GAME THEORETICAL METHODS IN PDES

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Nonlinear PDEs, mean value properties, and stochastic differential games are intrinsically connected. We will describe how the solutions to certain PDEs (of p-Laplacian type) can be interpreted as limits of values of a specific Tug-of-War game, when the step-size ϵ determining the allowed length of move of a token, decreases to 0. This approach originated in [PSSW] and [PS]; for the case of deterministic games see the review SIAM News article [K] and [KS, KS2].

How the linear elliptic equations arise in probability.

Let us begin with the case governed by the discrete Brownian motion. Consider an open bounded set $\Omega \subset \mathbb{R}^N$ and a non-empty portion of its boundary $\Gamma_1 \subset \partial\Omega$. Place a token at a point $x_0 \in \Omega$ and assume that at each step of the process, it is moved with equal probabilities $\frac{1}{2N}$, to one of the $2N$ symmetric positions $x_0 \pm \epsilon e_i$, $i : 1 \dots N$. Denote by $u_\epsilon(x_0)$ the probability that the first time the token exists Ω, it exits across Γ_1. By applying conditional probabilities, it is clear that u_ϵ satisfy the *mean value property*:

$$\frac{1}{2N}\sum_{i=1}^{N}\Big(u_\epsilon(x+\epsilon e_i) + u_\epsilon(x-\epsilon e_i)\Big) = u_\epsilon(x) \tag{1}$$

Further, it follows that as $\epsilon \to 0$, the functions u_ϵ converge uniformly in Ω to a continuous $u \in \mathcal{C}(\Omega)$ which is a *viscosity solution* to the problem $\Delta u = 0$ in Ω, $u = \chi_{\Gamma_1}$ on $\partial\Omega$.

More precisely, this means that: (i) for each $x_0 \in \Omega$ and each smooth test function ϕ satisfying $u(x)-\phi(x) > u(x_0)-\phi(x_0) = 0$ for all $x \neq x_0$ in a small neighbourhood of x_0, one has: $\Delta\phi(x_0) \leq 0$, (ii) the same condition holds if we replace u by $-u$. It is well known that the viscosity solutions to $\Delta u = 0$ coincide with the classical solutions. An avantage of working with the above, seemingly, more complex notion, is that the limiting properties of u_ϵ follow quite naturally from the mean value property (1). Namely, replacing the increments $u_\epsilon(x \pm \epsilon e_i) - u_\epsilon(x)$ in the discontinuous u_ϵ in (1), by the same increments in the smooth ϕ, applying Taylor's expansion and taking into account the assumed sign of $u - \phi$, yields the sign of $\Delta\phi$, wheras the first derivatives cancel out due to the symmetry in (1).

Heuristically, this can be seen by writing the Taylor expansion of u at a given point $x \in \Omega$ and averaging it on a ball $B_\epsilon(x)$. One obtains:

$$⨍_{B_\epsilon(x)} u = u(x) + \frac{\epsilon^2}{2(N+2)}\Delta u(x) + o(\epsilon^2), \tag{2}$$

which is a continuum version of (1) when the second term in the right hand side vanishes. Consequently, the function u must be *harmonic*, i.e. $\Delta u = 0$.

The p-Laplacian and its mean value property.

To apply a similar reasoning to a nonlinear problem, consider the homogeneous p-Laplacian:

$$\Delta_p^{\mathrm{H}} u = \Delta u + (p-2)\Delta_\infty u = |\nabla u|^{2-p}\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big), \qquad 1 < p < \infty, \tag{3}$$

where the *infinity-Laplacian* is given by: $\Delta_\infty u = \langle \nabla^2 u : \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\rangle$. Parallel to (2) one gets the expansion:

$$\frac{1}{2}\Big(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\Big) = u(x) + \frac{\epsilon^2}{2}\Delta_\infty u(x) + o(\epsilon^2). \tag{4}$$

Forming a linear combination of (2) and (4) with coefficients $\alpha = \frac{p-2}{p+N}$ and $\beta = \frac{2+N}{p+N}$, yields:

$$\frac{\alpha}{2}\Big(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\Big) + \beta \fint_{B_\epsilon(x)} u = u(x) + \frac{\epsilon^2}{2(p+N)}\Delta_p^{\mathrm{H}} u + o(\epsilon^2), \tag{5}$$

and so the equation (5) suggests that a *p-harmonic* function u, i.e. a function satisfying $\Delta_p^{\mathrm{H}} u = 0$, may be approximated by *p-harmonious* functions u_ϵ, defined by the mean value property:

$$\frac{\alpha}{2}\Big(\sup_{B_\epsilon(x)} u_\epsilon + \inf_{B_\epsilon(x)} u_\epsilon\Big) + \beta \fint_{B_\epsilon(x)} u_\epsilon = u_\epsilon(x). \tag{6}$$

As we shall see, the functions u_ϵ satisfying (6) have a probabilistic interpretation as values of Tug-of-War games with noise.

A Tug-of-War game with noise for $\Delta_p^{\mathbf{H}}$.

A Tug-of-War is a two-person, zero-sum game, i.e. two players compete and the gain of Player I equals the loss of Player II. Initially, a token is placed at a point $x_0 \in \Omega$. At each step of the process (the game) one of the three actions takes place: (i) with probability $\frac{\alpha}{2}$, Player I is allowed to play, and she moves the token from its current position x_n to her chosen position $x_{n+1} \in B_\epsilon(x_n)$, (ii) with probability $\frac{\alpha}{2}$, Player II moves the token to his chosen position in $B_\epsilon(x_n)$, (iii) with probability $\beta = 1 - \alpha \in [0,1]$, the token is moved randomly in the ball $B_\epsilon(x_n)$. The game stops when the token leaves Ω, whereas Player II pays to Player I the amount equal to the value of a given boundary pay-off function F at the exit token position x_τ (see Figures 1 and 2).

FIGURE 1. Player I and Player II compete in a Tug-of-War with random noise [Ka].

Players I and II play according to *strategies* σ_I and σ_{II} respectively, which are (Borel measurable) functions assigning to each finite history of the game $\mathbf{x}_n = (x_0, \ldots x_n)$ the next position x_{n+1} in the ϵ-neighborhood Ω_ϵ of Ω, where the player will move the token once she/he is allowed to play. These strategies determine a probability $\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}$ on the space of all possible game runs in $(\Omega_\epsilon)^\infty$. Since $\beta > 0$, the game ends $\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}$-a.e., so that we can define the stopping time $\tau(x_0, x_1, \ldots) = \inf\{n;\ x_n \notin \Omega\}$. The expected value of the game is then given by $\mathbb{E}^{x_0}_{\sigma_I,\sigma_{II}}[F_\tau] = \int_{\Omega^\infty} F(x_\tau)\ \mathrm{d}\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}$. Consequently, the minimum gain u_I that Player I can expect, and the maximum loss u_{II} of Player II in his best case scenario, are:

$$u_I(x_0) = \sup_{\sigma_I}\inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\sigma_I,\sigma_{II}}[F_\tau], \qquad u_{II}(x_0) = \inf_{\sigma_{II}}\sup_{\sigma_I} \mathbb{E}^{x_0}_{\sigma_I,\sigma_{II}}[F_\tau]. \tag{7}$$

The following main results were achieved in [PSSW] for $p = \infty$, and in [MPR, LPS] for $p \in [2, \infty)$:

Theorem A: The two game values in (7) coincide: $u_I = u_{II}$ and are equal to the p-harmonious function u_ϵ, which is the unique solution to the mean value law in (6) augmented by the boundary data: $u_\epsilon = F$ on $\mathbb{R}^n \setminus \Omega$.

Theorem B: As $\epsilon \to 0$, the game value u_ϵ converges uniformly in Ω to a function $u \in \mathcal{C}(\Omega)$, which is the unique viscosity solution to: $\Delta_p^{\mathrm{H}} u = 0$ in Ω and $u = F$ on $\partial\Omega$.

In brief, one obtains exactly the same as in the case of the discrete Brownian motion, whose value satisfied the averaging principle (1) and converged to a harmonic function. The equivalence of the notions of viscosity solution to $\Delta_p^{\mathrm{H}} u = 0$ and weak solution to $\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0$ has been proven in [JLM].

The key martingale calculation.

We now sketch the proof of $u_{II} \leq u_\epsilon$; a symmetric argument yields $u_\epsilon \leq u_I$, while $u_I \leq u_{II}$ is trivially true. We take advantage of the cancellation encoded in the mean value property (6) by showing that certain quantities related to u_ϵ are sub- and super-martingales.

Fix a small $\eta > 0$ and let $\bar\sigma_{II}$ be an "almost optimal" strategy for Player II, so that:

$$u_\epsilon(\bar\sigma_{II}(x_0, \ldots x_n)) \leq \inf_{B_\epsilon(x_n)} u_\epsilon + \frac{\eta}{2^{n+1}}.$$

Player I plays according to an arbitrary strategy σ_I. The key observation is that *the sequence of random variables* $\{u_\epsilon(x_n) + \frac{\eta}{2^n} \mid (x_0, \ldots x_n)\}_{n\geq 1}$ *is a supermartingale.*

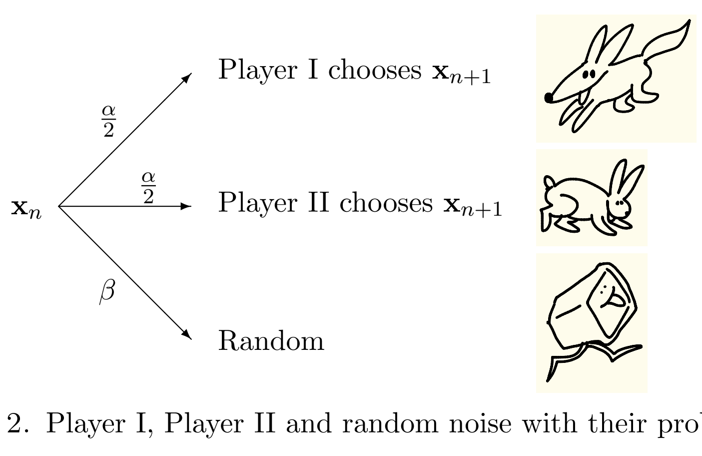

FIGURE 2. Player I, Player II and random noise with their probabilities [Ka].

To see this, compute the conditional expectation relative to the history $\mathbf{x}_n = (x_0 \dots x_n)$:

$$\begin{aligned}\mathbb{E}^{x_0}_{\sigma_I,\bar\sigma_{II}}\big\{u_\epsilon(x_{n+1}) + \frac{\eta}{2^{n+1}}\big\}(\mathbf{x}_n) &= \frac{\alpha}{2}u_\epsilon(\sigma_I(\mathbf{x}_n)) + \frac{\alpha}{2}u_\epsilon(\bar\sigma_{II}(\mathbf{x}_n)) + \beta \fint_{B_\epsilon(x_n)} u_\epsilon + \frac{\eta}{2^{n+1}}\\ &\leq \frac{\alpha}{2}\sup_{B_\epsilon(x_n)} u_\epsilon + \frac{\alpha}{2}\left(\inf_{B_\epsilon(x_n)} u_\epsilon + \frac{\eta}{2^{n+1}}\right) + \beta \fint_{B_\epsilon(x_n)} u_\epsilon + \frac{\eta}{2^{n+1}}\\ &= u_\epsilon(x_n) + (\frac{\alpha}{2}+1)\frac{\eta}{2^{n+1}} \leq u_\epsilon(x_n) + \frac{\eta}{2^n},\end{aligned}$$

where we first used the game's rules, then the sub-optimality of $\bar\sigma_{II}$, and further the formula (6) for u_ϵ. Applying Doob's optimal stopping time theorem we get the desired comparison result:

$$\begin{aligned}u_{II}(x_0) &\leq \sup_{\sigma_I}\mathbb{E}^{x_0}_{\sigma_I,\bar\sigma_{II}}[F_\tau] = \sup_{\sigma_I}\mathbb{E}^{x_0}_{\sigma_I,\bar\sigma_{II}}[u(x_\tau)] \leq \sup_{\sigma_I}\mathbb{E}^{x_0}_{\sigma_I,\bar\sigma_{II}}[u(x_\tau) + \frac{\eta}{2^\tau}]\\ &\leq \sup_{\sigma_I}\mathbb{E}^{x_0}_{\sigma_I,\bar\sigma_{II}}[u(x_0) + \frac{\eta}{2^0}] = u(x_0) + \eta, \qquad \text{for all } \eta > 0.\end{aligned}$$

Strategies and inequalities.

We have seen how probability tools can be used to study nonlinear PDEs, where the key technical ingredient was assigning sutiable *strategies* yielding the desired *inequalities* for game values. Below we sketch two further examples of this powerful technique.

The proof of *uniform convergence* in Theorem B relies on a variant of the Ascoli-Arzelá theorem valid for the discontinuous functions u_ϵ. The verification [MPR] of the appropriate 'equidiscontinuity' property requires estimating quantities $|u_\epsilon(x_0) - u_\epsilon(y_0)|$, say for $x_0 \in \Omega$, $y_0 \in \partial\Omega$. If F is Lipschitz, this reduces to estimating $|x_\tau - y_0|$, and the feasible strategy is that of Player II "pulling towards y_0", namely shifting the token by ϵ along the segment connecting its current position with y_0.

In [LPS2], the *local Harnack inequality* for p-harmonic functions for $p > 2$ is proven independent of the classical, yet technically challenging methods of De Giorgi or Moser. This is done via a uniform estimate on the oscillations of u_ϵ. Let $x_0, y_0 \in \Omega$ and let z be equidistanced from both points by a multiple of ϵ. Define strategies σ_i^* in which Player i cancells the earliest uncancelled move of her/his opponent, and otherwise "pulls towards z" as before, and let τ_i^* be the stopping time in which the game terminates when either Player i has played sufficiently many turns to place the token at z (modulo the random noise), or when the total amount of token's shifts by her/his opponent and by the random noise, has passed an undesired large treshold r. Let now σ_I, σ_{II} be two arbitrary strategies. By the symmetry of this construction, the bulk "nonlinear" parts in the two quantities: $\mathbb{E}^{x_0}_{\sigma_I,\sigma^*_{II}}[u_\epsilon(x_{\tau^*_{II}})]$ and $\mathbb{E}^{y_0}_{\sigma^*_I,\sigma_{II}}[u_\epsilon(x_{\tau^*_I})]$, corresponding to stopping the game due to the first condition, are equal. The remaining "linear" part in: $|\mathbb{E}^{x_0}_{\sigma_I,\sigma^*_{II}}[u_\epsilon(u_{\tau^*_{II}})] - \mathbb{E}^{y_0}_{\sigma^*_I,\sigma_{II}}[u_\epsilon(u_{\tau^*_I})]|$ can then be bounded by $\frac{|x_0-y_0|}{r}\mathrm{osc}(u_\epsilon, B_r(z))$, using a comparison with a cylinder walk. This concludes the proof, in view of: $|u_\epsilon(x_0) - u_\epsilon(y_0)| \leq \sup_{\sigma_I,\sigma_{II}} |\mathbb{E}^{x_0}_{\sigma_I,\sigma^*_{II}}[u_\epsilon(x_{\tau^*_{II}})] - \mathbb{E}^{y_0}_{\sigma^*_I,\sigma_{II}}[u_\epsilon(x_{\tau^*_I})]|$.

Further results.

Generalizations of Theorems A and B have been obtained in various contexts. For $p = \infty$, only the notion of a metric space is necessary to define the game, and indeed [PSSW] formulates its results for an arbitrary *length space* where the solutions to $\Delta_\infty u = 0$ are understood as *Absolutely Minimizing Lipschitz Extensions*. When $p \in [2, \infty)$, the game uses the notion of a metric and a measure, and it is amenable to the the recent extension to Heisenberg groups in [FLM]. For $p \in (1, \infty)$ one needs the additional notion of perpendicularity [PS]. We see that as $p \to 1$, the required complexity of structure increases. In the case $p = 1$ the game is naturally related to the mean curvature flow [KS] and functions of least gradient.

Other extensions include the obstacle problems [MRS], finite difference schemes [AS], equations with right hand side $f \neq 0$, mixed boundary data [APSS, CGR] and parabolic equations [KS2, MPR2].

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