CONVEX INTEGRATION FOR THE MONGE-AMPÈRE EQUATION IN TWO DIMENSIONS

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ABSTRACT. This paper concerns the questions of flexibility and rigidity of solutions to the Monge-Ampère equation which arises as a natural geometrical constraint in prestrained nonlinear elasticity. In particular, we focus on degenerate i.e. "flexible" weak solutions that can be constructed through methods of convex integration à la Nash & Kuiper and establish the related h-principle for the Monge-Ampère equation in two dimensions.

CONTENTS

1. INTRODUCTION.

In this paper we study the $\mathcal{C}^{1,\alpha}$ solutions to the Monge-Ampère equation in two dimensions:

$$(1.1) \qquad \mathcal{D}et\,\nabla^2 v := -\frac{1}{2}\mathrm{curl}\,\mathrm{curl}\,(\nabla v \otimes \nabla v) = f \qquad \text{in } \Omega \subset \mathbb{R}^2.$$

Our results concern the dichotomy of "rigidity vs. flexibility", in the spirit of the analogous results and techniques appearing in the contexts of: the low co-dimension isometric immersion problem [43, 34, 3, 4, 10], and the Onsager's conjecture for Euler equations [49, 13, 14, 9, 17].

In the first, main part of the paper we show that below the regularity threshold $\alpha < 1/7$, the very weak $\mathcal{C}^{1,\alpha}(\bar{\Omega})$ solutions to (1.1) as defined below, are dense in the set of all continuous functions (see Theorems 1.1 and 1.2). These flexibility statements are a consequence of the convex integration h-principle, that is a method proposed in [19] for solving certain partial differential relations and that turns out to be applicable to our setting of the Monge-Ampère equation as well.

Here, we directly adapt the iteration method of Nash and Kuiper [43, 34], in order to construct the oscillatory solutions to (1.1). [1]

In the second part of the paper we prove that the same class of very weak solutions fails the above flexibility in the regularity regime $\alpha > 2/3$. Our results are parallel with those concerning isometric immersions [3, 10, 45], Euler equations [9, 17], Perona-Malik equation [29, 30], the active scalar equation [23], and should also be compablue with results on the regularity of Sobolev solutions to the Monge-Ampère equation [45, 48, 35, 27] whose study is important in the context of nonlinear elasticity and with the rigidity results for the Monge-Ampère functions [25, 26].

1.1. **The weak determinant Hessian.** Let $\Omega \subset \mathbb{R}^2$ be an open set. Given a function $v \in W^{1,2}_{loc}(\Omega)$, we define its very weak Hessian (denoted by $\mathcal{H}^*_2$ in [24, 18]) as:

$$\mathcal{D}et\, \nabla^2 v = -\frac{1}{2}\text{curl curl}\,(\nabla v \otimes \nabla v),$$

understood in the sense of distributions. A straightforward approximation argument shows that if $v \in W^{2,2}_{loc}$ then $L^1_{loc}(\Omega) \ni \mathcal{D}et\, \nabla^2 v = \det \nabla^2 v$ a.e. in Ω, where $\nabla^2 v$ stands for the Hessian matrix field of v. We also remark that this notion of the very weak Hessian is distinct from the distributional Hessian $\text{Det}\nabla^2 v = \text{Det}\nabla(\nabla v)$ (denoted by $\mathcal{H}u$ in [24, 18]), that is defined through the distributional determinant Det:

$$\text{Det}\nabla\psi = -\text{div}\,\big(\psi_2 \nabla^\perp \psi_1\big) = \partial_2(\psi_2 \partial_1 \psi_1) - \partial_1(\psi_2 \partial_2 \psi_1) \quad \text{for} \quad \psi = (\psi_1, \psi_2) \in W^{1,4/3}(\Omega, \mathbb{R}^2).$$

Contrary to the distributional Hessian, the very weak Hessian is not continuous with respect to the weak topology. Indeed, an example of a sequence $v_n \in W^{1,2}(\Omega)$ is constructed in [24], where $\mathcal{D}et\, \nabla^2 v = -1$ while v_n converges weakly to 0. One consequence of the proof of our Theorem 1.1 below is that $\mathcal{D}et\, \nabla^2$ is actually weakly discontinuous everywhere in $W^{1,2}(\Omega)$ (see Corollary 6.2).

Here is our first main result:

Theorem 1.1. *Let $f \in L^{7/6}(\Omega)$ on an open, bounded, simply connected $\Omega \subset \mathbb{R}^2$. Fix an exponent:*

$$\alpha < \frac{1}{7}.$$

Then the set of $\mathcal{C}^{1,\alpha}(\bar{\Omega})$ solutions to (1.1) is dense in the space $\mathcal{C}^0(\bar{\Omega})$. More precisely, for every $v_0 \in \mathcal{C}^0(\bar{\Omega})$ there exists a sequence $v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$, converging uniformly to v_0 and satisfying:

$$(1.2) \qquad\qquad\qquad\qquad \mathcal{D}et\, \nabla^2 v_n = f \quad \text{in } \Omega.$$

When $f \in L^p(\Omega)$ and $p \in (1, \frac{7}{6})$, the same result is true for any $\alpha < 1 - \frac{1}{p}$.

In order to better understand Theorem 1.1, we point out a connection between the solutions to (1.1) and the isometric immersions of Riemannian metrics, motivated by a study of nonlinear elastic plates. Since on a simply connected domain Ω, the kernel of the differential operator curl curl consists of the fields of the form sym ∇w, a solution to (1.1) with the vanishing right hand side $f \equiv 0$ can be characterized by the criterion:

$$(1.3) \qquad\qquad \exists w : \Omega \to \mathbb{R}^2 \qquad \frac{1}{2}\nabla v \otimes \nabla v + \text{sym}\,\nabla w = 0 \quad \text{in } \Omega.$$

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[1]We remark that the recent work of De Lellis, Inaunen and Szekelyhidi [12] showed that the flexibility exponent $\frac{1}{7}$ can be improved to $\frac{1}{5}$ in the case of the isometric immersion problem in 2 dimensions. We expect similar improvement to be possible also in the present case of equation (1.1); this will be investigated in our future work.

The equation in (1.3) can be seen as an equivalent condition for the following 1-parameter family of deformations, given through the out-of-plane displacement v and the in-plane displacement w (albeit with different orders of magnitude ε and ε^2):

$$\phi_\varepsilon = id + \varepsilon v e_3 + \varepsilon^2 w : \Omega \to \mathbb{R}^3$$

to form a 2nd order infinitesimal isometry (bending), i.e. to induce the change of metric on the plate Ω whose 2nd order terms in ε disappear:

$$(\nabla \phi_\varepsilon)^T \nabla \phi_\varepsilon - \mathrm{Id}_2 = o(\varepsilon^2).$$

In this context, we take the cue about Theorem 1.1 from the celebrated work of Nash and Kuiper [43, 34], where they show the density of co-dimension one $\mathcal{C}^1$ isometric immersions of Riemannian manifolds in the set of short mappings. Since we are are now dealing with the 2nd order infinitesimal isometries rather than the exact isometries, the classical metric pull-back equation:

$$y^* g_e = h,$$

for a mapping y from (Ω, h) into $\mathbb{R}^3$ equipped with the standard Euclidean metric g_e, is replaced by the compatibility equation of the tensor $T(v, w) = \frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\, \nabla w$ with a matrix field A_0 that satisfies: $-\mathrm{curl}\,\mathrm{curl}\, A_0 = f$:

$$(1.4) \qquad\qquad T(v, w) = A_0$$

Note that there are many potential choices for A_0, for example one may take $A_0(x) = \lambda(x)\mathrm{Id}_2$ with $\Delta \lambda = -f$ in Ω. Again, equation (1.4) states precisely that the metric $(\nabla \phi_\varepsilon)^T \nabla \phi_\varepsilon$ agrees with the given metric $h = \mathrm{Id}_2 + 2\varepsilon^2 A_0$ on Ω, up to terms of order ε^2. The Gauss curvature κ of the metric h satisfies:

$$\kappa(h) = \kappa(\mathrm{Id}_2 + 2\varepsilon^2 A_0) = -\varepsilon^2 \mathrm{curl}\,\mathrm{curl}\, A_0 + o(\varepsilon^2),$$

while $\kappa((\nabla \phi_\varepsilon)^T \nabla \phi_\varepsilon) = -\varepsilon^2 \mathrm{curl}\,\mathrm{curl}\left(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\, w\right) + o(\varepsilon^2)$, so the problem (1.1) can also be interpreted as seeking for all appropriately regular out-of-plane displacements v that can be matched, by a higher order in-plane displacement perturbation w, to achieve the prescribed Gauss curvature f of Ω, at its highest order term.

In this paper, similarly as in the isometric immersion case, we show that solutions to (1.4) are ample. We design a scheme inspiblue by the work of Nash and Kuiper, which pushes a "short infinitesimal isometry", i.e. a couple (v_0, w_0) such that $T(v_0, w_0) < A_0$, towards an exact solution to (1.4) in successive small steps. Note that both $y^* g_e = (\nabla y)^T \nabla y$ and the term $\nabla v \otimes \nabla v$ in $T(v, w)$ have a quadratic structure, which is crucial in the analysis of [43, 34] and also of this paper. Here, not only the presence of the linear term $\mathrm{sym}\, \nabla w$ in $T(u, w)$ does not destroy the adaptation of the Nash-Kuiper scheme, but it actually allows for this construction to work.

1.2. Convex integration for the Monge-Ampère equation in two dimensions.
As we will see in section 4, Theorem 1.1 follows easily from the statement of our next main result:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Let $v_0 \in \mathcal{C}^1(\bar{\Omega})$, $w_0 \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$ and $A_0 \in \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R}^{2\times 2}_{sym})$, for some $\beta \in (0, 1)$, be such that:*

$$(1.5) \qquad\qquad \exists c_0 > 0 \qquad A_0 - \left(\frac{1}{2}\nabla v_0 \otimes \nabla v_0 + \mathrm{sym}\, \nabla w_0\right) > c_0 \mathrm{Id}_2 \qquad in\ \bar{\Omega}.$$

Then, for every exponent α in the range:

$$0 < \alpha < \min\left\{\frac{1}{7}, \frac{\beta}{2}\right\},$$

there exist sequences $v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $w_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ which converge uniformly to v_0 and w_0, respectively, and which satisfy:

$$(1.6) \qquad A_0 = \frac{1}{2}\nabla v_n \otimes \nabla v_n + \mathrm{sym}\,\nabla w_n \qquad in\ \bar{\Omega}.$$

The above result is the Monge-Ampère analogue of [10, Theorem 1], where the authors improved on the Nash-Kuiper method to obtain higher regularity within the flexibility regime. In our paper, we adapt similar methods to the system (1.6).

The term *convex integration* usually refers to a collection of approaches that allow for constructing anomalous solutions to nonlinear PDEs; in particular, flexibility type results for the isometric immersion problem were obtained via the above mentioned Nash and Kuiper's iteration scheme. From a geometric perspective, they are special cases of *h-principle*, a notion which was developed by Gromov in [19] for studying partial differential relations, see also [15]. From another perspective, one seeks weak solutions of a differential inclusion: $Lu(x) \in K$ in Ω, by investigating certain classes of sub-solutions, e.g. functions u that satisfy $Lu(x) \in \mathrm{conv}\,K$ where the original constraint set K is replaced by its convex hull conv K [50, 11, 41]. This approach leads to the density of very weak solutions, satisfying $Lu \in L^\infty(\Omega)$, in the set of sub-solutions. When K is a continuum, the regularity may be improved to $Lu \in \mathcal{C}^0(\Omega)$ by applying the correcting iterations.

Recently, similar techniques were advanced in the context of fluid dynamics and yielded many interesting results for the Euler equations. In [13], De Lellis and Székelyhidi proved existence of weak solutions with bounded velocity and pressure, their non-uniqueness and the existence of energy-decreasing solutions. In [14], using iteration methods à la Nash-Kuiper, the same authors proved existence of continuous periodic solutions of the 3-dimensional incompressible Euler equations, which dissipate the total kinetic energy. These results are to be contrasted with [9, 17], where it was shown that $\mathcal{C}^{0,\alpha}$ solutions of the Euler equations are energy conservative if $\alpha > 1/3$. There have been several improvements of [13, 14] since, towards a proof of the Onsager's conjecture which puts the Hölder regularity threshold for the energy conservation of the weak solutions to the Euler equations at $\mathcal{C}^{0,1/3}$ [20, 21, 22, 5, 6, 7, 8]. The stationary incompressible Euler equation has been studied in [8] where the existence of bounded anomalous solutions have been proved. The authors indicate that in 2 dimensions, the relaxation set corresponding to the appropriate subsolutions is smaller than in the case of the evolutionary equations. In this context, we noticed a connection between our reformulation of the Monge-Ampère equation and the steady state Euler equation, which lead to our modest Corollary 4.1.

In this paper we use a direct iteration method to construct exact solutions of (1.1). The recasting of the statement and the proof in the language of convex integration might shed more light on the structure of the Monge-Ampère equation, but it would not improve the results and therefore we do not address this task. We note, however, that constructing Lipschitz continuous piecewise affine approximating solutions to (1.6) for $A_0 \equiv 0$ is quite straightforward and could be used to prove a convex integration density result via the Baire category method as was done in [13] for the Euler equations (see also Figure 2.1 and the corresponding explanation).

1.3. **Rigidity versus flexibility.** The flexibility results obtained in view of the h-principle are usually coupled with the rigidity results for more regular solutions. Rigidity of isometric immersions of elliptic metrics for $\mathcal{C}^{1,\alpha}$ isometries [3, 13] with $\alpha > 2/3$, or the energy conservation of weak solutions of the Euler equations for $\mathcal{C}^{0,\alpha}$ solutions with $\alpha > 1/3$, are results of this type. For the Monge-Ampère equations, we recall two recent statements regarding solutions with Sobolev regularity: following the well known unpublished work by Šverák [48], we proved in [35] that if $v \in W^{2,2}(\Omega)$ is a solution to (1.1) with $f \in L^1(\Omega)$ and $f \geq c > 0$ in Ω, then in fact v must be $\mathcal{C}^1$ and globally convex (or concave). On the other hand, if $f = 0$ then [45] likewise $v \in \mathcal{C}^1(\Omega)$ and v must be developable (see also [25, 26, 27]). A clear statement of rigidity is still lacking for the general f, as is the case for isometric immersions, where rigidity results are usually formulated only for elliptic [10] or Euclidean metrics [45, 37, 27].

In this paper, we prove the rigidity properties of solutions to (1.1) in the Hölder regularity context when $f \equiv 0$. Namely, we prove:

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain and let:*

$$\frac{2}{3} < \alpha < 1.$$

If $v \in \mathcal{C}^{1,\alpha}(\bar\Omega)$ is a solution to $\mathcal{D}et\,\nabla^2 v = 0$ in $\bar\Omega$, then v must be developable. More precisely, for all $x \in \Omega$ either v is affine in a neighbourhood of x, or there exists a segment l_x joining $\partial\Omega$ on its both ends, such that ∇v is constant on l_x.

We also announce the following parallel rigidity result for $f \geq c > 0$, that will be the subject of the forthcoming paper [36]:

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain and let:*

$$\frac{2}{3} < \alpha < 1.$$

If $v \in \mathcal{C}^{1,\alpha}(\bar\Omega)$ is a solution to $\mathcal{D}et\,\nabla^2 v = f$ in $\bar\Omega$, where f is a positive Dini continuous function, then v is convex. In fact, it is also an Alexandrov solution to $\det \nabla^2 v = f$ in Ω.

In proving Theorem 1.3, we use a commutator estimate for deriving a degree formula in Proposition 7.1. Similar commutator estimates are used in [9] for the Euler equations and in [10] for the isometric immersion problem; this is not surprising, since the presence of a quadratic term plays a major role in all three cases, allowing for the efficiency of the convex integration and iteration methods. Let us also mention that it is still an open problem which value of α is the critical value for the rigidity-flexibility dichotomy, and it is conjectured to be $1/3, 1/2$ or $2/3$.

1.4. **Notation.** By $\mathbb{R}^{2\times2}_{sym}$ we denote the space of symmetric 2×2 matrices, and by $\mathbb{R}^{2\times2}_{sym,>}$ we denote the cone of symmetric, positive definite 2×2 matrices. The space of Hölder continuous functions $\mathcal{C}^{k,\alpha}(\bar\Omega)$ consists of restrictions of functions $f \in \mathcal{C}^{k,\alpha}(\mathbb{R}^2)$ to $\Omega \subset \mathbb{R}^2$. Then, the $\mathcal{C}^k(\bar\Omega)$ norm of such restriction is denoted by $\|f\|_k$, while its Hölder norm $\mathcal{C}^{k,\alpha}(\bar\Omega)$ is $\|f\|_{k,\alpha}$. By $C > 0$ we denote a universal constant which is independent of all parameters, unless indicated otherwise.

2. The $\mathcal{C}^1$ approximations - preliminary results.

In this and the next section we prove a weaker version of the result in Theorem 1.2. Namely:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Let $v_0 \in \mathcal{C}^\infty(\bar{\Omega})$, $w_0 \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$ and $A_0 \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{2 \times 2}_{sym})$ be such that:*

$$(2.1) \qquad \exists c_0 > 0 \qquad A_0 - \left(\frac{1}{2}\nabla v_0 \otimes \nabla v_0 + \mathrm{sym}\,\nabla w_0\right) > c_0 \mathrm{Id}_2 \qquad in\ \bar{\Omega}.$$

Then there exist sequences $v_n \in \mathcal{C}^1(\bar{\Omega})$ and $w_n \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$ which converge uniformly to v_0 and w_0 respectively, and which satisfy:

$$(2.2) \qquad A_0 = \frac{1}{2}\nabla v_n \otimes \nabla v_n + \mathrm{sym}\,\nabla w_n \qquad in\ \bar{\Omega}.$$

We start with a series of preliminary lemmas whose details we provide for the sake of completeness. The first lemma is an observation in convex integration, pertaining to solving an appropriate differential inclusion to be used for constructing the 1-dimensional oscillatory perturbations in v_n and w_n. As always, $C > 0$ is a universal constant, independent of all parameters, in particular independent of the function a below.

Lemma 2.2. *Let $a \in \mathcal{C}^\infty(\bar{\Omega})$ be a nonnegative function on an open and bounded set $\Omega \subset \mathbb{R}^2$. There exists a smooth 1-periodic field $\Gamma = (\Gamma_1, \Gamma_2) \in \mathcal{C}^\infty(\bar{\Omega} \times \mathbb{R}, \mathbb{R}^2)$ such that the following holds for all $(x, t) \in \bar{\Omega} \times \mathbb{R}$:*

$$(2.3) \qquad \begin{aligned} &\Gamma(x, t+1) = \Gamma(x, t), \\ &\frac{1}{2}|\partial_t \Gamma_1(x,t)|^2 + \partial_t \Gamma_2(x,t) = a(x)^2, \end{aligned}$$

together with the uniform bounds:

$$(2.4) \qquad \begin{aligned} &|\Gamma_1(x,t)| + |\partial_t \Gamma_1(x,t)| \leq Ca(x), \qquad |\Gamma_2(x,t)| + |\partial_t \Gamma_2(x,t)| \leq Ca(x)^2, \\ &|\nabla_x \Gamma_1(x,t)| \leq C|\nabla a(x)|, \qquad |\nabla_x \Gamma_2(x,t)| \leq C|a(x)||\nabla a(x)|. \end{aligned}$$

Proof. Firstly, note that there exists a smooth 1-periodic function $\gamma \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^2)$, such that for all $t \in \mathbb{R}$ there holds:

$$\gamma(t+1) = \gamma(t), \qquad \int_0^1 \gamma(t)\ \mathrm{d}t = (0,0),$$

$$\gamma(t) \in P := \left\{(s_1, s_2) \in \mathbb{R}^2;\ \frac{1}{2}s_1^2 + s_2 = 1,\ |s_1| \leq 2\right\}.$$

Existence of γ is a consequence of the fundamental lemma of convex integration, since the intended average $(0,0)$ lies in the convex hull of the parabola P (see Figure 2.1). Indeed, one can take:

$$\gamma(t) = \left(2\cos(2\pi t), -\cos(4\pi t)\right) \in P.$$

It is now enough to ensure that $\partial_t \Gamma_1 = a(x)\gamma_1(x)$ and $\partial_t \Gamma_2 = a(x)^2 \gamma_2(x)$ to obtain (2.3). Namely:

$$\Gamma_1(x,t) = \frac{a(x)}{\pi}\sin(2\pi t), \quad \Gamma_2(x,t) = -\frac{a(x)^2}{4\pi}\sin 4\pi t.$$

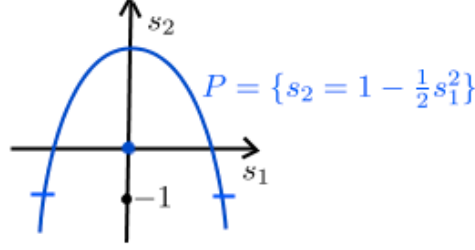We see directly that the bounds in (2.4) hold. ∎

FIGURE 2.1. The parabola P in the 1d convex integration problem of Lemma 2.2.

To compare with the problem of isometric immersions, note that in that context, a 1-dimensional convex integration lemma is similarly proved in [49, Figure2, p. 11], where instead of a parabola, the constraint set consists of a full circle.

We will also need a special case of [10, Lemma 3] about decomposition of positive definite symmetric matrices into rank-one matrices.

Lemma 2.3. *There exists a sufficiently small constant $r_0 > 0$ such that the following holds. For every positive definite symmetric matrix $G_0 \in \mathbb{R}^{2\times 2}_{sym,>}$, there are three unit vectors $\{\xi_k \in \mathbb{R}^2\}_{k=1}^3$ and three linear functions $\{\Phi_k : \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}\}_{k=1}^3$, such that: for any $G \in \mathbb{R}^{2\times 2}_{sym}$ we have*

$$(2.5) \qquad \forall G \in \mathbb{R}^{2\times 2}_{sym} \qquad G = \sum_{k=1}^3 \Phi_k(G)\xi_k \otimes \xi_k,$$

and that each Φ_k is strictly positive on the ball $B(G_0, r(G_0)) \subset \mathbb{R}^{2\times 2}_{sym}$ with radius $r(G_0) = \frac{r_0}{|G_0^{-1/2}|^2}$.

Proof. **1.** First, assume that $G_0 = \mathrm{Id}_2$. Set:

$$\zeta_1 = \frac{1}{\sqrt{12}}(2+\sqrt{2}, -2+\sqrt{2}), \qquad \zeta_2 = \frac{1}{\sqrt{12}}(-2+\sqrt{2}, 2+\sqrt{2}), \qquad \zeta_3 = \frac{1}{\sqrt{2}}(1,1).$$

In order to check that the following matrices form a basis of the 3-dimensional space $\mathbb{R}^{2\times 2}_{sym}$:

$$\zeta_1 \otimes \zeta_1 = \frac{1}{12}\begin{bmatrix} 6+4\sqrt{2} & -2 \\ -2 & 6-4\sqrt{2} \end{bmatrix}, \quad \zeta_2 \otimes \zeta_2 = \frac{1}{12}\begin{bmatrix} 6-4\sqrt{2} & -2 \\ -2 & 6+4\sqrt{2} \end{bmatrix}, \quad \zeta_3 \otimes \zeta_3 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

we validate that:

$$\det\left(\frac{1}{12}\begin{bmatrix} 6+4\sqrt{2} & 6-4\sqrt{2} & 6 \\ -2 & -2 & 6 \\ 6-4\sqrt{2} & 6+4\sqrt{2} & 6 \end{bmatrix}\right) \neq 0.$$

Consequently, there exist linear mappings $\{\Psi_k : \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}\}_{k=1}^3$ yielding the unique decomposition:

$$(2.6) \qquad \forall G \in \mathbb{R}^{2\times 2}_{sym} \qquad G = \sum_{k=1}^3 \Psi_k(G)\zeta_k \otimes \zeta_k.$$

Now, since $\mathrm{Id}_2 = \frac{3}{4}\zeta_1 \otimes \zeta_1 + \frac{3}{4}\zeta_2 \otimes \zeta_2 + \frac{1}{2}\zeta_3 \otimes \zeta_3$, the continuity of each function Ψ_k implies its positivity in a neighborhood of Id_2 of some appropriate radius r_0.

2. For an arbitrary $G_0 \in \mathbb{R}^{2\times 2}_{sym,>}$ we set:

$$\forall k = 1\ldots 3 \qquad \xi_k = \frac{1}{|G_0^{1/2}\zeta_k|}G_0^{1/2}\zeta_k \quad \text{and} \quad \Phi_k(G) = |G_0^{1/2}\zeta_k|^2 \Psi_k(G_0^{-1/2}GG_0^{-1/2}).$$

Then, in view of (2.6) we obtain (2.5):

$$\forall G \in \mathbb{R}^{2\times2}_{sym} \qquad G = G_0^{1/2}\Big(\sum_{k=1}^{3}\Psi_k(G_0^{-1/2}GG_0^{-1/2})\zeta_k \otimes \zeta_k\Big)G_0^{1/2} = \sum_{k=1}^{3}\Phi_k(G)\xi_k \otimes \xi_k,$$

Finally, if $|G - G_0| < r(G_0)$ then $|G_0^{-1/2}GG_0^{-1/2} - \text{Id}_2| \leq |G_0^{-1/2}|^2|G - G_0| < r_0$, and so indeed $\Phi_k(G) > 0$, since $\Psi_k(G_0^{-1/2}GG_0^{-1/2}) > 0$. ∎

The above result can be localized in the following manner, similar to [49, Lemma 3.3]:

Lemma 2.4. *There exists sequences of unit vectors* $\{\eta_k \in \mathbb{R}^2\}_{k=1}^{\infty}$ *and nonnegative smooth functions* $\{\phi_k \in \mathcal{C}_c^{\infty}(\mathbb{R}^{2\times2}_{sym,>})\}_{k=1}^{\infty}$, *such that:*

$$(2.7) \qquad\qquad \forall G \in \mathbb{R}^{2\times2}_{sym,>} \qquad G = \sum_{k=1}^{\infty}\phi_k(G)^2\eta_k \otimes \eta_k$$

and that:

   (i) *For all* $G \in \mathbb{R}^{2\times2}_{sym,>}$, *at most* N_0 *terms of the sum in (2.7) are nonzero. The constant* N_0 *is independent of* G.
   (ii) *For every compact* $K \subset \mathbb{R}^{2\times2}_{sym,>}$, *there exists a finite set of indices* $J(K) \subset \mathbb{N}$ *such that* $\phi_k(G) = 0$ *for all* $k \notin J(K)$ *and* $G \in K$.

Proof. **1.** Let r_0 be as in Lemma 2.3 and additionally ensure that:

$$(2.8) \qquad\qquad\qquad\qquad\qquad r_0 < \frac{1}{8}.$$

Recall that for each $G \in \mathbb{R}^{2\times2}_{sym,>}$ we have denoted $r(G) = \frac{r_0}{|G^{-1/2}|^2}$ and that $B(G, r(G)) \subset \mathbb{R}^{2\times2}_{sym,>}$. We first construct a locally finite covering of $\mathbb{R}^{2\times2}_{sym,>}$ with properties corresponding to (i) and (ii).

Since the set $\mathbb{R}^{2\times2}_{sym,>}$ is a cone, we have:

$$(2.9) \qquad\qquad \mathbb{R}^{2\times2}_{sym,>} = \bigcup_{k\in\mathbb{Z}}2^k\mathcal{C}_0, \quad \text{where} \quad \mathcal{C}_0 = \{G \in \mathbb{R}^{2\times2}_{sym,>};\ 1/2 \leq |G| \leq 1\}.$$

The collection $\{B(G, r(G))\}_{G\in\mathcal{C}_0}$ covers the sector $\mathcal{C}_0$ by balls that have uniformly bounded radii: $r(G) \leq r_0\frac{|G|}{\sqrt{2}} \leq r_0$. Hence, by the Besicovitch covering theorem, it has a countable subcovering $\mathcal{G}_0 = \bigcup_{\sigma=1}^{\sigma_0}\mathcal{G}_0^{\sigma}$, consisting of $\sigma_0 \in \mathbb{N}$ countable families $\{\mathcal{G}_0^{\sigma}\}_{\sigma=1}^{\sigma_0}$ of pairwise disjoint balls.

Note that for all $c > 0$ one has: $r(cG) = cr(G)$ and so: $B(cG, r(cG)) = cB(G, r(G))$. Consequently, the collections $\mathcal{G}_k^{\sigma} = \{2^k B;\ B \in \mathcal{G}_0^{\sigma}\}$ each consist of countably many pairwise disjoint balls, and $\mathcal{G}_k = \bigcup_{\sigma=1}^{\sigma_0}\mathcal{G}_k^{\sigma}$ is a covering of the dilated sector $2^k\mathcal{C}_0$, for every $k \in \mathbb{Z}$. Define:

$$(2.10) \qquad\qquad \forall\sigma = 1\ldots\sigma_0 \qquad \mathcal{G}_{even}^{\sigma} = \bigcup_{2|k}\mathcal{G}_k^{\sigma} \quad \text{and} \quad \mathcal{G}_{odd}^{\sigma} = \bigcup_{2|(k+1)}\mathcal{G}_k^{\sigma}.$$

Clearly, in view of (2.9), the $2\sigma_0$ families in (2.10) form a covering of $\mathbb{R}^{2\times2}_{sym,>}$, namely:

$$\mathcal{G} = \bigcup_{\sigma=1}^{\sigma_0}\mathcal{G}_{even}^{\sigma} \cup \bigcup_{\sigma=1}^{\sigma_0}\mathcal{G}_{odd}^{\sigma}.$$

We now prove that each of the families in $\mathcal{G}$ consists of pairwise disjoint balls. We argue by contradiction. Assume that:

$$\exists G \in B(G_1, r(G_1)) \cap B(G_2, r(G_2)) \qquad \text{for some} \qquad B(G_1, r(G_1)) \in \mathcal{G}^\sigma_{2k_1}, \quad B(G_2, r(G_2)) \in \mathcal{G}^\sigma_{2k_2}.$$

Without loss of generality we may take $k_1 = 0$ and $k_2 = k \geq 1$, so that:

$$\frac{1}{2} \leq |G_1| \leq 1 \quad \text{and} \quad 2^{2k-1} \leq |G_2| \leq 2^{2k}.$$

This yields a contradiction with (2.8), in view of:

$$2^{2k-1} - 1 \leq |G_2| - |G_1| \leq |G_2 - G_1| \leq |G_2 - G| + |G - G_1|$$

$$\leq r(G_2) + r(G_1) = r_0 \Big(\frac{1}{|G_2^{-1/2}|^2} + \frac{1}{|G_1^{-1/2}|^2} \Big) \leq \frac{r_0}{\sqrt{2}} (|G_2| + |G_1|) \leq r_0 (2^{2k} + 1),$$

2. Note that $\mathcal{G}$ can be assumed locally finite, by paracompactness. We write: $\mathcal{G} = \{B_i = B(G_i, r(G_i))\}_{i=1}^\infty$ and let $\{\theta_i \in \mathcal{C}^\infty_c(B_i)\}_{i=1}^\infty$ be a partition of unity subordinated to $\mathcal{G}$. For each $i \in \mathbb{N}$, let $\{\xi_{k,G_i}\}_{k=1}^3$ and $\{\Phi_{k,G_i}\}_{k=1}^3$ be the unit vectors and the linear functions as in Lemma 2.3. Then:

$$\forall G \in \mathbb{R}^{2\times 2}_{sym,>} \qquad G = \sum_{i \in \mathbb{N}} \theta_i(G) G = \sum_{i \in \mathbb{N}} \sum_{k=1}^3 \theta_i(G) \Phi_{k,G_i}(G) \xi_{k,G_i} \otimes \xi_{k,G_i},$$

and we see that (2.7) holds by taking:

$$\eta_{i,k} = \xi_{k,G_i} \quad \text{and} \quad \phi_{i,k} = \big(\theta_i \Phi_{k,G_i} \big).$$

Since supp $\phi_{i,k} \subset B_i$ and since each G belongs to at most $2\sigma_0$ balls B_i, we see that (i) holds with $N_0 = 6\sigma_0$. On the other hand, condition (ii) follows by local finiteness of $\mathcal{G}$. ∎

3. The $\mathcal{C}^1$ approximations - a proof of Theorem 2.1.

The first result in the approximating sequence construction is what corresponds to a 'step' in Nash and Kuiper's terminology.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set. Given are: functions $v \in \mathcal{C}^\infty(\bar{\Omega})$ and $w \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$, a nonnegative function $a \in \mathcal{C}^\infty(\bar{\Omega})$, and a unit vector $\eta \in \mathbb{R}^2$. Then, for every $\lambda > 1$ there exist approximations $\tilde{v}_\lambda \in \mathcal{C}^\infty(\bar{\Omega})$ and $\tilde{w}_\lambda \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$ satisfying the following bounds:*

(3.1)
$$\Big\| \big(\frac{1}{2} \nabla \tilde{v}_\lambda \otimes \nabla \tilde{v}_\lambda + \text{sym} \nabla \tilde{w}_\lambda \big) - \big(\frac{1}{2} \nabla v \otimes \nabla v + \text{sym} \nabla w + a^2 \eta \otimes \eta \big) \Big\|_0$$
$$\leq \frac{C}{\lambda} \|a\|_0 \big(\|\nabla a\|_0 + \|\nabla^2 v\|_0 \big) + \frac{C}{\lambda^2} \|\nabla a\|_0^2,$$

(3.2)
$$\|\tilde{v}_\lambda - v\|_0 \leq \frac{C}{\lambda} \|a\|_0 \quad \text{and} \quad \|\tilde{w}_\lambda - w\|_0 \leq \frac{C}{\lambda} \|a\|_0 (\|a\|_0 + \|\nabla v\|_0),$$

(3.3)
$$\forall x \in \bar{\Omega} \qquad |\nabla \tilde{v}_\lambda(x) - \nabla v(x)| \leq Ca(x) + \frac{C}{\lambda} \|\nabla a\|_0,$$

$$|\nabla \tilde{w}_\lambda(x) - \nabla w(x)| \leq Ca(x)(\|a\|_0 + \|\nabla v\|_0) + \frac{C}{\lambda} \Big(\|a\|_0 (\|\nabla a\|_0 + \|\nabla^2 v\|_0) + \|\nabla a\|_0 \|\nabla v\|_0 \Big).$$

Proof. Using the 1-periodic functions Γ_i from Lemma 2.2, we define $\tilde{v}_\lambda$ and $\tilde{w}_\lambda$ as λ-periodic perturbations of v, w in the direction η:

(3.4)
$$\tilde{v}_\lambda(x) = v(x) + \frac{1}{\lambda}\Gamma_1(x, \lambda x \cdot \eta)$$
$$\tilde{w}_\lambda(x) = w(x) - \frac{1}{\lambda}\Gamma_1(x, \lambda x \cdot \eta)\nabla v(x) + \frac{1}{\lambda}\Gamma_2(x, \lambda x \cdot \eta)\eta.$$

The error estimates in (3.2) follow immediately from (2.4). The pointwise error estimates (3.3) follow from (2.4) in view of:

$$\nabla\tilde{v}_\lambda(x) = \nabla v(x) + \frac{1}{\lambda}\nabla_x\Gamma_1(x, \lambda x \cdot \eta) + \partial_t\Gamma_1(x, \lambda x \cdot \eta)\eta,$$

$$\nabla\tilde{w}_\lambda(x) = \nabla w(x) - \frac{1}{\lambda}\nabla v(x) \otimes \nabla_x\Gamma_1(x, \lambda x \cdot \eta) - \partial_t\Gamma_1(x, \lambda x \cdot \eta)\eta \otimes \nabla v(x) - \frac{1}{\lambda}\Gamma_1(x, \lambda x \cdot \eta)\nabla^2 v(x)$$
$$+ \frac{1}{\lambda}\eta \otimes \nabla_x\Gamma_2(x, \lambda x \cdot \eta) + \partial_t\Gamma_2(x, \lambda x \cdot \eta)\eta \otimes \eta.$$

Finally, we compute:

$$\frac{1}{2}\nabla\tilde{v}_\lambda(x)\otimes\nabla\tilde{v}_\lambda(x) - \frac{1}{2}\nabla v(x) \otimes \nabla v(x)$$

$$= \boxed{\frac{1}{\lambda}\text{sym}\left(\nabla v(x) \otimes \nabla_x\Gamma_1(x, \lambda x \cdot \eta)\right) + \partial_t\Gamma_1(x, \lambda x \cdot \eta)\text{sym}\left(\nabla v(x) \otimes \eta\right)}$$

$$+ \boxed{\boxed{\frac{1}{2}|\partial_t\Gamma_1(x, \lambda x \cdot \eta)|^2\eta \otimes \eta}}$$

$$+ \frac{1}{\lambda}\partial_t\Gamma_1(x, \lambda x \cdot \eta)\text{sym}\left(\eta \otimes \nabla_x\Gamma_1(x, \lambda x \cdot \eta)\right) + \frac{1}{2\lambda^2}\nabla_x\Gamma_1(x, \lambda x \cdot \eta) \otimes \nabla_x\Gamma_1(x, \lambda x \cdot \eta),$$

and:

$$\text{sym}\,\nabla\tilde{w}_\lambda(x) - \text{sym}\,\nabla w(x)$$

$$= \boxed{-\frac{1}{\lambda}\text{sym}\left(\nabla v(x) \otimes \nabla_x\Gamma_1(x, \lambda x \cdot \eta)\right) - \partial_t\Gamma_1(x, \lambda x \cdot \eta)\text{sym}\left(\nabla v(x) \otimes \eta\right)}$$

$$- \frac{1}{\lambda}\Gamma_1(x, \lambda x \cdot \eta)\nabla^2 v(x) + \frac{1}{\lambda}\text{sym}\left(\eta \otimes \nabla_x\Gamma_2(x, \lambda x \cdot \eta)\right)$$

$$+ \boxed{\boxed{\partial_t\Gamma_2(x, \lambda x \cdot \eta)\eta \otimes \eta}}.$$

We see that the terms in boxes cancel out, while the terms in double boxes add up to $a(x)^2\eta \otimes \eta$ in virtue of (2.3). Consequently:

$$\left(\frac{1}{2}\nabla\tilde{v}_\lambda(x) \otimes \nabla\tilde{v}_\lambda(x) + \text{sym}\,\nabla\tilde{w}_\lambda(x)\right) - \left(\frac{1}{2}\nabla v(x) \otimes \nabla v(x) + \text{sym}\,\nabla w(x) + a(x)^2\eta \otimes \eta\right)$$

$$= \frac{1}{\lambda}\left(\partial_t\Gamma_1(x, \lambda x \cdot \eta)\text{sym}\left(\eta \otimes \nabla_x\Gamma_1(x, \lambda x \cdot \eta)\right) - \Gamma_1(x, \lambda x \cdot \eta)\nabla^2 v(x) + \text{sym}\left(\eta \otimes \nabla_x\Gamma_2(x, \lambda x \cdot \eta)\right)\right)$$

$$+ \frac{1}{2\lambda^2}\nabla_x\Gamma_1(x, \lambda x \cdot \eta) \otimes \nabla_x\Gamma_1(x, \lambda x \cdot \eta),$$

which implies (3.1) in view of the bounds in (2.4).   ∎

We now complete the 'stage' in the approximating sequence construction.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Let $v \in \mathcal{C}^\infty(\bar{\Omega})$, $w \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$ and $A \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{2\times 2}_{sym})$ be such that the deficit function $\mathcal{D}$ defined below is positive definite in $\bar{\Omega}$:*

$$(3.5) \qquad \exists c > 0 \qquad \mathcal{D} = A - \big(\frac{1}{2}\nabla v \otimes \nabla v + \text{sym}\,\nabla w\big) > c\text{Id}_2 \quad in\ \bar{\Omega}.$$

Fix $\varepsilon > 0$. Then there exist $\tilde{v} \in \mathcal{C}^\infty(\bar{\Omega})$ and $\tilde{w} \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$ such that the new deficit $\tilde{\mathcal{D}}$ is still positive definite, and bounded by ε together with the error in the approximations $\tilde{v}$, $\tilde{w}$, namely:

$$(3.6) \qquad \exists \tilde{c} > 0 \qquad \tilde{\mathcal{D}} = A - \big(\frac{1}{2}\nabla\tilde{v} \otimes \nabla\tilde{v} + \text{sym}\,\nabla\tilde{w}\big) > \tilde{c}\text{Id}_2 \quad in\ \bar{\Omega},$$

$$(3.7) \qquad \|\tilde{\mathcal{D}}\|_0 < \varepsilon \quad and \quad \|\tilde{v} - v\|_0 + \|\tilde{w} - w\|_0 < \varepsilon.$$

Moreover, we have the following uniform gradient error bounds:

$$(3.8) \quad \|\nabla\tilde{v} - \nabla v\|_0 \le CN_0^{1/2}\|\mathcal{D}\|_0^{1/2} \quad and \quad \|\nabla\tilde{w} - \nabla w\|_0 \le CN_0(\|\nabla v\|_0 + \|\mathcal{D}\|_0^{1/2})\|\mathcal{D}\|_0^{1/2},$$

where the constant $N_0 \in \mathbb{N}$ is as in Lemma 2.4.

Proof. **1.** Note that the image $\mathcal{D}(\bar{\Omega})$ is a compact subset of $\mathbb{R}^{2\times 2}_{sym,>}$. By Lemma 2.4 and rearranging the indices, if needed, so that $J(\mathcal{D}(\bar{\Omega})) = \{1 \ldots N\}$ in (ii), we get:

$$(3.9) \qquad \forall x \in \bar{\Omega} \qquad \mathcal{D}(x) = \sum_{k=1}^N b_k(x)^2 \eta_k \otimes \eta_k \quad where \quad b_k = \phi_k \circ \mathcal{D} \in \mathcal{C}^\infty(\bar{\Omega}).$$

Let now $a_k = (1-\delta)^{1/2}b_k$, with $\delta > 0$ so small that:

$$(3.10) \qquad \mathcal{D} - \sum_{k=1}^N a_k^2 \eta_k \otimes \eta_k = \delta\mathcal{D} \quad and \quad \delta\|\mathcal{D}\|_0 < \frac{\varepsilon}{2}.$$

We set $v_1 = v$, $w_1 = w$. For $k = 1 \ldots N$ we inductively define $v_{k+1} \in \mathcal{C}^\infty(\bar{\Omega})$ and $w_{k+1} \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$, by means of Proposition 3.1 applied to v_k, w_k, a_k, η_k and with $\lambda_k > 1$ sufficiently large as indicated below. We then finally set $\tilde{v} = v_{N+1}$ and $\tilde{w} = w_{N+1}$.

2. To prove the estimates (3.6) - (3.8), we start by observing that since by Lemma 2.4 (i) at most N_0 terms in the expansion (3.9) are nonzero, there holds:

$$(3.11) \qquad \begin{aligned} \sum_{k=1}^N a_k(x) &\le \sum_{k=1}^N b_k(x) \le N_0^{1/2}\Big(\sum_{k=1}^N b_k(x)^2\Big)^{1/2} = N_0^{1/2}\big(\text{Trace}\,\mathcal{D}(x)\big)^{1/2} \\ &\le N_0^{1/2}\big(\sqrt{2}\,|\mathcal{D}(x)|\big)^{1/2} \le CN_0^{1/2}\|\mathcal{D}\|_0^{1/2}. \end{aligned}$$

Further, by (3.1) and (3.10):

$$\tilde{\mathcal{D}} = \mathcal{D} - \left(\left(\frac{1}{2} \nabla \tilde{v} \otimes \nabla \tilde{v} + \mathrm{sym}\, \nabla \tilde{w} \right) - \left(\frac{1}{2} \nabla v \otimes \nabla v + \mathrm{sym}\, \nabla w \right) \right)$$

$$= \mathcal{D} - \sum_{k=1}^{N} \left(\left(\frac{1}{2} \nabla v_{k+1} \otimes \nabla v_{k+1} + \mathrm{sym}\, \nabla w_{k+1} \right) - \left(\frac{1}{2} \nabla v_k \otimes \nabla v_k + \mathrm{sym}\, \nabla w_k \right) \right)$$

$$= \left(\mathcal{D} - \sum_{k=1}^{N} a_k^2 \eta_k \otimes \eta_k \right)$$

$$\quad - \sum_{k=1}^{N} \left(\left(\frac{1}{2} \nabla v_{k+1} \otimes \nabla v_{k+1} + \mathrm{sym}\, \nabla w_{k+1} \right) - \left(\frac{1}{2} \nabla v_k \otimes \nabla v_k + \mathrm{sym}\, \nabla w_k + a_k^2 \eta_k \otimes \eta_k \right) \right)$$

$$= \delta \mathcal{D} + \sum_{k=1}^{N} \mathcal{O}\left(\frac{1}{\lambda_k} \left(\|a_k\|_0 \|\nabla a_k\|_0 + \|\nabla a_k\|_0^2 + \|a_k\|_0 \|\nabla^2 v_k\|_0 \right) \right).$$

Choosing at each step λ_k sufficiently large with respect to the given a_k and the already generated v_k, we may ensure the smallness of the error term in the right hand side above and hence the positive definiteness of $\tilde{\mathcal{D}}$ in (3.6), because of the uniform positive definiteness of: $\delta \mathcal{D} > c\delta \mathrm{Id}_2$ in $\bar{\Omega}$. Likewise, the first inequality in (3.7) follows already when the error is smaller than $\epsilon/2$.

The same reasoning proves the error bounds on $\tilde{v} - v$ and $\tilde{w} - w$ in (3.7), in view of (3.2):

$$\tilde{v}(x) - v(x) = \sum_{k=1}^{N} (v_{k+1}(x) - v_k(x)) = \sum_{k=1}^{N} \mathcal{O}\left(\frac{1}{\lambda_k} \|a_k\|_0 \right),$$

$$\tilde{w}(x) - w(x) = \sum_{k=1}^{N} (w_{k+1}(x) - w_k(x)) = \sum_{k=1}^{N} \mathcal{O}\left(\frac{1}{\lambda_k} \left(\|a_k\|_0^2 + \|\nabla a_k\|_0 \|\nabla v_k\|_0 \right) \right).$$

3. To obtain the first error bound in (3.8), use (3.3) and (3.11):

$$|\nabla \tilde{v}(x) - \nabla v(x)| \leq \sum_{k=1}^{N} |\nabla v_{k+1}(x) - \nabla v_k(x)| \leq C \sum_{k=1}^{N} a_k(x) + \sum_{k=1}^{N} \mathcal{O}\left(\frac{1}{\lambda_k} \|\nabla a_k\|_0^2 \right) \leq C N_0^{1/2} \|\mathcal{D}\|_0^{1/2},$$

where again, by adjusting λ_k at each step, we ensure the controllability of the error term with respect to the nonnegative quantity $N_0^{1/2} \|\mathcal{D}\|_0^{1/2}$. Likewise:

$$\forall k = 1 \ldots N \qquad |\nabla v_k(x)| \leq |\nabla v(x)| + \sum_{i=1}^{k-1} |\nabla v_{i+1}(x) - \nabla v_i(x)| \leq \|\nabla v\|_0 + C N_0^{1/2} \|\mathcal{D}\|_0^{1/2},$$

and obviously by (3.11):

$$a_k(x) \leq C N_0^{1/2} \|\mathcal{D}\|_0^{1/2},$$

which yield by (3.11):

$$\sum_{k=1}^{N} a_k(x) \left(\|a_k\|_0 + \|\nabla v_k\|_0 \right) \leq C \left(\|\nabla v\|_0 + N_0^{1/2} \|\mathcal{D}\|_0^{1/2} \right) \sum_{k=1}^{N} a_k(x) \leq C N_0 \left(\|\nabla v\|_0 + \|\mathcal{D}\|_0^{1/2} \right) \|\mathcal{D}\|_0^{1/2}.$$

Consequently and by (3.3), there follows the last gradient error bound in (3.8):

$$
\begin{aligned}
|\nabla \tilde{w}(x) - \nabla w(x)| &\leq \sum_{k=1}^{N} |\nabla w_{k+1}(x) - \nabla w_k(x)| \\
&\leq C \sum_{k=1}^{N} a_k(x) \big(\|a_k\|_0 + \|\nabla v_k\|_0 \big) \\
&\quad + \sum_{k=1}^{N} \mathcal{O}\Big(\frac{1}{\lambda_k} \big(\|a_k\|_0 \|\nabla a_k\|_0 + \|a_k\|_0 \|\nabla^2 v_k\|_0 + \|\nabla a_k\|_0 \|\nabla v_k\|_0 \big) \Big) \\
&\leq C N_0 \big(\|\nabla v\|_0 + \|\mathcal{D}\|_0^{1/2} \big) \|\mathcal{D}\|_0^{1/2}.
\end{aligned}
$$

This concludes the proof of the stage approximation construction. ∎

We now finally give:

Proof of Theorem 2.1.

1. Fix $\varepsilon > 0$. It suffices to construct $v \in \mathcal{C}^1(\bar{\Omega})$ and $w \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$ such that:

$$
\tag{3.12} A_0 = \frac{1}{2} \nabla v \otimes \nabla v + \operatorname{sym} \nabla w \qquad \text{in } \bar{\Omega}
$$

and:

$$
\tag{3.13} \|v - v_0\|_0 + \|w - w_0\|_0 < \varepsilon.
$$

The exact solution (v, w) of (3.12) will be obtained as the $\mathcal{C}^1$ limit of sequences of succesive approximations $\{v_k \in \mathcal{C}^\infty(\bar{\Omega}), \, w_k \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)\}_{k=0}^{\infty}$, where v_0 and w_0 are given in the statement of the Theorem and satisfy (2.1), while v_{k+1} and w_{k+1} are defined inductively by means of Proposition 3.2 applied to v_k, w_k and $\varepsilon_k > 0$, under the following requirement:

$$
\tag{3.14} \sum_{k=1}^{\infty} \varepsilon_k < \varepsilon \quad \text{and} \quad \sum_{k=1}^{\infty} \varepsilon_k^{1/2} < 1.
$$

In agreement with our notation convention, we introduce the k-th deficit $\mathcal{D}_k$, which is positive definite by (3.6):

$$
\forall k \geq 0 \qquad \mathcal{D}_k := A_0 - \Big(\frac{1}{2} \nabla v_k \otimes \nabla v_k + \operatorname{sym} \nabla w_k \Big) \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{2\times 2}_{sym,>}).
$$

By (3.7) it follows that:

$$
\|v_k - v\|_0 + \|w_k - w\|_0 \leq \sum_{i=0}^{k-1} \|v_{i+1} - v_i\|_0 + \sum_{i=0}^{k-1} \|w_{i+1} - w_i\|_0 < \sum_{i=1}^{k-1} \epsilon_i < \sum_{i=1}^{\infty} \epsilon_i.
$$

Thus, $\{v_k\}_{k=0}^{\infty}$ and $\{w_k\}_{k=0}^{\infty}$ converge uniformly in $\bar{\Omega}$, respectively, to v and w which satisfy (3.13) in view of (3.14).

2. We now show that this convergence is in $\mathcal{C}^1$. Indeed, by (3.7): $\|\mathcal{D}_k\|_0 < \epsilon_k$, so by (3.8):

$$
\tag{3.15} \|\nabla v_{k+m} - \nabla v_k\|_0 \leq \sum_{i=k}^{k+m-1} \|\nabla v_{i+1} - \nabla v_i\|_0 \leq C N_0^{1/2} \sum_{i=k}^{k+m-1} \|\mathcal{D}_i\|_0^{1/2} \leq C N_0^{1/2} \sum_{i=k}^{k+m-1} \varepsilon_i^{1/2}.
$$

In particular, in view of (3.14) the sequence $\{\|\nabla v_k\|_0\}_{k=0}^\infty$ is bounded, so we further have:

$$\|\nabla w_{k+m} - \nabla w_k\|_0 \leq \sum_{i=k}^{k+m-1} \|\nabla w_{i+1} - \nabla w_i\|_0$$

(3.16)

$$\leq CN_0 \sum_{i=k}^{k+m-1} \big(\|\nabla v_i\|_0 + \|\mathcal{D}_i\|_0^{1/2}\big)\|\mathcal{D}_i\|_0^{1/2} \leq \tilde{C}N_0 \sum_{i=k}^{k+m-1} \varepsilon_i^{1/2},$$

where the constant $\tilde{C}$ is independent of k and m. Through the above assertions (3.15) and (3.16), in view of the second condition in (3.14), we conclude that $\{v_k\}_{k=1}^\infty$ and $\{w_k\}_{k=0}^\infty$ are Cauchy sequences that converge in $\mathcal{C}^1(\bar{\Omega})$ to $v \in \mathcal{C}^1(\bar{\Omega})$ and $w \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$, respectively. Finally:

$$\|A_0 - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\,\nabla w\big)\|_0 = \lim_{k\to\infty} \|\mathcal{D}_k\|_0 \leq \lim_{k\to\infty} \varepsilon_k = 0$$

implies (3.12) and completes the proof of Theorem 2.1.   ∎

Remark 3.3. In addition to the uniform convergence postulated in Theorem 2.1, one also has:

$$\forall n \qquad \|\nabla v_n\|_0 \leq \|\nabla v_0\|_0 + CN_0^{1/2}.$$

Using notation as in the proof above and recalling (3.15) and (3.14), this bound follows by:

$$\|\nabla v - \nabla v_0\|_0 = \lim_{k\to\infty} \|\nabla v_k - \nabla v_0\|_0 \leq \lim_{k\to\infty} \Big(CN_0^{1/2} \sum_{i=0}^{k-1} \epsilon_i^{1/2}\Big) \leq CN_0^{1/2}.$$

4. The $\mathcal{C}^{1,\alpha}$ approximations - a proof of Theorem 1.1, preliminary results and some heuristics towards the proof of Theorem 1.2.

Theorem 1.1 follows easily from Theorem 1.2, that will be proved in the next section.

Proof of Theorem 1.1.
Since $\mathcal{C}^1(\bar{\Omega})$ is dense in $\mathcal{C}^0(\bar{\Omega})$, we may without loss of generality assume that $v_0 \in \mathcal{C}^1(\bar{\Omega})$. Set $w_0 = 0$ and $A_0 = (\lambda + c)\mathrm{Id} \in \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R}^{2\times2}_{sym})$ where c is a constant and λ is constructed as follows.
Extend the function f to $f \in L^p(\Omega_\epsilon)$ defined on an open smooth set $\Omega_\epsilon \supset \bar{\Omega}$ and solve:

$$-\Delta\lambda = f \quad \text{in } \Omega_\epsilon, \qquad \lambda = 0 \quad \text{on } \partial\Omega_\epsilon.$$

Since $\lambda \in W^{2,p}(\Omega_\epsilon)$, then Morrey's Theorem implies that $\lambda \in \mathcal{C}^{0,\beta}(\bar{\Omega})$ for every $\beta \in (0,1)$ when $p \geq 2$, and for $\beta = 2 - \frac{2}{p}$ when $p \in (1,2)$. Also, for c large enough, condition (1.5) on the positive definiteness of the defect is satisfied. On the other hand:

$$-\mathrm{curl}\,\mathrm{curl}A_0 = -\Delta(\lambda + c) = f,$$

so the result follows directly from Theorem 1.2, since $\frac{2-2/p}{2} \geq \frac{1}{7}$ is equivalent to $p \geq \frac{7}{6}$.   ∎

Our next simple Corollary concerns the steady-state Euler equations with the exchanged roles of the given pressure q and the unknown forcing term $\nabla^\perp g$.

Corollary 4.1. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Let $q \in \mathcal{C}^{0,\beta}(\bar{\Omega})$ for some $\beta \in (0,1)$ and fix $\epsilon > 0$. Then for every exponent α in the range: $0 < \alpha < \min\{\frac{1}{7}, \frac{\beta}{2}\}$, there exist sequences $\{u_n \in \mathcal{C}^{0,\alpha}(\bar{\Omega}, \mathbb{R}^2)\}_{n=1}^{\infty}$ and $\{g_n \in \mathcal{C}^{0,\alpha}(\bar{\Omega})\}_{n=1}^{\infty}$ solving in Ω the following system:*

$$\text{(4.1)} \qquad \operatorname{div}(u_n \otimes u_n) - \nabla q = \nabla^{\perp} g_n, \qquad \operatorname{div} u_n = 0,$$

and such that $u_n = \nabla^{\perp} v_n$ and $g_n = \operatorname{curl} w_n$, where each $v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $w_n \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$, while the sequence $\{v_n\}_{n=1}^{\infty}$ is dense in $\mathcal{C}^0(\bar{\Omega})$ and $\|w_n\|_0 < \epsilon$ for every $n \geq 1$.

Proof. As before, since $\mathcal{C}^1(\bar{\Omega})$ is dense in $\mathcal{C}^0(\bar{\Omega})$, it is enough to take $v_0 \in \mathcal{C}^1(\bar{\Omega})$ and approximate it by a sequence $\{v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})\}_{n=1}^{\infty}$ with the properties as in the statement of Corollary. Let $w_0 = 0$ and let $c > 0$ be a sufficiently large constant, so that $(q + c)\operatorname{Id}_2 - \nabla v_0 \otimes \nabla v_0$ is strictly positive definite in $\bar{\Omega}$. By Theorem 1.2, there exists sequences $v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $w_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ which converge uniformly to v_0 and w_0 and which satisfy:

$$(q + c)\operatorname{Id}_2 = \nabla v_n \otimes \nabla v_n + 2\operatorname{sym} \nabla w_n \quad \text{in } \bar{\Omega}.$$

Taking the cofactor of both sides in the above matrix identity, we get:

$$(q + c)\operatorname{Id}_2 = \nabla^{\perp} v_n \otimes \nabla^{\perp} v_n + 2\operatorname{cof}(\operatorname{sym} \nabla w_n).$$

Taking the row-wise divergence, we obtain (4.1) with $u_n = \nabla^{\perp} v_n$ and $g_n = \operatorname{curl} w_n$, since $\operatorname{div} \operatorname{cof} \nabla w_n = 0$, while $\left(\operatorname{div} \operatorname{cof} (\nabla w_n)^T\right)^{\perp} = -\nabla(\operatorname{curl} w_n)$. ∎

Towards a proof of Theorem 1.2 we will derive a sequence of approximation results, and then combine them with Theorem 2.1 in section 6. For completeness, we first prove a simple, useful:

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Given are functions: $f \in \mathcal{C}^N(\bar{\Omega}, \mathbb{R}^n)$ and $\psi \in \mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R}^m)$. Then:*

$$\forall k = 0 \ldots N \qquad \|\psi \circ f\|_k \leq M \|f\|_k,$$

where the constant $M > 0$ depends on the dimensions n, m, the differentiability order N, the domain Ω, the norm $\|\psi\|_N$ on the compact set $f(\bar{\Omega})$ and the norm $\|f\|_0$, but it does not depend on the higher norms of f.

Proof. The statement is obvious for $k = 0$. Fix $k \in \{1 \ldots N\}$ and let $m = (m_1, \cdots, m_k)$ be any k-tuple of nonnegative integers such that $\sum_{i=1}^{k} i m_i = k$. Denoting $|m| = \sum_{i=1}^{k} m_i$ and using the interpolation inequality [1]:

$$\forall i = 1 \ldots k \qquad \|f\|_i \leq M_0 \|f\|_0^{1-i/k} \|f\|_k^{i/k},$$

valid with a constant $M_0 > 0$ depending on n, N and Ω, we get:

$$\prod_{i=1}^{k} \|\nabla^i f\|_0^{m_i} \leq M_0^{|m|} \prod_{i=1}^{k} \|f\|_0^{m_i - i m_i/k} \|f\|_k^{i m_i/k} = M_0^{|m|} \|f\|_0^{|m|-1} \|f\|_k.$$

with $|m| := m_1 + \cdots + m_j$. Calculating the partial derivatives in $\nabla^k(\psi \circ f)$ by the Faà di Bruno formula, gives hence the desired estimate:

$$\|\nabla^k(\psi \circ f)\|_0 \leq M \sum_m \prod_{i=1}^{k} \|\nabla^i f\|_0^{m_i} \leq M \|f\|_k.$$

Above, the summation extends over all multiindices $m = (m_1, \cdots, m_k)$ with the properties listed at the beginning of the proof. ∎

We recall the following estimates which have been proved in [10]:

Lemma 4.3. *Let* $\varphi \in \mathcal{C}_c^\infty(B(0,1), \mathbb{R})$ *be a standard mollifier supported on the ball* $B(0,1) \subset \mathbb{R}^n$, *that is a nonnegative, smooth and radially symmetric function such that* $\int_{\mathbb{R}^n} \varphi = 1$. *Denote:*

$$\forall l \in (0,1) \qquad \varphi_l(x) = \frac{1}{l^n} \varphi(\frac{x}{l}).$$

Then, for every $f, g \in \mathcal{C}^0(\mathbb{R}^n)$ *there holds:*

$$(4.2) \qquad\qquad \forall k, j \geq 0 \qquad \|f * \varphi_l\|_{k+j} \leq \frac{C}{l^k} \|f\|_j$$

$$(4.3) \qquad\qquad \forall k = 0 \ldots 2 \qquad \|f * \varphi_l - f\|_k \leq C l^k \|f\|_2$$

$$(4.4) \qquad\qquad \forall \alpha \in (0,1] \qquad \|f * \varphi_l - f\|_0 \leq C l^\alpha \|f\|_{0,\alpha}$$

$$(4.5) \qquad\qquad \forall \alpha \in (0,1] \qquad \|f * \varphi_l\|_1 \leq \frac{C}{l^{1-\alpha}} \|f\|_{0,\alpha}$$

$$(4.6) \qquad \forall k \geq 0 \quad \forall \alpha \in (0,1] \qquad \|(fg) * \varphi_l - (f * \varphi_l)(g * \varphi_l)\|_k \leq \frac{C}{l^{k-2\alpha}} \|f\|_{0,\alpha} \|g\|_{0,\alpha},$$

with the uniform constants $C > 0$ *depending only on the smoothness exponents* k, j, α.

Proof. The estimate (4.2) follows directly from the definition of convolution, whereas (4.3) results by using Taylor's expansion of f at a given $x \in \mathbb{R}^n$ and discarding the contribution with the symmetric term $\nabla f(x) \cdot y$ which integrates to 0. The proof of (4.4) follows by:

$$|(f * \varphi_l - f)(x)| = \Big| \int_{\mathbb{R}^n} \varphi_l(y) |y|^\alpha \frac{f(x-y) - f(x)}{|y|^\alpha} \, \mathrm{d}y \Big| \leq C l^\alpha \|f\|_{0,\alpha} \int_{\mathbb{R}^n} \varphi_l(y) \mathrm{d}y \leq C l^\alpha \|f\|_{0,\alpha},$$

while for (4.5) we write:

$$|\nabla(f * \varphi_l)(x)| = \Big| \int_{\mathbb{R}^n} f(x-y) \frac{1}{l^{n+1}} \nabla \varphi_l(\frac{y}{l}) \, \mathrm{d}y \Big| = \frac{1}{l} \Big| \int_{\mathbb{R}^n} \frac{f(x-y) - f(x)}{|y|^\alpha} \frac{|y|^\alpha}{l} \frac{1}{l^n} \nabla \varphi_l(\frac{y}{l}) \, \mathrm{d}y \Big|$$

$$\leq C l^{\alpha-1} \|f\|_{0,\alpha} \int_{\mathbb{R}^n} \frac{1}{l^n} |\nabla \varphi_l(\frac{y}{l})| \, \mathrm{d}y \leq \frac{C}{l^{1-\alpha}} \|f\|_{0,\alpha}.$$

Finally, for the crucial commutator estimate (4.6) we refer to [10, Lemma 1]. ∎

A heuristic overview of the next two sections.
Let us attempt to follow the construction in sections 2 and 3, but with the goal of controlling the higher Hölder norms of the iterations, and hence also quantifying the growth of the $\mathcal{C}^2$ norms of v, w. Let $A \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^{2\times2}_{sym})$ be the target matrix field and let $v_1 \in \mathcal{C}^\infty(\bar\Omega)$, $w_1 \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^2)$ be given at an input of a 'stage'. As in Proposition 3.2, we decompose the defect $\mathcal{D} = A - (\frac{1}{2}\nabla v_1 \otimes \nabla v_1 + \mathrm{sym}\,\nabla w_1)$ into a linear combination $\sum_{k=1}^N a_k^2 \eta_k \otimes \eta_k$ of rank-one symmetric matrices with smooth coefficients given by Lemma 2.4. We define:

$$v_{k+1}(x) = v_k(x) + \frac{1}{\lambda} \Gamma_1(x, \lambda x \cdot \eta_k),$$

$$w_{k+1}(x) = w_k(x) - \frac{1}{\lambda} \Gamma_1(x, \lambda x \cdot \eta_k) \nabla v_k(x) + \frac{1}{\lambda} \Gamma_2(x, \lambda x \cdot \eta_k) \eta_k.$$

This yields, by applying Lemma 4.2 to $\psi(x) = x^2$ and $f = a_k$:

$$\forall m : 0 \ldots 3 \qquad \|\nabla^m v_{k+1} - \nabla^m v_k\|_0 \leq C \sum_{i+j=m;\ 0 \leq i,j \leq m} \|a_k\|_i \lambda^{j-1},$$

$$\forall m : 0 \ldots 2 \qquad \|\nabla^m w_{k+1} - \nabla^m w_k\|_0 \leq C \sum_{i+j=m;\ 0 \leq i,j \leq m} \|a_k\|_i \lambda^{j-1}$$
$$+ C \sum_{i+j+s=m;\ 0 \leq i,j,s \leq m} \|a_k\|_i \lambda^{j-1} \|\nabla^{s+1} v_k\|_0,$$

On the other hand, applying Lemma 4.2 to $\psi = \phi_k$ defined in Lemma 2.4 and to $f = \mathcal{D}$, we get:

$$\forall k : 1 \ldots N \qquad \|a_k\|_2 \leq C(\|v_1\|_3^2 + \|w_1\|_3 + \|A\|_2).$$

Now, in order to control the $\mathcal{C}^{1,\alpha}$ norm of v_{N+1} through interpolation, we need to control the norm $\|v_{N+1}\|_2$, which in turn depends on $\|a_k\|_2$. The above estimate shows that at the end of each stage, the $\mathcal{C}^2$ norm of a_k is determined by the $\mathcal{C}^3$ norms of the given v_1 and w_1 of the previous stage. Further, the $\mathcal{C}^2$ norm of w_{N+1} is only controlled by the $\mathcal{C}^3$ norm of v_0 and also of all the a_k's. One might hope to control $\|a_k\|_3$ if the deficit $\mathcal{D}$ is small enough, but the dependence of $\|w_{N+1}\|_2$ on $\|v_0\|_3$ cannot be easily bypassed. Recalling that we need infinitely many stages in the construction, this implies that a direct estimate cannot be obtained in this manner, unless we deal with analytic data similar as in [4]. We thus need to modify the previous simplistic approach.

The appropriate modification is achieved by introducing a mollification before each stage. This technique was first introduced in [10] for the isometric immersion problem, in order to control the loss of regularity through the stages and to improve on results in [4]. Indeed, we note that the loss of derivatives in the above estimates is accompanied by a similar gain in the powers of λ, in a manner that the total order of derivatives, plus the order of powers needed to control $\|v_{N+1}\|_2$ and $\|w_{N+1}\|_2$ is constant. If we replace v_1 and w_1 by their mollifications on the scale $l \sim \lambda^{-1}$, each derivative loss can be estimated by one power of λ, and $\|v_0\|_2$ and $\|w_0\|_2$ will control $\|v_{N+1}\|_2$ and $\|w_{N+1}\|_2$. One problem still remains to be taken care of: does the deficit $\mathcal{D}$ decrease at the end of each stage? As the calculation below will show, a mollification of order λ^{-1} does not suffice to this end, and we need to mollify at a larger scale of $l > \lambda^{-1}$.

This is indeed how we want proceed. In practice, we let the mollification scale to be $l = \delta/M$ and we treat ∇v "like a", controlling its j-th norm by δl^{-j}. We then "sacrifice" one l in order to gain one δ; instead of $\|\nabla(v * \varphi_l)\|_j \leq C\|v\|_1 l^{-j}$, we use $\|\nabla(v * \varphi_l)\|_j \leq C(\|v\|_2 l)l^{-j}$, choosing l such that $l\|v\|_2 < \delta$ and obtaining the desired bound (5.2).

Finally, note that the loss of N powers of $\lambda l > 1$ in the control of the $\mathcal{C}^2$ norms at the end of each stage, is the main reason why the described scheme does not deliver better than $\mathcal{C}^{1,1/7}$ estimates, even for the optimal $N = 3$ from the decomposition in Lemma 2.3.

5. The $\mathcal{C}^{1,\alpha}$ approximations - a 'step' and a 'stage' in a proof of Theorem 1.2.

In this section, we develop the approximation technique that will be used for a proof of Theorem 1.2 in the next section. The first result is a variant of Proposition 3.1 in which we accomplish the 'step' of the Nash-Kuiper construction with extra estimates on the higher derivatives.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded set. Given are functions: $v \in \mathcal{C}^3(\bar{\Omega})$, $w \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$, a nonnegative function $a \in \mathcal{C}^3(\bar{\Omega})$ and a unit vector $\eta \in \mathbb{R}^2$. Let $\delta, l \in (0,1)$ be two*

parameter constants such that:

$$(5.1) \qquad \|a\|_m \leq \frac{\delta}{l^m} \quad \forall m = 0 \ldots 3, \quad and \quad \|\nabla v\|_m \leq \frac{\delta}{l^m} \quad \forall m = 1, 2.$$

Then for every $\lambda > 1/l$ there exist approximating functions $\tilde{v}_\lambda \in \mathcal{C}^3(\bar{\Omega})$ and $\tilde{w}_\lambda \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ satisfying the following bounds, with a universal constant $C > 0$ independent of all parameters:

$$(5.2) \qquad \left\|\left(\frac{1}{2}\nabla\tilde{v}_\lambda \otimes \nabla\tilde{v}_\lambda + \mathrm{sym}\,\nabla\tilde{w}_\lambda\right) - \left(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\,\nabla w + a^2\eta \otimes \eta\right)\right\|_0 \leq C\frac{\delta^2}{\lambda l},$$

$$(5.3) \qquad \|\tilde{v}_\lambda - v\|_m \leq C\delta\lambda^{m-1} \qquad \forall m = 0 \ldots 3,$$

$$(5.4) \qquad \|\tilde{w}_\lambda - w\|_m \leq C\delta\lambda^{m-1}\big(1 + \|\nabla v\|_0\big) \qquad \forall m = 0 \ldots 2.$$

Proof. We define $\tilde{v}_\lambda$, $\tilde{w}_\lambda$ as in the proof of Proposition 3.1:

$$\tilde{v}_\lambda(x) = v(x) + \frac{1}{\lambda}\Gamma_1(x, \lambda x \cdot \eta),$$

$$\tilde{w}_\lambda(x) = w(x) - \frac{1}{\lambda}\Gamma_1(x, \lambda x \cdot \eta)\nabla v(x) + \frac{1}{\lambda}\Gamma_2(x, \lambda x \cdot \eta)\eta.$$

Firstly, (5.2) follows immediately from (3.1) in view of (5.1), because $\lambda l > 1$:

$$\frac{1}{\lambda}\|a\|_0\big(\|\nabla a\|_0 + \|\nabla^2 v\|_0\big) + \frac{1}{\lambda^2}\|\nabla a\|_0^2 \leq 2\frac{\delta}{\lambda}\frac{\delta}{l} + \frac{1}{\lambda^2}\frac{\delta^2}{l^2} \leq 3\frac{\delta^2}{\lambda l}.$$

To check (5.3), we compute directly as in Lemma 2.2:

$$\nabla^m(\tilde{v}_\lambda - v)\|_0 \leq \frac{C}{\lambda}\|\nabla^m\Gamma_1(x, \lambda x \cdot \eta)\|_0 \leq \frac{C}{\lambda}\sum_{i+j=m;\ 0\leq i,j\leq m}\|a\|_j\lambda^j \leq \frac{C}{\lambda}\sum_{i=0}^m\frac{\delta}{l^i}\lambda^{m-i} \leq C\delta\lambda^{m-1}$$

by (5.1) and noting again $\lambda l > 1$. Similarly:

$$\|\nabla^m(\tilde{w}_\lambda - w)\|_0 \leq \frac{C}{\lambda}\Big(\|\nabla^m\Gamma_2(x, \lambda x \cdot \eta)\|_0 + \|\nabla^m\Gamma_1(x, \lambda x \cdot \eta)\nabla v\|_0\Big)$$

$$\leq \frac{C}{\lambda}\Big(\sum_{i+j=m,\ 0\leq i,j\leq m}\|a^2\|_i\lambda^j + \sum_{i+j+s=m,\ 0\leq i,j,s\leq m}\|a\|_i\lambda^j\|\nabla v\|_s\Big)$$

$$\leq \frac{C}{\lambda}\Big(\sum_{i=1}^m\frac{\delta}{l^i}\lambda^{m-i} + \sum_{0\leq i+s\leq m,\ 0\leq i,s\leq m}\frac{\delta}{l^i}\lambda^{m-(i+s)}\frac{\delta}{l^s} + \sum_{i+j=m,\ 0\leq i,j\leq m}\frac{\delta}{l^i}\lambda^j\|\nabla v\|_0\Big)$$

$$\leq \frac{C}{\lambda}\Big(\sum_{i=1}^m\frac{\delta}{l^i}\lambda^{m-i}\Big)\big(1 + 1 + \|\nabla v\|_0\big) \leq C\delta\lambda^{m-1}(1 + \|\nabla v\|_0),$$

where we applied Lemma 4.2 to $\psi(x) = x^2$ and $f = a$ in view of (5.1) yielding $\|a\|_0 \leq 1$, so that: $\|a^2\|_i \leq C\|a\|_i \leq C\delta/l^i$. This achieves (5.4) and completes the proof of Proposition. ∎

We now accomplish the 'stage' in the Hölder regular approximation construction.

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain. Let $v \in \mathcal{C}^2(\bar{\Omega})$, $w \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ and $A \in \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R}^{2\times 2}_{sym})$ for some $\beta \in (0, 1)$, be such that the deficit $\mathcal{D}$ is appropriately small:*

$$(5.5) \qquad \mathcal{D} = A - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\,\nabla w\big), \qquad 0 < \|\mathcal{D}\|_0 < \delta_0 \ll 1.$$

Then, for every two parameter constants M, σ satisfying:

$$\text{(5.6)} \qquad\qquad M > \max\{\|v\|_2, \|w\|_2, 1\} \quad and \quad \sigma > 1,$$

there exists $\tilde{v} \in \mathcal{C}^2(\bar{\Omega})$ and $\tilde{w} \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ such that the following error bounds hold for $\tilde{v}$, $\tilde{w}$ and the new deficit $\tilde{\mathcal{D}} = A - \left(\frac{1}{2}\nabla\tilde{v} \otimes \nabla\tilde{v} + \text{sym}\,\nabla\tilde{w}\right)$:

$$\text{(5.7)} \qquad\qquad \|\tilde{\mathcal{D}}\|_0 \leq C\Big(\frac{\|A\|_{0,\beta}}{M^\beta}\|\mathcal{D}\|_0^{\beta/2} + \frac{1}{\sigma}\|\mathcal{D}\|_0\Big),$$

$$\text{(5.8)} \qquad \|\tilde{v} - v\|_1 \leq C\|\mathcal{D}\|_0^{1/2} \quad and \quad \|\tilde{w} - w\|_1 \leq C(1 + \|\nabla v\|_0)\|\mathcal{D}\|_0^{1/2},$$

$$\text{(5.9)} \qquad\qquad \|\tilde{v}\|_2 \leq CM\sigma^3 \quad and \quad \|\tilde{w}\|_2 \leq C(1 + \|\nabla v\|_0)M\sigma^3.$$

The constant $C > 0$ is universal and independent of all parameters.

Proof. Analogously to [10, Proposition 4], the proof is split into three parts.

1. Mollification. Let $\varphi \in \mathcal{C}_c^\infty(B(0,1))$ be the standard mollifier in 2d, as in Lemma 4.3. Since v, w and A can be extended on the whole $\mathbb{R}^2$, with all their relevant norms increased at most C times (C depends here on the curvature of the boundary $\partial\Omega$), we may define:

$$\mathfrak{v} = v * \varphi_l, \qquad \mathfrak{w} := w * \varphi_l, \qquad \mathfrak{A} := A * \varphi_l \qquad \text{with} \quad l = \frac{\|\mathcal{D}\|_0^{1/2}}{M} < 1.$$

Applying Lemma 4.3 and noting (5.6), we immediately get the following uniform error bounds for $\mathfrak{v}$, $\mathfrak{w}$, $\mathfrak{A}$ and for the induced deficit $\mathfrak{D} = \mathfrak{A} - \left(\frac{1}{2}\nabla\mathfrak{v} \otimes \nabla\mathfrak{v} + \text{sym}\,\nabla\mathfrak{w}\right)$:

$$\text{(5.10)} \quad \begin{aligned} \|\mathfrak{v} - v\|_1 + \|\mathfrak{w} - w\|_1 &\leq Cl(\|v\|_2 + \|w\|_2) \leq C\|\mathcal{D}\|_0^{1/2}, \\ \|\mathfrak{A} - A\|_0 &\leq Cl^\beta\|A\|_{0,\beta}, \\ \|\mathfrak{D}\|_m &\leq \|\mathcal{D} * \varphi_l\|_m + \|(\nabla v * \varphi_l) \otimes (\nabla v * \varphi_l) - (\nabla v \otimes \nabla v) * \varphi_l\|_m \\ &\leq \frac{C}{l^m}\|\mathcal{D}\| + \frac{C}{l^{m-2}}\|v\|_2^2 \leq \frac{C}{l^m}\|\mathcal{D}\|_0 \qquad \forall m = 0\ldots 3. \end{aligned}$$

In the proof of the last inequality above, we used (4.6) with the Hölder exponent $\alpha = 1$.

We note that so far we have simply exchanged the lower regularity fields v, w, A with their smooth approximations, at the expense of the error that, as we shall see below, is compatible with the that postulated in (5.7) - (5.9). The following estimate, however, reflects the advantage of averaging through mollification that results in the control of $\mathcal{C}^3$ norm of $\mathfrak{v}$ by the $\mathcal{C}^2$ norm:

$$\text{(5.11)} \qquad \forall m = 1, 2 \qquad \|\nabla\mathfrak{v}\|_m \leq \|\mathfrak{v}\|_{m+1} \leq \frac{C}{l^{m-1}}\|v\|_2 \leq \frac{C}{l^m}\|\mathcal{D}\|_0^{1/2},$$

where again we used Lemma 4.3 and (5.6). Note that the scaling bound (5.11) is consistent with the second requirement in (5.1) of Proposition 5.1. We also record the following simple bound:

$$\text{(5.12)} \qquad\qquad \|\mathfrak{w}\|_2 \leq C\|w\|_2 \leq CM.$$

2. Modification and positive definiteness. Contrary to the 'stage' construction in the proof of Proposition 3.2, we do not know whether the original defect $\mathcal{D}$ (and hence the induced defect $\mathfrak{D}$) is positive definite, so that Lemma 2.4 could be used. In any case, we need to keep the number of terms in the decomposition (3.9) into rank-one matrices as small as possible.

We now further modify $\mathfrak{w}$ in order to use the optimal decomposition in (2.5). Let r_0 be as in Lemma 2.3 and define:

$$\mathfrak{w}' = \mathfrak{w} - 2\frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0}id_2, \qquad \mathfrak{D}' = \mathfrak{A} - \big(\frac{1}{2}\nabla\mathfrak{v} \otimes \nabla\mathfrak{v} + \mathrm{sym}\,\nabla\mathfrak{w}'\big).$$

Clearly, by (5.10) we get:

$$(5.13) \qquad\qquad \|\mathfrak{w}' - \mathfrak{w}\|_2 \leq C(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0) \leq C\|\mathcal{D}\|_0.$$

Note now that:

$$\mathfrak{D}'(x) = 2\frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0}\mathrm{Id}_2 + \mathfrak{D}(x) = 2\frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0}\Big(\mathrm{Id}_2 + \frac{r_0}{2(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}\mathfrak{D}\Big) \qquad \forall x \in \bar{\Omega}.$$

By Lemma 2.3 we may apply (2.5) to the scaled defect $G = \mathrm{Id}_2 + \frac{r_0}{2(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}\mathfrak{D}$ and arrive at:

$$(5.14) \qquad \mathfrak{D}'(x) = \sum_{k=1}^{3} 2\frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0}\Phi_k(G(x))\xi_k \otimes \xi_k = \sum_{k=1}^{3} a_k^2(x)\xi_k \otimes \xi_k \qquad \forall x \in \bar{\Omega},$$

where $\Big\{a_k = \big(2\frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0}\Phi_k \circ G\big)^{1/2}\Big\}_{k=1}^{3}$ are positive smooth functions on $\bar{\Omega}$. We claim that:

$$(5.15) \qquad\qquad \forall k = 1\ldots3 \quad \forall m = 0\ldots3 \qquad \|a_k\|_m \leq \frac{C}{l^m}\|\mathcal{D}\|_0^{1/2}.$$

Indeed, for $m = 0$ this inequality follows directly by $\|\mathfrak{D}\|_0 \leq C\|\mathcal{D}\|_0$. For $m = 1\ldots3$ we use Lemma 4.2 to each $\psi = \Phi_k^{1/2}$ and $f = G$, where noting that $\|G\|_0 \leq C$ and recalling (5.10) yields:

$$(5.16) \quad \begin{aligned} \|a_k\|_m &\leq \Big(2\frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0}\Big)^{1/2}C\|G\|_m \\ &\leq C(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)^{1/2}\Big(C + \frac{r_0}{2(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}\|\mathfrak{D}\|_m\Big) \\ &\leq C\Big((\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)^{1/2} + \frac{1}{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)^{1/2}}\frac{1}{l^m}\|\mathcal{D}\|_0\Big) \leq C\Big(\|\mathcal{D}\|_0^{1/2} + \frac{1}{l^m}\|\mathcal{D}\|_0^{1/2}\Big) \end{aligned}$$

and hence achieves (5.15). Note that the scaling bound (5.15) is consistent with the first requirement in (5.1) of Proposition 5.1.

3. Iterating the one-dimensional oscillations. We set $v_1 = \mathfrak{v}$, $w_1 = \mathfrak{w}$ and inductively define $v_{k+1} \in \mathcal{C}^3(\bar{\Omega})$ and $w_{k+1} \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ for $k = 1, 2, 3$ by means of Proposition 5.1 applied to v_k, w_k, the function a_k and the unit vector ξ_k appearing in (5.14), with the parameters:

$$l_k = \frac{l}{\sigma^{k-1}} < 1, \qquad \lambda_k = \frac{\sigma}{l_k} = \frac{1}{l_{k+1}} > \frac{1}{l_k},$$

and with the remaining three parameters:

$$(5.17) \qquad\qquad \delta_3 \geq \delta_2 \geq \delta_1 = \max_{m=1,2}\big\{l^m\|\nabla\mathfrak{v}\|_m\big\} + \max_{m=0\ldots3,\, k=1\ldots3}\big\{l^m\|a_k\|_m\big\}$$

as indicated below. We then finally set: $\tilde{v} = v_4$ and $\tilde{w} = w_4$.

We start by checking that the assumptions of Proposition 5.1 are satisfied. Namely, we claim that $\delta_k, l_k \in (0, 1)$ together with:

$$(5.18) \qquad \|a_k\|_m \leq \frac{\delta_k}{l_k^m} \quad \forall m = 0\ldots3 \qquad \text{and} \qquad \|\nabla v_k\|_m \leq \frac{\delta_k}{l_k^m} \quad \forall m = 1, 2,$$

at each iteration step $k = 1, 2, 3$, if only the constant δ_0 in (5.5) is appropriately small.

Indeed, $\delta_1 \leq C\|\mathcal{D}\|_0^{1/2}$ in view of (5.11) and (5.15), so $\delta_1 < 1$ if only $\delta_0 \ll 1$. Further, by the definition (5.17) it follows that: $\|a_k\|_m = \frac{1}{l^m} l^m \|a_k\|_m \leq \frac{\delta_1}{l^m} \leq \frac{\delta_k}{l_k^m}$, so the first assertion in (5.18) holds. For the second assertion, we see directly that it holds when $k = 1$, as: $\|\nabla v_1\|_m = \frac{1}{l^m} l^m \|\nabla \mathfrak{v}\|_m \leq \frac{\delta_1}{l^m}$. On the other hand, using induction on k and exploiting (5.3), we get:

$$\|\nabla v_{k+1}\|_m \leq \|\nabla v_k\|_m + \|\nabla v_{k+1} - \nabla v_k\|_m \leq \frac{\delta_k}{l_k^m} + C\delta_k \lambda_k^m$$

$$\leq \delta_k \Big(\frac{1}{l_{k+1}^m} + \frac{C}{l_{k+1}^m}\Big) = C\frac{\delta_k}{l_{k+1}^m} \leq \frac{\delta_{k+1}}{l_{k+1}^m} \qquad \forall m = 1, 2 \quad \forall k = 1, 2.$$

The proof of (5.18) is now complete for the choice $\delta_{k+1} = C\delta_k$, where $C > 1$ is, as always, an appropriately large universal constant. Consequently: $\delta_2, \delta_3 \leq C\|\mathcal{D}\|_0^{1/2} < 1$ if only $\delta_0 \ll 1$.

4. We now directly verify the concluding estimates of Proposition 5.2. We have, in view of the definition of $\mathfrak{D}'$ and (5.14):

$$\tilde{\mathcal{D}} = A - \mathfrak{A} + \mathfrak{D}' + \Big(\frac{1}{2}\nabla v_1 \otimes \nabla v_1 + \text{sym} \nabla w_1\Big) - \Big(\frac{1}{2}\nabla v_4 \otimes \nabla v_4 + \text{sym} \nabla w_4\Big)$$

$$= A - \mathfrak{A} - \sum_{k=1}^{3} \Big(\big(\frac{1}{2}\nabla v_{k+1} \otimes \nabla v_{k+1} + \text{sym} \nabla w_{k+1}\big) - \big(\frac{1}{2}\nabla v_k \otimes \nabla v_k + \text{sym} \nabla w_k + a_k \xi_k \otimes \xi_k\big)\Big),$$

and thus by (5.10), (5.2) and the definition of l, there follows (5.7):

$$\|\tilde{\mathcal{D}}\|_0 \leq \|A - \mathfrak{A}\|_0 + C\sum_{k=1}^{3} \frac{\delta_k^2}{\lambda_k l_k} \leq C\Big(l^\beta \|A\|_{0,\beta} + \delta_3^2 \sum_{k=1}^{3} \frac{1}{\lambda_k l_k}\Big)$$

$$\leq C\Big(\frac{\|\mathcal{D}\|_0^{\beta/2}}{M^\beta}\|A\|_{0,\beta} + 3\frac{\delta_3^2}{\sigma}\Big) \leq C\Big(\frac{\|\mathcal{D}\|_0^{\beta/2}}{M^\beta}\|A\|_{0,\beta} + \frac{1}{\sigma}\|\mathcal{D}\|_0\Big).$$

We now check (5.8), using (5.10), (5.13) and (5.4):

$$\|\tilde{v} - v\|_1 \leq \|\mathfrak{v} - v\|_1 + \sum_{k=1}^{3} \|v_{k+1} - v_k\|_1 \leq C\|\mathcal{D}\|_0^{1/2} + C\sum_{k=1}^{3} \delta_k \leq C\|\mathcal{D}\|_0^{1/2}$$

$$\|\tilde{w} - w\|_1 \leq \|\mathfrak{w} - w\|_1 + \|\mathfrak{w}' - \mathfrak{w}\|_1 + \sum_{k=1}^{3} \|w_{k+1} - w_k\|_1$$

(5.19)

$$\leq C\Big(\|\mathcal{D}\|_0^{1/2} + \|\mathcal{D}\|_0 + \sum_{k=1}^{3} \delta_k(1 + \|\nabla v_k\|_0)\Big) \leq C\|\mathcal{D}\|_0^{1/2}\Big(1 + \sum_{k=1}^{3} \|\nabla v_k\|_0\Big)$$

$$\leq C\|\mathcal{D}\|_0^{1/2}\Big(1 + \|\nabla v\|_0 + \|\mathfrak{v} - v\|_1 + \sum_{k=1}^{2} \|v_{k+1} - v_k\|_1\Big)$$

$$\leq C\|\mathcal{D}\|_0^{1/2}\big(1 + \|\nabla v\|_0 + \|\mathcal{D}\|_0^{1/2}\big) \leq C\|\mathcal{D}\|_0^{1/2}\big(1 + \|\nabla v\|_0\big).$$

Finally, the first bound in (5.9) follows by (5.11) and (5.3):

$$\|\tilde{v}\|_2 \leq \|\mathfrak{v}\|_2 + \sum_{k=1}^{3} \|v_{k+1} - v_k\|_2 \leq \frac{C}{l}\|\mathcal{D}\|_0^{1/2} + C\sum_{k=1}^{3} \delta_k \lambda_k$$

$$\leq \frac{C}{l}\|\mathcal{D}\|_0^{1/2} + C\delta_3 \sum_{k=1}^{3} \frac{\sigma^k}{l} \leq \frac{C}{l}\|\mathcal{D}\|_0^{1/2}(1 + \sigma^3) \leq CM\sigma^3,$$

while the second bound is obtained by:

$$\|\tilde{w}\|_2 \leq \|\mathfrak{w}\|_2 + \|\mathfrak{w}' - \mathfrak{w}\|_2 + \sum_{k=1}^{3} \|w_{k+1} - w_k\|_2 \leq C\Big(M + \|\mathcal{D}\|_0 + \sum_{k=1}^{3} \delta_k \lambda_k(1 + \|\nabla v_k\|_0)\Big)$$

$$\leq C\Big(M + \delta_3 \sum_{k=1}^{3} \frac{\sigma^3}{l}(1 + \|\nabla v_k\|_0)\Big) \leq CM\Big(1 + \sigma^3 + \sigma^3 \sum_{k=1}^{3} \|\nabla v_k\|_0\Big)$$

$$\leq CM\sigma^3\Big(1 + \sum_{k=1}^{3} \|\nabla v_k\|_0\Big) \leq CM\sigma^3\big(1 + \|\nabla v\|_0\big).$$

in view of (5.12), (5.13) and reasoning as in (5.19).   ∎

6. The $\mathcal{C}^{1,\alpha}$ approximations - a proof of Theorem 1.2.

We are now in a position to state the final intermediary approximation result, parallel to [10, Theorem 1].

Theorem 6.1. *Assume that $\Omega \subset \mathbb{R}^2$ is an open, bounded domain. Given are functions $v \in \mathcal{C}^2(\bar{\Omega})$, $w \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ and $A \in \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R}^{2\times 2}_{sym})$ for some $\beta \in (0,1)$, such that the deficit $\mathcal{D}$ below is appropriately small:*

$$(6.1) \qquad \mathcal{D} = A - \big(\frac{1}{2}\nabla v \otimes \nabla v + \operatorname{sym}\nabla w\big), \qquad 0 < \|\mathcal{D}\|_0 < \delta_0 \ll 1.$$

Fix the exponent:

$$(6.2) \qquad\qquad\qquad 0 < \alpha < \min\Big\{\frac{1}{7}, \frac{\beta}{2}\Big\}.$$

Then, there exist $\bar{v} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $\bar{w} \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ such that:

$$(6.3) \qquad\qquad\qquad \frac{1}{2}\nabla\bar{v} \otimes \nabla\bar{v} + \operatorname{sym}\nabla\bar{w} = A,$$

$$(6.4) \qquad \|\bar{v} - v\|_1 \leq C\|\mathcal{D}\|_0^{1/2} \qquad and \qquad \|\bar{w} - w\|_1 \leq C(1 + \|\nabla\tilde{v}\|_0)\|\mathcal{D}\|_0^{1/2},$$

where $C > 0$ is a constant depending on α but independent of all other parameters.

Proof. The exact solution to (6.3) will be obtained as the $\mathcal{C}^{1,\alpha}$ limit of sequences of successive approximations $\{v_k \in \mathcal{C}^2(\bar{\Omega}), \ w_k \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)\}_{k=1}^{\infty}$.

1. Induction on stages. We set $v_0 = v$ and $w_0 = w$. Given v_k and w_k, define v_{k+1} and w_{k+1} by applying Proposition 5.2 with parameters σ and M_k that will be appropriately chosen below and that satisfy:

$$(6.5) \qquad\qquad M_k > \max\{\|v_k\|_2, \|w_k\|_2, 1\} \qquad and \qquad \sigma > 1.$$

Following our notational convention, we define the k-th deficit $\mathcal{D}_k = A - \left(\frac{1}{2}\nabla v_k \otimes \nabla v_k + \text{sym } \nabla w_k\right)$. In view of Proposition 5.2, we get:

$$(6.6) \qquad \|\mathcal{D}_{k+1}\|_0 \leq C\Big(\frac{\|A\|_{0,\beta}}{M_k^\beta}\|\mathcal{D}_k\|_0^{\beta/2} + \frac{1}{\sigma}\|\mathcal{D}_k\|_0\Big),$$

$$(6.7) \qquad \|v_{k+1} - v_k\|_1 \leq C\|\mathcal{D}_k\|_0^{1/2} \qquad \text{and} \qquad \|w_{k+1} - w_k\|_1 \leq C(1 + \|\nabla v_n\|_0)\|\mathcal{D}_k\|_0^{1/2},$$

$$(6.8) \qquad \|v_{k+1}\|_2 \leq C M_k \sigma^3 \qquad \text{and} \qquad \|w_{k+1}\|_2 \leq C(1 + \|\nabla v_k\|_0)M_k\sigma^3,$$

provided that (5.5) holds for each $\mathcal{D}_k$. We shall now validate this requirement, with the parameters:

$$(6.9) \qquad M_k = \Big(\mathfrak{C}(1 + \|\nabla v_0\|_0)\sigma^3\Big)^k M_0.$$

In fact, we will inductively prove that one can have:

$$(6.10) \qquad \|\mathcal{D}_k\|_0 \leq \frac{1}{\sigma^{sk}}\|\mathcal{D}\|_0 \qquad \text{with any} \qquad 0 < s < \min\Big\{1, \frac{6\beta}{2-\beta}\Big\}.$$

Fix s as indicated in (6.10). Clearly, (6.10) and (6.5) hold for $k = 0$. By (6.6) and the induction assumption we obtain the bound:

$$(6.11) \qquad \sigma^{s(k+1)}\frac{\|\mathcal{D}_{k+1}\|_0}{\|\mathcal{D}\|_0} \leq \frac{C\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2-1}\sigma^s}{M_0^\beta}\frac{1}{\mathfrak{C}^{k\beta}}\left(\frac{\sigma^{(1-\beta/2)(s-\frac{6\beta}{2-\beta})}}{(1+\|\nabla v_0\|_0)^\beta}\right)^k + C\sigma^{s-1}.$$

We see that in view of the condition on s in (6.10), both σ^{s-1} and $\sigma^{(1-\beta/2)(s-\frac{6\beta}{2-\beta})}$ are smaller than 1. Further, it is possible to choose $\sigma > 1$ so that the second term in (6.11) be smaller than $1/2$ and that the quotient term in parentheses above is also smaller than 1. Then, choose M_0 so that (6.5) holds for $k = 0$ together with:

$$\frac{C\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2-1}\sigma^s}{M_0^\beta} < \frac{1}{2}.$$

This results in the first term in (6.11) being smaller than $1/2$ if only $\mathfrak{C} \geq 1$. Consequently, we get $\sigma^{s(k+1)}\|\mathcal{D}_{k+1}\|_0/\|\mathcal{D}\|_0 \leq 1$ as needed in (6.10).

Observe now that by (6.7) and by the established (6.10):

$$(6.12) \qquad \begin{aligned} \forall k \geq 0 \qquad \|\nabla v_k\|_0 &\leq \|\nabla v_0\|_0 + \sum_{i=0}^{k-1}\|v_{i+1} - v_i\|_1 \leq \|\nabla v_0\|_0 + C\sum_{i=0}^{k-1}\|\mathcal{D}_i\|_0^{1/2} \\ &\leq \|\nabla v_0\|_0 + C\Big(\sum_{i=0}^{\infty}\frac{1}{\sigma^{si/2}}\Big)\|\mathcal{D}\|_0^{1/2} = \|\nabla v_0\|_0 + \frac{C}{1-\sigma^{-s/2}}\|\mathcal{D}\|_0^{1/2} \\ &\leq \|\nabla v_0\|_0 + C\|\mathcal{D}\|_0^{1/2}, \end{aligned}$$

if only, say, $\sigma^s > 4$ which can be easily achieved through the choice of σ. Now, by (6.8) and (6.12):

$$\frac{\|v_{k+1}\|_2}{M_{k+1}} \leq \frac{1}{\mathfrak{C}}\frac{C}{(1+\|\nabla v_0\|_0)} \qquad \text{and} \qquad \frac{\|w_{k+1}\|_2}{M_{k+1}} \leq \frac{1}{\mathfrak{C}}\frac{C(1+\|\nabla v_k\|_0)}{(1+\|\nabla v_0\|_0)} \leq \frac{1}{\mathfrak{C}}\frac{C(1+\|\nabla v_0\|_0 + \|\mathcal{D}\|_0^{1/2})}{(1+\|\nabla v_0\|_0)}.$$

Hence, taking the constant $\mathfrak{C} \gg 1$ large enough, we see that both quantities above can be made smaller than 1, proving therefore the requiblue (6.5).

2. $\mathcal{C}^{1,\alpha}$ control of the approximating sequences v_n and w_n. Let now α be an exponent as in (6.2). Choose s satisfying (6.10) and:

(6.13) $$\alpha(6+s) - s < 0.$$

It is an easy calculation that s satisfying (6.10) and (6.13) exists if and only if the exponent α is in the range (6.2). Indeed, (6.13) is equivalent to $\alpha < \frac{s}{6+s}$, while (6.10) is equivalent to:

$$0 < \frac{s}{6+s} < \min\left\{\frac{1}{7}, \frac{\beta}{2}\right\}.$$

We will prove that sequences $\{v_k, w_k\}_{k=0}^{\infty}$ are Cauchy in $\mathcal{C}^{1,\alpha}(\bar{\Omega})$. Firstly, by (6.7), (6.12),(6.10):

(6.14)
$$\|v_{k+1} - v_k\|_1 \le C\|\mathcal{D}_k\|_0^{1/2} \le \frac{C}{\sigma^{sk/2}}\|\mathcal{D}\|_0^{1/2},$$

$$\|w_{k+1} - w_k\|_1 \le C(1 + \|\nabla v_k\|_0)\|\mathcal{D}_k\|_0^{1/2} \le \frac{C}{\sigma^{sk/2}}\big(1 + \|\nabla v_0\|_0 + \|\mathcal{D}\|_0^{1/2}\big)\|\mathcal{D}\|_0^{1/2},$$

so we see right away that they are Cauchy in $\mathcal{C}^1(\bar{\Omega})$. On the other hand, by (6.8), (6.12), (6.10):

$$\|v_{k+1}-v_k\|_2 + \|w_{k+1}-w_k\|_2 \le C(1+\|\nabla v_k\|_0)M_k\sigma^3 \le C\big(1+\|\nabla v_0\|_0+\|\mathcal{D}\|_0^{1/2}\big)\big(\mathfrak{C}(1+\|\nabla v_0\|_0)\sigma^3\big)^k M_0,$$

so the sequences have the tendency to diverge in $\mathcal{C}^2(\bar{\Omega})$. Interpolating now the $\mathcal{C}^{1,\alpha}$ norm by [1]:

$$\|f\|_{0,\alpha} \le \|f\|_1^{\alpha}\|f\|_0^{1-\alpha},$$

we obtain:

(6.15)
$$\|\nabla(v_{k+1} - v_k)\|_{0,\alpha} + \|\nabla(w_{k+1} - w_k)\|_{0,\alpha} \le C_0^{\alpha}(C_0\sigma^3)^{k\alpha}M_0^{\alpha} \cdot C_0^{1-\alpha}\frac{1}{\sigma^{sk(1-\alpha)/2}}$$

$$= C_0 M_0^{\alpha}(C_0^{\alpha})^h\left(\sigma^{\frac{\alpha(6+s)-s}{2}}\right)^k,$$

where by C_0 we denoted an upper bound of all quantities involving C, v_0, $\mathcal{D}$. It is clear that choosing σ sufficiently large (so that $C_0\sigma^{3-s/2} < 1$), the resulting bound (6.15) implies that $\{\nabla v_k, \nabla w_k\}_{k=0}^{\infty}$ are Cauchy in $\mathcal{C}^{0,\alpha}(\bar{\Omega})$, provided that (6.13) holds. We see that the choice of exponent range in (6.2) so that the above construction technique works, is optimal.

3. Concluding, we see that $\{v_k, w_k\}_{k=0}^{\infty}$ converge to some $\bar{v} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $\bar{w} \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$. Since the defects in the approximating sequence obeys: $\lim_{k\to\infty}\|\mathcal{D}_k\|_0 = 0$ by (6.10), we immediately get (6.3). Additionally, by (6.14):

$$\|\bar{v} - v\|_1 \le \sum_{k=0}^{\infty}\|v_{k+1} - v_k\|_1 \le C\Big(\sum_{k=0}^{\infty}\frac{1}{\sigma^{sk/2}}\Big)\|\mathcal{D}\|_0^{1/2} = \frac{C}{1-\sigma^{-s/2}}\|\mathcal{D}\|_0^{1/2} \le C\|\mathcal{D}\|_0^{1/2}$$

$$\|\bar{w} - w\|_1 \le \sum_{k=0}^{\infty}\|w_{k+1} - w_k\|_1 \le C\Big(\sum_{k=0}^{\infty}\frac{1}{\sigma^{sk/2}}\Big)(1 + \|\nabla v\|_0)\|\mathcal{D}\|_0^{1/2} \le C(1 + \|\nabla v\|_0)\|\mathcal{D}\|_0^{1/2}.$$

completing the proof of (6.4).   ∎

We are now ready to give:

Proof of Theorem 1.2.

Fix a sufficiently small $\varepsilon > 0$. We will construct $\bar{v} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $\bar{w} \in \mathcal{C}^{1,\alpha}(\Omega, \mathbb{R}^2)$ such that:

$$(6.16) \qquad A_0 = \frac{1}{2}\nabla \bar{v} \otimes \nabla \bar{v} + \operatorname{sym} \nabla \bar{w} \qquad \text{in } \bar{\Omega}$$

and:

$$(6.17) \qquad \|\bar{v} - v_0\|_0 + \|\bar{w} - w_0\|_0 < \varepsilon.$$

In order to apply Theorem 6.1, we need to decrease the deficit $A_0 - \left(\frac{1}{2}\nabla v_0 \otimes \nabla v_0 + \operatorname{sym} \nabla w_0\right)$ so that it obeys (6.1). This will be done in three steps.

First, let $\tilde{v}_0 \in \mathcal{C}^\infty(\bar{\Omega})$, $\tilde{w}_0 \in \mathcal{C}^\infty(\Omega, \mathbb{R}^2)$ and $\tilde{A}_0 \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{2\times2}_{sym})$ be such that:

$$(6.18) \qquad \begin{aligned} &\|\tilde{v}_0 - v_0\|_1 + \|\tilde{w}_0 - w_0\|_1 + \|\tilde{A}_0 - A_0\|_0 < \varepsilon^2 \\ &\exists \tilde{c}_0 > 0 \qquad A_0 - \left(\frac{1}{2}\nabla \tilde{v}_0 \otimes \nabla \tilde{v}_0 + \operatorname{sym} \nabla \tilde{w}_0\right) > \tilde{c}_0 \operatorname{Id}_2 \qquad \text{in } \bar{\Omega}. \end{aligned}$$

Second, by Theorem 2.1 and Remark 3.3, there exists $v \in \mathcal{C}^1(\bar{\Omega})$ and $w \in \mathcal{C}^1(\Omega, \mathbb{R}^2)$ such that:

$$(6.19) \qquad \begin{aligned} &\tilde{A}_0 = \frac{1}{2}\nabla v \otimes \nabla v + \operatorname{sym} \nabla w \qquad \text{in } \bar{\Omega}, \\ &\|v - \tilde{v}_0\|_0 + \|w - \tilde{w}_0\|_0 < \varepsilon^2 \qquad \text{and} \qquad \|\nabla v - \nabla \tilde{v}_0\|_0 \le C. \end{aligned}$$

Third, let $\tilde{v} \in \mathcal{C}^2(\bar{\Omega})$ and $\tilde{w} \in \mathcal{C}^2(\Omega, \mathbb{R}^2)$ be such that:

$$(6.20) \qquad \|v - \tilde{v}\|_1 + \|w - \tilde{w}\|_1 < \varepsilon^2.$$

By (6.19), (6.20) and (6.18), we get:

$$(6.21) \qquad \begin{aligned} \|A_0 - &\left(\frac{1}{2}\nabla \tilde{v} \otimes \nabla \tilde{v} + \operatorname{sym} \nabla \tilde{w}\right)\|_0 \\ &\le \|A_0 - \tilde{A}_0\|_0 + \|\left(\frac{1}{2}\nabla \tilde{v} \otimes \nabla \tilde{v} + \operatorname{sym} \nabla \tilde{w}\right) - \left(\frac{1}{2}\nabla v \otimes \nabla v + \operatorname{sym} \nabla w\right)\|_0 \\ &\le \|A_0 - \tilde{A}_0\|_0 + \left(\|\nabla v\|_0 + \|\nabla \tilde{v}\|_0\right)\|\nabla v - \nabla \tilde{v}\|_0 + \|\nabla w - \nabla \tilde{w}\|_0 \\ &\le \varepsilon^2 + \left(2\|\nabla v_0\|_0 + 2\varepsilon^2 + C\right)\varepsilon^2 + \varepsilon^2 < \delta_0, \end{aligned}$$

as requiblue in Theorem 6.1, if only ε is small enough. We now apply Theorem 6.1 to $\tilde{v}$, $\tilde{w}$ and the original field A_0, and get $\bar{v} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $\bar{w} \in \mathcal{C}^{1,\alpha}(\Omega, \mathbb{R}^2)$ satisfying (6.16) and such that:

$$\begin{aligned} \|\bar{v} - v_0\|_0 + \|\bar{w} - w_0\|_0 &\le C\left(1 + \|\nabla \tilde{v}\|_0\right)\|A_0 - \left(\frac{1}{2}\nabla \tilde{v} \otimes \nabla \tilde{v} + \operatorname{sym} \nabla \tilde{w}\right)\|_0 + 3\varepsilon^2 \\ &\le C\left(1 + \varepsilon^2 + \|\nabla v_0\|_0\right)^2 \varepsilon^2 + 3\varepsilon^2, \end{aligned}$$

by (6.4), (6.21), (6.20), (6.19) and (6.18). Clearly (6.17) follows, if ε is small enough. $\blacksquare$

The following Corollary is of independent interest:

Corollary 6.2. *Let Ω, f, p, α be as in the statement of Theorem 1.1. Let $q \ge 2$. Then, for all $v_0 \in W^{1,q}(\Omega)$, there exists a sequence $v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ weakly converging to v_0 in $W^{1,q}(\Omega)$, and such that: $\mathcal{D}et \nabla^2 v_n = f$ in Ω.*

Proof. Let $\bar{v}_n \in \mathcal{C}^1(\bar{\Omega})$ converge to v_0 in $W^{1,q}(\Omega)$. For every $\bar{v}_n$, consider the approximating sequence $\{v_{n,k} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})\}_{k=1}^{\infty}$ as in Theorem 1.1, converging uniformly to $\bar{v}_n$. Define now $\{v_n\}$ to be an appropriate diagonal sequence, so that it converges to v_0 in $L^q(\Omega)$. We will check that $\{v_n\}$ is bounded in $W^{1,q}$.

The boundedness of $\|v_n\|_{L^q}$ is clear from the convergence statement. On the other hand, the proof of Theorem 1.2 gives, by (6.4), (6.18), (6.19), (6.20) and (6.21):

$$|\nabla v_n(x)| \le |\nabla \bar{v}_n(x)| + 2\epsilon^2 + C + C\delta_0^{1/2} \le |\nabla \bar{v}_n(x)| + C \qquad \forall x \in \Omega.$$

Consequently, $\|\nabla v_n\|_{L^q} \le \|\nabla \bar{v}_n\|_{L^q} + C \le C$, which concludes the proof. ∎

7. Rigidity results for $\alpha > 2/3$ - a proof of Theorem 1.3.

The crucial element in the proof of the rigidity Theorems 1.3 and 1.4 is the following result, that is the 'small slope analogue' of [10, Proposition 6]:

Proposition 7.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, simply connected domain. Assume that for some $\alpha \in (2/3, 1)$, the function $v \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ is a solution to:*

$$\mathcal{D}et \, \nabla^2 v = f \qquad in \, \bar{\Omega},$$

where $f \in L^p(\Omega)$ and $p > 1$. Then the following degree formula holds true, for every open subset U compactly contained in Ω and every $g \in L^{\infty}(\mathbb{R}^2)$ with supp $g \subset \mathbb{R}^2 \setminus \nabla v(\partial U)$:

$$(7.1) \qquad \int_U (g \circ \nabla v) f = \int_{\mathbb{R}^2} g(y) \deg(\nabla v, U, y) \, \mathrm{d}y.$$

Above, $\deg(\psi, U, y)$ denotes the Brouwer degree of a continuous function $\psi : \bar{U} \to \mathbb{R}^2$ at a point $y \in \mathbb{R}^2 \setminus \psi(\partial U)$.

Proof. **1.** Fix U and g as in the statement of the Proposition. We refer to [38] for the definition and properties of the Brouwer degree; recall first that $\deg(\nabla v, U, \cdot)$ is well defined on the open set $\mathbb{R}^2 \setminus \nabla v(\partial U)$. In fact, this function is constant on each connected component $\{U_i\}_{i=0}^{\infty}$ of $\mathbb{R}^2 \setminus \nabla v(\partial U)$ and it equals 0 on the only unbounded component $U_0 \subset \mathbb{R}^2 \setminus \nabla v(\bar{U})$. Thus, without loss of generality, we may assume that g is compactly supported and that: supp $g \subset \bigcup_{k=1}^{\infty} U_k$. By compactness, there must be: supp $g \subset \bigcup_{k=1}^{N} U_k$ for some N, and consequently the integral in the right hand side of (7.1) is well defined.

Let now $\{g_i \in \mathcal{C}_c^{\infty}(\bigcup_{k=1}^{N} U_k)\}_{i=1}^{\infty}$ be a sequence pointwise converging to g and such that $\|g_i\|_0 \le \|g\|_{L^{\infty}}$ for all i. It is sufficient to prove the formula (7.1) for each g_i and pass to the limit by the dominated convergence theorem. To simplify the notation, we drop the index i, and so in what follows we assume that $g \in \mathcal{C}_c^{\infty}(\mathbb{R}^2 \setminus \nabla v(\partial U))$.

As in the proof of Theorem 1.1, let $A \in W^{2,p}(\Omega) \cap \mathcal{C}^{0,\beta}(\bar{\Omega})$ be such that $\operatorname{curl} \operatorname{curl} A = -f$. Here, we take $\beta = \min\{2 - \frac{2}{p}, \alpha\} \in (0, 1)$. Consequently, in view of the simple connectedness of Ω, there exists $w \in \mathcal{C}^{1,\beta}(\bar{\Omega}, \mathbb{R}^2)$ such that:

$$A = \frac{1}{2} \nabla v \otimes \nabla v + \operatorname{sym} \nabla w.$$

For a standard 2d mollifier $\varphi \in \mathcal{C}_c^{\infty}(B(0,1))$ as in Lemma 4.3, define:

$$\forall l \in (0,1) \qquad v_l = v * \varphi_l, \quad w_l = w * \varphi_l, \quad A_l = A * \varphi_l,$$

and apply the degree formula (change of variable formula [16, 2]) to the smooth functions g and ∇v_l, noting that for sufficiently small l, there holds: $g \in \mathcal{C}_c^\infty\big(\mathbb{R}^2 \setminus \nabla v_l(\partial U)\big)$:

$$(7.2) \qquad \int_U (g \circ \nabla v_l) \det \nabla^2 v_l = \int_{\mathbb{R}^2} g(y) \deg(\nabla v_l, U, y) \, \mathrm{d}y.$$

We see that ∇v_l converge uniformly to ∇v, so by [28, Proposition 2.1] we obtain that for l sufficiently small, and for all $y \in \mathrm{supp}\, g$, there holds: $\deg(\nabla v, U, y) = \deg(\nabla v_l, U, y)$. Thus:

$$\lim_{l \to 0} \int_{\mathbb{R}^2} g(y) \deg(\nabla v_l, U, y) \, \mathrm{d}y = \int_{\mathbb{R}^2} g(y) \deg(\nabla v, U, y) \, \mathrm{d}y.$$

Another proof of integrability of the Brouwer degree, in a more general context, can be found in [44]. Now, to conclude the proof in view of (7.2), it suffices to show that:

$$(7.3) \qquad \lim_{l \to 0} \int_U (g \circ \nabla v_l) \det \nabla^2 v_l = \int_U (g \circ \nabla v) f.$$

2. Following [10, 9] we use a commutator estimate to get (7.3). As $f = -\mathrm{curl\,curl} A$, we have:

$$\Big| \int_U (g \circ \nabla v_l) \det \nabla^2 v_l - (g \circ \nabla v) f \Big|$$

$$(7.4) \qquad \leq \Big| \int_U (g \circ \nabla v_l)\big(\det \nabla^2 v_l + \mathrm{curl\,curl} A_l \big) \Big| + \Big| \int_U (g \circ \nabla v_l) \mathrm{curl\,curl}(A_l - A) \Big|$$

$$+ \Big| \int_U \big((g \circ \nabla v_l) - (g \circ \nabla v) \big) f \Big|.$$

The second term above is bounded by $C \int_U |\nabla^2 A_l - \nabla^2 A| \leq C\|A_l - A\|_{W^{2,p}(\Omega)}$, hence it converges to 0. The third term also converges to 0 by the dominated convergence theorem, since $g \circ \nabla v_l$ converges to $g \circ \nabla v$. In order to deal with the first term in (7.4), observe that $\det \nabla^2 v_l = -\mathrm{curl\,curl}\big(\frac{1}{2}\nabla v_l \otimes \nabla v_l + \mathrm{sym}\,\nabla w_l\big)$ and integrate by parts, in view of $g \circ \nabla v_l = 0$ on ∂U:

$$\Big| \int_U (g \circ \nabla v_l)\big(\det \nabla^2 v_l + \mathrm{curl\,curl} A_l \big) \Big|$$

$$(7.5) \qquad = \Big| \int_U \Big\langle \nabla^\perp (g \circ \nabla v_l), \mathrm{curl}\big(\frac{1}{2}\nabla v_l \otimes \nabla v_l + \mathrm{sym}\,\nabla w_l - A_l\big) \Big\rangle \Big|$$

$$\leq C\|\nabla g\|_0 \|\nabla^2 v_l\|_0 \big\|\nabla v_l \otimes \nabla v_l - (\nabla v \otimes \nabla v) * \varphi_l\big\|_1$$

$$\leq C \frac{1}{l^{1-\alpha}} \|\nabla v\|_{0,\alpha} \cdot \frac{1}{l^{1-2\alpha}} \|\nabla v\|_{0,\alpha}^2 = C \frac{1}{l^{2-3\alpha}} \|\nabla v\|_{0,\alpha}^3,$$

where we used Lemma 4.3. Clearly, for $\alpha > 2/3$ the right hand side in (7.5) converges to 0 as $l \to 0$. By (7.4), this implies (7.3) and concludes the proof. ∎

Below, we present all the details of the proof of Theorem 1.3. The proof of Theorem 1.4 will be postponed to [36].

Proof of Theorem 1.3.
1. By Proposition 7.1 it follows that for all open sets $U \subset \bar{U} \subset \Omega$:

$$(7.6) \qquad \deg(\nabla v, U, y) = 0 \qquad \forall y \in \mathbb{R}^2 \setminus \nabla v(\partial U).$$

We would like to conclude [46, 47] that the image set $\nabla v(U)$ is of measure 0. This will result in the developability of v, by the main statement of [32]. However, we note [39] that for each $\alpha \in (0, 1)$,

there exists a map in $C^{0,\alpha}(\Omega, \mathbb{R}^2)$ whose local degree vanishes everywhere, but whose image is onto the unit square. This example can be constructed through a similar approach as in [40, Section 5]. Therefore, we will additionally exploit the gradient structure of ∇v, using ideas of [31, Chapter 2], in combination with the commutator estimate technique of the proof of Proposition 7.1.

Let $v_l = v * \varphi_l$ be as in the proof of Proposition 7.1 and for every $\delta > 0$ define:

$$u_{l,\delta}(x_1, x_2) = \nabla v_l(x_1, x_2) + \delta(-x_2, x_1), \qquad u_\delta(x_1, x_2) = \nabla v(x_1, x_2) + \delta(-x_2, x_1).$$

Fix an open set U with smooth boundary and compactly contained in Ω. Let $g \in \mathcal{C}_c^\infty(\mathbb{R}^2 \backslash \nabla v(\partial U))$, and use the change of variable formula to g and $u_{l,\delta}$:

$$(7.7) \qquad \int_U (g \circ u_{l,\delta})(\det \nabla^2 v_l + \delta^2) = \int_{\mathbb{R}^2} g(y) \deg(u_{l,\delta}, U, y) \; \mathrm{d}y,$$

where we noted that $\det \nabla u_{l,\delta} = \det \nabla^2 v_l + \delta^2$. The integral in the right hand side of (7.7) is well defined for sufficiently small l and δ, because then $y \in \operatorname{supp} g$ implies $y \notin u_{l,\delta}(\partial U)$.

Passing to the limit, we immediately obtain:

$$(7.8) \qquad \lim_{l \to 0} \int_{\mathbb{R}^2} g(y) \deg(u_{l,\delta}, U, y) \; \mathrm{d}y = \int_{\mathbb{R}^2} g(y) \deg(u_\delta, U, y) \; \mathrm{d}y,$$

while to the left hand side of (7.7) we apply the following estimate:

$$\left| \int_U (g \circ u_{l,\delta})(\det \nabla^2 v_l + \delta^2) - (g \circ u_\delta)\delta^2 \right| \leq \left| \int_U (g \circ u_{l,\delta}) \det \nabla^2 v_l \right| + \left| \int_U (g \circ u_{l,\delta} - g \circ u_\delta)\delta^2 \right|.$$

The second term above clearly converges to 0 as $l \to 0$, because $u_{l,\delta}$ converge to u_δ. The first term also converges to 0 as $\alpha > 2/3$, where we reason exactly as in (7.4) and (7.5), keeping in mind that $f = 0$. We hence conclude:

$$\lim_{l \to 0} \int_U (g \circ u_{l,\delta})(\det \nabla^2 v_l + \delta^2) = \int_U (g \circ u_\delta)\delta^2.$$

In view of (7.8) and (7.7) this implies:

$$\forall 0 < \delta \ll 1 \qquad \int_U (g \circ u_\delta)\delta^2 = \int_{\mathbb{R}^2} g(y) \deg(u_\delta, U, y) \; \mathrm{d}y.$$

Consequently:

$$(7.9) \qquad \forall 0 < \delta \ll 1 \quad \forall y \in u_\delta(U) \setminus u_\delta(\partial U) \qquad \deg(u_\delta, U, y) \geq 1.$$

2. We now claim that:

$$(7.10) \qquad \nabla v(U) \subset \nabla v(\partial U).$$

To prove (7.10) we argue by contradiction, assuming that for some $x_0 \in U$ there is: $y_0 = \nabla v(x_0) \in \nabla v(U) \setminus \nabla v(\partial U)$. Note that for δ small enough, we have: $y_0 \notin u_\delta(\partial U)$, because u_δ converges uniformly to ∇v as $\delta \to 0$. We distinguish two cases:

(i) There exist sequences $\{x_k \in U\}_{k=1}^\infty$ and $\delta_k \to 0^+$ as $k \to \infty$ such that $y_0 = u_{\delta_k}(x_k)$ for all k. In view of (7.9) we get: $\deg(u_{\delta_k}, U, y_0) \geq 1$, contradicting (7.6).

(ii) For all δ small enough, $y_0 \notin u_\delta(\bar{U})$. In this case, there must be: $\deg(u_\delta, U, y_0) = 0$. But on the other hand, there exists a ball $B(y_0, 2r) \subset \mathbb{R}^2 \setminus \nabla v(\partial U)$, so also $B(y_0, r) \subset \mathbb{R}^2 \setminus u_\delta(\partial U)$ for all small δ. Consequently, continuity of the degree yields that $\deg(u_\delta, U, z) = 0$, for every $z \in B(y_0, r)$. In particular: $\deg(u_\delta, U, u_\delta(x_0)) = 0$, because $\lim_{\delta \to 0} u_\delta(x_0) = \nabla v(x_0) = y_0$. This finally contradicts (7.9), as $u_\delta(x_0) \in u_\delta(U) \setminus u_\delta(\partial U)$.

Our claim (7.10) is now established. Since the set $\nabla v(\partial U)$ is the image of a Hausdorff 1d set ∂U under a $\mathcal{C}^{0,\alpha}$, $\alpha > 1/2$, deformation ∇v, it has Lebesgue measure 0 (see [10, Lemma 4]). Thus $\nabla v(U)$ must have measure 0 for every smooth U compactly contained in Ω. The same then must be true for the entire set Ω, i.e.: $|\nabla v(\Omega)| = 0$ and we consequently obtain:

$$(7.11) \qquad \text{Int}\big(\nabla v(\Omega)\big) = \emptyset.$$

3. By [33, Corollary 1.1.2.], condition (7.11) implies that every point $y \in \Omega$ has a convex open neighbourhood Ω_y such that for every point $x \in \Omega_y$ there is a line L_x passing through x so that ∇v is constant on $L_x \cap \Omega_y$. The same result in the present dimensionality has been first established in [32], see also footnote on pg. 875 in [33] for an explanation.

We now prove that v is developable. Fix $x_0 \in \Omega$ and let $[y, z] \subset \bar{\Omega}$ be the maximal segment passing through x_0 on which $\nabla v = \nabla v(x_0)$ is constant. Assume that $[y, z]$ does not extend to the boundary $\partial \Omega$, i.e. $y \in \Omega$. We will prove that then ∇v must be constant in an open neighbourhood of x_0. In fact, we will show that:

$$(7.12) \qquad V = \text{Int}\Big((\nabla v)^{-1}\big(\nabla v(x_0)\big)\Big) \supset (y, z).$$

Let $(p, q) = L_y \cap \Omega_y$. By the maximality of $[y, z]$, the segment (p, q) is not an extension of (is not parallel to) $[y, z]$. Also, $\nabla v = \nabla v(x_0)$ on (p, q). Take any $y_1 \in (y, z) \cap \Omega_y$ and define the open triangle $T = \text{Int}\big(\text{span}\{p, q, y_1\}\big)$. It is easy to notice that every line passing through any point $x \in T$ must intersect at least one of the segments (p, q) or (y, y_1). Since $T \subset \Omega_y$, it follows that $\nabla v(x) = \nabla v(x_0)$. Hence:

$$(y, y_1) \subset T \subset V$$

and, in particular, the set V in (7.12) is nonempty.

To prove (7.12) assume, by contradiction, that there exists $y_2 \in [y_1, z]$ so that:

$$(7.13) \qquad (y, y_2) \subset V \quad \text{but} \quad (y, y_3) \not\subset V \quad \forall y_3 \in (y_2, z).$$

Now, the intersection $\Omega_{y_2} \cap V$ contains an open arc C crossing the segment $(y, y_2) \cap \Omega_{y_2}$. As above, we argue that every point in a sufficienty small open neighbourhood of the segment $I = (y, z) \cap \Omega_{y_2}$ must have the property that every line passing through it intersects C or I, where $\nabla v = \nabla v(x_0)$. Consequently $I \subset V$, contradicting (7.13) and establishing (7.12). ∎

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