31,32}, \nabla\eta_3(x')\big\rangle + \big\langle (EA^T)_{31,32}, \nabla\eta_3(x')\big\rangle ~\mathrm{d}x,$$

where we used the convergence (3.12) and the following strong convergences in $L^3(\Omega^1)$: of A^h to A by (3.5), of $(R^h)^T\bar{R}^h$ to Id by (3.3), and of $\nabla\eta_3(y^{h'})$ to $\nabla\eta_3(x')$ in view of the Sobolev embedding and the strong convergence in $W^{1,2}(\Omega^1,\mathbb{R}^2)$ in (3.6).

Concluding, the first two terms in (7.4) converge to:

$$\int_{\Omega^1} \Big\langle E''\nabla v, \nabla\eta_3(x')\Big\rangle - \Big\langle E_{33}\nabla v, \nabla\eta_3(x')\Big\rangle ~\mathrm{d}x$$

in view of the structure of the limiting tensor A in (3.5). Since the right hand side of (7.2) converges to $\bar{R}_{33}\int_\Omega f(x')\eta_3(x')$ by (3.6), passing to the limit in all terms of (7.2) yields the desired equality (7.1) and thus proves the lemma. ■

Lemma 7.2. *For every $\eta \in \mathcal{C}^2_b(\mathbb{R}^2,\mathbb{R}^2)$, it follows that:*

$$\begin{aligned}&\int_{\Omega^1} \Big\langle (E'' - E_{33}\mathrm{Id}_2) : (x_3 + v(x'))\nabla_{tan}\eta(x')\Big\rangle ~\mathrm{d}x \\ &\qquad + \int_{\Omega^1} \Big\langle (E'' - E_{33}\mathrm{Id}_2) : \nabla v(x') \otimes \eta(x')\Big\rangle ~\mathrm{d}x ~+~ \lim_{h\to 0}\frac{1}{h}\int_{\Omega^1} \Big\langle E^h_{13,23}, \nabla\eta_3(y^{h'})\Big\rangle ~\mathrm{d}x = 0.\end{aligned} \tag{7.5}$$

Proof. **1.** Let $\eta \in \mathcal{C}^2_b(\mathbb{R}^2,\mathbb{R}^2)$ be a given test function, and define $\eta_3(x') = -\mathrm{div}~\eta(x')$. Given θ^h as in Lemma 5.2, with:

$$\lim_{h\to 0}\omega_h = \lim_{h\to 0} h\omega_h = +\infty \quad \text{ and } \quad \lim_{h\to 0} h^{1+\frac{1-\gamma}{2}}\omega_h = 0 ~~\text{for some fixed } \gamma \in (0,1), \tag{7.6}$$

consider the divergence-free test functions $\phi^h \in \mathcal{C}^1_b(\mathbb{R}^3,\mathbb{R}^3)$:

$$\phi^h(x',x_3) = \left[\begin{array}{c} \theta^{h'}\left(\frac{x_3}{h}\right)\theta^h\left(\frac{x_3}{h}\right)\eta(x') \\ \frac{h}{2}(\theta^h\left(\frac{x_3}{h}\right))^2\eta_3(x')\end{array}\right].$$

Denoting ∇_{tan} the gradient in the tangential directions e_1, e_2, we have:

$$\nabla\phi^h(x',x_3) = \left[\begin{array}{c|c} \theta^{h\prime}\left(\frac{x_3}{h}\right)\theta^h\left(\frac{x_3}{h}\right)\nabla_{tan}\eta(x') & \frac{1}{h}\left(\theta^{h\prime\prime}\left(\frac{x_3}{h}\right)\theta^h\left(\frac{x_3}{h}\right) + (\theta^h\left(\frac{x_3}{h}\right))^2\right)\eta(x') \\ \hline \frac{h}{2}(\theta^h\left(\frac{x_3}{h}\right))^2\nabla_{tan}\eta_3(x') & \theta^{h\prime}\left(\frac{x_3}{h}\right)\theta^h\left(\frac{x_3}{h}\right)\eta_3(x') \end{array}\right].$$

2. Applying now the equilibrium equation (3.13) with $\phi = \phi^h$, we obtain:

(7.7)
$$\begin{aligned}
&\int_{\Omega^1}\left\langle\left((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\right)'' - \left((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\right)_{33}\mathrm{Id}_2 : \theta^{h\prime}\left(\frac{y_3^h}{h}\right)\theta^h\left(\frac{y_3^h}{h}\right)\nabla_{tan}\eta(y^{h\prime})\right\rangle \\
&\qquad + \frac{1}{h}\int_{\Omega^1}\left\langle\left((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\right)_{13,23}, \left(\theta^{h\prime\prime}\left(\frac{x_3}{h}\right)\theta^h\left(\frac{x_3}{h}\right) + (\theta^h\left(\frac{x_3}{h}\right))^2\right)\eta(y^{h\prime})\right\rangle \\
&\qquad + \frac{h}{2}\int_{\Omega^1}\left\langle\left((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\right)_{31,32}, (\theta^h\left(\frac{y_3^h}{h}\right))^2\nabla_{tan}\eta(y^{h\prime})\right\rangle \\
&= h\int_{\Omega^1}\left\langle f(x')(\bar R^h)_{31,32}, \theta^{h\prime}\left(\frac{y_3^h}{h}\right)\theta^h\left(\frac{y_3^h}{h}\right)\eta(y^{h\prime})\right\rangle\,\mathrm{d}x \\
&\qquad + \frac{h^2}{2}\int_{\Omega^1} f(x')(\bar R^h)_{33}(\theta^h\left(\frac{y_3^h}{h}\right))^2\eta_3(y^{h\prime})\,\mathrm{d}x.
\end{aligned}$$

In what follows, we will check convergence as $h \to 0$ of each of the five terms in the identity (7.7). We first easily notice that the two terms in the right hand side converge to 0, as they are bounded by:

$$\begin{aligned}
C\int_{\Omega^1}|f(x')|\Big(h\left|\theta^h\left(\frac{y_3^h}{h}\right)\right| + h^2\left|\theta^h\left(\frac{y_3^h}{h}\right)\right|^2\Big)\,\mathrm{d}x &\le C\int_{\Omega^1}|f(x')|\left(|y_3^h| + |y_3^h|^2\right)\,\mathrm{d}x \\
&\le C\|f\|_{L^2(\Omega^1)}\left(\|y_3^h\|_{L^2(\Omega^1)} + \|y_3^h\|^2_{L^4(\Omega^1)}\right).
\end{aligned}$$

Since $\frac{y_3^h}{h}$ has a strong limit in $W^{1,2}(\Omega^1)$ by (3.6), it results that $\|y_3^h\|_{L^2}$ and $\|y_3^h\|_{L^4}$ converge to 0.

3. The third term in (7.7) is bounded by the following expression, in view of (5.2), (3.12), (3.6) and (3.11):

$$\begin{aligned}
Ch\int_{\Omega^1}\chi_h|E^h|(\theta^h\left(\frac{y_3^h}{h}\right))^2\,\mathrm{d}x &+ Ch\int_{\Omega^1}(1-\chi_h)|E^h|(\theta^h\left(\frac{y_3^h}{h}\right))^2\,\mathrm{d}x \\
&\le Ch\int_{\Omega^1}\chi_h|E^h|\left|\frac{y_3^h}{h}\right|^2\,\mathrm{d}x + Ch\omega_h^2\int_{\Omega^1\setminus B_h}|E^h|\,\mathrm{d}x \\
&\le Ch\|\chi_h E^h\|_{L^2}\left\|\frac{y_3^h}{h}\right\|^2_{L^4} + Ch\omega_h^2 h^{2-\gamma} \le Ch + C\left(h^{1+\frac{1-\gamma}{2}}\omega_h\right)^2
\end{aligned}$$

which converges to 0 by (7.6).

4. We will now investigate the first term in (7.7). Integrated on $\Omega^1\setminus B_h$, it is bounded by:

$$C\omega_h\int_{\Omega^1}(1-\chi_h)|E^h|\,\mathrm{d}x \le C\omega_h h^{2-\gamma} \le Ch^{1+\frac{1-\gamma}{2}}\omega_h,$$

by (3.11) and hence it converges to 0 through (7.6). The same term integrated on B_h equals now the following sum:

$$\begin{aligned}
&\int_{\Omega^1} \left(\theta^{h\prime}\left(\frac{y_3^h}{h}\right) - 1\right)\theta^h\left(\frac{y_3^h}{h}\right)\cdot \\
&\qquad \cdot\Big\langle \left((\bar R^h)^T R^h \chi_h E^h (R^h)^T \bar R^h\right)'' - \left((\bar R^h)^T R^h \chi_h E^h (R^h)^T \bar R^h\right)_{33}\mathrm{Id}_2 : \nabla_{tan}\eta(y^{h\prime})\Big\rangle \,\mathrm{d}x \\
&+ \int_{\Omega^1} \theta^h\left(\frac{y_3^h}{h}\right)\cdot \\
&\qquad \cdot\Big\langle \left((\bar R^h)^T R^h \chi_h E^h (R^h)^T \bar R^h\right)'' - \left((\bar R^h)^T R^h \chi_h E^h (R^h)^T \bar R^h\right)_{33}\mathrm{Id}_2 : \nabla_{tan}\eta(y^{h\prime})\Big\rangle \,\mathrm{d}x.
\end{aligned} \tag{7.8}$$

The first term in (7.8) goes to 0, as it is bounded by:

$$C\int_{\left\{\frac{|y_3^h|}{h}\geq\omega_h\right\}} \left|\frac{y_3^h}{h}\right| |\chi_h E^h|\,\mathrm{d}x \leq C\left\|\frac{y_3^h}{h}\right\|_{L^4(\Omega^1)} \left|\left\{x\in\Omega^1;\ \frac{|y_3^h|}{h}\geq\omega_h\right\}\right|^{1/4} \|\chi_h E^h\|_{L^2(\Omega^1)} \leq \frac{C}{\omega_h^{1/2}},$$

in view of (5.2), (3.7), (3.12) and recalling (7.6). The second term of (7.8) converges to:

$$\int_{\Omega^1} \Big\langle E'' - E_{33}\mathrm{Id}_2 : (x_3 + v(x'))\nabla_{tan}\eta(x')\Big\rangle \,\mathrm{d}x \tag{7.9}$$

because of (3.12) and through the following strong convergences: convergence of $\nabla_{tan}\eta(y^{h\prime})$ to $\nabla_{tan}\eta(x')$ in $L^5(\Omega^1)$ by (3.6), of $(\bar R^h)^T R^h$ to Id in $L^{20}(\Omega)$ by (3.3), and of $\theta^h\left(\frac{y_3^h}{h}\right)$ to $(x_3+v(x'))$ in $L^5(\Omega^1)$. The last convergence can be seen from:

$$\begin{aligned}
\int_{\Omega^1}\left|\theta^h\left(\frac{y_3^h}{h}\right) - (x_3+v(x'))\right|^5 \mathrm{d}x &\leq C\int_{\Omega^1}\left|\theta^h\left(\frac{y_3^h}{h}\right) - \frac{y_3^h}{h}\right|^5 \mathrm{d}x + C\int_{\Omega^1}\left|\frac{y_3^h}{h} - (x_3+v(x'))\right|^5 \mathrm{d}x \\
&\leq C\int_{\left\{\frac{|y_3^h|}{h}\geq\omega_h\right\}}\left|\frac{y_3^h}{h}\right|^5 \mathrm{d}x + o(1) \leq \frac{C}{\omega_h^{1/3}} + o(1) \leq o(1)
\end{aligned}$$

by (3.6), (3.7) and (7.6). Concluding, we obtain that the first term in (7.7) converges to the expression in (7.9).

5. Regarding the second term in (7.7), using (3.10), (5.2), (3.1) and (3.7) we note that:

$$\begin{aligned}
&\Big|\int_{\Omega^1}\left(\theta^{h\prime\prime}\left(\frac{y_3^h}{h}\right)\theta^h\left(\frac{y_3^h}{h}\right) + \theta^{h\prime}\left(\frac{y_3^h}{h}\right)^2 - 1\right)\Big\langle\left((\bar R^h)^T R^h E^h (R^h)^T \bar R^h\right)_{13,23}, \eta(y^{h\prime})\Big\rangle \,\mathrm{d}x\Big| \\
&\qquad\leq \frac{C}{h}\int_{\left\{x\in\Omega^1;\ \frac{|y_3^h(x)|}{h}\geq\omega_h\right\}}\left(\frac{1}{\omega_h}\omega_h + 1\right)|E^h|\,\mathrm{d}x \\
&\qquad\leq \frac{C}{h}\int_{\left\{\frac{|y_3^h|}{h}\geq\omega_h\right\}}\frac{1}{h^2}W(\nabla u^h(x', hx_3)) + |G^h|\,\mathrm{d}x \\
&\qquad\leq \frac{C}{h}\left(h^2 + \|G^h\|_{L^2(\Omega^1}\left|\left\{x\in\Omega^1;\ \frac{|y^h(x)|}{h}\geq\omega_h\right\}\right|^{1/2}\right) \leq \frac{C}{h}\left(h^2 + \frac{1}{\omega_h}\right),
\end{aligned}$$

which converges to 0 by (7.6). The remaining part of the second term in (7.7) is:

$$\begin{aligned} &\frac{1}{h}\int_{\Omega^1}\Big\langle \Big((\bar{R}^h)^T R^h E^h (R^h)^T \bar{R}^h\Big)_{13,23}, \eta(y^{h'})\Big\rangle \,\mathrm{d}x \\ &= \int_{\Omega^1}\Big\langle \Big(A^h E^h (R^h)^T \bar{R}^h\Big)_{13,23}, \eta(y^{h'})\Big\rangle \,\mathrm{d}x + \int_{\Omega^1}\Big\langle \Big(E^h (A^h)^T\Big)_{13,23}, \eta(y^{h'})\Big\rangle \,\mathrm{d}x \\ &\quad + \frac{1}{h}\int_{\Omega^1}\Big\langle (E^h)_{13,23}, \eta(y^{h'})\Big\rangle \,\mathrm{d}x, \end{aligned} \tag{7.10}$$

where we used the decomposition (7.3). Now, exactly as in the proof of Lemma 7.1 and recalling the block structure of the limiting tensor A in (3.5), we see that (7.10) converges to:

$$\begin{aligned} &\int_{\Omega^1}\Big\langle (AE)_{13,23}, \eta(x')\Big\rangle \,\mathrm{d}x + \int_{\Omega^1}\Big\langle (EA^T)_{13,23}, \eta(x')\Big\rangle \,\mathrm{d}x + \frac{1}{h}\int_{\Omega^1}\Big\langle (E^h)_{13,23}, \eta(y^{h'})\Big\rangle \,\mathrm{d}x \\ &= \int_{\Omega^1}\Big\langle \big(E'' - E_{33}\mathrm{Id}_2\big)\nabla v, \eta(x')\Big\rangle \,\mathrm{d}x + \frac{1}{h}\int_{\Omega^1}\Big\langle (E^h)_{13,23}, \eta(y^{h'})\Big\rangle \,\mathrm{d}x. \end{aligned}$$

In conclusion, passing to the limit in (7.7) clearly yields (7.5) and achieves the lemma. ■

Proof of the second Euler-Lagrange equation (1.13).
Let now $\xi \in \mathcal{C}_b^3(\mathbb{R}^2, \mathbb{R})$. Applying Lemma 7.1 with $\eta_3 = \xi$, and Lemma 7.2 with $\eta = \nabla\xi$, it follows:

$$-\int_{\Omega^1}\Big\langle E'' - E_{33}\mathrm{Id}_2 : (x_3 + v(x')\nabla^2\xi\Big\rangle \,\mathrm{d}x = \bar{R}_{33}\int_{\Omega} f(x')\xi(x')\,\mathrm{d}x'. \tag{7.11}$$

By the first Euler-Lagrange equation in (6.4) applied with $\eta = v\nabla\xi \in W^{2,2}(\Omega, \mathbb{R}^2)$, we see that:

$$\int_{\Omega^1}\Big\langle E'' - E_{33}\mathrm{Id}_2 : \nabla v \otimes \nabla\xi + v(x')\nabla^2\xi\Big\rangle \,\mathrm{d}x = 0.$$

Thus, (7.11) becomes:

$$\int_{\Omega^1}\Big\langle E'' - E_{33}\mathrm{Id}_2 : \nabla v \otimes \nabla\xi\Big\rangle \,\mathrm{d}x - \int_{\Omega^1}\Big\langle E'' - E_{33}\mathrm{Id}_2 : x_3\nabla^2\xi\Big\rangle \,\mathrm{d}x = \bar{R}_{33}\int_{\Omega} f(x')\xi(x').$$

The equality in (1.13) follows now from the above in view of (4.6), and by the density of test functions $\xi \in \mathcal{C}_b^3$ in $W^{2,2}(\Omega, \mathbb{R})$. ■

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