QUANTITATIVE IMMERSABILITY OF RIEMANN METRICS AND THE INFINITE HIERARCHY OF PRESTRAINED SHELL MODELS

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Abstract. We propose results that relate the following two contexts:

(i) Given a Riemann metric G on $\Omega^1 = \omega \times (-\frac{1}{2}, \frac{1}{2})$, we study the question of what is the infimum of the averaged pointwise deficit of an immersion from being an orientation-preserving isometric immersion of $G_{|\Omega^h}$ on $\Omega^h = \omega \times (-\frac{h}{2}, \frac{h}{2})$, over all weakly regular immersions. This deficit is measured by the non-Euclidean energies $\mathcal{E}^h$, which can be seen as modifications of the classical nonlinear three-dimensional elasticity.

(ii) We complete the scaling analysis of $\mathcal{E}^h$, in the context of dimension reduction as $h \to 0$, and the derivation of Γ-limits of the scaled energies $h^{-2n}\mathcal{E}^h$, for all $n \geq 1$. We show the energy quantisation, in the sense that the even powers $2n$ of h are indeed the only possible ones (all of them are also attained).

For each n, we identify conditions for the validity of the scaling h^{2n}, in terms of the vanishing of Riemann curvatures of G up to appropriate orders, and in terms of the matched isometry expansions. We also establish the asymptotic behaviour of the minimizing immersions as $h \to 0$.

1. Introduction

In this paper, we propose results that address and relate the following two contexts:

(i) Quantitative analysis of immersability of Riemann metrics.
(ii) Dimension reduction in non-Euclidean elasticity of prestrained thin films.

It is a well-known fact that a three-dimensional Riemann metric G has a smooth isometric immersion in $\mathbb{R}^3$, if an only if its curvature tensor $R(G) = \{R_{ab,cd}\}_{a,b,c,d=1\ldots 3}$ vanishes identically. The smoothness requirement may be replaced by the orientation-preservation of a Lipschitz continuous immersion; then condition $R(G) = 0$ automatically yields smoothness and uniqueness, up to rigid motions. When $R(G) \neq 0$, one may pose the question of what is the infimum of the average pointwise deficit from being an orientation-preserving isometric immersion, over all, weakly regular, immersions. We study this question on a family of thin films $\big\{\Omega^h = \omega \times (-\frac{h}{2}, \frac{h}{2})\big\}_{h\to 0}$ around a given two-dimensional midplate ω, where the said deficit is measured by the energy: $\mathcal{E}^h(u) = \fint_{\Omega^h} \mathrm{dist}^2((\nabla u)G^{-1/2}, SO(3))$. Our first goal is to determine the possible scalings: $\inf \mathcal{E}^h \sim h^\beta$, as $h \to 0$, in terms of powers β of the thickness h. We are then interested in identifying properties of G, that correspond to each scaling range, in function of the curvatures. Finally, we want to predict the asymptotics of the minimizing immersions as $h \to 0$.

Similar questions arise in the context of the so-called prestrained elasticity. A prestrained elastic body is a three-dimensional object, modeled in its reference configuration by a domain and a Riemann metric G, which is induced by mechanisms such as growth, plasticity or thermal expansion. The body wants to realize the distances between its constitutive cell elements, which are set by G, by deforming its shape. This realization, taking place in the flat three-dimensional space, is impossible unless $R(G) = 0$, which is precisely equivalent to having the stored non-Euclidean energy of deformations infimize to zero (one can prove that the zero infimum is always attained as a minimum). In the variational description of thin prestrained films Ω^h, we thus study the nonlinear energies: $\big\{\mathcal{E}^h(u) =$

$\fint_{\Omega^h} W((\nabla u)G^{-1/2})\}_{h\to 0}$ and, as above, want to determine the viable scalings of their infima, their singular limits as $h \to 0$, and the asymptotic behaviour of the three-dimensional minimizing shapes.

In our previous works [28, 6] we analyzed the scenario: $\inf \mathcal{E}^h \sim h^2$, whereas in [29, 30] we showed that the next limiting energy level beyond h^2 is: $\inf \mathcal{E}^h \sim h^4$, arising when $\{R_{12,ab}\}_{a,b=1\ldots 3} = 0$ on ω. Then we observed that the further scaling level is: $\inf \mathcal{E}^h \sim h^6$ and that it corresponds to $R(G) = 0$ on ω. In the present paper, we complete this analysis and provide the derivation of the Γ-limits $\mathcal{I}_{2n}$ to scaled energies $h^{-2n}\mathcal{E}^h$, for all $n \geq 1$. We prove the previously conjectured energy quantisation so that h^{2n} are indeed the only possible scalings, all of them attained (by $G = e^{x_3^n} Id_3$). The obtained singular limits $\{\mathcal{I}_{2n}\}_{n\geq 1}$ should be compared with the hierarchy of plate models in the classical nonlinear elasticity [9], as follows. The energy $\mathcal{I}_2$ consists of pure bending, quantifying the curvature under the midplate isometric immersion constraint. This is a Kirchhoff-like model, relative to the ambient metric G. The next energy $\mathcal{I}_4$ consists of linearised first order bending and second order stretching; this is a von Karman-like model, augmented by terms carrying the relevant components of the Riemann tensor $R(G)$. Each higher order energy $\mathcal{I}_{2n}$ consists of linearised bending augmented by the the order-related covariant derivatives of $R(G)$ on the midplate. This is a linear elasticity-like model, in the present context valid in the quantized scaling regimes $n \geq 3$, whereas in the classical case appearing in the regimes h^β for all $\beta > 4$.

Before we describe the set-up and the obtained results, we point out that recently, there has been a sustained interest in studying shape formation driven by internal prestrain, through the experimental, modelling via formal methods, numerics, and analytical arguments [36, 18, 14, 7]. Beyond the context of the present paper, higher energies $\inf \mathcal{E}^h \sim h^\beta$ with $\beta \in (0, 2)$, have been shown to result from the interaction of the metric with boundary conditions or external forces, leading to the "wrinkling-like" effects. Indeed, our setting pertains to the "no wrinkling" regime where $\beta \geq 2$ and the reduced prestrain $G_{2\times 2}$ on ω, admits a $W^{2,2}$ isometric immersion in $\mathbb{R}^3$. While the systematic description of the singular limits at scalings $\beta < 2$ is not yet available, there exists a variety of studies of emerging patterns: compression- driven blistering [15, 3, 4], buckling [10, 11, 12], origami patterns [5, 39], conical singularities [33, 34, 35], coarsening patterns [1, 2, 38]. In [24, 25, 27], derivations similar to our results were carried out under a different assumption on the asymptotic behavior of the prestrain (here, constant), which in particular allowed for the energy scalings h^β in non-even regimes of $\beta > 2$. General results have been presented in the abstract setting of Riemannian manifolds [20, 19, 32]. On the frontier of experimental modelling of shape formation, we point to [17, 16, 40, 21, 13] and to references therein.

1.1. **The set-up of the problem.** Let $\omega \subset \mathbb{R}^2$ be an open, bounded, connected set with Lipschitz boundary. We consider a family of thin hyperelastic sheets occupying the reference domains:

$$\Omega^h = \omega \times \Big(-\frac{h}{2}, \frac{h}{2}\Big) \subset \mathbb{R}^3, \qquad 0 < h \ll 1.$$

A typical point in Ω^h is denoted by $x = (x_1, x_2, x_3) = (x', x_3)$. We often use the unit-thickness plate Ω^1 as the referential rescaling of each Ω^h via: $\Omega^h \ni (x', x_3) \mapsto (x', x_3/h) \in \Omega^1$.

The films Ω^h are characterized by given smooth incompatibility (Riemann metric) tensor:

$$G \in \mathcal{C}^\infty(\bar{\Omega}^1, \mathbb{R}^{3\times 3}_{\mathrm{sym,pos}}),$$

and we want to study the singular limit behaviour, as $h \to 0$, of the following energy functionals:

$$\mathcal{E}^h(u^h) = \frac{1}{h}\int_{\Omega^h} W\big(\nabla u^h(x)G(x)^{-1/2}\big)\,\mathrm{d}x = \int_{\Omega^1} W\big(\nabla u^h(x', hx_3)G(x', hx_3)^{-1/2}\big)\,\mathrm{d}x, \tag{1.1}$$

defined on vector fields $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ that are interpreted as deformations of Ω^h. Above, $G(x)^{-1/2}$ stands for the inverse of the square root of $G(x)$. When $G = Id_3$, the functionals $\mathcal{E}^h$

are the classical Hookean nonlinear elastic energies of deformations, with the density W obeying properties listed below.

In the present general setting, $\mathcal{E}^h(u^h)$ is designed to measure the deviation of u^h from being an (equidimensional) isometric immersion of G on Ω^h. Indeed, by polar decomposition theorem:

$$FG^{-1/2} \in SO(3) \quad \text{if and only if:} \quad F^{\mathsf{T}}F = G \ \text{ and } \ \det F > 0. \tag{1.2}$$

The Borel-regular, homogeneous density $W : \mathbb{R}^{3\times 3} \to [0,\infty]$ is thus assumed to satisfy:

(i) $W(RF) = W(F)$ for all $R \in SO(3)$ and $F \in \mathbb{R}^{3\times 3}$,
(ii) $W(F) = 0$ for all $F \in SO(3)$,
(iii) $W(F) \geq C\,\mathrm{dist}^2\big(F, SO(3)\big)$ for all $F \in \mathbb{R}^{3\times 3}$, with some uniform constant $C > 0$,
(iv) there exists a neighbourhood $\mathcal{U}$ of $SO(3)$ such that W is finite and $\mathcal{C}^2$ regular on $\mathcal{U}$.

By a more refined analysis [28] one can prove the global counterpart of the pointwise statement (1.2), namely that: $\inf_{W^{1,2}} \mathcal{E}^h = 0$ if an only if all the components of the Riemann curvature tensor of G vanish identically: $\{R_{ab,cd}\}_{a,b,c,d=1\ldots 3} = 0$ on Ω^h.

In this paper, we determine the possible energy scalings: $\inf \mathcal{E}^h \sim h^\beta$ in the limit of vanishing thickness $h \to 0$, and study the corresponding variational limits (Γ-limits) $\mathcal{I}_\beta$ of $h^{-\beta}\mathcal{E}^h$, in the regime $\beta > 4$ that has not been analyzed before. We thus complete the discussion of weakly prestrained films, started in our previous works [28, 6, 29, 30], that covered the range $\beta \in [2,4]$. The energies $\mathcal{I}_\beta$ are typically of the form $\mathcal{I} = \|Tensor(y)\|^2_{\mathcal{Q}_2}$ defined on the appropriate sets of limiting deformations/displacements y of the midplate ω. They quantify the resulting curvatures in $Tensor(y)$ relative to G at the level induced by β, and in the following weighted L^2 space on ω:

$$\mathbb{E} \doteq \big(L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \|\cdot\|_{\mathcal{Q}_2}\big), \qquad \|F\|_{\mathcal{Q}_2} = \Big(\int_\omega \mathcal{Q}_2(x', F(x'))\,\mathrm{d}x'\Big)^{1/2}. \tag{1.3}$$

Above, the quadratic form $\mathcal{Q}_2$ carries the two-dimensional reduction of the lowest nonzero term in the Taylor expansion of W close to its energy well $SO(3)$. More precisely, we define:

$$\begin{aligned} &\mathcal{Q}_3(F) = D^2W(Id_3)(F,F) \\ &\mathcal{Q}_2(x', F_{2\times 2}) = \min\Big\{\mathcal{Q}_3\big(G(x',0)^{-1/2}\tilde{F}G(x',0)^{-1/2}\big);\ \tilde{F} \in \mathbb{R}^{3\times 3} \text{ with } \tilde{F}_{2\times 2} = F_{2\times 2}\Big\}. \end{aligned} \tag{1.4}$$

The quadratic form $\mathcal{Q}_3$ is defined for all $F \in \mathbb{R}^{3\times 3}$, while each $\mathcal{Q}_2(x', \cdot)$ is defined on $F_{2\times 2} \in \mathbb{R}^{2\times 2}$. Both $\mathcal{Q}_3$ and all $\mathcal{Q}_2$ are nonnegative definite and depend only on the symmetric parts of their arguments, in view of assumptions on W. The minimization problem in (1.4) has thus the unique solution among symmetric matrices $\tilde{F}$, which for each $x' \in \omega$ is determined via the linear function denoted by $c(x', \cdot)$. More precisely, there exists the following map:

$$F_{2\times 2} \mapsto c(x', F_{2\times 2}) \in \mathbb{R}^3 \quad \text{with:} \quad \mathcal{Q}_2(x', F_{2\times 2}) = \mathcal{Q}_3\Big(G(x',0)^{-1/2}\big(F^*_{2\times 2} + c\otimes e_3\big)G(x',0)^{-1/2}\Big), \tag{1.5}$$

where $F^*_{2\times 2}$ is the 3×3 matrix with the principle minor $F_{2\times 2}$ and all other entries equal 0.

1.2. **Description of the main results of this paper.** As already pointed out, we will be concerned with the regimes of curvatures of G, yielding the incompatibility rate, quantified by $\inf \mathcal{E}^h$, of order higher than h^4 in the thickness $h \to 0$. We first recall the following result from [30]:

$$\lim_{h\to 0} \frac{1}{h^4} \inf \mathcal{E}^h = 0 \quad \Leftrightarrow \quad R_{ab,cd}(x', 0) = 0 \quad \text{for all } x' \in \omega, \quad \text{for all } \ a,b,c,d = 1\ldots 3. \tag{1.6}$$

The above conditions are further equivalent to existence of smooth vector fields $y_0, \vec{b}_1, \vec{b}_2 : \bar{\omega} \to \mathbb{R}^3$, defined uniquely up to rigid motions, such that for the smooth $\mathbb{R}^{3\times 3}$ matrix fields on $\bar{\omega}$:

$$B_0 = \big[\partial_1 y_0,\ \partial_2 y_0,\ \vec{b}_1\big], \qquad B_1 = \big[\partial_1 \vec{b}_1,\ \partial_2 \vec{b}_1,\ \vec{b}_2\big],$$

there holds:

$$
\begin{aligned}
& B_0^{\mathsf T} B_0 = G(x',0) \quad \text{with} \quad \det B_0 > 0, \\
\text{and} \quad & (B_0^{\mathsf T} B_1)_{\mathrm{sym}} = \frac{1}{2}\partial_3 G(x',0), \\
\text{and} \quad & \big((\nabla y_0)^{\mathsf T}\nabla \vec b_2\big)_{\mathrm{sym}} + (\nabla \vec b_1)^{\mathsf T}\nabla \vec b_1 = \frac{1}{2}\partial_{33} G(x',0)_{2\times 2}.
\end{aligned}
\tag{1.7}
$$

Note that the last equality above implies that we can uniquely define a new smooth vector and matrix fields: $\vec b_3 : \bar\omega \to \mathbb{R}^3$ and $B_2 = \big[\partial_1 \vec b_2,\ \partial_2 \vec b_2,\ \vec b_3\big]$, so that: $(B_0^{\mathsf T} B_2)_{\mathrm{sym}} + B_1^{\mathsf T} B_1 = \frac{1}{2}\partial_{33} G(x',0)$. This condition, together with the first two equalities in (1.7) is jointly equivalent to:

$$
\Big(\sum_{k=0}^{2} \frac{x_3^k}{k!} B_k\Big)^{\mathsf T}\Big(\sum_{k=0}^{2} \frac{x_3^k}{k!} B_k\Big) = G(x',x_3) + \mathcal{O}(h^3) \quad \text{on } \Omega^h, \quad \text{as } h\to 0. \tag{1.8}
$$

In conclusion, the following three conditions: the two conditions in (1.6) and the one in (1.8), are equivalent. Our first main result generalizes this statement to all even order powers $2(n+1)$ in the infimum energy scaling, for any $n \geq 2$. Moreover, these scalings exhaust all possibilities in the remaining regime: $\inf \mathcal{E}^h \sim h^\beta$ with $\beta > 4$:

Theorem 1.1. *The following three statements are equivalent, for each fixed integer $n \geq 2$:*

(i) $R_{12,12}(x',0) = R_{12,13}(x',0) = R_{12,23}(x',0) = 0$ *for all* $x' \in \omega$, *and* $\partial_3^{(k)} R_{i3,j3}(x',0) = 0$ *for all* $x' \in \omega$, *all* $k = 0 \dots n-2$ *and all* $i,j = 1 \dots 2$.

(ii) $\inf \mathcal{E}^h \leq C h^{2(n+1)}$.

(iii) *There exist smooth fields* $y_0, \{\vec b_k\}_{k=1}^{n+1} : \bar\omega \to \mathbb{R}^3$ *such that calling* $\big\{B_k = \big[\partial_1 \vec b_k,\ \partial_2 \vec b_k,\ \vec b_{k+1}\big]\big\}_{k=1}^n$, *in addition to* $B_0 = \big[\partial_1 y_0,\ \partial_2 y_0,\ \vec b_1\big]$ *satisfying* $\det B_0 > 0$, *we have:*

$$
\Big(\sum_{k=0}^{n} \frac{x_3^k}{k!} B_k\Big)^{\mathsf T}\Big(\sum_{k=0}^{n} \frac{x_3^k}{k!} B_k\Big) = G(x',x_3) + \mathcal{O}(h^{n+1}) \quad \textit{on } \Omega^h, \quad \textit{as } h\to 0. \tag{1.9}
$$

$$
\textit{Equivalently:} \ \sum_{k=0}^{m} \binom{m}{k} B_k^{\mathsf T} B_{m-k} - \partial_3^{(m)} G(x',0) = 0 \ \ \textit{for all } m = 0\dots n, \textit{ for all } x' \in \omega.
$$

We further show compactness and the lower bound, at any of the new viable scaling levels $\inf \mathcal{E}^h \sim h^{2(n+1)}$, completing thus the analysis done for $n=0$ in [28, 6] and for $n=1$ in [29, 30]:

Theorem 1.2. *Fix $n \geq 2$ and assume that any of the equivalent conditions in Theorem 1.1 holds. Let the sequence of deformations $\{u^h \in W^{1,2}(\Omega^h,\mathbb{R}^3)\}_{h\to 0}$ satisfy: $\mathcal{E}^h(u^h) \leq C h^{2(n+1)}$. Then, the following convergences hold up to a subsequence which we do not relabel:*

(i) *There exists $\bar R^h \in SO(3)$, $c^h \in \mathbb{R}^3$ such that the displacements $\{V^h \in W^{1,2}(\omega,\mathbb{R}^3)\}_{h\to 0}$ in:*

$$
V^h(x') = \frac{1}{h^n} \fint_{-h/2}^{h/2} (\bar R^h)^{\mathsf T}\big(u^h(x',x_3) - c^h\big) - \Big(y_0(x') + \sum_{k=1}^{n} \frac{x_3^k}{k!} \vec b_k(x')\Big)\,\mathrm{d}x_3
$$

converge as $h \to 0$, strongly in $W^{1,2}(\omega,\mathbb{R}^3)$, to the limiting displacement:

$$
V \in \mathcal{V}_{y_0} = \Big\{V \in W^{2,2}(\omega,\mathbb{R}^3);\ \big((\nabla y_0)^{\mathsf T}\nabla V\big)_{\mathrm{sym}} = 0 \quad \textit{a.e. in } \omega\Big\}. \tag{1.10}
$$

(ii) *The above condition $V \in \mathcal{V}_{y_0}$ automatically defines $\vec p \in W^{1,2}(\omega,\mathbb{R}^3)$ such that:*

$$
\big(B_0^{\mathsf T}\big[\nabla V,\ \vec p\,\big]\big)_{\mathrm{sym}} = 0 \qquad \textit{a.e. in } \omega,
$$

and we have: $\liminf_{h\to 0} \frac{1}{h^{2(n+1)}}\mathcal{E}^h(u^h) \geq \mathcal{I}_{2(n+1)}(V)$*, where:*

$$\begin{aligned}\mathcal{I}_{2(n+1)}(V) =& \frac{1}{24}\cdot \Big\|(\nabla y_0)^{\mathsf{T}}\nabla\vec p + (\nabla V)^{\mathsf{T}}\nabla \vec b_1 + \alpha_n\big[\partial_3^{(n-1)}R_{i3,j3}\big]_{i,j=1\ldots 2}\Big\|^2_{\mathcal{Q}_2}\\ &+ \beta_n\cdot \Big\|\mathbb{P}_{\mathcal{S}_{y_0}^\perp}\big(\big[\partial_3^{(n-1)}R_{i3,j3}\big]_{i,j=1\ldots 2}\big)\Big\|^2_{\mathcal{Q}_2}\\ &+ \gamma_n\cdot \Big\|\mathbb{P}_{\mathcal{S}_{y_0}}\big(\big[\partial_3^{(n-1)}R_{i3,j3}\big]_{i,j=1\ldots 2}\big)\Big\|^2_{\mathcal{Q}_2}.\end{aligned}\tag{1.11}$$

Above, $\mathcal{S}_{y_0}$ *is the following closed subspace of the Hilbert space* $\mathbb{E}$ *in (1.3):*

$$\mathcal{S}_{y_0} = \mathrm{closure}_{\mathbb{E}}\Big\{\big((\nabla y_0)^{\mathsf{T}}\nabla w\big)_{\mathrm{sym}};\ w\in W^{1,2}(\omega,\mathbb{R}^3)\Big\},$$

whereas $\mathbb{P}_{\mathcal{S}_{y_0}}$ *and* $\mathbb{P}_{\mathcal{S}_{y_0}^\perp}$ *denote, respectively, the orthogonal projections onto the space* $\mathcal{S}_{y_0}$ *and its orthogonal complement* $\mathcal{S}_{y_0}^\perp$ *in* $\mathbb{E}$*. The coefficients in (1.11) are:*

$$\begin{aligned}&\alpha_n = \begin{cases} 0 & \textit{for } n \textit{ odd}\\ \dfrac{3}{2^n(n+3)(n+1)!} & \textit{for } n \textit{ even}\end{cases},\\ &\beta_n = \frac{1}{2^{2n+3}(2n+3)\big((n+1)!\big)^2}\cdot\begin{cases} 1 & \textit{for } n \textit{ odd}\\ \dfrac{n^2}{(n+3)^2} & \textit{for } n \textit{ even}\end{cases},\\ &\gamma_n = \frac{1}{2^{2n+3}(2n+3)\big((n+1)!\big)^2}\cdot\begin{cases} \dfrac{(n+1)^2}{(n+2)^2} & \textit{for } n \textit{ odd}\\ \dfrac{n^2}{(n+3)^2} & \textit{for } n \textit{ even}\end{cases}.\end{aligned}\tag{1.12}$$

(iii) *There holds on* ω*:*

$$\begin{aligned}2\cdot\big[\partial_3^{(n-1)}R_{i3,j3}(\cdot,0)\big]_{i,j=1\ldots 2} =& 2\big((\nabla y_0)^{\mathsf{T}}\nabla\vec b_{n+1}\big)_{\mathrm{sym}} + \sum_{k=1}^n\binom{n+1}{k}(\nabla\vec b_k)^{\mathsf{T}}\nabla\vec b_{n+1-k}\\ &- \partial_3^{(n+1)}G(\cdot,0)_{2\times 2}.\end{aligned}\tag{1.13}$$

We list a few related observations:

(i) When $G = Id_3$, then each functional in (1.11) reduces to the classical linear elasticity. We have: $y_0 = id$, $\vec b_1 = e_3$ and $\mathcal{V} = \big\{(\alpha x^\perp + \vec\beta, v);\ \alpha\in\mathbb{R},\ \vec\beta\in\mathbb{R}^2,\ v\in W^{2,2}(\omega)\big\}$, and for $V\in\mathcal{V}$ there holds: $\vec p = (-\nabla v, 0)$. Consequently: $\mathcal{I}_{2(n+1)}(V) = \frac{1}{24}\int_\omega \mathcal{Q}_2\big(x', \nabla^2 v\big)\,\mathrm{d}x'$ yields the classical biharmonic energy, in function of the out-of-plane scalar displacement v.

(ii) In the present geometric context, the bending term is given by: $(\nabla y_0)^{\mathsf{T}}\nabla\vec p + (\nabla V)^{\mathsf{T}}\nabla\vec b_1$. It is of order $h^n x_3$ and it interacts with the curvature $\big[\partial_3^{(n-1)}R_{i3,j3}(\cdot,0)\big]_{i,j=1\ldots 2}$, which is of order x_3^{n+1}. The interaction occurs only when the two terms have the same parity in x_3, namely at even n, so that $\alpha_n = 0$ for all n odd. The two remaining terms in (1.11) measure the (squared) L^2 norm of $\big[\partial_3^{(n-1)}R_{i3,j3}(\cdot,0)\big]_{i,j=1\ldots 2}$, with distinct weights assigned to the $\mathcal{S}_{y_0}$ and $\big(\mathcal{S}_{y_0}\big)^\perp$ projections, again according to the parity of n.

(iii) In the same line, the quantity $\inf_{\mathcal{V}_{y_0}} \mathcal{I}_{2(n+1)}$ is precisely the square of an appropriate weighted L^2 norm of $\big[\nabla^{(n-1)} R_{ab,cd}\big]$ restricted to ω, as it is quadratic in $\big[\partial_3^{(n-1)} R_{i3,j3}(\cdot,0)\big]_{i,j=1\dots 2}$ and:

$$\inf_{\mathcal{V}_{y_0}} \mathcal{I}_{2(n+1)} \geq \gamma_n \cdot \big\|\big[\partial_3^{(n-1)} R_{i3,j3}(\cdot,0)\big]_{i,j=1\dots 2}\big\|^2_{\mathcal{Q}_2},$$
$$\inf_{\mathcal{V}_{y_0}} \mathcal{I}_{2(n+1)} \leq \mathcal{I}_{2(n+1)}(0) \leq \big(\frac{1}{24}\alpha_n^2 + \beta_n\big) \cdot \big\|\big[\partial_3^{(n-1)} R_{i3,j3}(\cdot,0)\big]_{i,j=1\dots 2}\big\|^2_{\mathcal{Q}_2}.$$

We note that: $\frac{1}{24}\alpha_n^2 + \beta_n = \big(2^{2n+3}(2n+3)((n+1)!)^2\big)^{-1}$. Also, it follows from Theorem 1.3 that: $\inf \mathcal{I}_{2(n+1)} = \min \mathcal{I}_{2(n+1)}$.

(iv) The formula in (1.13) relates the quantities appearing in conditions (i) and (iii) of Theorem 1.1. The curvature $\big[\partial_3^{(n-1)} R_{i3,j3}(\cdot,0)\big]_{i,j=1\dots 2}$ is thus precisely the coefficient of the discrepancy of the order h^{n+1} in (1.9) at the 2×2 minor, scaled by the $(n+1)!/2$ factor.

(v) The finite strain space $\mathcal{S}_{y_0}$ can be identified, in particular, in the following two cases. When $y_0 = id_2$, then $\mathcal{S}_{y_0} = \{\mathbb{S} \in L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}});\ \mathrm{curl}^{\mathsf{T}}\mathrm{curl}\,\mathbb{S} = 0\}$. When the Gauss curvature $\kappa((\nabla y_0)^{\mathsf{T}}\nabla y_0) = \kappa\big(G_{2\times 2}\big) > 0$ on $\bar\omega$, then $\mathcal{S}_{y_0} = L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, as shown in [26].

Our next result proves the upper bound that is consistent with Theorem 1.1 and yields the Γ-convergence of the rescaled energies $h^{-2(n+1)}\mathcal{E}^h$ to the dimensionally reduced limits $\mathcal{I}_{2(n+1)}$ in (1.11):

Theorem 1.3. *Fix $n \geq 2$ and assume that any of the equivalent conditions in Theorem 1.1 hold. Then for every $V \in \mathcal{V}_{y_0}$ in (1.10), there exists a sequence $\{u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)\}_{h\to 0}$ so that:*

$$\frac{1}{h^n}\fint_{-h/2}^{h/2} u^h(x',x_3) - \Big(y_0 + \sum_{k=1}^n \frac{x_3^k}{k!}\vec b_k\Big)\,\mathrm{d}x_3 \to V \quad \textit{as } h\to 0, \quad \textit{strongly in } W^{1,2}(\omega,\mathbb{R}^3), \tag{1.14}$$

and that: $\lim_{h\to 0} \frac{1}{h^{2(n+1)}}\mathcal{E}^h(u^h) = \mathcal{I}_{2(n+1)}(V)$, where the limiting energy functional is as in (1.11).

It is worth noting the following self-evident application of Theorems 1.1, 1.2 and 1.3:

Corollary 1.4. *Under either of the equivalent conditions in (1.6), assume that for some $n \geq 2$ there holds: $\partial_3^{(m)}\big[R_{i3,j3}(\cdot,0)\big]_{i,j=1\dots 2} = 0$ on ω, for all $m = 0\dots n-2$, but $\partial_3^{(n-1)}\big[R_{i3,j3}(\cdot,0)\big]_{i,j=1\dots 2} \not\equiv 0$. Then there exist $c, C > 0$ such that:*

$$ch^{2(n+1)} \leq \inf \mathcal{E}^h \leq Ch^{2(n+1)}. \tag{1.15}$$

Moreover, the scaled energies $\frac{1}{h^{2(n-1)}}\mathcal{E}^h$, Γ-converge to the limiting functional $\mathcal{I}_{2(n+1)}$ in (1.11), effectively defined on the space $\mathcal{V}_{y_0}$ of first order infinitesimal isometries in (1.10).

For completeness, we note that the conformal metrics of the form: $G(x',x_3) = e^{2\phi(x_3)}Id_3$ provide a class of examples for the viability of all scalings in (1.15). Indeed, the trace midplate metric $e^{2\phi(0)}Id_2$ has a smooth isometric immersion $y_0 = e^{\phi(0)}id_2 : \omega \to \mathbb{R}^2$, and the only possibly nonzero Riemann curvatures of G are given by: $R_{12,12} = -\phi'(x_3)^2 e^{2\phi(x_3)}$, $R_{13,13} = R_{23,23} = -\phi''(x_3)e^{2\phi(x_3)}$. By Corollary 1.4 we see that $\inf \mathcal{E}^h \sim h^{2n}$ if and only if $\phi^{(k)}(0) = 0$ for $k = 1\dots n-1$ and $\phi^{(n)}(0) \neq 0$.

1.3. **The structure of the paper.** In sections 2 and 3 we work under the assumption (iii) of Theorem 1.1. First, in Lemma 2.1, we give an easy proof of the implication (iii)$\Rightarrow$ (ii). The obtained energy-consistent deformation field is further used as the local change of variables allowing for the application of the nonlinear rigidity estimate [8] in the present context. This is done in Lemma 2.2 and Corollary 2.3, and it yields an approximation of an arbitrary energy-consistent deformation gradient ∇u^h, by a non-symmetric square root of the n-th order Taylor expansion of the metric G, derived from the expansion guaranteed in (iii). Both the approximation error and the L^2 norm of the gradient of the rotation field component are energy-controlled. In Lemma 2.4 we then prove the compactness part of Theorem 1.2. In Lemma 2.5 we conclude a preliminary lower bound estimate, involving a version of the functional $\mathcal{I}_{2(n+1)}$, whose curvature terms are still expressed in terms of the expansion fields in (iii), as suggested in the right hand side of (1.13).

In section 3, we develop a geometric line of arguments, serving to prove (in Corollary 3.6) the identity (1.13) under assumption (iii). In Lemmas 3.1 and 3.2, we partially reprove the equivalent conditions valid at the previously analyzed scalings h^2 and h^4. These statements are then generalized in Lemma 3.5, where we show the implication (iii) $\Rightarrow$ (i), resulting also in the existence of a one order higher approximate field $\vec{b}_{n+1}$, that is given solely through the Christoffel symbols of G on ω.

In section 4 we finally prove Theorem 1.1, showing equivalence of the stated three conditions, by induction on $n \geq 2$. We also finish the proof of Theorem 1.2 by: improving the lower bound from section 2, identifying its curvature components via (1.13), and separating the bending and the excess terms. In section 5 we prove Theorem 1.3, constructing a energy-consistent recovery sequence.

1.4. **Notation.** Given a matrix $F \in \mathbb{R}^{n\times n}$, we denote its transpose by F^{T} and its symmetric part by $F_{\mathrm{sym}} = \frac{1}{2}(F + F^{\mathsf{T}})$. The space of symmetric $n \times n$ matrices is denoted by $\mathbb{R}^{n\times n}_{\mathrm{sym}}$, whereas $\mathbb{R}^{n\times n}_{\mathrm{sym,pos}}$ stands for the space of symmetric, positive definite $n \times n$ matrices. By $SO(n) = \{R \in \mathbb{R}^{n\times n};\ R^{\mathsf{T}} = R^{-1} \text{ and } \det R = 1\}$ we mean the group of special rotations; its tangent space at Id_n consists of skew-symmetric matrices: $T_{Id_n}SO(n) = so(n) = \{F \in \mathbb{R}^{n\times n};\ F_{\mathrm{sym}} = 0\}$. We use the matrix norm $|F| = (\mathrm{trace}(F^{\mathsf{T}}F))^{1/2}$, which is induced by the inner product $\langle F_1 : F_2\rangle = \mathrm{trace}(F_1^{\mathsf{T}}F_2)$. The 2×2 principal minor of $F \in \mathbb{R}^{3\times 3}$ is denoted by $F_{2\times 2}$. Conversely, for a given $F_{2\times 2} \in \mathbb{R}^{2\times 2}$, the 3×3 matrix with principal minor equal $F_{2\times 2}$ and all other entries equal to 0, is denoted by $F^*_{2\times 2}$. Unless specified otherwise, all limits are taken as the thickness parameter h vanishes: $h \to 0$. By C we denote any universal positive constant, independent of h.

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2. A proof of Theorem 1.2: compactness and a preliminary lower bound

In this section, assuming condition (iii) of Theorem 1.1, we derive the compactness and (a version of) the lower bound in Theorem 1.2. We first observe the implication $(iii) \Rightarrow (ii)$ in Theorem 1.1:

Lemma 2.1. *Assume that condition (iii) in Theorem 1.1 holds, for some $n \geq 2$. Then we have:*

$$\inf \mathcal{E}^h \leq Ch^{2(n+1)}.$$

Proof. Define $u^h(x', x_3) = y_0 + \sum_{k=1}^{n+1} \frac{x_3^k}{k!}\vec{b}_k$, so that:

$$\nabla u^h(x', x_3) = \sum_{k=0}^{n} \frac{x_3^k}{k!}B_k + \frac{x_3^{n+1}}{(n+1)!}\big[\partial_1\vec{b}_{n+1},\ \partial_2\vec{b}_{n+1},\ 0\big].$$

Consequently, $(\nabla u^h)G^{-1/2}$ is positive definite for all small h, and modulo a rotation field it equals the following matrix field on Ω^h, where we used the assumption (1.9):

$$\begin{aligned}\sqrt{\big((\nabla u^h)G^{-1/2}\big)^{\mathsf T}\big((\nabla u^h)G^{-1/2}\big)} &= \sqrt{Id_3 + G^{-1/2}\big((\nabla u^h)^{\mathsf T}\nabla u^h - G\big)G^{-1/2}} \\ &= \sqrt{Id_3 + \mathcal{O}(h^{n+1})} = Id_3 + \mathcal{O}(h^{n+1}).\end{aligned}$$

This implies: $\mathcal{E}^h(u^h) = \frac{1}{h}\int_{\Omega^h} W\big(Id_3 + \mathcal{O}(h^{n+1})\big)\,\mathrm{d}x \leq Ch^{2(n+1)}$, as claimed. ∎

Recalling results (1.6) and (1.8) quoted from [30], we already see that $\lim_{h\to 0}\frac{1}{h^4}\inf\mathcal{E}^h = 0$ automatically implies: $\inf\mathcal{E}^h \leq Ch^6$. Before addressing compactness at h^{2n} with $h \geq 3$, we develop the nonlinear rigidity estimates applicable in the present context.

Lemma 2.2. *Assume that condition (iii) in Theorem 1.1 holds, for some $n \geq 2$. Let $V \subset \omega$ be an open, Lipschitz subdomain such that y_0 is injective on V. Denote $V^h = V \times (-\frac{h}{2}, \frac{h}{2})$. Then for every $u^h \in W^{1,2}(V^h, \mathbb{R}^3)$ there exists $\bar R^h \in SO(3)$ such that:*

$$\frac{1}{h}\int_{V^h}\Big|\nabla u^h - \bar R^h\sum_{k=0}^{n}\frac{x_3^k}{k!}B_k\Big|^2\,\mathrm{d}x \leq C\Big(\frac{1}{h}\int_{V^h} W\big((\nabla u^h)G^{-1/2}\big)\,\mathrm{d}x + h^{2n+1}|V^h|\Big).$$

The constant C is uniform for all $V^h \subset \Omega^1$ that are bi-Lipschitz equivalent with controlled Lipschitz constants.

Proof. Define $Y = y_0 + \sum_{k=1}^{n+1}\frac{x_3^k}{k!}\vec b_k$, and observe that for h sufficiently small, Y is a smooth diffeomorphism of V^h onto its image $U^h \subset \mathbb{R}^3$. Consider the change of variables $v^h = u^h \circ Y^{-1} \in W^{1,2}(U^h, \mathbb{R}^3)$ and apply the fundamental geometric rigidity estimate [8], yielding existence of $\bar R^h \in SO(3)$ with:

$$\int_{U^h}|\nabla v^h - \bar R^h|^2 \leq C\int_{U^h}\operatorname{dist}^2\big(\nabla v^h, SO(3)\big).$$

Changing variable in the left hand side gives:

$$\begin{aligned}\int_{U^h}|\nabla v^h - \bar R^h|^2 &= \int_{V^h}(\det\nabla Y)\cdot\big|(\nabla u^h)(\nabla Y)^{-1} - \bar R^h\big|^2 \geq C\int_{V^h}\big|\nabla u^h - \bar R^h\nabla Y\big|^2 \\ &= C\int_{V^h}\Big|\nabla u^h - \bar R^h\Big(\sum_{k=0}^{n}\frac{x_3^k}{k!}B_k\Big)\Big|^2 + C\int_{V^h}\mathcal{O}(h^{2(n+1)}).\end{aligned}$$

Changing now variable in the right hand side and using $(\nabla Y)G^{-1/2} \in SO(3)\big(Id_3 + \mathcal{O}(h^{n+1})\big)$, as established in Lemma 2.1, results in:

$$\begin{aligned}\int_{U^h}\operatorname{dist}^2\big(\nabla v^h, SO(3)\big) &= \int_{V^h}(\det\nabla Y)\cdot\operatorname{dist}^2\big((\nabla u^h)(\nabla Y)^{-1}, SO(3)\big) \\ &\leq C\int_{V^h}\operatorname{dist}^2\big((\nabla v^h)G^{-1/2}, SO(3)(\nabla Y)G^{-1/2}\big) \\ &\leq C\int_{V^h}\operatorname{dist}^2\big((\nabla v^h)G^{-1/2}, SO(3)\big) + C\int_{V^h}\mathcal{O}(h^{2(n+1)}).\end{aligned}$$

Combining the three displayed inequalities above proves the result. ∎

The well-known approximation technique [9] together with the arguments in [29, Corollary 2.3], yield the following estimate, whose proof we leave to the reader:

Corollary 2.3. *Assume condition (iii) in Theorem 1.1, for some $n \geq 2$. Then, given a sequence $\{u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)\}_{h\to 0}$ such that $\mathcal{E}^h(u^h) \leq Ch^{2(n+1)}$, there exists $\{R^h \in W^{1,2}(\omega, SO(3))\}_{h\to 0}$ with:*

$$\frac{1}{h}\int_{\Omega^h} \big|\nabla u^h - R^h \sum_{k=0}^n \frac{x_3^k}{k!}B_k\big|^2 \,\mathrm{d}x \leq Ch^{2(n+1)} \quad \text{ and } \quad \int_\omega |\nabla R^h(x')|^2 \,\mathrm{d}x' \leq Ch^{2n}.$$

We now show the compactness part of Theorem 1.2:

Lemma 2.4. *Assume condition (iii) in Theorem 1.1, for some $n \geq 2$. Let the sequence of deformations $\{u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)\}_{h\to 0}$ satisfy: $\mathcal{E}^h(u^h) \leq Ch^{2(n+1)}$. Then:*

- (i) *The averaged displacements V^h converge, up to a subsequence, to the first order isometry V as in Theorem 1.2 (i).*
- (ii) *The scaled strains $\frac{1}{h}\big((\nabla y_0)^{\mathsf{T}}\nabla V^h\big)_{\mathrm{sym}}$ converge, up to a subsequence, weakly in $L^2(\omega, \mathbb{R}^{2\times 2})$ to some $\mathbb{S} \in \mathcal{S}_{y_0}$.*

Proof. **1.** Define the following rotation: $\bar R^h = \mathbb{P}_{SO(3)} \fint_{\Omega^h} (\nabla u^h)\Big(\sum_{k=0}^n \frac{x_3^k}{k!}B_k\Big)^{-1} \,\mathrm{d}x$. In order to observe that the above definition is legitimate, we write:

$$\begin{split}\mathrm{dist}^2\Big(\fint_{\Omega^h} (\nabla u^h)\Big(\sum_{k=0}^n \frac{x_3^k}{k!}B_k\Big)^{-1} \,\mathrm{d}x, SO(3)\Big) &\leq \Big|\fint_{\Omega^h} (\nabla u^h)\Big(\sum_{k=0}^n \frac{x_3^k}{k!}B_k\Big)^{-1} \,\mathrm{d}x - R^h(x')\Big|^2 \\ &\leq 2\fint_{\Omega^h} \big|(\nabla u^h)\Big(\sum_{k=0}^n \frac{x_3^k}{k!}B_k\Big)^{-1} - R^h\big|^2 \,\mathrm{d}x + 2\big|\big(\fint_\omega R^h \,\mathrm{d}x'\big) - R^h(x')\big|^2,\end{split}$$

and upon integrating $\mathrm{d}x'$ on the domain ω while noting Corollary 2.3, obtain:

$$\mathrm{dist}^2\Big(\fint_{\Omega^h} (\nabla u^h)\Big(\sum_{k=0}^n \frac{x_3^k}{k!}B_k\Big)^{-1} \,\mathrm{d}x, SO(3)\Big) \leq Ch^{2(n+1)} + Ch^{2n} \leq Ch^{2n}.$$

Consequently, there also follows:

$$\Big|\fint_{\Omega^h} (\nabla u^h)\Big(\sum_{k=0}^n \frac{x_3^k}{k!}B_k\Big)^{-1} \,\mathrm{d}x - \bar R^h\Big|^2 \leq Ch^{2n}, \qquad \fint_\omega |R^h - \bar R^h|^2 \,\mathrm{d}x' \leq Ch^{2n}. \tag{2.1}$$

Set now $c^h \in \mathbb{R}^3$ so that $\int_\omega V^h \,\mathrm{d}x' = 0$. We get:

$$\begin{split}\nabla V^h = {} & \frac{1}{h^n}\fint_{-h/2}^{h/2} (\bar R^h)^{\mathsf{T}}\big[\partial_1 u^h, \partial_2 u^h\big] - (\bar R^h)^{\mathsf{T}} R^h \Big(\sum_{k=0}^n \frac{x_3^k}{k!}B_k\Big)_{3\times 2} \,\mathrm{d}x_3 \\ & + S^h \fint_{-h/2}^{h/2} \Big(\sum_{k=0}^n \frac{x_3^k}{k!}B_k\Big)_{3\times 2} \,\mathrm{d}x_3,\end{split} \tag{2.2}$$

where we define the following matrix fields whose convergence (up to a subsequence) results from the second bound in (2.1) and from Corollary 2.3:

$$S^h = \frac{1}{h^n}\Big((\bar R^h)^{\mathsf{T}} R^h - Id_3\Big) \rightharpoonup S \quad \text{ weakly in } W^{1,2}(\omega, \mathbb{R}^{3\times 3}). \tag{2.3}$$

We also note that $S \in so(3)$ a.e. in ω. Since the first term in the right hand side of (2.2) converges to 0 in $L^2(\omega)$, in virtue of Corollary 2.3, we conclude the following convergence, up to a subsequence:

$$\nabla V^h \to (SB_0)_{3\times 2} = S\nabla y_0 \quad \text{ strongly in } L^2(\omega, \mathbb{R}^{3\times 2}).$$

It also follows that the limit $S\nabla y_0 \in W^{1,2}(\omega,\mathbb{R}^{3\times 2})$. A further application of the Poincare inequality to the mean-zero displacements V^h, yields their strong convergence (up to a subsequence in $W^{1,2}(\omega,\mathbb{R}^3)$) to some $V\in W^{2,2}(\omega,\mathbb{R}^3)$ satisfying $\nabla V = (SB_0)_{3\times 2}$. By skew-symmetry of S, it follows that $(\nabla y_0)^{\mathsf{T}}\nabla V$ is skew a.e. in ω, proving (i).

2. We observe that the first term in the right hand side of (2.2) has its $L^2(\omega)$ norm bounded by Ch^2, in view of the first estimate in Corollary 2.3. Consequently, in the decomposition of $\frac{1}{h}\big((\nabla y_0)^{\mathsf{T}}\nabla V^h\big)_{\mathrm{sym}}$, parallel to that in (2.2), the corresponding first term has a weakly converging subsequence. The remaining second term equals:

$$\frac{1}{h}\Big((\nabla y_0)^{\mathsf{T}} S^h\big(\nabla y_0+\mathcal{O}(h^2)\big)\Big)_{\mathrm{sym}} = \frac{1}{h}(\nabla y_0)^{\mathsf{T}} S^h_{\mathrm{sym}}\nabla y_0 + \mathcal{O}(h|S^h|).$$

The $L^2(\omega)$ norm of the second term above clearly converges to 0, whereas the first term obeys:

$$\frac{1}{h}S^h_{\mathrm{sym}} = -\frac{h^{n-1}}{2}(S^h)^{\mathsf{T}}S^h \to 0 \quad \text{strongly in } \ L^2(\omega,\mathbb{R}^{3\times 3}). \tag{2.4}$$

This ends the proof of the claim. ■

We are now ready to derive the lower bound on the scaled energies $h^{-2(n+1)}\mathcal{E}^h(u^h)$, in terms of the expansion fields $y_0, \{\vec b_k\}_{k=1}^{n+1}$ in condition (iii) of Theorem 1.1:

Lemma 2.5. *In the context of Lemma 2.4, there holds:*

$$\begin{split}\liminf_{h\to 0}\ & \frac{1}{h^{2(n+1)}}\mathcal{E}^h(u^h)\\ & \geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_2\bigg(x',\ \mathbb{S}-\delta_{n+1}\big((\nabla y_0)^{\mathsf{T}}\nabla\vec b_{n+1}\big)_{\mathrm{sym}} + x_3\big((\nabla y_0)^{\mathsf{T}}\nabla\vec p + (\nabla V)^{\mathsf{T}}\nabla \vec b_1\big)\\ & \qquad + \frac{x_3^{n+1}}{2(n+1)!}\Big(2\big((\nabla y_0)^{\mathsf{T}}\nabla\vec b_{n+1}\big)_{\mathrm{sym}} + \sum_{k=1}^n\binom{n+1}{k}(\nabla\vec b_k)^{\mathsf{T}}\nabla\vec b_{n+1-k} - \partial_3^{(n+1)}G(x',0)\Big)\bigg)\,\mathrm{d}x,\end{split}$$

with the coefficient δ_{n+1} given by:

$$\delta_{n+1} = \begin{cases}\dfrac{1}{(n+2)!2^{n+1}} & \textit{for } \ n \ \textit{odd}\\ 0 & \textit{for } \ n \ \textit{even}\end{cases}. \tag{2.5}$$

Proof. **1.** By Corollary 2.3, the following matrix fields $\{\mathcal{Z}^h\in L^2(\Omega^1,\mathbb{R}^{3\times 3})\}_{h\to 0}$ have a converging subsequence, weakly in $L^2(\Omega^1,\mathbb{R}^{3\times 3})$:

$$\mathcal{Z}^h(x',x_3) = \frac{1}{h^{n+1}}\Big(\nabla u^h(x',hx_3) - R^h(x')\sum_{k=0}^n\frac{h^k x_3^k}{k!}B_k(x')\Big)\rightharpoonup\mathcal{Z}. \tag{2.6}$$

We write: $(R^h)^{\mathsf{T}}\nabla u^h(x',hx_3) = \sum_{k=0}^n \frac{h^k x_3^k}{k!}B_k + h^{n+1}(R^h)^{\mathsf{T}}\mathcal{Z}^h$ and observe that:

$$\begin{split}\mathcal{E}^h(u^h) & = \int_{\Omega^1} W\big((R^h)^{\mathsf{T}}\nabla u^h(x',hx_3)G(x',hx_3)^{-1/2}\big)\,\mathrm{d}x\\ & \geq \int_{\{|\mathcal{Z}^h|^2\leq\frac{1}{h}\}} W\Big(\sqrt{Id_3 + G(x',hx_3)^{-1/2}J^hG(x',hx_3)^{-1/2}}\Big)\,\mathrm{d}x,\end{split}\tag{2.7}$$

where the intermediary field J^h has the following expansion, on the set $\{|\mathcal{Z}^h|^2 \leq \frac{1}{h}\} \subset \Omega^1$:

$$\begin{aligned}
J^h(x', x_3) &= \Big(\sum_{k=0}^{n} \frac{h^k x_3^k}{k!} B_k\Big)^{\mathsf{T}} \Big(\sum_{k=0}^{n} \frac{h^k x_3^k}{k!} B_k\Big) - G(x', hx_3) \\
&\quad + 2h^{n+1}\Big(\Big(\sum_{k=0}^{n} \frac{h^k x_3^k}{k!} B_k\Big)^{\mathsf{T}} (R^h)^{\mathsf{T}} \mathcal{Z}^h\Big)_{\mathrm{sym}} + h^{2(n+1)} (\mathcal{Z}^h)^{\mathsf{T}} \mathcal{Z}^h \\
&= \frac{h^{n+1} x_3^{n+1}}{(n+1)!} \Big(\sum_{k=1}^{n} \binom{n+1}{k} (B_k)^{\mathsf{T}} B_{n+1-k} - \partial_3^{(n+1)} G(x', 0)\Big) \\
&\quad + 2h^{n+1}\big(B_0^{\mathsf{T}} (R^h)^{\mathsf{T}} \mathcal{Z}^h\big)_{\mathrm{sym}} + o(h^{n+1})
\end{aligned}$$

Consequently, we get from (2.7) and Taylor expanding W at Id_3:

$$\frac{1}{h^{2(n+1)}} \mathcal{E}^h(u^h) \geq \frac{1}{8} \int_{\{|\mathcal{Z}^h|^2 \leq \frac{1}{h}\}} \mathcal{Q}_3\Big(G(x', hx_3)^{-1/2} \big(\frac{1}{h^{n+1}} J^h + o(1)\big) G(x', hx_3)^{-1/2}\Big) \,\mathrm{d}x.$$

Since $B_0^{\mathsf{T}} (R^h)^{\mathsf{T}} \mathcal{Z}^h$ converges weakly in $L^2(\Omega^1, \mathbb{R}^{3\times 3})$, up to a subsequence, to $B_0^{\mathsf{T}} \bar{R}^{\mathsf{T}} \mathcal{Z}$, for some $\bar{R} \in SO(3)$ (which is an accumulation point of $\bar{R}^h$ in the proof of Lemma 2.4), the above results in:

$$\begin{aligned}
&\liminf_{h\to 0} \frac{1}{h^{2(n+1)}} \mathcal{E}^h(u^h) \\
&\geq \frac{1}{2} \int_{\Omega^1} \mathcal{Q}_3\Big(\frac{x_3^{n+1}}{2(n+1)!} G(x', 0)^{-1/2} \Big(\sum_{k=1}^{n} \binom{n+1}{k} (B_k)^{\mathsf{T}} B_{n+1-k} - \partial_3^{(n+1)} G(x', 0)\Big) G(x', 0)^{-1/2} \\
&\qquad + G(x', 0)^{-1/2} \big(B_0^{\mathsf{T}} \bar{R}^{\mathsf{T}} \mathcal{Z}\big)_{\mathrm{sym}} G(x', 0)^{-1/2}\Big) \,\mathrm{d}x
\end{aligned} \tag{2.8}$$

2. We need to identify the relevant 2×2 minor of the limiting term $\big(B_0^{\mathsf{T}} \bar{R}^{\mathsf{T}} \mathcal{Z}\big)_{\mathrm{sym}}$ in (2.8). We apply the finite difference technique [9] and consider the following fields $\{f^{s,h} \in W^{1,2}(\Omega^1, \mathbb{R}^3)\}_{s>0, h\to 0}$:

$$\begin{aligned}
f^{s,h}(x) &= \fint_0^s h(\bar{R}^h)^{\mathsf{T}} \mathcal{Z}^h(x', x_3 + t) + S^h \sum_{k=0}^{n} \frac{h^k (x_3 + t)^k}{k!} B_k(x') \,\mathrm{d}t \; e_3 \\
&= \frac{1}{h^{n+1} s} (\bar{R}^h)^{\mathsf{T}} \big(u^h(x', h(x_3 + s)) - u^h(x', hx_3)\big) - \frac{1}{h^n} \fint_0^s \sum_{k=0}^{n} \frac{h^k (x_3 + t)^k}{k!} \vec{b}_{k+1} \,\mathrm{d}t.
\end{aligned}$$

where S^h is defined in (2.3). Recall that, as proved in Lemma 2.4, $\nabla V = (SB_0)_{3\times 2}$ and that S is a.e. in $so(3)$. It follows that the vector $\vec{p}$ defined in Theorem 1.2 (ii) must coincide with $SB_0 e_3 = S\vec{b}_1$. Consequently, using the first definition above it now easily follows that:

$$f^{s,h} \to S\vec{b}_1 = \vec{p} \quad \text{strongly in } \; L^2(\Omega^1, \mathbb{R}^3). \tag{2.9}$$

Using the second definition, we further compute the in-plane derivatives of $f^{s,h}$ for $j = 1 \ldots 2$:

$$\begin{aligned}
\partial_j f^{s,h}(x) &= \frac{1}{sh^{n+1}} (\bar{R}^h)^{\mathsf{T}} \big(\partial_j u^h(x', h(x_3 + s)) - \partial_j u^h(x', hx_3)\big) - \frac{1}{h^n} \fint_0^s \sum_{k=0}^{n} \frac{h^k (x_3 + t)^k}{k!} \partial_k \vec{b}_{k+1} \,\mathrm{d}t \\
&= \frac{1}{s} (\bar{R}^h)^{\mathsf{T}} \Big(\mathcal{Z}^h(x', x_3 + s) - \mathcal{Z}^h(x', x_3)\Big) e_j \\
&\qquad + \frac{1}{sh^{n+1}} (Id_3 + h^n S^h) \sum_{k=1}^{n} \frac{h^k}{k!} \big((x_3 + s)^k - x_3^k\big) B_k e_j - \frac{1}{h^n} \fint_0^s \sum_{k=0}^{n} \frac{h^k (x_3 + t)^k}{k!} \partial_j \vec{b}_{k+1} \,\mathrm{d}t.
\end{aligned}$$

The first term in the right hand side above converges to $\frac{1}{s}\bar{R}^{\mathsf{T}}\big(\mathcal{Z}(x',x_3+s)-\mathcal{Z}(x',x_3)\big)e_j$, weakly in $L^2(\Omega^1,\mathbb{R}^3)$, whereas the last two terms may be rewritten as:

$$\begin{aligned}&\frac{1}{sh^{n+1}}(Id_3+h^nS^h)\sum_{k=1}^n\frac{h^k}{k!}\big((x_3+s)^k-x_3^k\big)\partial_j\vec{b}_k-\frac{1}{sh^{n+1}}\sum_{k=1}^{n+1}\frac{h^k}{k!}\big((x_3+s)^k-x_3^k\big)\partial_j\vec{b}_k\\ &=\frac{1}{sh}S^h\sum_{k=1}^n\frac{h^k}{k!}\big((x_3+s)^k-x_3^k\big)\partial_j\vec{b}_k-\frac{1}{s(n+1)!}\big((x_3+s)^{n+1}-x_3^{n+1}\big)\partial_j\vec{b}_{n+1}\\ &\qquad\rightharpoonup S\partial_j\vec{b}_1-\frac{1}{s(n+1)!}\big((x_3+s)^{n+1}-x_3^{n+1}\big)\partial_j\vec{b}_{n+1}\quad\text{ weakly in }\; W^{1,2}(\Omega^1,\mathbb{R}^3).\end{aligned}$$

In conclusion, and recalling (2.9), we obtain the following convergence, weakly in $W^{1,2}(\omega,\mathbb{R}^3)$:

$$\partial_jf^{s,h}(x)\rightharpoonup\frac{1}{s}\bar{R}^{\mathsf{T}}\big(\mathcal{Z}(x',x_3+s)-\mathcal{Z}(x',x_3)\big)e_j+S\partial_j\vec{b}_1-\frac{1}{s(n+1)!}\big((x_3+s)^{n+1}-x_3^{n+1}\big)\partial_j\vec{b}_{n+1}=\partial_j\vec{p}.$$

We thus see that:

$$\bar{R}^{\mathsf{T}}\Big(\mathcal{Z}(x',x_3)-\mathcal{Z}(x',0)\Big)e_j=x_3\big(\partial_j\vec{p}-S\partial_j\vec{b}_1\big)+\frac{1}{(n+1)!}x_3^{n+1}\partial_j\vec{b}_{n+1}\qquad\text{for }\; j=1\dots2,$$

which finally yields:

$$\begin{aligned}\Big(B_0^{\mathsf{T}}\bar{R}^{\mathsf{T}}\mathcal{Z}(x',x_3)\Big)_{2\times2}&=\Big(B_0^{\mathsf{T}}\bar{R}^{\mathsf{T}}\mathcal{Z}(x',0)\Big)_{2\times2}\\ &\quad+x_3\Big((\nabla y_0)^{\mathsf{T}}\nabla\vec{p}+(\nabla V)^{\mathsf{T}}\nabla\vec{b}_1\Big)+\frac{1}{(n+1)!}x_3^{n+1}(\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_{n+1}.\end{aligned}\tag{2.10}$$

3. We now compute the symmetric part of the trace term $\big(B_0^{\mathsf{T}}\bar{R}^{\mathsf{T}}\mathcal{Z}(x',0)\big)_{2\times2,\mathrm{sym}}$ and conclude the proof of the Lemma. It follows from (2.2) and the definition of $\mathcal{Z}^h$ in (2.6) that:

$$\nabla V^h=h\int_{-1/2}^{1/2}(\bar{R}^h)^{\mathsf{T}}\mathcal{Z}^h_{3\times2}\,\mathrm{d}x_3+S^h\big(\nabla y_0+\mathcal{O}(h^2)\big)$$

In virtue of (2.6), (2.10) and (2.4), we obtain convergence, weakly in $\mathbb{E}$:

$$\frac{1}{h}\big((\nabla y_0)^{\mathsf{T}}\nabla V^h\big)_{\mathrm{sym}}\rightharpoonup\Big((\nabla y_0)^{\mathsf{T}}\bar{R}^{\mathsf{T}}\mathcal{Z}(x',0)_{3\times2}\Big)_{\mathrm{sym}}+\int_{-1/2}^{1/2}\frac{x_3^{n+1}}{(n+1)!}\,\mathrm{d}x_3\big((\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_{n+1}\big)_{\mathrm{sym}},$$

which allows to conclude, by Lemma 2.4 (ii):

$$\big(B_0^{\mathsf{T}}\bar{R}^{\mathsf{T}}\mathcal{Z}(x',0)\big)_{2\times2,\mathrm{sym}}=\mathbb{S}-\delta_{n+1}\big((\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_{n+1}\big)_{\mathrm{sym}}.\tag{2.11}$$

This ends the proof of Lemma, in virtue of (2.8), (2.10), (2.11) and recalling definitions (1.4). ■

3. Relations between (i) and (iii) of Theorem 1.1 and a proof of Theorem 1.2 (iii)

In this section we show the relation between the defining quantities appearing in conditions (i) and (iii) of Theorem 1.1. Equivalence of (i) and (iii) at $n=2$ has been shown in [30], building on the previous results in [6, 29], while the proof of the general case will be carried out by induction on $n\geq2$. We start by introducing some notation that allows for a systematic approach.

Define the smooth matrix fields $\{\Gamma_a:\bar{\Omega}^1\to\mathbb{R}^{3\times3}\}_{a=1\dots3}$ by setting their coefficients $(\Gamma_a)_{bc}=\Gamma_{ac}^b$ to be the usual Christoffel symbols $\Gamma_{ac}^b=\frac{1}{2}\sum_{m=1}^3G^{bm}\big(\partial_aG_{mc}+\partial_cG_{ma}-\partial_mG_{ac}\big)$ of the metric G.

Recall the standard notation for the coefficients of the inverse: $(G^{-1})_{ab} = G^{ab}$. Since the Levi-Civita connection is torsion-free, it follows that $\Gamma_a e_b = \Gamma_b e_a$ for all $a, b = 1\ldots3$ and also, the Riemann curvature tensor is expressed by, for all $c, d = 1\ldots3$:

$$\big[R^a_{b,cd}\big]_{a,b=1\ldots3} = \big(\partial_c\Gamma_d + \Gamma_c\Gamma_d\big) - \big(\partial_d\Gamma_c + \Gamma_d\Gamma_c\big),$$
$$\big[R_{ab,cd}\big]_{a,b=1\ldots3} = G\big[R^a_{b,cd}\big]_{a,b=1\ldots3}\ .$$

Given a matrix field $F : \Omega^1 \to \mathbb{R}^{3\times3}$, we define: $\nabla_a F = \partial_a F + \Gamma_a F$ for each $a = 1\ldots3$, so that $(\nabla_a F)e_b$ coincides with the usual covariant derivative of vector fields: $\nabla_a(Fe_b)$. It also follows that:

$$\nabla_c\nabla_d F - \nabla_d\nabla_c F = \big[R^a_{b,cd}\big]_{a,b=1\ldots3}F \qquad \text{and} \qquad \nabla_a(F_1F_2) = (\nabla_a F_1)F_2 + F_1\partial_a F_2.$$

We now partially reprove the mentioned statements at $n = 1, 2$ for completeness of presentation.

Lemma 3.1. *Assume that there exist smooth fields $y_0, \vec b_1 : \bar\omega \to \mathbb{R}^3$ such that the matrix field: $B_0 = \big[\partial_1 y_0, \partial_2 y_0, \vec b_1\big]$ has positive determinant and such that:*

$$B_0^{\mathsf T}B_0 = G(x',0) \ \ \textit{and} \ \ \big((\nabla y_0)^{\mathsf T}\nabla\vec b_1\big)_{\rm sym} = \frac12\partial_3 G(x',0)_{2\times2} \qquad \textit{for all} \ \ x' \in \omega.$$

Then:

(i) $\partial_i B_0 = B_0\Gamma_i$ *for all* $i = 1\ldots2$, *and in particular:* $\partial_i\vec b_1 = B_0\Gamma_3 e_i$.

(ii) $R^a_{b,ij}(x',0) = R_{ab,ij}(x',0) = 0$ *for all* $x' \in \omega$ *and all* $a, b = 1\ldots3$, $i, j = 1\ldots2$.

(iii) *There exists a unique smooth field $\vec b_2 : \bar\omega \to \mathbb{R}^3$ such that defining the matrix field $B_1 = \big[\partial_1\vec b_1,\ \partial_2\vec b_1,\ \vec b_2\big]$, there holds: $\big(B_0^{\mathsf T}B_1\big)_{\rm sym} = \frac12\partial_3 G(x',0)$ for all $x' \in \omega$. Moreover:*

$$B_1 = B_0\Gamma_3 \qquad \textit{and} \qquad \partial_i\vec b_2 = B_0\nabla_i\Gamma_3 e_3 \qquad \textit{for all} \ \ i = 1\ldots2.$$

Proof. **1.** One easily calculates, by a repeated use of the assumed identities, that: $\langle\partial_i y_0, \partial_j\vec b_1\rangle = \partial_j G_{i3} - \langle\partial_{ij}y_0, \vec b_1\rangle = \partial_j G_{i3} - \partial_i G_{j3} + \langle\partial_j y_0, \partial_i\vec b_1\rangle$ and thus: $\partial_3 G_{ij} = \langle\partial_i y_0, \partial_j\vec b_1\rangle + \langle\partial_j y_0, \partial_i\vec b_1\rangle = \partial_j G_{i3} - \partial_i G_{j3} + 2\langle\partial_j y_0, \partial_i\vec b_1\rangle$, for all $i, j = 1\ldots2$, where all the identities are taken on $\omega\times\{0\}$. Thus:

$$\langle\partial_j y_0, \partial_i\vec b_1\rangle = \frac12\big(\partial_3 G_{ij} + \partial_i G_{j3} - \partial_j G_{i3} = \big(G\Gamma_3\big)_{ji} \qquad \text{for all} \ \ i, j = 1\ldots2. \tag{3.1}$$

Secondly: $\langle\partial_j y_0, \partial_{ik}y_0\rangle = \partial_i G_{jk} - \langle\partial_k y_0, \partial_{ij}y_0\rangle = \partial_i G_{jk} - \partial_j G_{ik} + \langle\partial_i y_0, \partial_{jk}y_0\rangle = \partial_i G_{jk} - \partial_j G_{ik} + \partial_k G_{ij} - \langle\partial_j y_0, \partial_{ik}y_0\rangle$, which results in:

$$\langle\partial_j y_0, \partial_{ik}y_0\rangle = \frac12\big(\partial_i G_{jk} + \partial_k G_{ij} - \partial_j G_{ik}\big) = \big(G\Gamma_i\big)_{jk} \qquad \text{for all} \ \ i, j, k = 1\ldots2.$$

Thirdly, from (3.1) we obtain:

$$\langle\vec b_1, \partial_{ik}y_0\rangle = \partial_i G_{k3} - \langle\partial_i\vec b_1, \partial_k y_0\rangle = \frac12\big(\partial_i G_{k3} + \partial_k G_{i3} - \partial_3 G_{ik}\big) = \big(G\Gamma_i\big)_{3k} \qquad \text{for all} \ \ i, k = 1\ldots2.$$

Finally: $\langle\vec b_1, \partial_i\vec b_1\rangle = \frac12\partial_i G_{33} = \big(G\Gamma_i\big)_{33}$, so that the last two identities yield:

$$B_0^{\mathsf T}\partial_i B_0 = G\Gamma_i \qquad \text{for all} \ \ i = 1\ldots2, \qquad \text{on} \ \ \omega\times\{0\}.$$

This proves (i) and further: $\partial_i\vec b_1 = B_0\Gamma_i e_3 = B_0\Gamma_3 e_i$, as claimed.

2. Using (i) we compute:

$$\begin{aligned} 0 &= \partial_{ij}B_0 - \partial_{ji}B_0 = \partial_i\big(B_0\Gamma_j\big) - \partial_j\big(B_0\Gamma_i\big) = B_0\Gamma_i\Gamma_j + B_0\partial_i\Gamma_j - \big(B_0\Gamma_j\Gamma_i + B_0\partial_j\Gamma_i\big) \\ &= -B_0\big[R^k_{s,ij}(\cdot,0)\big]_{k,s=1\ldots3} \qquad \text{for all} \ \ i, j = 1\ldots2. \end{aligned}$$

which implies (ii). For (iii), uniqueness of $\vec{b}_2$ is obvious, while $\vec{b}_2 = B_0\Gamma_3 e_3$ follows from the requested defining identity, in view of (3.1). The covariant derivative formula is a consequence of (i). ∎

Lemma 3.2. *Assume that there exist smooth fields $y_0, \vec{b}_1, \vec{b}_2 : \bar{\omega} \to \mathbb{R}^3$ such that the matrix field: $B_0 = \big[\partial_1 y_0,\ \partial_2 y_0,\ \vec{b}_1\big]$ has positive determinant and that together with $B_1 = \big[\partial_1\vec{b}_1,\ \partial_2\vec{b}_1,\ \vec{b}_2\big]$ it satisfies:*

$$\begin{aligned} & B_0^{\mathsf{T}} B_0 = G(x',0) \ \ \textit{and} \ \ \big(B_0^{\mathsf{T}} B_1\big)_{\mathrm{sym}} = \frac{1}{2}\partial_3 G(x',0) \\ & \big((\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_2\big)_{\mathrm{sym}} + (\nabla\vec{b}_1)^{\mathsf{T}}\nabla\vec{b}_1 = \frac{1}{2}\partial_{33}G(x',0)_{2\times 2} \qquad \textit{for all} \ \ x'\in\omega. \end{aligned}$$

Then:

(i) *$R_{ab,cd}(x',0) = 0$ for all $x'\in\omega$ and all $a,b,c,d = 1\ldots 3$.*

(ii) *There exists a unique smooth field $\vec{b}_3 : \bar{\omega}\to\mathbb{R}^3$ such that defining the matrix field $B_2 = \big[\partial_1\vec{b}_2,\ \partial_2\vec{b}_2,\ \vec{b}_3\big]$, there holds: $\big(B_0^{\mathsf{T}}B_2\big)_{\mathrm{sym}} + B_1^{\mathsf{T}}B_1 = \frac{1}{2}\partial_{33}G(x',0)$ for all $x'\in\omega$. Moreover:*

$$B_2 = B_0\nabla_3\Gamma_3 \qquad \textit{and} \qquad \partial_i\vec{b}_3 = B_0\nabla_i\nabla_3\Gamma_3 e_3 \qquad \textit{for all} \ \ i = 1\ldots 2.$$

Proof. Observe first that for all $a,b = 1\ldots 3$ we have:

$$\begin{aligned} \langle\partial_{33}Ge_a, e_b\rangle & = \partial_3\big(\langle G\Gamma_3 e_a, e_b\rangle + \langle G\Gamma_3 e_b, e_a\rangle\big) \\ & = \langle\nabla_3\Gamma_a e_3, Ge_b\rangle + \langle\nabla_3\Gamma_b e_3, Ge_a\rangle + 2\langle G\Gamma_3 e_a, \Gamma_3 e_b\rangle. \end{aligned} \tag{3.2}$$

Consequently, and using Lemma 3.1 (iii), the last assumed condition is equivalent to:

$$\begin{aligned} 0 & = \langle B_0 e_i, \partial_j\vec{b}_2\rangle + 2\langle\partial_i\vec{b}_1, \partial_j\vec{b}_1\rangle + \langle B_0 e_j, \partial_i\vec{b}_2\rangle - \langle\partial_{33}Ge_i, e_j\rangle \\ & = \langle Ge_i, \nabla_j\Gamma_3 e_3\rangle + 2\langle G\Gamma_3 e_i, \Gamma_3 e_j\rangle + \langle Ge_j, \nabla_i\Gamma_3 e_3\rangle \\ & \qquad - \Big(\langle\nabla_3\Gamma_j e_3, Ge_i\rangle + \langle\nabla_3\Gamma_i e_3, Ge_j\rangle + 2\langle G\Gamma_3 e_j, \Gamma_3 e_j\rangle\Big) \\ & = \langle Ge_i, \big[R^a_{3j3}(\cdot,0)\big]_{a=1\ldots 3}\rangle + \langle Ge_j, \big[R^a_{3j3}(\cdot,0)\big]_{a=1\ldots 3}\rangle \\ & = 2R_{i3,j3}(\cdot,0) \quad \text{on} \ \ \omega, \quad \text{for all} \ \ i,j = 1\ldots 2. \end{aligned}$$

The above proves (i), in virtue of Lemma 3.1 (ii) that guarantees $R_{ab,ij}(\cdot,0) = 0$ for all $a,b = 1\ldots 3$, $i,j = 1\ldots 2$. To show (ii), we observe that by Lemma 3.1 and by (i):

$$B_2 e_i = \partial_i\vec{b}_2 = B_0\nabla_i\Gamma_3 e_3 = B_0\nabla_3\Gamma_i e_3 = B_0\nabla_3\Gamma_3 e_i \quad \text{for all} \ \ i,j = 1\ldots 2$$

and also, $(B_0^{\mathsf{T}}B_2)_{\mathrm{sym}} + B_1^{\mathsf{T}}B_1 - \frac{1}{2}\partial_{33}G(x',0) = (B_0^{\mathsf{T}}B_2)_{\mathrm{sym}} + \Gamma_3^{\mathsf{T}}G\Gamma_3 - \Big((G\nabla_3\Gamma_3)_{\mathrm{sym}} + \Gamma_3^{\mathsf{T}}G\Gamma_3\Big)$, in view of (3.2), so $B_2 = B_0\nabla_3\Gamma_3$ satisfies the defining relation. Finally, $\partial_i\vec{b}_3 = \partial_i\big(B_0\nabla_3\Gamma_3 e_3\big) = B_0\nabla_i\nabla_3\Gamma_3 e_3$ results from Lemma 3.1 (i). ∎

We state the following two useful observations:

Lemma 3.3. *For all $n\geq 0$ there holds:*

$$\partial_3^{(n+1)}G(x',0) = 2\Big(G\nabla_3^{(n)}\Gamma_3\Big)_{\mathrm{sym}} + \sum_{k=1}^{n}\binom{n+1}{k}(\nabla_3^{(k-1)}\Gamma_3)^{\mathsf{T}}G\nabla_3^{(n-k)}\Gamma_3 \quad \textit{for all} \ \ x'\in\omega.$$

Proof. The proof follows by induction. For $n = 0$, the statement is obviously true. Assume that it is true for some $n-1$, then:

$$
\begin{aligned}
\partial_3^{(n+1)} G(x', 0) &= \partial_3 \Big(2\big(G\nabla_3^{(n-1)}\Gamma_3\big)_{\mathrm{sym}} + \sum_{k=1}^{n-1} \binom{n}{k} \big(\nabla_3^{(k-1)}\Gamma_3\big)^{\mathsf{T}} G \nabla_3^{(n-1-k)}\Gamma_3 \Big) \\
&= G\nabla_3^{(n)}\Gamma_3 + \big(\nabla_3^{(n)}\Gamma_3\big)^{\mathsf{T}} G + \Gamma_3^{\mathsf{T}} G \nabla_3^{(n-1)}\Gamma_3 + \big(\nabla_3^{(n-1)}\Gamma_3\big)^{\mathsf{T}} G\Gamma_3 \\
&\quad + \sum_{k=1}^{n-1} \binom{n}{k} \Big(\big(\nabla_3^{(k-1)}\Gamma_3\big)^{\mathsf{T}} G\nabla_3^{(n-k)}\Gamma_3 + \big(\nabla_3^{(k)}\Gamma_3\big)^{\mathsf{T}} G\nabla_3^{(n-k-1)}\Gamma_3 \Big) \\
&= G\nabla_3^{(n)}\Gamma_3 + \big(\nabla_3^{(n)}\Gamma_3\big)^{\mathsf{T}} G + \Gamma_3^{\mathsf{T}} G \nabla_3^{(n-1)}\Gamma_3 + \big(\nabla_3^{(n-1)}\Gamma_3\big)^{\mathsf{T}} G\Gamma_3 \\
&\quad + \sum_{k=1}^{n-1} \left(\binom{n}{k} + \binom{n}{k-1} \right) \big(\nabla_3^{(k-1)}\Gamma_3\big)^{\mathsf{T}} G\nabla_3^{(n-k)}\Gamma_3 \\
&\quad - \binom{n}{0} \Gamma_3^{\mathsf{T}} G \nabla_3^{(n-1)}\Gamma_3 + \binom{n}{n-1} \big(\nabla_3^{(n-1)}\Gamma_3\big)^{\mathsf{T}} G\Gamma_3 .
\end{aligned}
$$

Collecting all the terms and recalling that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ implies the result. ∎

Lemma 3.4. *Assume that $R_{12,12}(x',0) = R_{12,13}(x',0) = R_{12,23}(x',0) = 0$ for all $x' \in \omega$ and also that $\partial_3^{(k)} R_{i3,j3}(x',0) = 0$ for all $k = 0 \dots n$, all $i,j = 1 \dots 2$ and all $x' \in \omega$. Then all the mixed partial derivatives of both $R_{ab,cd}$ and $R^a_{b,cd}$, of any order up to n, are zero on ω, for all $a,b,c,d = 1 \dots 3$.*

Proof. The proof proceeds by induction on n. For $n = 0$ the result is obviously true. Assume that it is true for some $n \geq 0$ and let the result assumption at $n+1$ hold. Then:

$$
\begin{aligned}
&\partial^{(k)} R^a_{b,cd}(x',0) = \partial^{(k)} R_{ab,cd}(x',0) = 0 \qquad \text{for all } \ k = 0 \dots n, \quad a,b,c,d = 1 \dots 3, \\
&\partial_3^{(n+1)} R_{i3,j3}(x',0) = 0 \qquad\qquad\qquad\qquad\quad\ \text{for all } \ i,j = 1 \dots 2, \quad x' \in \omega,
\end{aligned}
$$

and we need to show that any partial derivatives of order $n+1$, of the Riemann tensor's components is zero on ω. This is certainly true for partial derivatives containing ∂_i for some $i = 1 \dots 2$, so it suffices to prove the claim for $\partial_3^{(n+1)}$. Below, we consider various combinations of indices $i,j = 1 \dots 2$ and $a,b = 1 \dots 3$. Firstly:

$$
\partial_3^{(n+1)} R^a_{b,ij} = \partial_3^{(n)} \nabla_3 R^a_{b,ij} = \partial_3^{(n)} \big(-\nabla_i R^a_{b,j3} - \nabla_j R^a_{b,31} \big) = \partial_3^{(n)} \big(-\partial_i R^a_{b,j3} - \partial_j R^a_{b,31} \big) = 0, \tag{3.3}
$$

where we used the induction assumption in the first and the third equalities and the second Bianchi identity in the second one. Secondly:

$$
\partial_3^{(n+1)} R_{ab,ij} = \partial_3^{(n+1)} \big\langle \big[G_{ap}\big]_{p=1\dots3}, \big[R^p_{b,ij}\big]_{p=1\dots3} \big\rangle = \big\langle \big[G_{ap}\big]_{p=1\dots3}, \big[\partial_3^{(n+1)} R^p_{b,ij}\big]_{p=1\dots3} \big\rangle = 0, \tag{3.4}
$$

where we used the induction assumption and (3.3) in the last equality. Thirdly:

$$
\partial_3^{(n+1)} R^a_{b,i3} = \partial_3^{(n+1)} \big\langle \big[G^{ap}\big]_{p=1\dots3}, \big[R_{pb,i3}\big]_{p=1\dots3} \big\rangle = \big\langle \big[G^{ap}\big]_{p=1\dots3}, \big[\partial_3^{(n+1)} R_{pb,i3}\big]_{p=1\dots3} \big\rangle = 0, \tag{3.5}
$$

by using (3.4) and the result assumption at $n+1$, in the last equality. Finally: $\partial_3^{(n+1)} R_{ab,cd} = 0$ by (3.4) and the result assumption. ∎

The following is the main result of this section:

Lemma 3.5. *Fix $n \geq 2$. Assume that there exist smooth $y_0, \{\vec b_k\}_{k=1}^n : \bar\omega \to \mathbb{R}^3$ such that the matrix fields: $B_0 = \big[\partial_1 y_0,\ \partial_2 y_0,\ \vec b_1\big]$ with positive determinant and $\{B_k = \big[\partial_1 \vec b_k,\ \partial_2 \vec b_k,\ \vec b_{k+1}\big]\}_{k=1}^{n-1}$, satisfy:*

$$\sum_{k=0}^{m} \binom{m}{k} B_k^{\mathsf{T}} B_{m-k} - \partial_3^{(m)} G(x',0) = 0 \qquad \textit{for all } \ m = 0\dots n-1,$$
$$2\big((\nabla y_0)^{\mathsf{T}}\nabla \vec b_n\big)_{\mathrm{sym}} + \sum_{k=1}^{n-1}\binom{n}{k}(\nabla \vec b_k)^{\mathsf{T}}\nabla \vec b_{n-k} = \partial_3^{(n)} G(x',0)_{2\times 2} \qquad \textit{for all } \ x' \in \omega.$$

Then:

- (i) *The condition in Theorem 1.1 (i) holds.*
- (ii) *There exists a unique smooth field $\vec b_{n+1} : \bar\omega \to \mathbb{R}^3$ such that defining the matrix field $B_n = \big[\partial_1 \vec b_n,\ \partial_2 \vec b_n,\ \vec b_{n+1}\big]$, there holds: $\sum_{k=0}^{n}\binom{n}{k} B_k^{\mathsf{T}} B_{n-k} = \partial_3^{(n)} G(x',0)$ for all $x' \in \omega$. Moreover:*

$$B_n = B_0 \nabla_3^{(n-1)}\Gamma_3 \qquad \textit{and} \qquad \partial_i \vec b_{n+1} = B_0 \nabla_i \nabla_3^{(n-1)}\Gamma_3 e_3 \qquad \textit{for all } \ i = 1\dots 2.$$

Proof. **1.** The proof proceeds by induction. The statement at $n = 2$ has been shown in Lemma 3.2. We now assume it to be true for some $n \geq 2$. By Lemma 3.4, we get:

(3.6) All mixed partial derivatives up to order $n-2$, of all components of the Riemann curvature tensor, are 0 at $\omega \times \{0\}$.

Since $B_k = \big[\partial_1 \vec b_n,\ \partial_2 \vec b_n,\ \vec b_{n+1}\big]$ with $\vec b_{n+1}$ as in (ii), and recalling Lemma 3.3, we obtain for all $x' \in \omega$:

$$\begin{aligned}
&2\big((\nabla y_0)^{\mathsf{T}}\nabla \vec b_{n+1}\big)_{\mathrm{sym}} + \sum_{k=1}^{n}\binom{n+1}{k}(\nabla \vec b_k)^{\mathsf{T}}\nabla \vec b_{n+1-k} - \partial_3^{(n+1)} G(x',0)_{2\times 2} \\
&\quad = 2\big((\nabla y_0)^{\mathsf{T}}\nabla \vec b_{n+1}\big)_{\mathrm{sym}} + \sum_{k=1}^{n}\Big(B_k^{\mathsf{T}} B_{n+1-k}\Big)_{2\times 2} \\
&\qquad\qquad - \Big(2\big(G\nabla_3^{(n)}\Gamma_3\big)_{\mathrm{sym}} + \sum_{k=1}^{n}\binom{n+1}{k}\big(\nabla_3^{(k-1)}\Gamma_3\big)^{\mathsf{T}} G \nabla_3^{(n-k)}\Gamma_3\Big)_{2\times 2} \\
&\quad = 2\Big((\nabla y_0)^{\mathsf{T}}\nabla \vec b_{n+1} - G\nabla_3^{(n)}\Gamma_3\Big)_{\mathrm{sym}} = 2\Big[\langle Ge_i, \nabla_j \nabla_3^{(n-1)}\Gamma_3 e_3 - \nabla_3^{(n)}\Gamma_3 e_j\rangle\Big]_{i,j=1\dots 2,\ \mathrm{sym}} \\
&\quad = 2\Big[\langle Ge_i, \nabla_j \nabla_3^{(n-1)}\Gamma_3 e_3 - \nabla_3^{(n)}\Gamma_j e_3\rangle\Big]_{i,j=1\dots 2,\ \mathrm{sym}}.
\end{aligned} \tag{3.7}$$

By (3.6) we can consecutively swap the order of all the covariant derivatives on $\omega \times \{0\}$ in:

$$\nabla_j \nabla_3^{(n-1)}\Gamma_3 = \nabla_3 \nabla_j \nabla_3^{(n-2)}\Gamma_3 = \nabla_3^{(2)}\nabla_j\nabla_3^{(n-3)}\Gamma_3 = \nabla_3^{(3)}\nabla_j \nabla_3^{(n-4)}\Gamma_3 = (\dots) = \nabla_3^{(n-1)}\nabla_j \Gamma_3,$$

so that:

$$\nabla_j \nabla_3^{(n-1)}\Gamma_3 - \nabla_3^{(n)}\Gamma_j = \nabla_3^{(n-1)}\big(\nabla_j\Gamma_3 - \nabla_3\Gamma_j\big) = \nabla_3^{(n-1)}\big[R^a_{b,j3}(x',0)\big]_{a,b=1\dots 3}. \tag{3.8}$$

In conclusion, using (3.6) again, the formula in (3.7) becomes:

$$\begin{aligned}
&2\big((\nabla y_0)^{\mathsf{T}}\nabla\vec b_{n+1}\big)_{\rm sym} + \sum_{k=1}^{n}\binom{n+1}{k}(\nabla\vec b_k)^{\mathsf{T}}\nabla\vec b_{n+1-k} - \partial_3^{(n+1)}G(x',0)_{2\times 2}\\
&\qquad = \Big[\langle Ge_i, \big[\partial_3^{(n-1)}R^a_{3,j3}(x',0)\big]_{a=1\ldots 3}\rangle + \langle Ge_j, \big[\partial_3^{(n-1)}R^a_{3,i3}(x',0)\big]_{a=1\ldots 3}\rangle\Big]_{i,j=1\ldots 2}\\
&\qquad = 2\Big[\partial_3^{(n-1)}R_{i3,j3}(x',0)\Big]_{i,j=1\ldots 2} \qquad \text{for all } \ x'\in\omega,
\end{aligned}\tag{3.9}$$

proving (i) in view of the second assumption at $n+1$.

2. For (ii), observe that B_{n+1} is indeed uniquely defined, by choosing $\vec b_{n+2} = B_{n+1}e_3$ such that:

$$\sum_{k=0}^{n+1}\binom{n+1}{k}B_k^{\mathsf{T}}B_{n+1-k} = \partial_3^{(n+1)}G(x',0) \quad \text{ for all } \ x'\in\omega,$$

since the principal 2×2 minors of both sides in the above formula coincide by assumption. Further, by (3.8) and the already established (i) at $n+1$, we get:

$$\begin{aligned}
B_{n+1}e_i = \partial_i\vec b_{n+1} &= \partial_i\Big(B_0\nabla_3^{(n-1)}\Gamma_3 e_3\Big) = B_0\nabla_i\nabla_3^{(n-1)}\Gamma_3 e_3\\
&= B_0\nabla_3^{(n)}\Gamma_i e_3 + \nabla_3^{(n-1)}\big[R^a_{3,i3}(x',0)\big]_{a=1\ldots 3} = B_0\nabla_3^{(n)}\Gamma_i e_3\\
&= B_0\nabla_3^{(n)}\Gamma_3 e_i \qquad \text{for all } \ i=1\ldots 2 \quad \text{and all } \ x'\in\omega.
\end{aligned}$$

Hence, there must be $\vec b_{n+1} = B_0\nabla_3^{(n)}\Gamma_3$, as claimed. This ends the proof of the Lemma. ■

We note that the argument in the proof above leading to (3.9), automatically gives:

Corollary 3.6. *For any $n\geq 1$, condition (iii) in Theorem 1.1 implies the formula (1.13).*

4. The end of proof of Theorem 1.2 and a proof of Theorem 1.1

The following statement concludes the proof of Theorem 1.2, assuming (iii) of Theorem 1.1:

Lemma 4.1. *In the context of Lemma 2.4, there holds: $\liminf_{h\to 0}\frac{1}{h^{2(n+1)}}\mathcal{E}^h(u^n)\geq \mathcal{I}_{2(n+1)}(V)$.*

Proof. By Lemma 2.5 and Corollary 3.6, we get:

$$\begin{aligned}
\liminf_{h\to 0}\frac{1}{h^{2(n+1)}}\mathcal{E}^h(u^h) \geq \frac12\int_{\Omega^1}\mathcal{Q}_2\Big(x', \mathbb{S} - \delta_{n+1}\big((\nabla y_0)^{\mathsf{T}}\nabla\vec b_{n+1}\big)_{\rm sym} + x_3\big((\nabla y_0)^{\mathsf{T}}\nabla\vec p + (\nabla V)^{\mathsf{T}}\nabla\vec b_1\big)\\
+ \frac{x_3^{n+1}}{(n+1)!}\big[\partial_3^{(n-1)}R_{i3,j3}(x',0)\big]_{i,j=1\ldots 2}\Big)\,\mathrm{d}x.
\end{aligned}$$

Denoting the x'-dependent tensor terms at different powers of x_3 in the integrand above by I, II and III, and recalling the definition of δ_{n+1} in (2.5), the right hand side becomes:

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega^1} \mathcal{Q}_2\big(x',\ I + x_3 II + x_3^{n+1} III\big)\,\mathrm{d}x \\
&\quad = \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(x',\ I + \big(\int_{-1/2}^{1/2} x_3^{n+1}\,\mathrm{d}x_3\big) III\Big) + \frac{1}{24}\mathcal{Q}_2\Big(x',\ II + 12\big(\int_{-1/2}^{1/2} x_3^{n+2}\,\mathrm{d}x_3\big) III\Big) \\
&\qquad + \frac{1}{2}\Big(\big(\int_{-1/2}^{1/2} x_3^{2n+2}\,\mathrm{d}x_3\big) - \big(\int_{-1/2}^{1/2} x_3^{n+1}\,\mathrm{d}x_3\big)^2 - 12\big(\int_{-1/2}^{1/2} x_3^{n+2}\,\mathrm{d}x_3\big)^2\Big)\mathcal{Q}_2\big(x',\ III\big)\,\mathrm{d}x' \\
&\quad = \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(x',\ \mathbb{S} - \delta_{n+1}\big((\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_{n+1}\big)_{\mathrm{sym}} + \delta_{n+1}\big[\partial_3^{(n-1)} R_{i3,j3}(x',0)\big]_{i,j=1\ldots 2}\Big)\,\mathrm{d}x' \\
&\qquad + \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(x',\ (\nabla y_0)^{\mathsf{T}}\nabla\vec{p} + (\nabla V)^{\mathsf{T}}\nabla\vec{b}_1 + \alpha_n\big[\partial_3^{(n-1)} R_{i3,j3}(x',0)\big]_{i,j=1\ldots 2}\Big)\,\mathrm{d}x' \\
&\qquad + \gamma_n \int_\omega \mathcal{Q}_2\Big(x',\ \big[\partial_3^{(n-1)} R_{i3,j3}(x',0)\big]_{i,j=1\ldots 2}\Big)\,\mathrm{d}x',
\end{aligned} \tag{4.1}
$$

where by a direct calculation one easily checks that the numerical coefficients α_n and γ_n have the form (1.12). Further, since $\mathbb{S} - \delta_{n+1}\big((\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_{n+1}\big)_{\mathrm{sym}} \in \mathcal{S}_{y_0}$, the first term in the right hand side above is bounded from below by:

$$
\begin{aligned}
\frac{1}{2}\mathrm{dist}^2_{\mathcal{Q}_2}\Big(\delta_{n+1}\big[\partial_3^{(n-1)} R_{i3,j3}(x',0)\big]_{i,j=1\ldots 2},\ \mathcal{S}_{y_0}\Big) &= \frac{\delta_{n+1}^2}{2}\mathrm{dist}^2_{\mathcal{Q}_2}\Big(\big[\partial_3^{(n-1)} R_{i3,j3}(x',0)\big]_{i,j=1\ldots 2},\ \mathcal{S}_{y_0}\Big) \\
&= \frac{\delta_{n+1}^2}{2}\Big\|\mathbb{P}_{\mathcal{S}_{y_0}^\perp}\big(\big[\partial_3^{(n-1)} R_{i3,j3}(x',0)\big]_{i,j=1\ldots 2}\big)\Big\|^2_{\mathcal{Q}_2}.
\end{aligned}
$$

Decomposing the third term into:

$$
\gamma_n\Big\|\mathbb{P}_{\mathcal{S}_{y_0}^\perp}\big(\big[\partial_3^{(n-1)} R_{i3,j3}(x',0)\big]_{i,j=1\ldots 2}\big)\Big\|^2_{\mathcal{Q}_2} + \gamma_n\Big\|\mathbb{P}_{\mathcal{S}_{y_0}}\big(\big[\partial_3^{(n-1)} R_{i3,j3}(x',0)\big]_{i,j=1\ldots 2}\big)\Big\|^2_{\mathcal{Q}_2},
$$

the claim follows by checking that: $\dfrac{\delta_{n+1}^2}{2} + \gamma_n = \beta_n$ in (1.12). ■

We are now ready to give:

A proof of Theorem 1.1. The proof is carried out by induction on $n \geq 2$. When $n = 2$, then (i) is equivalent with (iii) by facts recalled in the preliminary discussion in section 1.2. Condition (iii) implies (ii) by Lemma 2.1, whereas (ii) implies (i) again in view of (1.6).

Assume now the equivalence of the three conditions at some $n \geq 2$. We want to show the equivalence at $n+1$. Condition (i) implies (iii) by Corollary 3.6. Condition (iii) implies (ii) by Lemma 2.1. Finally, assuming (ii) at $n+1$ allows to write:

$$
\begin{aligned}
0 = \lim_{h\to 0}\frac{1}{h^{2(n+1)}}\inf\mathcal{E}^h &= \lim_{h\to 0}\frac{1}{h^{2(n+1)}}\mathcal{E}^h(u^h) \geq \mathcal{I}_{2(n+1)}(V) \\
&\geq \gamma_n \cdot \Big\|\big[\partial_3^{(n-1)} R_{i3,j3}(x',0)\big]_{i,j=1\ldots 2}\Big\|^2_{\mathcal{Q}_2},
\end{aligned}
$$

for some infimizing sequence $\{u^h \in W^{1,2}(\Omega^h,\mathbb{R}^3)\}_{h\to 0}$ and a resulting V from Theorem 1.2. This establishes (i) at $n+1$, in view of the inductive assumption. ■

For completeness, we state the following auxiliary observations:

Lemma 4.2. *In the context of Theorem 1.2, we have:*

(i) *The bending term* $(\nabla y_0)^\mathsf{T}\nabla\vec p + (\nabla V)^\mathsf{T}\nabla\vec b_1$ *is symmetric and it equals:*

$$\Big[\Big\langle \Gamma_j e_i, \left[\begin{array}{c}(\nabla V)^\mathsf{T}\vec b_1\\ 0\end{array}\right]\Big\rangle - \langle\partial_{ij}V, \vec b_1\rangle\Big]_{i,j=1\dots 2}.$$

(ii) *Under any of the equivalent conditions in Theorem 1.1 at* $n+1$*, we have:*

$$\mathrm{Ker}\ \mathcal{I}_{2(n+1)} = \big\{Sy_0 + c;\ S\in so(3),\ c\in\mathbb{R}^3\big\},$$

and the following coercivity estimate holds:

$$\mathrm{dist}^2_{W^{2,2}(\omega,\mathbb{R}^3)}\big(V,\ \mathrm{Ker}\ \mathcal{I}_{2(n+1)}\big) \leq C\mathcal{I}_{2(n+1)}(V) \qquad \textit{for all } V\in\mathcal{V}_{y_{y_0}}$$

with a constant $C>0$ *that depends on* G,ω *and* W *but is independent of* V*.*

Proof. The symmetry of the bending term in (i) follows from:

$$\begin{aligned}\langle\partial_i y_0, \partial_j\vec p\rangle + \langle\partial_i V, \partial_j\vec b_1\rangle &= \partial_j\big(\langle\partial_i y_0, \vec p\rangle + \langle\partial_i V, \vec b_1\rangle\big) - \big(\langle\partial_{ij}y_0, \vec p\rangle + \langle\partial_{ij}V, \vec b_1\rangle\big)\\ &= -\langle\partial_{ij}y_0, \vec p\rangle - \langle\partial_{ij}V, \vec b_1\rangle \qquad \text{for all } i,j = 1\dots 2.\end{aligned}$$

The coercivity statement in (ii) has been proved in [30, Theorems 8.2, 8.3]. ∎

5. A proof of Theorem 1.3

In this section, we prove the upper bound result of Theorem 1.3. In view of the already established Theorem 1.1, it suffices to show:

Lemma 5.1. *Fix* $n\geq 2$ *and assume condition (iii) in Theorem 1.1. Let* $V\in\mathcal{V}_{y_0}$ *be a first order isometry displacement as in (1.10). Then, there exists a sequence* $\{u^h\in W^{1,2}(\Omega^h,\mathbb{R}^3)\}_{h\to 0}$ *of deformations satisfying (1.14), and such that:* $\displaystyle\lim_{h\to 0}\frac{1}{h^{2(n+1)}}\mathcal{E}^h(u^h) = \mathcal{I}_{2(n+1)}(V)$*.*

Proof. **1.** Denote $Y(x',x_3) = y_0 + \sum_{k=1}^{n+1}\frac{x_3^k}{k!}\vec b_k$ and define:

$$\begin{aligned}u^h(x',x_3) &= Y(x',x_3) + h^n v^h(x') + h^{n+1}w^h(x') + h^n x_3\vec p^h(x') + h^{n+1}x_3\vec q^h(x')\\ &\quad + \frac{x_3^{n+2}}{(n+2)!}\vec k_0(x') + h^n\frac{x_3^2}{2}\vec r^h(x') \qquad \text{for all } (x',x_3)\in\Omega^h.\end{aligned}\tag{5.1}$$

We now introduce terms in the above expansion. For a fixed small $\varepsilon>0$, the truncated sequence $\{v^h\in W^{2,\infty}(\omega,\mathbb{R}^3)\}_{h\to 0}$ is chosen according to the standard construction in [8] (see also references therein), in a way that:

$$\begin{aligned}&v^h\to V \quad \text{strongly in } W^{2,2}(\omega,\mathbb{R}^3) \quad \text{as } h\to 0,\\ &h^n\|v^h\|_{W^{2,\infty}(\omega,\mathbb{R}^3)}\leq\varepsilon \quad \text{and} \quad \lim_{h\to 0}\frac{1}{h^{2n}}\big|\{x'\in\omega;\ v^h(x')\neq V(x')\}\big| = 0.\end{aligned}\tag{5.2}$$

The sequence $\{\vec p^h\in W^{1,\infty}(\omega,\mathbb{R}^3)\}_{h\to 0}$ is defined by:

$$B_0^\mathsf{T}\vec p^h = \left[\begin{array}{c}-(\nabla v^h)^\mathsf{T}\vec b_1\\ 0\end{array}\right] \qquad \text{so that} \qquad \Big(B_0^\mathsf{T}\big[\nabla v^h,\ \vec p^h\big]\Big)_{\mathrm{sym}} = \Big((\nabla y_0)^\mathsf{T}\nabla v^h\Big)^*_{\mathrm{sym}}.\tag{5.3}$$

The sequence $\{w^h \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^3)\}_{h\to 0}$ is such that, recalling (2.5):

$$
\begin{split}
& \Big((\nabla y_0)^{\mathsf T}\nabla w^h\Big)_{\mathrm{sym}} \to -\delta_{n+1}\mathbb{P}_{\mathcal{S}_{y_0}}\Big(\big[\partial_3^{(n-1)}R_{i3j3}\big]_{i,j=1\ldots 2}\Big) \\
& \qquad\qquad\qquad\qquad \text{strongly in } \mathbb{E} = L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \quad \text{as } h\to 0, \\
& \lim_{h\to 0} h^{1/2}\|w^h\|_{W^{2,\infty}(\omega,\mathbb{R}^3)} = 0.
\end{split}
\tag{5.4}
$$

Finally, $\vec k_0 \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^3)$ and $\{\vec q^{\,h} \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^3)\}_{h\to 0}$, $\{\tilde r^h \in L^\infty(\omega, \mathbb{R}^3)\}_{h\to 0}$ are defined by:

$$
\begin{split}
2B_0^{\mathsf T}\vec k_0 = & \, c\Big(x', 2\big((\nabla y_0)^{\mathsf T}\nabla \vec b_{n+1}\big)_{\mathrm{sym}} + \sum_{k=1}^{n}\binom{n+1}{k}(\nabla\vec b_k)^{\mathsf T}\nabla\vec b_{n+1-k} - \partial_3^{(n+1)}G(x',0)_{2\times 2}\Big) \\
& - \begin{bmatrix} 2\displaystyle\sum_{k=0}^{n}\binom{n+1}{k}(\nabla\vec b_{n+1-k})^{\mathsf T}\nabla\vec b_{k+1} \\ \displaystyle\sum_{k=1}^{n}\binom{n+1}{k}(\nabla\vec b_{k+1})^{\mathsf T}\nabla\vec b_{n+2-k}\end{bmatrix} + \begin{bmatrix} 2\partial_3^{(n+1)}G(x',0)_{31,32} \\ \partial_3^{(n+1)}G(x',0)_{33}\end{bmatrix}, \\
B_0^{\mathsf T}\vec q^{\,h} = & \, c\Big(x', \big((\nabla y_0)^{\mathsf T}\nabla w^h\big)_{\mathrm{sym}}\Big) - \begin{bmatrix}(\nabla w^h)^{\mathsf T}\vec b_1 \\ 0\end{bmatrix}, \\
B_0^{\mathsf T}\tilde r^h = & \, c\Big(x', (\nabla y_0)^{\mathsf T}\nabla\vec p^{\,h} + (\nabla v^h)^{\mathsf T}\nabla\vec b_1\Big) - \begin{bmatrix}(\nabla v^h)^{\mathsf T}\vec b_2 \\ \langle\vec p^{\,h}, \vec b_2\rangle\end{bmatrix}.
\end{split}
\tag{5.5}
$$

Further, we choose $\{\vec r^{\,h} \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^3)\}_{h\to 0}$ to satisfy, in view of (5.2):

$$
\lim_{h\to 0}\|\vec r^{\,h} - \tilde r^h\|_{L^2(\omega,\mathbb{R}^3)} = 0 \quad \text{and} \quad \lim_{h\to 0} h^{1/2}\|\vec r^{\,h}\|_{W^{1,\infty}(\omega,\mathbb{R}^3)} = 0.
\tag{5.6}
$$

2. By (5.4) and (5.6) we easily deduce (1.14). Compute now, for all rescaled variables $(x', x_3) \in \Omega^1$:

$$
\begin{split}
\nabla u^h(x', hx_3) = & \, h^n\big[\nabla v^h, \ \vec p^{\,h}\big] + \sum_{k=0}^{n}\frac{h^k x_3^k}{k!}B_k + \frac{h^{n+1}x_3^{n+1}}{(n+1)!}\big[\partial_1\vec b_{n+1}, \ \partial_2\vec b_{n+1}, \ \vec k_0\big] \\
& + h^{n+1}x_3\big[\nabla\vec p^{\,h}, \ \vec r^{\,h}\big] + h^{n+1}\big[\nabla w^h, \ \vec q^{\,h}\big] + \mathcal{O}(h^{n+2})\big(1 + |\nabla\vec q^{\,h}| + |\nabla\vec r^{\,h}|\big).
\end{split}
$$

Consequently, it follows that for h small enough we have:

$$
\mathrm{dist}\big((\nabla u^h)G^{-1/2}, SO(3)\big) \le C\big(|\nabla u^h - B_0| + h\big) \le C\epsilon,
$$

which justifies writing, by Taylor's expansion of W and taking $\epsilon \ll 1$:

$$
\begin{split}
W\big((\nabla u^h)G^{-1/2}\big) & = W\Big(\sqrt{Id_3 + G^{-1/2}\big((\nabla u^h)^{\mathsf T}\nabla u^h - G\big)G^{-1/2}}\Big) \\
& = W\Big(Id_3 + \frac{1}{2}G^{-1/2}\big((\nabla u^h)^{\mathsf T}\nabla u^h - G\big)G^{-1/2} + \mathcal{O}\big(|(\nabla u^h)^{\mathsf T}\nabla u^h - G|^2\big)\Big) \\
& = W\Big(Id_3 + \frac{1}{2}G(x',0)^{-1/2}\big((\nabla u^h)^{\mathsf T}\nabla u^h - G\big)G(x',0)^{-1/2} \\
& \qquad\qquad + \mathcal{O}\big(h|(\nabla u^h)^{\mathsf T}\nabla u^h - G|\big) + \mathcal{O}\big(|(\nabla u^h)^{\mathsf T}\nabla u^h - G|^2\big)\Big) \\
& = \frac{1}{8}\mathcal{Q}_3\Big(G(x',0)^{-1/2}\big((\nabla u^h)^{\mathsf T}\nabla u^h - G\big)G(x',0)^{-1/2}\Big) \\
& \qquad\qquad + \mathcal{O}\big(h|(\nabla u^h)^{\mathsf T}\nabla u^h - G|^2\big) + \mathcal{O}\big(|(\nabla u^h)^{\mathsf T}\nabla u^h - G|^3\big).
\end{split}
$$

This implies that:

$$
\begin{aligned}
&\frac{1}{h^{2n+2}}\mathcal{E}^h(u^h)\\
&\qquad = \frac{1}{8}\int_{\Omega^1} \mathcal{Q}_3\Big(\frac{1}{h^{n+1}}G(x',0)^{-1/2}\big((\nabla u^h)^\mathsf{T}\nabla u^h(x',hx_3) - G(x',hx_3)\big)G(x',0)^{-1/2}\Big)\,\mathrm{d}x\\
&\qquad\quad + \int_{\Omega^1} \frac{1}{h^{2n+2}}\mathcal{O}\big(h|(\nabla u^h)^\mathsf{T}\nabla u^h - G|^2\big) + \frac{1}{h^{2n+2}}\mathcal{O}\big(|(\nabla u^h)^\mathsf{T}\nabla u^h - G|^3\big)\,\mathrm{d}x.
\end{aligned}\tag{5.7}
$$

We thus compute, for all $(x',x_3)\in\Omega^1$:

$$
\begin{aligned}
(\nabla u^h)^\mathsf{T}\nabla u^h(x',hx_3) - G(x',hx_3) &= 2h^n\big((\nabla y_0)^\mathsf{T}\nabla v^h\big)^*_{\mathrm{sym}}\\
&+ \frac{h^{n+1}x_3^{n+1}}{(n+1)!}\bigg(\sum_{k=1}^n\binom{n+1}{k}B_k^\mathsf{T}B_{n+1-k} + 2\big(B_0^\mathsf{T}\big[\partial_1\vec b_{n+1},\ \partial_2\vec b_{n+1},\ \vec k_0\big]\big)_{\mathrm{sym}} - \partial_3^{(n+1)}G(x',0)\bigg)\\
&+ 2h^{n+1}x_3\big(B_0^T\big[\nabla\vec p^h,\ \vec r^h\big]\big)_{\mathrm{sym}} + 2h^{n+1}x_3\big(B_1^T\big[\nabla\vec v^h,\ \vec p^h\big]\big)_{\mathrm{sym}}\\
&+ 2h^{n+1}\big(B_0^T\big[\nabla w^h,\ \vec q^h\big]\big)_{\mathrm{sym}} + \mathcal{R}_h,
\end{aligned}
$$

where:

$$
\mathcal{R}^h = o(h^{n+1}) + \mathcal{O}(h^{n+2})\big(|\nabla\vec v^h| + |\nabla^2\vec v^h|\big) + \mathcal{O}(h^{2n})|\nabla^2\vec v^h|^2 + \mathcal{O}(h^{2n+2})|\nabla^2\vec v^h|^2.
$$

3. We now estimate the two last (error) terms in the right hand side of (5.7). Observe that:

$$
\begin{aligned}
|(\nabla u^h)^\mathsf{T}\nabla u^h - G| &= \mathcal{O}(h^{n+1})\big(1 + |\nabla v^h| + |\nabla w^h| + |\vec p^h| + |\nabla\vec p^h| + |\vec q^h| + |\vec r^h|\big)\\
&\quad + \mathcal{R}^h + \mathcal{O}(h^n)|\big((\nabla y_0)^\mathsf{T}\nabla v^h\big)_{\mathrm{sym}}|\\
&= \mathcal{O}(h^{n+1})\big(1 + |\nabla v^h| + |\nabla^2 v^h| + h^{-1/2}o(1)\big) + \mathcal{O}(h^n)|\big((\nabla y_0)^\mathsf{T}\nabla v^h\big)_{\mathrm{sym}}|\\
&\quad + \mathcal{O}(h^{2n})|\nabla v^h|^2 + \mathcal{O}(h^{2n+2})|\nabla^2 v^h|^2,
\end{aligned}
$$

where we have repeatedly used (5.3), (5.4), (5.5) and (5.6), Consequently:

$$
\begin{aligned}
\frac{1}{h^{2n+2}}\mathcal{O}\big(|(\nabla u^h)^\mathsf{T}\nabla u^h - G|^3\big) &= \mathcal{O}(h^{n+1})\big(1 + |\nabla v^h|^3 + |\nabla^2 v^h|^3 + h^{-3/2}o(1)\big) + \mathcal{O}(h^{4n+4})|\nabla^2 v^h|^6\\
&\quad + \mathcal{O}(h^{4n-2})|\nabla v^h|^6 + \mathcal{O}(h^{n-2})|\big((\nabla y_0)^\mathsf{T}\nabla v^h\big)_{\mathrm{sym}}|^3.
\end{aligned}
$$

The first two terms in the right hand side above converge to 0 in $L^1(\omega^1)$ by (5.2) and (5.4). The L^1 norm of the third term is bounded by $Ch^{4n-2}\|\nabla v^h\|^4_{W^{1,2}}$ and thus converges to 0 as well. The final fourth term is bounded, in virtue of (5.2) by:

$$
\begin{aligned}
\frac{1}{h^2}\int_\omega\big|\big((\nabla y_0)^\mathsf{T}\nabla v^h\big)_{\mathrm{sym}}\big|^2\,\mathrm{d}x &\le \frac{C}{h^2}\big(\|\nabla v^h\|^2_{L^\infty} + \|\nabla^2 v^h\|^2_{L^\infty}\big)\int_{\{v^h\neq V\}}\mathrm{dist}^2(x',\{v^h = V\})\,\mathrm{d}x'\\
&\le \frac{C\epsilon^2}{h^{2n+2}}\int_{\{v^h\neq V\}}\mathrm{dist}^2(x',\{v^h = V\})\,\mathrm{d}x' \le \frac{C\epsilon^2}{h^{2n+2}}\big|\{v^h\neq V\}\big|^2\\
&\le \frac{C\epsilon^2}{h^{2n+2}}h^{4n}\cdot o(1)\to 0 \quad \text{as } h\to 0.
\end{aligned}\tag{5.8}
$$

This completes the convergence analysis of the second error term in (5.7). For the first term, we get:

$$
\begin{aligned}
\frac{1}{h^{2n+2}}\mathcal{O}\big(h|(\nabla u^h)^\mathsf{T}\nabla u^h - G|^2\big) &= \mathcal{O}(h)\big(1 + |\nabla v^h|^2 + |\nabla^2 v^h|^2 + h^{-1}o(1)\big) + \mathcal{O}(h^{2n-1})|\nabla v^h|^4\\
&\quad + \mathcal{O}(h^{2n+3})|\nabla^2 v^h|^4 + \frac{1}{h}\mathcal{O}\big(|\big((\nabla y_0)^\mathsf{T}\nabla v^h\big)_{\mathrm{sym}}|^2\big).
\end{aligned}
$$

As before, the first three terms converge to 0 in $L^1(\omega)$, whereas convergence of the last term follows by (5.8). Concluding, and since $\frac{1}{h^{n+1}}\mathcal{R}^h$ converges to 0 in $L^2(\Omega^1)$, the limit in (5.7) becomes:

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h^{2n+2}}\mathcal{E}^h(u^h)\\
&\quad=\lim_{h\to 0}\frac{1}{8}\int_{\Omega^1}\mathcal{Q}_3\Big(\frac{1}{h^{n+1}}G(x',0)^{-1/2}\big((\nabla u^h)^{\mathsf{T}}\nabla u^h(x',hx_3)-G(x',hx_3)\big)G(x',0)^{-1/2}\Big)\,\mathrm{d}x\\
&\quad=\lim_{h\to 0}\frac{1}{8}\int_{\Omega^1}\mathcal{Q}_3\Big(G(x',0)^{-1/2}K^h(x',x_3)G(x',0)^{-1/2}\Big)\,\mathrm{d}x,
\end{aligned}\tag{5.9}$$

where for a.e. $(x',x_3)\in\Omega^1$ we define:

$$\begin{aligned}
K^h(x',x_3) = {} & \frac{2}{h}\big((\nabla y_0)^{\mathsf{T}}\nabla v^h\big)^*_{\mathrm{sym}}\\
& + \frac{x_3^{n+1}}{(n+1)!}\bigg(\sum_{k=1}^{n}\binom{n+1}{k}B_k^{\mathsf{T}}B_{n+1-k} + 2\big(B_0^{\mathsf{T}}\big[\partial_1\vec b_{n+1},\ \partial_2\vec b_{n+1},\ \vec k_0\big]\big)_{\mathrm{sym}} - \partial_3^{(n+1)}G(x',0)\bigg)\\
& + 2x_3\Big(B_0^T\big[\nabla\vec p^h,\ \vec r^h\big] + \big(B_1^T\big[\nabla\vec v^h,\ \vec p^h\big]\big)_{\mathrm{sym}}\Big) + 2\big(B_0^T\big[\nabla w^h,\ \vec q^h\big]\big)_{\mathrm{sym}}.
\end{aligned}$$

In view of (5.8) and since $\|\vec r^h - \tilde r^h\|_{L^2}$ converges to 0 as requested in (5.6), the compatibility in the definition (5.5) now yields from (5.9):

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h^{2n+2}}\mathcal{E}^h(u^h)\\
&\quad=\lim_{h\to 0}\frac{1}{2}\int_{\Omega^1}\mathcal{Q}_2\bigg(x',\frac{x_3^{n+1}}{2(n+1)!}\Big(2\big((\nabla y_0)^{\mathsf{T}}\nabla\vec b_{n+1}\big)_{\mathrm{sym}} + \sum_{k=1}^{n}\binom{n+1}{k}(\nabla\vec b_k)^{\mathsf{T}}\nabla\vec b_{n+1-k}\\
&\qquad\qquad - \partial_3^{(n+1)}G(x',0)_{2\times 2}\Big)\\
&\qquad\qquad + x_3\Big((\nabla y_0)^{\mathsf{T}}\nabla\vec p^h + (\nabla v^h)^{\mathsf{T}}\nabla\vec b_1\Big) + \big((\nabla y_0)^{\mathsf{T}}\nabla w^h\big)_{\mathrm{sym}}\bigg)\,\mathrm{d}x'.
\end{aligned}\tag{5.10}$$

Decomposing the integrand above as in the proof of Lemma 4.1, formula (4.1) and recalling convergences in (5.2), (5.4), we conclude that the right hand side of (5.10) equals $\mathcal{I}_{2(n+1)}(V)$, as claimed. ■

It is worth observing that directly from Theorems 1.2 and 1.3 we obtain:

Corollary 5.2. *Each functional $\mathcal{I}_{2(n+1)}$ attains its infimum and there holds:*

$$\lim_{h\to 0}\frac{1}{h^{2(n+1)}}\inf\mathcal{E}^h = \min\ \mathcal{I}_{2(n+1)}.$$

The infima in the left hand side are taken over $W^{1,2}(\Omega,\mathbb{R}^3)$ deformations u^h, whereas the minimum in the right hand side is taken over admissible displacements $V\in\mathcal{V}_{y_0}$.

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