FLEXIBILITY OF THE ISOMETRIC IMMERSION SYSTEM IN ARBITRARY DIMENSION AND CODIMENSION AND THE ENERGY SCALING OF PRESTRAINED THIN FILMS

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ABSTRACT. We prove that, for a given $\mathcal{C}^{r,\beta}$-regular Riemann metric posed on a d-dimensional domain, every short immersion into the Euclidean space $\mathbb{R}^{d+k}$, can be uniformly approximated by exact isometric immersions of regularity $\mathcal{C}^{1,\alpha}$ for any $\alpha < \alpha_0 = \min\{\frac{r+\beta}{2}, \frac{1}{1+d(d+1)/k}\}$. Our theorem recovers several previously known results as special cases. The novelty thereof lies in providing a unified flexibility statement for arbitrary dimensions d and codimensions k, while also treating the so far uncharted range $k \in (1, \frac{d(d+1)}{2} - d + 1) \setminus \{d\}$, where no corresponding general result was previously available. Our threshold flexibility exponent α_0 agrees with that obtained in [26] for the closely related Monge-Ampère system. As an application, we prove a new estimate in the quantitative immersability of thin prestrained films, setting the scaling exponent of the infimum of non-Euclidean energies in presence of an arbitrary prestrain metric, and in the limit of the film's vanishing thickness, at $\frac{4\alpha_0}{\alpha_0+1}$.

1. INTRODUCTION

In this paper, we prove a new result in the theory of *convex integration* for the system of *isometric immersions* and, as an application, derive a new estimate in the *quantitative immersability* of thin *non-Euclidean films*. The central departing question is whether, for a given Riemannian metric g posed on a domain $\omega \subset \mathbb{R}^d$, every short immersion into $\mathbb{R}^{d+k}$, where $k \geq 1$ denotes the codimension, can be approximated by exact isometric immersions u:

$$\begin{aligned} &(\nabla u)^T \nabla u = g \quad \text{in } \ \omega \subset \mathbb{R}^d. \\ &\text{for } \ u : \omega \to \mathbb{R}^{d+k}, \quad g : \omega \to \mathbb{R}^{d\times d}_{\mathrm{sym},>}. \end{aligned} \tag{1.1}$$

The seminal works by Nash [34, 35] and Kuiper [25] established that, already in codimension $k = 1$, any short immersion in the sense that $(\nabla u)^T \nabla u < g$ on the chart, is the uniform limit of solutions to (1.1) of regularity $\mathcal{C}^1$. The analogous *flexibility result* in the Hölder classes $\mathcal{C}^{1,\alpha}$, for every $\alpha < \frac{1}{1+d(d+1)}$, was later provided by Conti, De Lellis and Szekelyhidi [12]. Since then, considerable effort has been devoted to determining the maximal Hölder exponent α_0 for which flexibility persists and it is natural to expect that α_0 should increase with k. A number of partial results support this intuition. In codimension $k = 1$, the best result currently available is due to Inauen [19] and it is set at $\alpha_0 = \frac{1}{2d-1}$. At the other extreme, Lewicka proved in [29] that for $d = 2, k = 2$, flexibility holds for every $\alpha < 1$, i.e. up to $\alpha_0 = 1$. This full flexibility phenomenon was subsequently extended in the context of the closely related Monge-Ampère system in [6], for the codimension $k = \frac{d^2-d+2}{2}$. We also mention the result by Källen [24], showing *full flexibility* for (1.1) whenever k is sufficiently large. For the intermediate case

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$k = d$, Cao and Szekelyhidi [8] established flexibility up to $\alpha_0 = \frac{1}{2+d}$. Despite these advances, no general result has so far provided a consistent unified description of α_0 for all d and k.

codimension k	exponent α_0	reference
1	0	[34, 35, 25]
$2d_*$	1	[24]
1	$1/(1+2d_*)$	[12]
1	$1/(2d-1)$	[5, 19]
d	$1/(2+d)$	[8]
$d_* - d + 1$	1	[29, 6]
$k \geq 1$	$1/(1+2d_*/k)$	this paper

TABLE 1. Development of convex integration results for system of isometric immersions (1.1) in the general dimension $d \geq 2$ and the metric $g \in \mathcal{C}^2$.

This is what we achieve in the present paper. We prove flexibility of the system (1.1), namely the density of its $\mathcal{C}^{1,\alpha}$ solutions in the set of subsolutions (the *short immersions*) for all $\alpha < \alpha_0 = \frac{1}{1+2d_*/k}$. Our theorem recovers several previously known results as special cases. When $k = 1$, it yields the exponent obtained in [12], although it does not reach the improved threshold in [19]. When $k \to \infty$, it is consistent with the full flexibility in [24], while remaining weaker than the sharp findings of [29, 6]. When $d = k$, it includes the result of [8]. The novelty of our work lies also in treating the so far uncharted range $k \in (1, \frac{d(d+1)}{2} - d + 1) \setminus \{d\}$, where no corresponding general theorem was previously available. Moreover, our exponent agrees with that obtained in [26] for the Monge-Ampère system. Our main result is the following:

Theorem 1.1. *Let $g \in \mathcal{C}^{r,\beta}(\bar{\omega}, \mathbb{R}^{d\times d}_{\mathrm{sym},>})$ be defined on the closure of an open set $\omega \subset \mathbb{R}^d$, diffeomorphic to B_1, for some $r+\beta > 0$. Then, for every immersion $u \in \mathcal{C}^1(\bar{\omega}, \mathbb{R}^{d+k})$ satisfying:*

$$(\nabla u)^T \nabla u < g \quad \textit{in } \bar{\omega},$$

for every $\epsilon > 0$ and for every regularity exponent α in:

$$0 < \alpha < \min\Big\{\frac{r+\beta}{2}, \frac{1}{1+2d_*/k}\Big\} \quad \textit{where} \quad d_* = \frac{d(d+1)}{2}, \tag{1.2}$$

there exist $\tilde{u} \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^{d+k})$, such that:

$$\|\tilde{u} - u\|_0 \leq \epsilon \quad \textit{and} \quad (\nabla \tilde{u})^T \nabla \tilde{u} = g \quad \textit{in } \bar{\omega}. \tag{1.3}$$

We point out that the first condition in (1.3) can be replaced by $\|\tilde{u} - u\|_{0,\bar{\alpha}} \leq \epsilon$ for any fixed independent $\bar{\alpha} \in (0,1)$, so that, in fact, a subsolution u to (1.1) is the limit in $\mathcal{C}^{0,\bar{\alpha}}(\bar{\omega}, \mathbb{R}^{d+k})$ of the exact isometric immersions $\tilde{u}$ of regularity $\mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^{d+k})$ with the exponent α limited according to (1.2). It is currently an important open problem how to combine the various convex integration techniques that have been developed by the authors in recent years, see Table 1. At present, these methods appear to be tailored to the specific dimension and codimension scenarios, and it remains unclear how to interpolate between the best results available at $k = 1$ and $k = \frac{d(d+1)}{2} - d + 1 = d_* - d + 1$, or whether further improvements can be achieved by a suitable synthesis of these approaches. We refer the reader to [29] for a broader overview of the field and a historical account of the quest for the optimal flexibility exponent α_0.

As an application, we additionally prove a new estimate in the *quantitative immersability* of thin *non-Euclidean films*. More precisely, given $\omega \subset \mathbb{R}^d$, define the family of "thin films":

$$\Omega^h = \big\{(x,z);\ x \in \omega,\ z \in B_h \subset \mathbb{R}^k\big\} \subset \mathbb{R}^{d+k}, \tag{1.4}$$

parametrised by their "thickness" $h \ll 1$. Consider the Riemannian metric g on Ω^1 and pose the problem of minimizing the following energy functionals, as $h \to 0$:

$$\mathcal{E}_g^h(u) = \frac{1}{h^k}\int_{\Omega^h} W\big((\nabla u)g^{-1/2}\big)\det g^{1/2}\ \mathrm{d}(x,z) \qquad \text{for all } \ u \in H^1(\Omega^h, \mathbb{R}^{d+k}). \tag{1.5}$$

In general, the energy density $W : \mathbb{R}^{(d+k)\times(d+k)} \to [0,\infty]$ is assumed to be $\mathcal{C}^2$-regular in the vicinity of $\mathrm{SO}(d{+}k)$, equal to 0 at Id_{d+k}, and frame-invariant in the sense that $W(RF) = W(F)$ for all $R \in \mathrm{SO}(d{+}k)$. When $W \sim \mathrm{dist}^2(\cdot, \mathrm{SO}(d{+}k))$, the energy in (1.5) measures the averaged pointwise deficit of a deformation u from being an orientation preserving isometric immersion of g on Ω^h. For $d = 2$, $k = 1$, it models the *elastic energy* of deformations of a prestrained film Ω^h, and various techniques have been applied to its study [30], including questions on asymptotics of the minimizing configurations as $h \to 0$, in function of the scaling exponent θ in: $\inf \mathcal{E}_g^h \sim h^\theta$. Our second main result is the following:

Theorem 1.2. *Let $g \in \mathcal{C}^{r,\beta}(\bar\Omega^1, \mathbb{R}^{(d+k)\times(d+k)}_{\mathrm{sym},>})$ be defined on the thin films Ω^h in (1.4) where $\omega \subset \mathbb{R}^d$ is diffeomorphic to B_1, and where $r + \beta > 0$. Then, there holds:*

$$\inf \mathcal{E}_g^h \le C h^{\frac{4\alpha}{\alpha+1}} \qquad \text{for all } \alpha \text{ in the range (1.2)},$$

with C depending only on ω, g, d, k, α but not on h.

We point out that for $d = 2, k = 1$, the asymptotic behaviour as $h \to 0$ of the minimizing sequences to (1.5) under the assumption that the prestrain metric g is smooth, has been fully classified in all viable scaling regimes corresponding to h^θ with $\theta \ge 2$ (see [30]). Since the current state of the art sets the flexibility exponent at $\alpha_0 = \frac{1}{3}$ in virtue of [5], resulting in $\frac{4\alpha_0}{\alpha_0+1} = 1$, Theorem 1.2 consequently implies the following uncharted scaling regime in which questions on the asymptotics and the arising types of singularities should be studied:

$$\inf \mathcal{E}_g^h \sim h^\theta \quad \text{ where } \quad 1 \le \theta < 2.$$

On the other hand, for $d = 2, k = 2$, flexibility holds all the way up to $\alpha_0 = 1$ in virtue of [29], so that $\frac{4\alpha_0}{\alpha_0+1} = 2$, which yields $\inf \mathcal{E}_g^h \le C h^\theta$ for all $\theta < 2$. Consequently, there is no "gap" between the applicability of the membrane energy and the Kirchhoff bending energy, similarly to the situation of thin rods (enjoying the same codimension $k = 2$).

1.1. **The main technical contribution.**

Our result relies on the *stage* construction, whose iteration via the Nash-Kuiper algorithm in Theorem 1.4 yields Theorem 1.1. To ease the notation, we define the *defect* relative to a matrix field g and a vector field u:

$$\mathcal{D}(g,u) \doteq g - (\nabla u)^T \nabla u.$$

Theorem 1.3. [STAGE] *Let $g \in \mathcal{C}^{r,\beta}(\bar\omega, \mathbb{R}^{d\times d}_{\mathrm{sym},>})$ be defined on the closure of an open set $\omega \subset \mathbb{R}^d$ diffeomorphic to B_1, for some regularity exponents:*

$$0 < r + \beta \le 2. \tag{1.6}$$

Then, for every $\underline{\gamma}>0$, there exists $\underline{\delta}\in(0,1)$ and $\underline{\sigma}>1$, depending only on $\underline{\gamma},\omega,g,d,k$, such that the following holds. Given any $u\in\mathcal{C}^2(\bar\omega,\mathbb{R}^{d+k})$ and any δ,λ,σ such that:

$$\delta\leq\underline{\delta},\qquad \lambda\delta^{1/2}\geq 1,\qquad \sigma\geq\underline{\sigma},\qquad \sigma^{J+3}\delta^{1/2}\leq 1, \tag{1.7$_1$}$$

$$\frac{1}{2\underline{\gamma}}\mathrm{Id}_d\leq(\nabla u)^T\nabla u\leq 2\underline{\gamma}\mathrm{Id}_d\quad \text{in } \bar\omega, \tag{1.7$_2$}$$

$$\|\mathcal{D}(g-\delta H_0,u)\|_0\leq\frac{r_0}{4}\delta\quad\text{and}\quad\|u\|_2\leq\lambda\delta^{1/2}, \tag{1.7$_3$}$$

where r_0,H_0 are independent quantities in Lemma 2.2, there exists $\tilde u\in\mathcal{C}^2(\bar\omega,\mathbb{R}^{d+k})$ satisfying:

$$\|\tilde u-u\|_1\leq C\delta^{1/2},\qquad \|\tilde u\|_2\leq C\lambda\sigma^J\delta^{1/2}, \tag{1.8$_1$}$$

$$\Big\|\mathcal{D}\Big(g-\frac{\delta}{\sigma^S}H_0,\tilde u\Big)\Big\|_0\leq\frac{r_0}{5}\frac{\delta}{\sigma^S}+\frac{\|g\|_{r,\beta}}{\lambda^{r+\beta}}, \tag{1.8$_2$}$$

with constants C depending only on $\underline{\gamma},\omega,g,d,k$ and where the integers J,S are given through:

$$Jk=Sd_*=lcm(d_*,k). \tag{1.9}$$

The above yields the decay rate S of $\mathcal{D}$ and the blow-up rate J of ∇^2u, having the quotient:

$$\frac{J}{S}=\frac{lcm(d_*,k)}{k}\cdot\frac{d_*}{lcm(d_*,k)}=\frac{d_*}{k}.$$

The Hölder regularity of the limiting immersion deduced from iterating the *stage*, depends only on the aforementioned quotient and the regularity of g, through the formula in (1.11). We now quote the statement of the *Nash-Kuiper iteration scheme* from [29, Theorem 1.3], where it was proved for the specific case $d=k=2$. However, the proof given there is independent of the particular values of $d,k\geq 2$. We note that the effective assumption on the codimension $k\geq 2$ is only needed to employ an alternative *step* formula, namely Nash's spirals (rather than Kuiper's corrugations as in Lemma 2.3), in order to construct an initial immersion satisfying the specific scaling law involving its first and second derivatives, and the norm of the defect. Since the codimension $k=1$ case has been treated in [19], Theorem 1.1 follows from:

Theorem 1.4. [NASH-KUIPER'S ITERATION] *Let $\underline{u}\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^{d+k})$ be an immersion, defined on the closure of an open set $\omega\subset\mathbb{R}^d$ diffeomorphic to B_1, together with $g\in\mathcal{C}^{r,\beta}(\bar\omega,\mathbb{R}^{d\times d}_{\mathrm{sym},>})$ for some regularity exponents in (1.6). Assume that:*

$$\mathcal{D}(g,\underline{u})>0\quad\text{on } \bar\omega.$$

Assume further that for some $S,J,p>0$ and for some parameters:

$$\underline{\delta}\in(0,1),\quad \underline{\gamma}>1,\quad \underline{\sigma}>1$$

depending only on $\omega, \underline{u}, g, S, J, p$, the following holds:

$$\left[\begin{array}{l}
\textit{Given any } u \in \mathcal{C}^2(\bar\omega, \mathbb{R}^{d+k}) \textit{ and any } \delta, \lambda, \sigma \textit{ such that:} \\
\qquad \delta \leq \underline{\delta}, \qquad \lambda\delta^{1/2} \geq 1, \qquad \sigma \geq \underline{\sigma}, \qquad \sigma^p\delta^{1/2} \leq 1, \\
\qquad \dfrac{1}{2\underline{\gamma}}\mathrm{Id}_d \leq (\nabla u)^T \nabla u \leq 2\underline{\gamma}\mathrm{Id}_d \quad \textit{in } \bar\omega, \\
\qquad \|\mathcal{D}(g - \delta H_0, u)\|_0 \leq \dfrac{r_0}{4}\delta \quad \textit{and} \quad \|u\|_2 \leq \lambda\delta^{1/2}, \\
\textit{with } r_0, H_0 \textit{ as in Lemma 2.2, there exists } \tilde u \in \mathcal{C}^2(\bar\omega, \mathbb{R}^{d+k}) \textit{ satisfying:} \\
\qquad \|\tilde u - u\|_1 \leq C\delta^{1/2}, \qquad \|\tilde u\|_2 \leq C\lambda\sigma^J\delta^{1/2}, \\
\qquad \Big\|\mathcal{D}\Big(g - \dfrac{\delta}{\sigma^S}H_0, \tilde u\Big)\Big\|_0 \leq \dfrac{r_0}{5}\dfrac{\delta}{\sigma^S} + \dfrac{\|g\|_{r,\beta}}{\lambda^{r+\beta}}, \\
\textit{with constants } C \textit{ depending only on } \omega, \underline{u}, g, S, J, p.
\end{array}\right] \tag{1.10}$$

Then, for every $\epsilon > 0$ and every exponent α in the range:

$$0 < \alpha < \min\Big\{\frac{r+\beta}{2}, \frac{1}{1 + 2J/S}\Big\}, \tag{1.11}$$

there exists an immersion $\bar u \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^{d+k})$ such that:

$$\|\bar u - \underline{u}\|_0 \leq \epsilon \quad \textit{and} \quad \mathcal{D}(g, \bar u) = 0 \quad \textit{in } \bar\omega. \tag{1.12}$$

Again, the first condition in (1.12) may be replaced by $\|\bar u - \underline{u}\|_{0,\bar\alpha} \leq \epsilon$, for any fixed independent $\bar\alpha \in (0,1)$. Indeed, $\bar u$ is obtained as the limit of a sequence of immersions $\{u_n \in \mathcal{C}^2(\bar\omega, \mathbb{R}^{d+k})\}_{n\to\infty}$, converging in $\mathcal{C}^{1,\alpha}$ and obtained by successive applications of the stage statement (1.10). For a given $\epsilon > 0$, the first immersion u_0 is chosen to satisfy:

$$\|u_0 - \underline{u}\|_0 \leq \frac{\epsilon}{2}, \qquad \|\nabla u_0 - \nabla \underline{u}\|_0 \leq C,$$

with constant C depending only on $\omega, \underline{u}, g$, whose magnitude is essentially the order 1 quantity $\|\mathcal{D}(g, \underline{u})\|_0^{1/2}$. The subsequent immersions satisfy then the inductive bounds:

$$\|u_{n+1} - u_n\|_1 \leq C\delta_n^{1/2}, \quad \mathcal{D}(g, u_n) \leq C\delta_n \quad \text{with} \quad C\sum_{n=0}^{\infty}\delta_n^{1/2} \leq \frac{\epsilon}{2}.$$

Hence, $\{u_n\}_{n\to\infty}$ is a Cauchy sequence in $\mathcal{C}^1$ (it is such also in $\mathcal{C}^{1,\alpha}$) and the limiting $\bar u$ satisfies:

$$\|\bar u - \underline{u}\|_0 \leq \epsilon, \qquad \|\nabla \bar u - \nabla \underline{u}\|_0 \leq C,$$

which implies, by the interpolation inequality with respect to $\bar\alpha < 1$:

$$\|\bar u - \underline{u}\|_{0,\bar\alpha} \leq C\|\bar u - \underline{u}\|_1^{\bar\alpha}\|\bar u - \underline{u}\|_0^{1-\bar\alpha} \leq C\epsilon^{1-\bar\alpha}.$$

This implies that a subsolution $\bar u$ is the limit in $\mathcal{C}^{0,\bar\alpha}(\bar\omega, \mathbb{R}^{d+k})$ of the exact isometric immersions $\bar u$ of regularity $\mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^{d+k})$, with the exponent α limited according to the range (1.2).

We also remark that the same result as in Theorem 1.1, remains valid in any target space $\mathbb{R}^n$ replacing $\mathbb{R}^{d+k}$, for $n \geq d+k$. Our proofs only require existence of and the propagation bounds on k normal vector fields, which are guaranteed by having $n - d \geq k$. The same statement as in Theorem 1.3 also holds and one concludes the final result by a version of Theorem 1.4.

1.2. **Comparison with the literature.** Our present result contributes to the growing body of work on the flexibility of $\mathcal{C}^{1,\alpha}$ isometric immersions and solutions to related PDEs. The study of flexibility in the context of the system (1.1) originates from the classical Nash-Kuiper theorem [34, 25] about $\mathcal{C}^1$ isometric immersions of a d-dimensional Riemannian manifold into $\mathbb{R}^{d+1}$ (in our terminology, this corresponds to codimension $k=1$). The Hölder-regular isometric immersions were studied by Borisov [1, 2], who conjectured that the Nash-Kuiper theorem extended to $\mathcal{C}^{1,\alpha}$-regularity for $\alpha < \frac{1}{1+d^2+d}$ and $\alpha < \frac{1}{5}$ for $d=2$. He provided a proof for the exponent $\alpha < \frac{1}{7}$, dimension $d=2$ and the analytic metric g in [3]. Borisov's conjecture was confirmed by Conti, De Lellis and Székelyhidi in [12], whereas the case $\alpha < \frac{1}{5}$, $d=2$ was resolved by De Lellis, Inauen and Székelyhidi in [13]. These results were generalized to compact manifolds by Cao and Székelyhidi in [10]. Locally, it has been recently improved to $\alpha < \frac{1}{d^2-d+1}$ by Cao, Hirsch and Inauen in [5] and to $\alpha < \frac{1}{2d-1}$ by Inauen in [19].

If the codimension k is larger than 1, one can construct more regular isometric immersions. In this direction, Nash first showed [35] that any manifold d-dimensional Riemannian manifold with metric $g \in \mathcal{C}^r$, $r \geq 3$, admits a $\mathcal{C}^r$-regular isometric immersion into $\mathbb{R}^{d+k}$ for sufficiently large k. The codimension bounds were later established and refined by Gromov [15] and Günther [16], while the case of $g \in \mathcal{C}^{r,\beta}$ with $r+\beta > 2$, was was treated by Jacobowitz [23]. When the metric's regularity is lower, i.e. $r+\beta < 2$, Källén [24] constructed a $\mathcal{C}^{1,\alpha}$ immersion for any $\alpha < \frac{r+\beta}{2}$ when $k \geq 2d_*(d+1)$ is sufficiently large. See also [14, 7, 9] for further results about isometric immersions in high codimension. Recently, Lewicka proved in [29] that in dimension $d=2$ and codimension $k \geq 2$, every subsolution to (1.1) with the metric $g \in \mathcal{C}^{r,\beta}$ admits an approximating sequence of $\mathcal{C}^{1,\alpha}$ isometric immersions, for all $\alpha < \min\{\frac{r+\beta}{2}, 1\}$.

When $d=2, k=1$, we quote the following rigidity results. Firstly, according to the classical statements due to Cohn-Vossen [11] and Herglotz [18], in the resolution of the Weyl problem [39], a $\mathcal{C}^2$ isometric immersion of $\mathbb{S}^2$ into $\mathbb{R}^3$ must be a rigid motion, i.e. a composition of a rotation and a translation. Second, any $\mathcal{C}^2$ surface with positive Gauss's curvature must be locally convex; for generalizations of this statement we refer to [17], and to [36, Chapter II], [37, Chapter IX] where the requirement of the $\mathcal{C}^2$ regularity is replaced by the requirement that the immersion is $\mathcal{C}^1$ and that the measure on $\mathbb{S}^2$ induced by the Gauss map has bounded variation. Third, using geometric arguments, the same rigidity has been proved by Borisov in a series of papers [1, 2], for $\mathcal{C}^{1,\alpha}$ isometric immersions at $\alpha > \frac{2}{3}$, of $\mathcal{C}^2$ metrics, with a simpler analytic proof of this result obtained in [12]. In particular, we see that $\frac{2}{3}$ is an upper bound on the range of Hölder exponents that can be reached using convex integration for isometric immersions of 2-dimensional metrics into $\mathbb{R}^{2+1}$. We point out that no such rigidity statement is possible for the isometric immersions already into $\mathbb{R}^4$, in view of [29].

The parallel version of the Nash-Kuiper scheme as in the papers cited above, was first developed to handle flexibility of $\mathcal{C}^{1,\alpha}$ weak solutions to the Monge-Ampère equation by Lewicka and Pakzad in [33], where the result was proved for $\alpha < \frac{1}{7}$. This regularity exponent was later increased to $\alpha < \frac{1}{5}$ and $\alpha < \frac{1}{3}$ by Cao and Szekelyhidi [8], and by Cao, Hirsch and Inauen [4], respectively. Further, in [26] Lewicka introduced the Monge-Ampère system and obtained flexibility of its solutions for any $\alpha < \min\{\frac{\beta}{2}, \frac{1}{1+2d_*/k}\}$ given the right hand side field $A \in \mathcal{C}^{0,\beta}$, where the dimension d and k were arbitrary; and also for any $\alpha < \min\{\frac{\beta}{2}, 1\}$ provided that $k \geq 2d_*$. For the special case $d=2$, weak solutions of progressively higher regularity, depending on the value of k, were obtained in the sequence of papers [27, 28, 20]. In [21], Inauen and Lewicka established the full flexibility of the Monge-Ampére system for $d=k=2$, and in [6] their result was extended to arbitrary d and the corresponding $k = d_* - d + 1$.

1.3. **Organization of the paper.** Section 2 begins by recalling the standard mollification estimates and the lemma on the decomposition of the defect into rank-one primitive defects. We then introduce the new *step* construction, based on the Kuiper corrugation ansatz from [26], lifted here to the fully nonlinear setting of (1.1). Its proof is deferred to section 5. We also establish several auxiliary lemmas concerning the propagation of the normal frame to the immersions generated in the course of the *stage*; these are proved in section 6.

Section 3 is devoted to the proof of Theorem 1.3, which through the Nash-Kuiper iteration scheme stated in Theorem 1.4, yields Theorem 1.1. The modified immersion $\tilde{u}$ produced in a single stage is obtained through $N = lcm(d_*, k)$ steps (i.e. the least common multiple of the Janet dimension and the codimension). Each step applies the Kuiper corrugation to the effect of replacing one rank-one primitive component of the defect $\mathcal{D}$ by higher order error terms. Since $\mathcal{D}$ consists of d_* such components, every time the step counter goes over a multiple of d_*, the defect decreases by one factor of σ. On the other hand, whenever the step counter goes over a multiple of k, the size of $\nabla^2 u$ increases, also essentially by one factor of σ. Consequently, after N steps one obtains the fundamental blow-up/decay ratio $J/S = d_*/k$. Throughout the construction, each step necessitates the update of the normal frame to the modified immersion, and components of $\nabla^2 u$ are then estimated separately along the various normal directions, exploiting the bounds on the propagation of the normal frame given in the auxiliary lemmas.

Finally, in section 4 we provide the proof of Theorem 1.2 by applying a suitable version of the Kirchhoff-Love extension, formulated with respect to the ambient metric g, to the Hölder regular isometric immersions constructed in the preceding sections.

1.4. **Notation.** By $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ we denote the space of symmetric $d \times d$ matrices, and $\mathbb{R}^{d\times d}_{\mathrm{sym},>}$ stands for the set of such matrices which are positive definite. The space of Hölder continuous vector fields $\mathcal{C}^{m,\alpha}(\bar{\omega}, \mathbb{R}^k)$ consists of restrictions of all $f \in \mathcal{C}^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^k)$ to the closure of an open, bounded set $\omega \subset \mathbb{R}^d$. The $\mathcal{C}^m(\bar{\omega}, \mathbb{R}^k)$ norm of this restriction is denoted by $\|f\|_m$, while its Hölder norm in $\mathcal{C}^{m,\alpha}(\bar{\omega}, \mathbb{R}^k)$ is $\|f\|_{m,\alpha}$. By $\nabla^{(m)} f$ we denote the table-valued field of all partial derivatives of f of order m. If $d = k$, the symmetric part of ∇f is: $\mathrm{sym}\nabla f = (\nabla f + (\nabla f)^T)/2$. By C we denote a universal constant that may change from line to line.

2. The preparatory statements

In this section, we collect several technical ingredients that will be used in the proof of Theorem 1.3. The first lemma concerns the convolution and commutator estimates from [12].

Lemma 2.1. *Let $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^d, \mathbb{R})$ be a standard mollifier that is nonnegative, radially symmetric, supported on the unit ball $B(0,1) \subset \mathbb{R}^d$ and such that $\int_{\mathbb{R}^d} \phi \, \mathrm{d}x = 1$. Denote:*

$$\phi_l(x) \doteq \frac{1}{l^d}\phi(\frac{x}{l}) \quad \text{for all } \ l \in (0,1], \ x \in \mathbb{R}^d.$$

Then, for every $f, g \in \mathcal{C}^0(\mathbb{R}^d, \mathbb{R})$, every $m \geq 0$ and $r \in \{0,1\}$, $\beta \in (0,1]$, there holds:

$$\|\nabla^{(m)}(f * \phi_l)\|_0 \leq \frac{C}{l^m}\|f\|_0, \tag{2.1$_1$}$$

$$\|f - f * \phi_l\|_0 \leq l^{r+\beta}\|f\|_{r,\beta}, \tag{2.1$_2$}$$

$$\|\nabla^{(m)}\big((fg) * \phi_l - (f * \phi_l)(g * \phi_l)\big)\|_0 \leq Cl^{2-m}\|\nabla f\|_0\|\nabla g\|_0, \tag{2.1$_3$}$$

with constants $C > 0$ depending only on d and m.

The next lemma provides a decomposition of symmetric matrices into linear combinations of rank-one *primitive matrices.*

Lemma 2.2. *Let $\{\eta_i\in\mathbb{R}^d\}_{i=1}^{d_*}$ be the unit vectors such that $\mathbb{R}^{d\times d}_{\mathrm{sym}} = span\{\eta_i\otimes\eta_i\}_{i=1}^{d_*}$, namely:*

$$H=\sum_{i=1}^{d_*}\bar a_i(H)\eta_i\otimes\eta_i \quad \text{ for all } H\in\mathbb{R}^{d\times d}_{\mathrm{sym}},$$

where $\big\{\bar a_i:\mathbb{R}^{d\times d}_{\mathrm{sym}}\to\mathbb{R}\big\}_{i=1}^{d_*}$ are the corresponding linear "coordinate" functions. Denoting:

$$H_0\doteq\sum_{i=1}^{d_*}\eta_i\otimes\eta_i, \tag{2.2}$$

there exists $r_0\in(0,1)$, depending only on d, such that:

$$|\bar a_i(H)-1|\leq\frac12 \quad \text{ for all } H\in B(H_0,r_0)\subset\mathbb{R}^{d\times d}_{\mathrm{sym}},\ i=1\dots d_*$$

The above result is elementary. Define $\{\eta_i\}_{i=1}^{d_*}$ as $\big\{\xi_{ij}\doteq\frac{e_i+e_j}{|e_i+e_j|}\big\}_{i,j=1\dots d,\ i\leq j}$. Then, it is not hard to show that the set $\{\xi_{ij}\otimes\xi_{ij}\}_{i,j=1\dots d,\ i\leq j}$ provides a basis of $\mathbb{R}^{d\times d}_{\mathrm{sym}}$. Since the set's cardinality matches the dimension of $\mathbb{R}^{d\times d}_{\mathrm{sym}}$, its elements must be linearly independent.

We now present the *step* in our convex integration algorithm, in which a single codimension is used to cancel one rank-one defect of the form $a(x)^2\eta\otimes\eta$. The construction utilizes the ansatz that is the basic building block of the convex integration step for the Monge-Ampère system in [26, Lemma 2.1]. The observation that a similar ansatz also applies in the present fully nonlinear setting of (1.1), was first made in [5, Lemma 3.1] and led to a simplification of the analysis as well as an improvement of the regularity exponent pertaining to codimension $k=1$, and revealed the further close connection between the Monge-Ampère and the isometric immersion systems. The proof will of the lemma below be given in section 5. We also note that the version of the ansatz, given in [5, Lemma 3.1] and [29, Lemma 2.5], featuring an unbalanced form of the leading terms in (2.3), is not sufficient for our purposes.

Lemma 2.3. [STEP: KUIPER'S CORRUGATION] *Let $u\in\mathcal{C}^2(\mathbb{R}^d,\mathbb{R}^{d+k})$ be an immersion and let $E_u\in\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^{d+k})$ be a unit normal vector field to u, so that:*

$$(\nabla u)^TE_u=0,\quad |E_u|=1 \quad \text{ in } \mathbb{R}^d.$$

For a given unit vector $\eta\in\mathbb{R}^d$ we set $t=\langle x,\eta\rangle$, and denote:

$$\Gamma(t)=\sqrt2\sin t,\quad \bar\Gamma(t)=-\frac14\sin(2t),\quad \bar{\bar\Gamma}(t)=\frac14\cos(2t),\quad \bar{\bar{\bar\Gamma}}(t)=1-\frac12\cos(2t).$$

Then, for every $\lambda>0$ and $a\in\mathcal{C}^2(\mathbb{R}^d,\mathbb{R})$, the vector field $\tilde u\in\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^{d+k})$ in:

$$\begin{aligned}&\tilde u(x)=u(x)+\frac{\Gamma(\lambda t)}{\lambda}a(x)E_u(x)+\frac{\bar\Gamma(\lambda t)}{\lambda}a(x)^2T_u(x)\eta+\frac{\bar{\bar\Gamma}(\lambda t)}{\lambda^2}a(x)T_u(x)\nabla a(x),\\ &\text{where: } T_u=(\nabla u)\big((\nabla u)^T\nabla u\big)^{-1}\in\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^{(d+k)\times d}),\end{aligned}$$

satisfies the following identity:

$$\begin{aligned}&(\nabla\tilde u)^T\nabla\tilde u-(\nabla u)^T\nabla u-a^2\eta\otimes\eta\\ &=-\frac{2\Gamma(\lambda t)}{\lambda}a\,\big[\langle\partial^2_{ij}u,E_u\rangle\big]_{i,j=1\dots d}+\frac{2\bar{\bar\Gamma}(\lambda t)}{\lambda^2}a\nabla^2a+\frac{\bar{\bar{\bar\Gamma}}(\lambda t)}{\lambda^2}\nabla a\otimes\nabla a+\mathcal{R}.\end{aligned}\tag{2.3}$$

The error term $\mathcal{R} = \mathcal{R}(\lambda, a, \eta, \nabla u, E_u)$ above is written as:

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4,$$

whose respective components, grouped by the powers of λ^{-1}, are given by:

$$\begin{aligned}
\mathcal{R}_0 &= \bar{\Gamma}'(\lambda t)^2 a^4 |T_u\eta|^2 \eta\otimes\eta,\\
\mathcal{R}_1 &= -\frac{2\bar{\Gamma}(\lambda t)}{\lambda} a^2\ \big[\langle \partial^2_{ij}u, T_u\eta\rangle\big]_{i,j=1\dots d} + \frac{2\bar{\Gamma}(\lambda t)\Gamma'(\lambda t)}{\lambda} a\ \mathrm{sym}\big(\nabla(a^2T_u\eta)^T(E_u\otimes\eta)\big)\\
&\quad + \frac{2\Gamma(\lambda t)\bar{\Gamma}'(\lambda t)}{\lambda} a^2\ \mathrm{sym}\big(\nabla(aE_u)^T T_u(\eta\otimes\eta)\big) + \frac{2\bar{\Gamma}(\lambda t)\bar{\Gamma}'(\lambda t)}{\lambda} a^2\ \mathrm{sym}\big(\nabla(a^2T_u\eta)^T T_u(\eta\otimes\eta)\big)\\
&\quad + \frac{2\bar{\Gamma}'(\lambda t)\bar{\bar{\Gamma}}'(\lambda t)}{\lambda} a^3\ \big\langle T_u\eta, T_u\nabla a\big\rangle \eta\otimes\eta,\\
\mathcal{R}_2 &= -\frac{2\bar{\bar{\Gamma}}(\lambda t)}{\lambda^2} a\ \big[\langle \partial^2_{ij}u, T_u\nabla a\rangle\big]_{i,j=1\dots d} + \frac{2\bar{\bar{\Gamma}}(\lambda t)\Gamma'(\lambda t)}{\lambda^2} a\ \mathrm{sym}\big(\nabla(aT_u\nabla a)^T(E_u\otimes\eta)\big)\\
&\quad + \frac{2\bar{\Gamma}'(\lambda t)\bar{\bar{\Gamma}}(\lambda t)}{\lambda^2} a^2\ \mathrm{sym}\big(\nabla(aT_u\nabla a)^T T_u(\eta\otimes\eta)\big) + \frac{\Gamma(\lambda t)^2}{\lambda^2} a^2(\nabla E_u)^T\nabla E_u\\
&\quad + \frac{2\Gamma(\lambda t)\bar{\Gamma}(\lambda t)}{\lambda^2}\mathrm{sym}\big(\nabla(aE_u)^T\nabla(a^2T_u\eta)\big) + \frac{2\Gamma(\lambda t)\bar{\bar{\Gamma}}'(\lambda t)}{\lambda^2} a\ \mathrm{sym}\big(\nabla(aE_u)^T T_u(\nabla a\otimes\eta)\big)\\
&\quad + \frac{\bar{\Gamma}(\lambda t)^2}{\lambda^2}\nabla(a^2T_u\eta)^T\nabla(a^2T_u\eta) + \frac{2\bar{\Gamma}(\lambda t)\bar{\bar{\Gamma}}'(\lambda t)}{\lambda^2} a\ \mathrm{sym}\big(\nabla(a^2T_u\eta)^T T_u(\nabla a\otimes\eta)\big)\\
&\quad + \frac{\bar{\bar{\Gamma}}'(\lambda t)^2}{\lambda^2} a^2\ |T_u\nabla a|^2\eta\otimes\eta,\\
\mathcal{R}_3 &= \frac{2\bar{\Gamma}(\lambda t)\bar{\bar{\Gamma}}(\lambda t)}{\lambda^3}\mathrm{sym}\big(\nabla(a^2T_u\eta)^T\nabla(aT_u\nabla a)\big) + \frac{2\Gamma(\lambda t)\bar{\bar{\Gamma}}(\lambda t)}{\lambda^3}\mathrm{sym}\big(\nabla(aE_u)^T\nabla(aT_u\nabla a)\big)\\
&\quad + \frac{2\bar{\bar{\Gamma}}(\lambda t)\bar{\bar{\Gamma}}'(\lambda t)}{\lambda^3} a\ \mathrm{sym}\big(\nabla(aT_u\nabla a)^T T_u(\nabla a\otimes\eta)\big)\\
\mathcal{R}_4 &= \frac{\bar{\bar{\Gamma}}(\lambda t)^2}{\lambda^4}\nabla(aT_u\nabla a)^T\nabla(aT_u\nabla a).
\end{aligned}$$

Finally, we collect several lemmas concerning propagation of the normal frame to the immersions constructed inductively in our convex integration algorithm. The proofs, which follow the arguments of [29, Section 3] in the case $d = k = 2$, are presented in section 6.

Lemma 2.4. *Let $u \in \mathcal{C}^1(\bar{\omega}, \mathbb{R}^{d+k})$ defined on the closure of an open, bounded $\omega \subset \mathbb{R}^d$, satisfy:*

$$\frac{1}{\gamma}\mathrm{Id}_d \leq (\nabla u)^T\nabla u \leq \gamma\mathrm{Id}_d \quad \textit{in } \bar{\omega}, \tag{2.4}$$

for some $\gamma > 1$. Then there holds:

$$\|\nabla u\|_0 \leq (d\gamma)^{1/2} \quad \textit{and} \quad \frac{1}{\gamma^d} \leq \det((\nabla u)^T\nabla u) \leq \gamma^d \textit{ in } \bar{\omega}.$$

Lemma 2.5. *Let $u \in \mathcal{C}^{n+2}(\bar{\omega}, \mathbb{R}^{d+k})$, defined on the closure of an open, bounded set $\omega \subset \mathbb{R}^d$, satisfy (2.4) for some $\gamma > 1$. Assume additionally that:*

$$\|\nabla^{(m)}\nabla^{(2)}u\|_0 \leq \bar{C}\mu^{m+1}A \quad \textit{for all } m = 0\dots n,$$

with some constants $\mu > 1$, $A < 1$ and $\bar{C} > 1$. Then, the following bounds are valid for the tangent field $T_u = (\nabla u)\big((\nabla u)^T\nabla u\big)^{-1} \in \mathcal{C}^{n+1}(\bar{\omega}, \mathbb{R}^{(d+k)\times d})$:

$$\|T_u\|_0 \leq C \quad \textit{and} \quad \|\nabla^{(m)}T_u\|_0 \leq C\mu^m A \quad \textit{for all } m = 1\dots n+1,$$

where C depends only on γ, $\bar{C}$ and d and n, but not on μ, A.

Concerning existence of the normal frame, the following has been proved in [7, Lemma 3.5]:

Lemma 2.6. *Let $u \in \mathcal{C}^{n+1}(\bar{\omega}, \mathbb{R}^{d+k})$ be defined on the closure of $\omega \subset \mathbb{R}^d$ diffeomorphic to B_1, satisfying (2.4) with some $\gamma > 1$. Then, there exist $\{E_u^i \in \mathcal{C}^n(\bar{\omega}, \mathbb{R}^{d+k})\}_{i=1}^k$, with:*

$$(\nabla u)^T E_u^i = 0, \quad |E_u^i| = 1, \quad \langle E_u^i, E_u^j\rangle = 0 \quad \text{in } \bar{\omega}, \quad \text{for all } i, j = 1 \dots k,\ i \neq j, \tag{2.5}$$

and obeying the bounds:

$$\|\nabla^{(m)} E_u^i\|_0 \leq C\big(1 + \|\nabla u\|_m\big) \quad \text{for all } m = 1 \dots n,\ i = 1 \dots k. \tag{2.6}$$

The constant C above depends only on ω, γ, d, k and n.

The final lemma of this section provides the key construction and estimates on the propagation of normal vectors from a given, to a nearby immersion.

Lemma 2.7. *Let $u \in \mathcal{C}^{n+2}(\bar{\omega}, \mathbb{R}^{d+k})$ be an immersion defined on the closure of an open, bounded set $\omega \subset \mathbb{R}^d$, and satisfying (2.4) for some $\gamma > 1$. Let $\{E_u^i \in \mathcal{C}^{n+1}(\bar{\omega}, \mathbb{R}^{d+k})\}_{i=1}^k$ be a normal frame of u, such that (2.5) holds. Fix another vector field $v \in \mathcal{C}^{n+2}(\bar{\omega}, \mathbb{R}^{d+k})$.*

(i) *There exists $\rho \in (0,1)$ depending only on γ, d, such that if $\|\nabla v - \nabla u\|_0 \leq \rho$ then a normal frame $\{E_v^i \in \mathcal{C}^{n+1}(\bar{\omega}, \mathbb{R}^{d+k})\}_{i=1}^k$ of v, satisfying:*

$$(\nabla v)^T E_v^i = 0, \quad |E_v^i| = 1, \quad \langle E_v^i, E_v^j\rangle = 0 \quad \text{in } \bar{\omega}, \quad \text{for all } i, j = 1 \dots k,\ i \neq j,$$

can be defined via the following formulas:

$$\begin{aligned} & E_v^i = \frac{\nu_v^i - \sum_{j<i} \langle \nu_v^i, E_v^j\rangle E_v^j}{|\nu_v^i - \sum_{j<i} \langle \nu_v^i, E_v^j\rangle E_v^j|}, && \text{for all } i = 1 \dots k, \\ & \text{where } \nu_v^i = \big(\mathrm{Id}_{d+k} - T_v(\nabla v - \nabla u)^T\big) E_u^i && \text{for all } i = 1 \dots k, \\ & \text{and} \quad T_v = (\nabla v)((\nabla v)^T \nabla v)^{-1} \in \mathcal{C}^{n+1}(\bar{\omega}, \mathbb{R}^{(d+k)\times d}). \end{aligned} \tag{2.7}$$

(ii) *If, in addition to $\|\nabla v - \nabla u\|_0 \leq \rho$, there holds:*

$$\begin{aligned} & \|\nabla^{(m)}(\nabla v - \nabla u)\|_0 \leq \bar{C}\mu^m A \quad \text{for } m = 0 \dots n+1, \\ & \|\nabla^{(m)} \nabla^{(2)} u\|_0 \leq \bar{C}\mu^{m+1}, \quad \|\nabla^{(m+1)} E_u^i\|_0 \leq \bar{C}\mu^{m+1} \quad \text{for } m = 0 \dots n,\ i = 1 \dots k, \end{aligned}$$

with some constants $\mu > 1, A < 1$ and $\bar{C} > 1$, then $\{E_v^i\}_{i=1}^k$ given in (2.7) obey:

$$\|\nabla^{(m)}(E_v^i - E_u^i)\|_0 \leq C\mu^m A \quad \text{for all } m = 0 \dots n+1,\ i = 1 \dots k, \tag{2.8}$$

where C depends only on $\gamma, \bar{C}, \omega$ and n, d, k, but not on μ, A.

3. The stage construction and a proof of Theorem 1.3

The proof consists of several steps in an inductive construction below. By C we always denote a constant that is larger than 1 and depends only on $\underline{\gamma}$, ω, d, k, although it may change from line to line. In general, we will use the following counters:

$$\begin{aligned}
&i : 0\dots N = lcm(d_*, k): && \text{consecutive induction counter}\\
&s : 0\dots S: && \text{counter on multiplicities of } d_*\\
&j : 0\dots J: && \text{counter on multiplicities of } k\\
&\theta : 1\dots k: && \text{counter on codimensions}\\
&\rho : 1\dots d_*: && \text{counter on defect decomposition modes}\\
&m \geq 0: && \text{number of derivatives}\\
&p, q : 1\dots d: && \text{differentiation directions.}
\end{aligned}$$

Our induction procedure comprises one preparatory step (at counter value $i = 0$) in which the given quantites u, g are mollified to their smooth approximations u_0, g_0; and $N = lcm(d_*, k)$ subsequent steps (from the counter value $i = 1$ to $i = N$), in which the preceding immersion u_{i-1} is adapted to a new immersion u_i. Since each of these finitely many steps "loses" only finitely many derivatives between those whose bounds are required at $i - 1$ and those whose bounds are achieved at i – as is the case, for example, in Lemmas 2.6 and 2.7 – all constants C appearing in the proof depend only on finitely many derivatives of the initial quantites u_0, g_0. To avoid cluttering the presentation and obscuring the argument, we shall write statements such as "for all $m \geq 0$" or "for all $m \geq 1$", in place of the more precise "for all $m = 0\dots K$", instead "for all $m = 1\dots K$". Here, K is always sufficiently large and depends through a rather complicated formula on the current values of the counters i, s, j, θ, ρ, as well as on d_* and k.

Proof of Theorem 1.3

1. (Preparing the data) We will use the following assumptions on $u \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^{d+k})$:

$$\frac{1}{2\underline{\gamma}}\mathrm{Id}_d \leq (\nabla u)^T\nabla u \leq 2\underline{\gamma}\mathrm{Id}_d \quad \text{in } \bar{\omega}, \tag{3.1$_1$}$$

$$\|\mathcal{D}(g - \delta H_0, u)\|_0 \leq \frac{r_0}{4}\delta \qquad \text{and} \qquad \|u\|_2 \leq \lambda\delta^{1/2}, \tag{3.1$_2$}$$

where $\underline{\gamma} > 1$ is given, r_0, H_0 are as in Lemma 2.2, and δ, λ, σ are any chosen parameters satisfying a slightly weakened version of the assumption $(1.7)_1$:

$$0 < \delta \leq \underline{\delta}, \qquad \lambda\delta^{1/2} \geq 1, \qquad \sigma \geq \underline{\sigma}, \qquad \sigma^{J+2}\delta^{1/2} \leq 1. \tag{3.2}$$

The full assumption $(1.7)_1$ will be used in the final Step 7 of the proof. The constants $\underline{\delta} < 1$ and $\underline{\sigma} > 1$ will be chosen, respectively, sufficiently small and sufficiently large, in the course of the arguments below, and will depend only on $\omega, \underline{\gamma}, d, k$ and J, S that are defined in (1.9) (thus effectively also depending only on d, k). We set:

$$\lambda_0 = \lambda, \quad \delta_0 = \delta, \quad l = \frac{1}{\bar{C}\lambda_0},$$
$$u_0 = u * \phi_l \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{d+k}), \quad g_0 = g * \phi_l \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{(d+k)\times(d+k)}_{\mathrm{sym},>}),$$

for $\bar{C} > 1$ that depends only on d, k and is suffuciently large to ensure that:

$$\|(\nabla u_0)^T\nabla u_0 - ((\nabla u)^T\nabla u) * \phi_l\|_0 \leq Cl^2\|\nabla^2 u\|_0^2 \leq \frac{C}{\bar{C}^2\lambda_0^2}\lambda_0^2\delta_0 \leq \frac{r_0}{12}\delta_0, \tag{3.3}$$

in view of $(2.1)_3$ and $(3.1)_2$. Further, Lemma 2.1 and $(3.1)_2$ yield that:

$$\begin{aligned}
&\|g_0 - g\|_0 \leq l^{r+\beta}\|g\|_{r,\beta} \leq \frac{\|g\|_{r,\beta}}{\lambda_0^{r+\beta}}\\
&\|u_0 - u\|_1 \leq l\|u\|_2 \leq l\lambda\delta^{1/2} \leq \delta_0^{1/2}\\
&\|\nabla^{(m)}\nabla^{(2)}u_0\|_0 \leq \frac{C}{l^m}\|\nabla^{(2)}u_0\|_0 \leq \frac{C}{l^m}\lambda\delta^{1/2} \leq C\lambda_0^{m+1}\delta^{1/2} \quad \text{ for all } m \geq 0.
\end{aligned} \tag{3.4}$$

In particular, recalling Lemma 2.4, the second bound above implies:

$$\begin{aligned}
\|(\nabla u_0)^T\nabla u_0 - (\nabla u)^T\nabla u\|_0 &\leq \|\nabla(u_0 - u)\|_0\big(2\|\nabla u\|_0 + \|\nabla(u_0 - u)\|_0\big)\\
&\leq \delta_0^{1/2}(2(2\underline{\gamma}d)^{1/2} + 1),
\end{aligned}$$

so taking δ_0 sufficiently small (i.e. setting $\underline{\delta}$ small enough) we ensure that u_0 is an immersion:

$$\frac{1}{3\underline{\gamma}}\mathrm{Id}_d \leq (\nabla u_0)^T\nabla u_0 \leq 3\underline{\gamma}\mathrm{Id}_d \quad \text{ in } \bar{\omega}, \tag{3.5}$$

By Lemma 2.6 we obtain the (sufficiently regular) normal frame $\{E_0^\theta \in \mathcal{C}(\bar{\omega}, \mathbb{R}^{d+k})\}_{\theta=1}^k$, namely:

$$(\nabla u_0)^T E_0^\theta = 0, \quad |E_0^\theta| = 1, \quad \langle E_0^\theta, E_0^{\bar{\theta}}\rangle = 0 \quad \text{in } \bar{\omega}, \quad \text{for all } \theta, \bar{\theta} = 1\ldots k, \ \theta \neq \bar{\theta},$$

so that the following bounds hold:

$$\begin{aligned}
&\|\nabla^{(m)}E_0^\theta\|_0 \leq C(1 + \|\nabla u_0\|_m) \leq C\lambda_0^m\delta_0^{1/2} \quad \text{ for all } \theta = 1\ldots k, \ m \geq 1,\\
&\|\nabla^{(m)}\langle\partial^2_{pq}u_0, E_0^\theta\rangle\|_0 \leq C\lambda_0^{m+1}\delta_0^{1/2} \quad \text{ for all } \theta = 1\ldots k, \ m \geq 0,
\end{aligned} \tag{3.6}$$

where in the first bound we used the second assumption that (3.2), while in the second bound we used (3.4). Finally, writing:

$$\mathcal{D}(g_0 - \delta_0 H_0, u_0) = \mathcal{D}(g - \delta_0 H_0, u) * \phi_l - \big((\nabla u_0)^T\nabla u_0 - ((\nabla u)^T\nabla u) * \phi_l\big)$$

we estimate by $(2.1)_1$, $(2.1)_3$, $(3.1)_2$, for all $m \geq 1$:

$$\begin{aligned}
&\|\nabla^{(m)}\Big(\mathcal{D}(g - \delta_0 H_0, u) * \phi_l\Big)\|_0 \leq \frac{C}{l^m}\frac{r_0\delta_0}{4} = C\lambda_0^m\delta_0,\\
&\|\nabla^{(m)}\big((\nabla u_0)^T\nabla u_0 - ((\nabla u)^T\nabla u) * \phi_l\big)\|_0 \leq Cl^{2-m}\|\nabla^2 u\|_0^2 \leq C\lambda_0^m\delta_0,
\end{aligned} \tag{3.7}$$

so that, in virtue of (3.3) and $(3.1)_2$ we get:

$$\begin{aligned}
&\|\mathcal{D}(g_0 - \delta_0 H_0, u_0)\|_0 \leq \frac{r_0}{4}\delta_0 + \frac{r_0}{12}\delta_0 = \frac{r_0}{3}\delta_0,\\
&\|\nabla^{(m)}\mathcal{D}(g_0 - \delta_0 H_0, u_0)\|_0 \leq C\lambda_0^m\delta_0 \quad \text{ for all } m \geq 1.
\end{aligned} \tag{3.8}$$

2. (Setting up the induction) Starting with u_0, we will inductively define the immersions:

$$\{u_i \in \mathcal{C}(\bar{\omega}, \mathbb{R}^{d+k})\}_{i=1}^N, \quad \text{ where } N = lcm(d_*, k),$$

and their (sufficiently regular) normal frames $\{E_i^\theta \in \mathcal{C}(\bar{\omega}, \mathbb{R}^{d+k})\}_{\theta=1}^k$, that is:

$$(\nabla u_i)^T E_i^\theta = 0, \quad |E_i^\theta| = 1, \quad \langle E_i^\theta, E_i^{\bar{\theta}}\rangle = 0 \quad \text{in } \bar{\omega} \quad \text{for all } \theta, \bar{\theta} = 1\ldots k, \ \theta \neq \bar{\theta}, \ i = 1\ldots N.$$

The consecutive frequencies $\{\lambda_i\}_{i=1}^N$ are given by the formula:

$$\lambda_i = \lambda_{i-1} \cdot \begin{cases} \sigma & \text{if } k \mid i-1\\ \sigma^{1/2} & \text{if } d_* \mid i-1 \text{ and } i = 2\ldots N\\ 1 & \text{otherwise.}\end{cases}$$

In particular, $\lambda_1 = \lambda_0\sigma$ and, given $j = 0\dots J-1$, $s = 0\dots S-1$ it follows that:

$$\lambda_i = \lambda_0\sigma^{1+j+s/2} \qquad \text{for all } i \in (jk,(j+1)k] \cap (sd_*, (s+1)d_*]. \tag{3.9}$$

The consecutive defect measures $\{\delta_s\}_{s=1}^S$ are defined as a decreasing S-tuple, through:

$$\delta_s = \frac{\delta_0}{\sigma^s}\tilde{C}_s \tag{3.10}$$

with the help of an increasing S-tuple of constants $\tilde{C}_1 \dots \tilde{C}_S$, all greater than 1 and depending only on $\omega, \underline{\gamma}, d, k$. In particular, setting $\tilde{C}_0 = 1$, we request that:

$$\sigma \geq \frac{\tilde{C}_s}{\tilde{C}_{s-1}} \geq 1 \quad \text{for all } \ s = 1\dots S, \tag{3.11}$$

which can always be achieved by setting $\underline{\sigma}$ large enough. To define the immersions u_i, we recall Lemma 2.3 and the oscillatory profiles Γ, $\bar{\Gamma}$, $\bar{\bar{\Gamma}}$ therein. First, for $i = 1\dots N$ we uniquely write:

$$i = jk + \theta = sd_* + \rho \qquad \text{with } \ \theta = 1\dots k, \ \rho = 1\dots d_*$$

and then set, in an induction step from $i-1$ to i:

$$\begin{aligned} u_i = u_{i-1} + \frac{\Gamma(\lambda_i\langle x,\eta_\rho\rangle)}{\lambda_i} a_\rho^s E_{i-1}^\theta + \frac{\bar{\Gamma}(\lambda_i\langle x,\eta_\rho\rangle)}{\lambda_i}(a_\rho^s)^2 T_{i-1}\eta_\rho + \frac{\bar{\bar{\Gamma}}(\lambda_i\langle x,\eta_\rho\rangle)}{\lambda_i^2} a_\rho^s T_{i-1}\nabla a_\rho^s \\ \text{where: } T_i = (\nabla u_i)\big((\nabla u_i)^T\nabla u_i\big)^{-1}. \end{aligned} \tag{3.12}$$

The maps $a_\rho^s \in \mathcal{C}^?(\bar{\omega})$ for $s = 0\dots S-1$, $\rho = 1\dots d_*$ will be defined in Step 3 through an application of Lemma 2.2 to the defect $\mathcal{D}(g_0 - \delta_s H_0, u_{sd_*})$. The unit vector $\eta_\rho \in \mathbb{R}^d$ is as in Lemma 2.2. In Steps 3-6 below we will prove that for all $i = 1\dots N$ there holds:

$$\frac{1}{3\underline{\gamma}}\mathrm{Id}_d \leq (\nabla u_i)^T\nabla u_i \leq 3\underline{\gamma}\mathrm{Id}_d \quad \text{ in } \ \bar{\omega}, \tag*{$(3.13)_1$}$$

$$\|u_i - u_{i-1}\|_1 \leq C\delta_0^{1/2}, \tag*{$(3.13)_2$}$$

$$\begin{aligned} &\|\nabla^{(m)}\nabla^{(2)}u_i\|_0 + \|\nabla^{(m)}\nabla E_i^\theta\|_0 \leq C\lambda_0\lambda_i^m\sigma^j\delta_0^{1/2} \\ &\qquad\qquad \text{for all } (j-1)k < i \leq jk, \ j = 1\dots J, \ \theta = 1\dots k, \ m \geq 0, \end{aligned} \tag*{$(3.13)_3$}$$

$$\begin{aligned} &\left.\begin{aligned} &\|\nabla^{(m)}\langle\partial^2_{pq}u_i, E_i^\theta\rangle\|_0 \leq C\lambda_0\lambda_i^m\sigma^j\delta_0^{1/2} && \text{for } \ jk \leq i < jk+\theta \\ &\|\nabla^{(m)}\langle\partial^2_{pq}u_i, E_i^\theta\rangle\|_0 \leq C\lambda_0\lambda_i^m\sigma^{j+1}\delta_0^{1/2} && \text{for } \ jk+\theta \leq i \leq (j+1)k \end{aligned}\right\} \\ &\qquad\qquad \text{for all } \ j = 0\dots J-1, \ \theta = 1\dots k, \ m \geq 0, \end{aligned} \tag*{$(3.13)_4$}$$

$$\|\nabla^{(m)}\mathcal{D}(g_0 - \delta_s H_0, u_{sd_*})\|_0 \leq \frac{r_0}{5}\lambda_{sd_*}^m\delta_s \quad \text{ for all } s = 1\dots S, \ m \geq 0. \tag*{$(3.13)_5$}$$

Observe that, writing the first assertion in $(3.13)_3$ as:

$$\|\nabla^{(m)}\nabla^{(2)}u_i\|_0 \leq C\lambda_i^{m+1}A \qquad \text{with } \ A = \frac{\lambda_0}{\lambda_i}\sigma^j\delta_0^{1/2}$$

and noting that $A \leq \sigma^J\delta_0^{1/2} < 1$ by the last assumption in (3.2), we use Lemma 2.5 to get:

$$\begin{aligned} \|T_i\|_0 \leq C, \quad \|\nabla^{(m)}T_i\|_0 \leq C\lambda_0\lambda_i^{m-1}\sigma^j\delta_0^{1/2} \\ \text{for all } \ (j-1)k < i \leq jk, \ j = 1\dots J, \ m \geq 1. \end{aligned} \tag{3.14}$$

Also, we note the simplified version of the bound above and the second bound in $(3.13)_3$:

$$\|\nabla^{(m)}E_i^\theta\|_0 + \|\nabla^{(m)}T_i\|_0 \leq C\lambda_i^m \quad \text{for all } \ m \geq 0, \tag{3.15}$$

which clearly holds at $m=0$, whereas at $m\geq 1$ we use that $\sigma^j\delta_0^{1/2}\leq 1$ in virtue of the last assumption in (3.2). The statements $(3.13)_1$-$(3.13)_5$ will be proved inductively. For the induction base, observe that at $i=0$, the assertion $(3.13)_1$ already holds by (3.5), while $(3.13)_3$ holds by the last claim in (3.4) and the first one in (3.6). Whereas, the second assertion in there validates the first inequality in $(3.13)_4$, meanwhile the second inequality is void at $i=0$. Finally, the estimate $(3.13)_5$ holds for $s=0$ in the form of (3.8).

3. (The defect decomposition) Assume $(3.13)_5$ at some $s=1\dots S-1$ or assume (3.8) if $s=0$. We want to apply Lemma 2.2 to the scaled defect:

$$H=\frac{1}{\delta_s}\mathcal{D}(g_0-\delta_{s+1}H_0,u_{sd_*}).$$

To check that $H(x)\in B(H_0,r_0)$ for all $x\in\bar\omega$, we note that $H-H_0=\big(\mathcal{D}(g_0-\delta_sH_0,u_{sd_*})-\delta_{s+1}H_0\big)/\delta_s$ which implies:

$$\|H-H_0\|_0\leq\frac{\|\mathcal{D}(g_0-\delta_sH_0,u_{sd_*})\|_0}{\delta_s}+\frac{\delta_{s+1}}{\delta_s}H_0\leq\frac{r_0}{3}+\frac{\tilde C_{s+1}}{\tilde C_s\sigma}|H_0|<\frac{r_0}{2}$$

where the first term was estimated by $(3.13)_5$ or (3.8), and for the second term is bounded in view of the following assumption, complementary to that of (3.11):

$$\frac{\tilde C_s}{\tilde C_{s-1}}\cdot\frac{6|H_0|}{r_0}<\underline{\sigma}\quad\text{for all}\ \ s=1\dots S. \tag{3.16}$$

Therefore, by Lemma 2.2 we obtain the decomposition:

$$\begin{aligned}&H=\sum_{\rho=1}^{d_*}\bar a_\rho(H)\eta_\rho\otimes\eta_\rho\qquad\text{with}\ \ \|\bar a_\rho-1\|_0\leq\frac12\ \ \text{for all}\ \ \rho=1\dots d_*,\\&\|\nabla^{(m)}\bar a_\rho(H)\|_0\leq C\|\nabla^{(m)}H\|_0\leq C\frac{\|\nabla^{(m)}\mathcal{D}(g_0-\delta_sH_0,u_{sd_*})\|_0}{\delta_s}\leq C\lambda_{sd_*}^m\qquad\text{for all}\ \ m\geq 1.\end{aligned}$$

Note that the above bounds are independent of $\tilde C_{s+1}$ or δ_{s+1} and only utilize properties of quantities obtained in the so-far inductive process, as long as (3.16) holds. We now define:

$$(a_\rho^s)^2=\delta_s\bar a_\rho(H),\qquad\text{so that:}\ \ \mathcal{D}(g_0-\delta_{s+1}H_0,u_{sd_*})=\sum_{\rho=1}^{d_*}(a_\rho^s)^2\eta_\rho\otimes\eta_\rho, \tag{3.17}$$

and observe the following bounds:

$$\begin{aligned}&\|(a_\rho^s)^2-\delta_s\|_0\leq\frac{\delta_s}{2}\qquad\text{and}\qquad\|a_\rho^s-\delta_s^{1/2}\|_0=\Big\|\frac{(a_\rho^s)^2-\delta_s}{a_\rho^s+\delta_s^{1/2}}\Big\|_0\leq\frac{\delta_s/2}{\delta_s^{1/2}}=\frac{\delta_s^{1/2}}{2},\\&\|\nabla^{(m)}(a_\rho^s)^2\|_0\leq C\lambda_{sd_*}^m\delta_s\qquad\text{and}\qquad\|\nabla^{(m)}a_\rho^s\|_0\leq C\lambda_{sd_*}^m\delta_s^{1/2}\qquad\text{for all}\ \ m\geq 1,\end{aligned} \tag{3.18}$$

where the last bound follows by the Faá di Bruno inequality in:

$$\begin{aligned}\|\nabla^{(m)}a_\rho^s\|_0&\leq C\Big\|\sum_{p_1+2p_2+\ldots+mp_m=m}(a_\rho^s)^{2(\frac12-p_1-\ldots-p_m)}\prod_{t=1}^m|\nabla^{(t)}(a_\rho^s)^2|^{p_t}\Big\|_0\\&\leq C\|a_\rho^s\|_0\sum_{p_1+2p_2+\ldots+mp_m=m}\prod_{t=1}^m\Big\|\frac{\nabla^{(t)}(a_\rho^s)^2}{(a_\rho^s)^2}\Big\|_0^{p_t}\leq C\delta_s^{1/2}\sum_{p_1+2p_2+\ldots+mp_m=m}\prod_{t=1}^m\Big\|\frac{\delta_s\lambda_{sd_*}^t}{\delta_s/2}\Big\|_0^{p_t}\\&\leq C\lambda_{sd_*}^m\delta_s^{1/2},\end{aligned}$$

Again, we point out that constants in (3.18) depend only on $\omega, \underline{\gamma}, d, k$ through quantities obtained in the so-far inductive process, but are independent of $\tilde{C}_{s+1}$ or δ_{s+1}.

4. (Proof of $(3.13)_1$, $(3.13)_2$ and $(3.13)_3$) Fix $i = 1 \dots N$. We will prove the indicated assertions, assuming that they hold up to the counter $i-1$. We write:

$$i = jk + \theta = sd_* + \rho \quad \text{ with } \ \theta = 1 \dots k, \ \rho = 1 \dots d_*, \ j = 0 \dots J-1, \ s = 0 \dots S-1, \tag{3.19}$$

and apply (3.12), (3.18), and $(3.13)_3$ (3.14) in the form $\|\nabla^{(v)} E^\theta_{i-1}\|_0 + \|\nabla^{(v)} T_{i-1}\|_0 \leq C\lambda^v_{i-1}$, valid in view of the last assumption in (3.2), to estimate:

$$\begin{aligned} \|\nabla^{(m)}(u_i - u_{i-1})\|_0 \leq C \sum_{t+w+v=m} & \Big(\lambda_i^{t-1} \|\nabla^{(w)} a^s_\rho\|_0 \|\nabla^{(v)} E^\theta_{i-1}\|_0 \\ & + \lambda_i^{t-1} \|\nabla^{(w)} (a^s_\rho)^2\|_0 \|\nabla^{(v)} T^\theta_{i-1}\|_0 + \lambda_i^{t-2} \|\nabla^{(w+1)} (a^s_\rho)^2\|_0 \|\nabla^{(v)} T^\theta_{i-1}\|_0\Big) \\ \leq C \sum_{t+w+v=m} & \Big(\lambda_i^{t-1} \delta_s^{1/2} \lambda^w_{sd_*} \lambda^v_{i-1} + \lambda_i^{t-1} \delta_s \lambda^w_{sd_*} \lambda^v_{i-1} + \lambda_i^{t-2} \delta_s \lambda^{w+1}_{sd_*} \lambda^v_{i-1}\Big) \\ \leq C\lambda_i^{m-1} \delta_s^{1/2} & \quad \text{ for all } \ m \geq 0, \end{aligned} \tag{3.20}$$

since both λ_{sd_*} and λ_{i-1} are not greater than λ_i. In particular, there follows $(3.13)_2$, namely:

$$\|u_i - u_{i-1}\|_1 \leq C\delta_s^{1/2} \leq C\delta_0^{1/2}.$$

Using now the induction assumption and (3.4), we obtain:

$$\|u_i - u\|_1 \leq \|u_i - u_0\|_1 + \|u_0 - u\|_1 \leq C\delta_0^{1/2},$$

which yields $(3.13)_1$ upon setting $\underline{\delta}$ sufficiently small to get, when $\delta_0 = \delta \leq \underline{\delta}$:

$$\begin{aligned} \|(\nabla u_i)^T \nabla u_i - (\nabla u)^T \nabla u\|_0 & \leq \|\nabla(u_i - u)\|_0 \big(2\|\nabla u\|_0 + \|\nabla(u_i - u)\|_0\big) \\ & \leq C\delta_0^{1/2} (2(2\underline{\gamma} d)^{1/2} + 1) \leq C\underline{\delta}^{1/2} \leq \frac{1}{6\underline{\gamma}}, \end{aligned}$$

where we use Lemma 2.4 and recall $(3.1)_1$. The first inequality in $(3.13)_3$ follows by (3.20) in:

$$\|\nabla^{(m)} \nabla^{(2)} (u_i - u_{i-1})\|_0 \leq C\lambda_i^{m+1} \delta_s^{1/2} \leq C\lambda_0 \lambda_i^m \sigma^{j+1} \delta_0^{1/2} \quad \text{ for all } \ m \geq 0,$$

because by (3.9) and (3.10) there holds:

$$\lambda_i^{m+1} \delta_s^{1/2} = \lambda_0 \lambda_i^m \delta_0^{1/2} \frac{\lambda_i}{\lambda_0} \big(\frac{\delta_s}{\delta_0}\big)^{1/2} = \lambda_0 \lambda_i^m \delta_0^{1/2} \sigma^{1+j+s/2} \frac{\tilde{C}_s^{1/2}}{\sigma^{s/2}} \leq C\lambda_0 \lambda_i^m \sigma^{j+1} \delta_0^{1/2}. \tag{3.21}$$

To define the normal frame $\{E_i^\theta\}_{\theta=1}^k$ and prove the second inequality in $(3.13)_3$, we use Lemma 2.7 we first validate its assumptions. By (3.20) and the induction assumption, we get:

$$\begin{aligned} & \|\nabla^{(m)}(\nabla u_i - \nabla u_{i-1})\|_0 \leq C\lambda_i^m \delta_s^{1/2} \qquad \text{ for all } \ m \geq 0, \\ & \|\nabla^{(m)} \nabla^{(2)} u_{i-1}\|_0 + \|\nabla^{(m+1)} E^\theta_{i-1}\|_0 \leq C\lambda^{m+1}_{i-1} \leq C\lambda_i^{m+1} \quad \text{ for all } \ \theta = 1 \dots k, \ m \geq 0. \end{aligned}$$

Thus, taking $A = \delta_s^{1/2}$ and $\mu = \lambda_i$, the estimate (2.8) becomes:

$$\|\nabla^{(m)}(E_i^\theta - E^\theta_{i-1})\|_0 \leq C\lambda_i^m \delta_s^{1/2} \quad \text{ for all } \ \theta = 1 \dots k, \ m \geq 0, \tag{3.22}$$

which completes the proof of $(3.13)_3$, because $\lambda_i^{m+1} \delta_s^{1/2} \leq \lambda_0 \lambda_i^m \sigma^{j+1} \delta_0^{1/2}$, as shown in (3.21).

5. (Proof of $(3.13)_4$) We fix the index $i = 1 \dots N$ and express it as in (3.19), with the appropriate θ and ρ. Now, fix any $\bar\theta = 1 \dots k$ and estimate derivatives of the three terms in:

$$\langle \partial^2_{pq} u_i, E_i^{\bar\theta}\rangle = I + II + III, \quad \text{where } \ I = \langle \partial^2_{pq} u_{i-1}, E_{i-1}^{\bar\theta}\rangle, \quad II = \langle \partial^2_{pq} u_i, E_i^{\bar\theta} - E_{i-1}^{\bar\theta}\rangle,$$
$$III = \langle \partial^2_{pq}(u_i - u_{i-1}), E_{i-1}^{\bar\theta}\rangle.$$

For the first term I, note that $jk \le i-1 < (j+1)k$, so the induction assumption on $(3.13)_4$ implies that for any $\bar\theta$ there holds:

$$\|\nabla^{(m)} I\|_0 \le C\lambda_0 \lambda_{i-1}^m \sigma^{j+1}\delta_0^{1/2} \le C\lambda_0\lambda_i^m \sigma^{j+1}\delta_0^{1/2} \quad \text{for all } \ m \ge 0.$$

On the other hand, when $jk < i = jk + \theta < jk + \bar\theta$ then, trivially $jk \le i-1 < jk+\bar\theta$, so:

$$\|\nabla^{(m)} I\|_0 \le C\lambda_0 \lambda_{i-1}^m \sigma^{j}\delta_0^{1/2} \le C\lambda_0\lambda_i^m \sigma^{j}\delta_0^{1/2} \quad \text{for all } \ m \ge 0.$$

For the terms II, we use the already proven bound on $\|\nabla^{(m+2)}u_i\|_0$ and (3.22) to get:

$$\begin{aligned}
\|\nabla^{(m)} II\|_0 &\le C \sum_{t+v=m} \|\nabla^{(t+2)} u_i\|_0 \|\nabla^{(v)}(E_i^{\bar\theta} - E_{i-1}^{\bar\theta})\|_0 \\
&\le C \sum_{t+v=m} \lambda_0 \lambda_i^t \sigma^{j+1}\delta_0^{1/2} \lambda_i^v \delta_s^{1/2} \le C\lambda_0\lambda_i^m\sigma^j\delta_0^{1/2}(\sigma\delta_s^{1/2}) \\
&\le C\lambda_0\lambda_i^m \sigma^j \delta_0^{1/2} \quad \text{for all } \ m \ge 0,
\end{aligned}$$

as $\sigma\delta_s^{1/2} \le \sigma\delta_0^{1/2} \le 1$ by the last assumption in (3.2). For the term III, we use (3.12) to get:

$$\begin{aligned}
III = {} & A + B + C + D \quad \text{where } \ A = \partial^2_{pq}\Big(\frac{\Gamma(\lambda_i\langle x, \eta_\rho\rangle)}{\lambda_i} a_\rho^s\Big)\langle E_{i-1}^\theta, E_{i-1}^{\bar\theta}\rangle, \\
& B = \partial_p\Big(\frac{\Gamma(\lambda_i\langle x, \eta_\rho\rangle)}{\lambda_i} a_\rho^s\Big)\langle \partial_q E_{i-1}^\theta, E_{i-1}^{\bar\theta}\rangle + \partial_q\Big(\frac{\Gamma(\lambda_i\langle x, \eta_\rho\rangle)}{\lambda_i} a_\rho^s\Big)\langle \partial_p E_{i-1}^\theta, E_{i-1}^{\bar\theta}\rangle \\
& \qquad + \frac{\Gamma(\lambda_i\langle x, \eta_\rho\rangle)}{\lambda_i} a_\rho^s \langle \partial_{pq} E_{i-1}^\theta, E_{i-1}^{\bar\theta}\rangle, \\
& C = \partial_p\Big(\frac{\bar\Gamma(\lambda_i\langle x, (\eta_\rho)^2\rangle)}{\lambda_i}(a_\rho^s)^2\Big)\langle (\partial_q T_{i-1})\eta_\rho, E_{i-1}^{\bar\theta}\rangle + \partial_q\Big(\frac{\bar\Gamma(\lambda_i\langle x, \eta_\rho\rangle)}{\lambda_i}(a_\rho^s)^2\Big)\langle (\partial_p T_{i-1})\eta_\rho, E_{i-1}^{\bar\theta}\rangle \\
& \qquad + \frac{\bar\Gamma(\lambda_i\langle x, \eta_\rho\rangle)}{\lambda_i}(a_\rho^s)^2 \langle (\partial^2_{pq} T_{i-1})\eta_\rho, E_{i-1}^{\bar\theta}\rangle, \\
& D = \Big\langle (\partial_p T_{i-1})\partial_q\Big(\frac{\bar{\bar\Gamma}(\lambda_i\langle x, \eta_\rho\rangle)}{2\lambda_i^2}\nabla(a_\rho^s)^2\Big), E_{i-1}^{\bar\theta}\Big\rangle \\
& \quad + \Big\langle (\partial_q T_{i-1})\partial_p\Big(\frac{\bar{\bar\Gamma}(\lambda_i\langle x, \eta_\rho\rangle)}{2\lambda_i^2}\nabla(a_\rho^s)^2\Big), E_{i-1}^{\bar\theta}\Big\rangle + \frac{\bar{\bar\Gamma}(\lambda_i\langle x, \eta_\rho\rangle)}{2\lambda_i^2}\langle (\partial^2_{pq} T_{i-1})\nabla(a_\rho^s)^2, E_{i-1}^{\bar\theta}\rangle.
\end{aligned}$$

Note that the term A is nonzero only when $\theta = \bar\theta$, in which case we get by (3.18) and (3.21):

$$\begin{aligned}
\|\nabla^{(m)} A\|_0 &\le C \sum_{t+v=m+2} \lambda_i^{t-1}\|\nabla^{(v)} a_\rho^s\|_0 \le C \sum_{t+v=m+2} \lambda_i^{t-1}\lambda_{sd_*}^v \delta_s^{1/2} \\
&\le C\lambda_i^{m+1}\delta_s^{1/2} \le C\lambda_0\lambda_i^m\sigma^{j+1}\delta_0^{1/2} \quad \text{for all } \ m \ge 0.
\end{aligned}$$

When $jk < i = jk + \theta < jk + \bar\theta$, then certainly $\theta \ne \bar\theta$, so the above term is not present in III. We now show that all other terms estimate by $C\lambda_0\lambda_i^m\sigma^j\delta_0^{1/2}$ in their m-th derivatives, for any

$\bar\theta = 1\dots k$. Indeed, from (3.18) and $(3.13)_3$, (3.15) at $i-1$ we get:

$$\begin{aligned}
\|\nabla^{(m)}B\|_0 &\leq C\sum_{t+w+v=m}\ \sum_{\bar t+\bar w=t+1}\lambda_i^{\bar t-1}\|\nabla^{(\bar w)}a_\rho^s\|_0\|\nabla^{(w+1)}E_{i-1}^\theta\|_0\|\nabla^{(v)}E_{i-1}^{\bar\theta}\|_0\\
&\quad + C\sum_{t+w+v+\bar v=m}\lambda_i^{t-1}\|\nabla^{(w)}a_\rho^s\|_0\|\nabla^{(v+2)}E_{i-1}^\theta\|_0\|\nabla^{(\bar v)}E_{i-1}^{\bar\theta}\|_0\\
&\leq C\sum_{t+v+w=m}\ \sum_{\bar t+\bar w=t+1}\lambda_i^{\bar t-1}\lambda_{sd_*}^{\bar w}\delta_s^{1/2}\lambda_0\lambda_{i-1}^{w}\sigma^{j+1}\delta_0^{1/2}\lambda_{i-1}^{v}\\
&\quad + C\sum_{t+w+v+\bar v=m}\lambda_i^{t-1}\lambda_{sd_*}^{w}\delta_s^{1/2}\lambda_0\lambda_{i-1}^{v+1}\sigma^{j+1}\delta_0^{1/2}\lambda_{i-1}^{\bar v}\\
&\leq C\lambda_0\lambda_i^m\sigma^j\delta_0^{1/2}(\sigma\delta_s^{1/2}) \leq C\lambda_0\lambda_i^m\sigma^j\delta_0^{1/2} \qquad \text{for all } \ m\geq 0,
\end{aligned}$$

since $\sigma\delta_s^{1/2}\leq 1$. Similarly, replacing the application of $(3.13)_3$ by (3.14), implies:

$$\begin{aligned}
\|\nabla^{(m)}C\|_0 &\leq C\sum_{t+w+v=m}\ \sum_{\bar t+\bar w=t+1}\lambda_i^{\bar t-1}\|\nabla^{(\bar w)}(a_\rho^s)^2\|_0\|\nabla^{(w+1)}T_{i-1}\|_0\|\nabla^{(v)}E_{i-1}^{\bar\theta}\|_0\\
&\quad + C\sum_{t+w+v+\bar v=m}\lambda_i^{t-1}\|\nabla^{(w)}(a_\rho^s)^2\|_0\|\nabla^{(v+2)}T_{i-1}\|_0\|\nabla^{(\bar v)}E_{i-1}^{\bar\theta}\|_0\\
&\leq C\lambda_0\lambda_i^m\sigma^j\delta_0^{1/2}(\sigma\delta_s) \leq C\lambda_0\lambda_i^m\sigma^j\delta_0^{1/2} \qquad \text{for all } \ m\geq 0.
\end{aligned}$$

Finally, for the term D we get, as above:

$$\begin{aligned}
\|\nabla^{(m)}D\|_0 &\leq C\sum_{t+w+v=m}\ \sum_{\bar t+\bar w=t+1}\lambda_i^{\bar t-2}\|\nabla^{(\bar w+1)}(a_\rho^s)^2\|_0\|\nabla^{(w+1)}T_{i-1}\|_0\|\nabla^{(v)}E_{i-1}^{\bar\theta}\|_0\\
&\quad + C\sum_{t+w+v+\bar v=m}\lambda_i^{t-2}\|\nabla^{(w+1)}(a_\rho^s)^2\|_0\|\nabla^{(v+2)}T_{i-1}\|_0\|\nabla^{(\bar v)}E_{i-1}^{\bar\theta}\|_0\\
&\leq C\sum_{t+v+w=m}\ \sum_{\bar t+\bar w=t+1}\lambda_i^{\bar t-2}\lambda_{sd_*}^{\bar w+1}\delta_s\lambda_0\lambda_{i-1}^{w}\sigma^{j+1}\delta_0^{1/2}\lambda_{i-1}^{v}\\
&\quad + C\sum_{t+w+v+\bar v=m}\lambda_i^{t-2}\lambda_{sd_*}^{w+1}\delta_s\lambda_0\lambda_{i-1}^{v+1}\sigma^{j+1}\delta_0^{1/2}\lambda_{i-1}^{\bar v}\\
&\leq C\lambda_0\lambda_i^m\sigma^j\delta_0^{1/2}(\sigma\delta_s) \leq C\lambda_0\lambda_i^m\sigma^j\delta_0^{1/2} \qquad \text{for all } \ m\geq 0.
\end{aligned}$$

This ends the proof of $(3.13)_4$ at the counter value i.

6. (Proof of $(3.13)_5$) Fix the index $s=0\dots S-1$ and assume that $(3.13)_1$ - $(3.13)_4$ hold for all $i=1\dots(s+1)d_*$, together with $(3.13)_5$ if $s\geq 1$ or (3.8) in case $s=0$. To prove $(3.13)_5$ at the counter value $s+1$, we use the decomposition (3.17) and write:

$$\begin{aligned}
&\mathcal{D}(g_0-\delta_{s+1}H_0, u_{(s+1)d_*})\\
&= \mathcal{D}(g_0-\delta_{s+1}H_0, u_{sd_*}) - \Big((\nabla u_{(s+1)d_*})^T\nabla u_{(s+1)d_*} - (\nabla u_{sd_*})^T\nabla u_{sd_*}\Big) = \sum_{\rho=1}^{d_*} I_\rho, \qquad (3.23)
\end{aligned}$$

$$\text{where: }\ I_\rho = (a_\rho^s)^2\eta_\rho\otimes\eta_\rho - \Big((\nabla u_{sd_*+\rho})^T\nabla u_{sd_*+\rho} - (\nabla u_{sd_*+\rho-1})^T\nabla u_{sd_*+\rho-1}\Big).$$

Recalling (2.3) in Lemma 2.3, we see that each term I_ρ for $\rho = 1\dots d_*$ has the following form, where we denote $i = sd_* + \rho = jk + \theta$ for the uniquely defined $j = 0\dots J-1$ and $\theta = 1\dots k$:

$$I_\rho = A + B + C - \mathcal{R}, \quad \text{where:} \;\; A = \frac{2\Gamma(\lambda_i\langle x, \eta_\rho\rangle)}{\lambda_i} a_\rho^s \big[\langle \partial^2_{pq} u_{i-1}, E^\theta_{i-1}\rangle\big]_{p,q=1\dots d}$$

$$B = -\frac{2\bar{\bar{\Gamma}}(\lambda_i\langle x, \eta_\rho\rangle)}{\lambda_i^2} a_\rho^s \nabla^2 a_\rho^s, \qquad C = -\frac{\bar{\bar{\bar{\Gamma}}}(\lambda_i\langle x, \eta_\rho\rangle)}{\lambda_i^2} \nabla a_\rho^s \otimes \nabla a_\rho^s,$$

and where $\mathcal{R}$ comprises the terms $\mathcal{R}_0\dots\mathcal{R}_4$ given in Lemma 2.3 with the following defining quantites: $\lambda = \lambda_i$, $a = a_\rho^s$, $\eta = \eta_\rho$, $u = u_{i-1}$, $E_u = E^\theta_{i-1}$. We now estimate, in virtue of (3.18), and of $(3.13)_4$ used with $jk \leq i-1 < jk+\theta$:

$$\begin{aligned}
\|\nabla^{(m)} A\|_0 &\leq C \sum_{t+w+v=m} \lambda_i^{t-1} \|\nabla^{(w)} a_\rho^s\|_0 \sum_{p,q=1\dots d} \|\nabla^{(v)} \langle \partial^2_{pq} u_{i-1}, E^\theta_{i-1}\rangle\|_0 \\
&\leq C \sum_{t+w+v=m} \lambda_i^{t-1} \lambda_{sd_*}^w \delta_s^{1/2} \lambda_0 \lambda_{i-1}^v \sigma^j \delta_0^{1/2} \leq C\lambda_0 \lambda_i^{m-1} \delta_s^{1/2} \sigma^j \delta_0^{1/2} \\
&= C\lambda_i^m \delta_s \sigma^j \frac{\lambda_0}{\lambda_i} \Big(\frac{\delta_0}{\delta_s}\Big)^{1/2} \leq C\lambda^m_{(s+1)d_*} \frac{\delta_s}{\sigma} \quad \text{ for all } \; m \geq 0.
\end{aligned}$$

The last bound is due to $\lambda_i \leq \lambda_{(s+1)d_*}$ and to (3.9), (3.10) in:

$$\sigma^j \frac{\lambda_0}{\lambda_i} \Big(\frac{\delta_0}{\delta_s}\Big)^{1/2} = \frac{\sigma^j}{\sigma^{1+j+s/2}} \frac{\sigma^{s/2}}{\tilde{C}_s^{1/2}} \leq \frac{1}{\sigma}.$$

The next two terms B and C are similarly estimated by:

$$\begin{aligned}
\|\nabla^{(m)} (B+C)\|_0 &\leq C \sum_{t+w+v=m} \lambda_i^{t-2} \big(\|\nabla^{(w)} a_\rho^s\|_0 \|\nabla^{(v+2)} a_\rho^s\|_0 + \|\nabla^{(w+1)} a_\rho^s\|_0 \|\nabla^{(v+1)} a_\rho^s\|_0 \big) \\
&\leq C \sum_{t+w+v=m} \lambda_i^{t-2} \lambda_{sd_*}^{w+v+2} \delta_s \leq C\lambda_i^m \frac{\delta_s}{(\lambda_i/\lambda_{sd_*})^2} \\
&\leq C\lambda^m_{(s+1)d_*} \frac{\delta_s}{\sigma} \quad \text{ for all } \; m \geq 0,
\end{aligned}$$

because $\lambda_i/\lambda_{sd_*} \geq \sigma^{1/2}$ from the definition of the progression of frequencies leading to (3.9). We proceed to estimating derivatives of the remainder terms in $\mathcal{R}$. For $\mathcal{R}_0$, we observe:

$$\begin{aligned}
\|\nabla^{(m)} \mathcal{R}_0\|_0 &\leq C \sum_{t+w+v=m} \lambda_i^t \big(\|\nabla^{(w)} (a_\rho^s)^4\|_0 \|\nabla^{(v)} |T_{i-1}|^2\|_0 \leq C \sum_{t+w+v=m} \lambda_i^t \lambda_{sd_*}^w \delta_s^2 \lambda_{i-1}^v \\
&\leq C\lambda_i^m \delta_s^2 \leq C\lambda^m_{(s+1)d_*} \frac{\delta_s}{\sigma} \quad \text{ for all } \; m \geq 0,
\end{aligned}$$

where we used (3.18), (3.15) and the fact that $\sigma\delta_s \leq \sigma\delta_0 \leq 1$, evident from the last of the conditions listed in (3.2). For the remaining terms constituting $\mathcal{R}_1\dots\mathcal{R}_4$, we note that each of them contains the product of at least two quantities involving a_ρ^s (or ∇a_ρ^s), which jointly contribute the multiplier δ_s in the estimate; and the products of either another instance of a_ρ^s or one of: $\nabla^2 u$ or ∇E^θ_{i-1}, each of which contribute the muliplier bounded from above by $\delta_0^{1/2}$. By assuring $\delta_0^{1/2}$ to be sufficiently small with respect to the powers of σ accumulated in the estimate of each term, through the last assumption in (3.2), we arrive at:

$$\|\nabla^{(m)} \mathcal{R}\|_0 \leq C\lambda^m_{(s+1)d_*} \frac{\delta_s}{\sigma} \quad \text{ for all } \; m \geq 0,$$

Consequently, it follows that each $\nabla^{(m)} I_\rho$, for $\rho = 1 \dots d_*$, obeys the same bound as above, hence by (3.23) we may conclude that:

$$\begin{aligned}
&\|\nabla^{(m)}\mathcal{D}(g_0 - \delta_{s+1}H_0, u_{(s+1)d_*})\|_0 \leq C\lambda^m_{(s+1)d_*}\frac{\delta_s}{\sigma} \\
&= \frac{r_0}{5}\lambda^m_{(s+1)d_*}\delta_{s+1}\cdot\frac{5C}{r_0}\frac{\tilde{C}_s}{\tilde{C}_{s+1}} \leq \frac{r_0}{5}\lambda^m_{(s+1)d_*}\delta_{s+1} \qquad \text{for all } \ m \geq 0.
\end{aligned}$$

The last estimate is valid in view of (3.10), provided that we define $\tilde{C}_{s+1}$ so that:

$$\tilde{C}_{s+1} \geq \frac{5C}{r_0}\tilde{C}_s.$$

This ends the proof of $(3.13)_5$ at the counter value $s+1$, and the proof of all the inductive estimates $(3.13)_1$ - $(3.13)_5$.

7. (The reparametrisation and the end of proof) We now define:

$$\tilde{u} = u_N,$$

whereupon $(3.13)_1$ - $(3.13)_5$ and (3.4), (3.10) imply:

$$\begin{aligned}
&\|u - \tilde{u}\|_1 \leq C\delta_0^{1/2}, \qquad \|\nabla^2\tilde{u}\|_0 \leq C\lambda_0\sigma^J\delta_0^{1/2}, \\
&\Big\|\mathcal{D}\Big(g - \frac{\tilde{C}_S\delta_0}{\sigma^S}H_0, \tilde{u}\Big)\Big\|_0 \leq \frac{r_0}{5}\frac{\tilde{C}_S\delta_0}{\sigma^S} + \frac{\|g\|_{r,\beta}}{\lambda_0^{r+\beta}}.
\end{aligned} \tag{3.24}$$

These are essentially the claimed bounds $(1.8)_1$, $(1.8)_2$, up to the constant $\tilde{C}_S$, that depends only on $\underline{\gamma}, \omega, d, k$. Given σ, we define the associated reparametrised relative frequency:

$$\sigma_{new} = \frac{\sigma}{\tilde{C}_S^{1/S}},$$

whereupon the first two bounds in (3.24) remain the same and the last one becomes exactly $(1.8)_2$. We claim that the assumptions in $(1.7)_1$ for σ_{new} imply assumptions in (3.2) for σ:

$$\begin{aligned}
&\sigma = \tilde{C}_S^{1/S}\sigma_{new} \geq \sigma_{new} \geq \underline{\sigma}, \\
&\sigma^{J+2}\delta^{1/2} = \tilde{C}_S^{(J+2)/S}\sigma_{new}^{J+2}\delta^{1/2} \leq \sigma_{new}^{J+3}\delta^{1/2} \leq 1,
\end{aligned}$$

provided that $\tilde{C}_S^{(J+2)/S} \leq \sigma_{new}$, which is secured by taking a sufficiently large $\underline{\sigma} \geq \tilde{C}_S^{(J+2)/S}$. The proof is done. ■

4. Energy scaling bound for non-Euclidean films and a proof of Theorem 1.2

Given a solution $u \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^{d+k})$ of (1.1) corrresponding to the d-dimensional Riemannian metric $g(\cdot, 0)_{d\times d}$ on $\bar{\omega}$ – that is, the restriction induced by the ambient $(d+k)$-dimensional metric g, to the midplate $\bar{\omega}$ – whose existence is guaranteed by Theorem 1.1, we will explicitly define the family of immersions:

$$\{u^h \in H^1(\Omega^h, \mathbb{R}^{d+k})\}_{h\to 0}$$

via a suitable version of the Kirchhoff-Love extension, adapted to the metric g. We will then show that, with a constant C depending only on $\omega, g, d, k, \alpha, u$, but not on h, there holds:

$$\mathcal{E}_g^h(u^h) \leq Ch^{\frac{4\alpha}{\alpha+1}} \qquad \text{for all } \ h \ll 1.$$

We use the following notation. For a matrix $g \in \mathbb{R}^{(d+k)\times(d+k)}_{\mathrm{sym},>}$, we denote by $g_{d\times d}$, $g_{d\times k}$, $g_{k\times d}$, $g_{k\times k}$, the four block submatrices determined by the first d and last k rows and columns, respectively. Clearly $g_{d\times k} = (g_{k\times d})^T$, and both matrices $g_{d\times d}$ and $g_{k\times k}$ are symmetric and positive definite, satisfying the formula:

$$\big((g^{-1})_{k\times k}\big)^{-1} = g_{k\times k} - g_{k\times d}(g_{d\times d})^{-1} g_{d\times k}. \tag{4.1}$$

Proof of Theorem 1.2

1. Fix $0 < \alpha < \min\big\{\frac{r+\beta}{2}, q(d,k)\big\}$, where $q = q(d,k) \leq 1$ is the maximal regularity exponent achievable through any variant of Theorem 1.1. In the present paper, the range (1.2) implies $q = \frac{1}{1+2d_*/k}$. Then, there exists $u \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^{d+k})$ satisfying:

$$(\nabla u)^T \nabla u = g(\cdot, 0)_{d\times d} \quad \text{ in } \bar\omega.$$

Without loss of generality, $r + \beta \leq 2$. We will use the regularisation as in Lemma 2.1:

$$u_\epsilon * \phi_\epsilon \in \mathcal{C}^2(\bar\omega, \mathbb{R}^{d+k}), \qquad \text{where } \ \epsilon = h^t$$

for some exponent $t > 0$ that will be chosen later. It follows by a direct inspection that:

$$\begin{aligned} &\|\nabla u_\epsilon\|_0 \leq C, \qquad \|\nabla^2 u_\epsilon\|_0 \leq C\epsilon^{\alpha-1}, \\ &\|g(\cdot,0)_{d\times d} * \phi_\epsilon - g(\cdot,0)_{d\times d}\|_0 \leq C\epsilon^{r+\beta} \leq C\epsilon^{2\alpha}, \\ &\|(\nabla u_\epsilon)^T\nabla u_\epsilon - g(\cdot,0)_{d\times d} * \phi_\epsilon\|_0 \leq C\epsilon^{2\alpha}, \end{aligned} \tag{4.2}$$

where in the last bound we used the following extension of $(2.1)_3$ from [12, Lemma 2.1]:

$$\|\nabla^{(m)}\big((fg)*\phi_l - (f*\phi_l)(g*\phi_l)\big)\|_0 \leq C l^{2\alpha - m}\|f\|_{0,\alpha}\|g\|_{0,\alpha},$$

with constants C independent of ϵ. Note that in (4.2) we also used that $2\alpha \leq r+\beta$. By Lemma 2.6 we further obtain existence of the normal frame $\{E^i_\epsilon \in \mathcal{C}^1(\bar\omega, \mathbb{R}^{d+k})\}_{i=1}^k$, for each $\epsilon \ll 1$:

$$(\nabla u_\epsilon)^T E^i_\epsilon = 0, \quad |E^i_\epsilon| = 1, \quad \langle E^i_\epsilon, E^j_\epsilon\rangle = 0 \quad \text{in } \ \bar\omega, \qquad \text{for all } \ i,j = 1\dots k, \ i \neq j,$$

with the following uniform bounds, due to the fact that the corresponding immersability constant γ of u_ϵ is likewise uniform for all small $\epsilon > 0$:

$$\|\nabla E^i_\epsilon\|_0 \leq C\big(1 + \|\nabla u_\epsilon\|_1\big) \leq C\epsilon^{\alpha-1} \qquad \text{for all } \ i = 1\dots k. \tag{4.3}$$

Since the proof of Lemma 2.6 essentially constructs the normal frame by extension and the Gramm-Schmidt normalisation, starting from the chosen orthonormal frame to $\nabla u_\epsilon(x)$ at a fixed chosen $x \in \omega$, it additionally follows the orientation preservation namely:

$$\det\big[\ \nabla u_\epsilon \ \big| \ E^1_\epsilon \ \ \dots \ \ E^k_\epsilon \ \big] > 0 \quad \text{ in } \ \bar\omega. \tag{4.4}$$

2. We define the Cosserat vectors $\{b^i_\epsilon \in \mathcal{C}^1(\bar\omega, \mathbb{R}^{d+k})\}_{i=1}^k$ through the following formulas:

$$B_\epsilon \doteq \big[\ b^1_\epsilon \ \ b^2_\epsilon \ \ \dots \ \ b^k_\epsilon \ \big] = \big[\ \nabla u_\epsilon \ \big| \ E^1_\epsilon \ \ \dots \ \ E^k_\epsilon \ \big] A_\epsilon \in \mathcal{C}^1(\bar\omega, \mathbb{R}^{(d+k)\times k}),$$

where the matrix of coefficients $A_\epsilon \in \mathcal{C}^1(\bar\omega, \mathbb{R}^{(d+k)\times k})$ is defined so that:

$$\begin{aligned} &(\nabla u^h)^T\nabla u^h(\cdot,0) = \left[\begin{array}{c|c} (\nabla u_\epsilon)^T\nabla u_\epsilon & g(\cdot,0)_{d\times k} * \phi_\epsilon \\ \hline g(\cdot,0)_{k\times d} * \phi_\epsilon & g(\cdot,0)_{k\times k} * \phi_\epsilon \end{array}\right] \\ &\text{and} \qquad \det\nabla u^h(\cdot,0) > 0 \quad \text{ on } \ \bar\omega, \end{aligned} \tag{4.5}$$

for the Kirchhoff-Love extensions u^h of the mollified fields u_ϵ given by:

$$u^h(x,z) = u_\epsilon(x) + \sum_{i=1}^k z_i b^i_\epsilon(x) \quad \text{ for all } (x,z)\in\Omega^h.$$

To utilize the defining condition (4.5), we first compute:

$$\nabla u^h(x,z) = \Big[\ \nabla u_\epsilon(x) + \textstyle\sum_{i=1}^k z_i \nabla b^i_\epsilon(x)\ \Big|\ B_\epsilon(x)\ \Big] \in \mathbb{R}^{(d+k)\times(d+k)} \quad \text{ for all } (x,z)\in\Omega^h.$$

Equating the $d\times k$ and $k\times k$ minors in (4.5), the first condition in there yields:

$$(\nabla u_\epsilon)^T B_\epsilon = g(\cdot,0)_{d\times k} * \phi_\epsilon, \qquad B_\epsilon^T B_\epsilon = g(\cdot,0)_{k\times k} * \phi_\epsilon \tag{4.6}$$

Expanding with the help of the coefficient matrix fields A_ϵ, we obtain:

$$\begin{aligned}
(\nabla u_\epsilon)^T B_\epsilon &= \big[\ (\nabla u_\epsilon)^T\nabla u_\epsilon\ \big|\ 0\ \big]\, A_\epsilon = \big((\nabla u_\epsilon)^T\nabla u_\epsilon\big)(A_\epsilon)_{d\times k},\\
B_\epsilon^T B_\epsilon &= A_\epsilon^T \operatorname{diag}\big[(\nabla u_\epsilon)^T\nabla u_\epsilon, \mathrm{Id}_k\big] A_\epsilon\\
&= (A_\epsilon)^T_{d\times k}\big((\nabla u_\epsilon)^T\nabla u_\epsilon\big)(A_\epsilon)_{d\times k} + (A_\epsilon)^T_{k\times k}(A_\epsilon)_{k\times k},
\end{aligned}$$

so that (4.6) becomes:

$$\begin{aligned}
&(A_\epsilon)_{d\times k} = \big((\nabla u_\epsilon)^T\nabla u_\epsilon\big)^{-1}\big(g(\cdot,0)_{d\times k} * \phi_\epsilon\big),\\
&(A_\epsilon)^T_{k\times k}(A_\epsilon)_{k\times k} = g(\cdot,0)_{k\times k} * \phi_\epsilon - \big(g(\cdot,0)_{k\times d} * \phi_\epsilon\big)\big((\nabla u_\epsilon)^T\nabla u_\epsilon\big)^{-1}\big(g(\cdot,0)_{d\times k} * \phi_\epsilon\big).
\end{aligned} \tag{4.7}$$

We see that $(A_\epsilon)_{d\times k}\in\mathcal{C}^1(\bar\omega,\mathbb{R}^{d\times k})$ is indeed well defined, as $\big((\nabla u_\epsilon)^T\nabla u_\epsilon\big)^{-1}$ is well defined in virtue of the last two formulas in (4.2). Similarly, $(A_\epsilon)_{k\times k}\in\mathcal{C}^1(\bar\omega,\mathbb{R}^{k\times k}_{\mathrm{sym},>})$ may be taken as the unique symmetric, positive definite square root of the second right hand side in (4.7), which is positive definite as its difference from the following matrix field is infinitesimal as $\epsilon\to 0$:

$$g(\cdot,0)_{k\times k} - g(\cdot,0)_{k\times d}\big(g(\cdot,0)_{d\times d}\big)^{-1} g(\cdot,0)_{d\times k} = \big((g(\cdot,0)^{-1})_{k\times k}\big)^{-1}$$

by (4.2), where above we have used the formula (4.1). Towards the second requirement in (4.5), we apply the column operations to obtain that in $\bar\omega$ there holds:

$$\begin{aligned}
\det\nabla u^h(\cdot,0) &= \det\big[\ \nabla u_\epsilon\ \big|\ B_\epsilon\ \big] = \det\Big[\ \nabla u_\epsilon\ \Big|\ \big[\ \nabla u_\epsilon \quad E^1_\epsilon \quad \ldots \quad E^k_\epsilon\ \big]\, A_\epsilon\ \Big]\\
&= \det\Big[\ \nabla u_\epsilon\ \Big|\ \big[\ E^1_\epsilon \quad \ldots \quad E^k_\epsilon\ \big]\,(A_\epsilon)_{k\times k}\ \Big]\\
&= \det\big[\ \nabla u_\epsilon\ \big|\ E^1_\epsilon \quad \ldots \quad E^k_\epsilon\ \big]\cdot \det\operatorname{diag}\big[\mathrm{Id}_d, (A_\epsilon)_{k\times k}\big] > 0,
\end{aligned}$$

by (4.4) and since $(A_\epsilon)_{k\times k}$ is defined as positive definite. Finally, by (4.2) we directly note:

$$\|A_\epsilon\|_0 \le C, \qquad \|\nabla (A_\epsilon)_{d\times k}\|_0 + \|\nabla\big((A_\epsilon)^T_{k\times k}(A_\epsilon)_{k\times k}\big)\|_0 \le C\epsilon^{\alpha-1},$$

which then implies in virtue of our construction, the derivative bound:

$$\|\nabla A_\epsilon\|_0 \le C\epsilon^{\alpha-1}.$$

We therefore conclude, in view of (4.3), that:

$$\begin{aligned}
&\|B_\epsilon\|_0 \le C\big(\|\nabla u_\epsilon\|_0 + 1\big)\|A_\epsilon\|_0 \le C,\\
&\|\nabla B_\epsilon\|_0 \le C\big(\|\nabla^2 u_\epsilon\|_0 + C\epsilon^{\alpha-1}\big)\|A_\epsilon\|_0 + C\big(\|\nabla u_\epsilon\|_0+1\big)\|\nabla A_\epsilon\|_0 \le C\epsilon^{\alpha-1}.
\end{aligned} \tag{4.8}$$

3. Since $\det\big((\nabla u^h) g^{-1/2}(\cdot,0)\big)$ is positive and bounded away from 0 on $\bar\omega$, uniformly in $\epsilon\ll 1$, we use the polar decomposition theorem to get, for some $SO(d+k)$-valued function Q^h:

$$(\nabla u^h) g^{-1/2}(\cdot,0) = Q^h\Big(g^{-1/2}\big((\nabla u^h)^T\nabla u^h\big)g^{-1/2}(\cdot,0)\Big)^{1/2}.$$

Observe that, in view of (4.5) and (4.2):

$$\begin{aligned}
&\big\|g^{-1/2}\big((\nabla u^h)^T\nabla u^h\big)g^{-1/2}(\cdot,0) - \mathrm{Id}_d\big\|_0 \\
&= \big\|g^{-1/2}\big((\nabla u^h)^T\nabla u^h - g\big)g^{-1/2}(\cdot,0)\big\|_0 \\
&\leq C\big(\|(\nabla u_\epsilon)^T\nabla u_\epsilon - g(\cdot,0)\|_0 + \|g(\cdot,0) - g(\cdot,0)*\phi_\epsilon\|_0\big) \leq C\epsilon^{2\alpha}
\end{aligned} \tag{4.9}$$

so consequently $(\nabla u^h)g^{-1/2}(x,0)$ for all $x\in\bar\omega$, remains in a small neighbourhood of $SO(d+k)$. In that (sufficiently small) neighbourhood, we may then use the assumed properties of the energy density W. We also observe that the same statement holds for $(\nabla u^h)g^{-1/2}$ on the entire domain Ω^h, provided that in the expression $h\epsilon^{\alpha-1} = h^{1+t(\alpha-1)}$ the exponent is positive, namely:

$$1 + t(\alpha-1) > 0. \tag{4.10}$$

This is because (4.8) implies:

$$\begin{aligned}
&|(\nabla u^h)g^{-1/2}(x,z) - (\nabla u^h)g^{-1/2}(x,0)| \\
&\leq Ch(\|\nabla B_\epsilon\|_0 + 1) \leq Ch\epsilon^{\alpha-1} \quad \text{ for all } \ (x,z)\in\Omega^h,
\end{aligned} \tag{4.11}$$

We are now ready to estimate, by the frame invariance of W and by (4.9), 4.11):

$$\begin{aligned}
&\mathcal{E}_g^h(u^h) \leq C\int_{\Omega^1} W\big((\nabla u^h)g^{-1/2}(x,hz)\big)\;\mathrm{d}(x,z) \\
&= C\int_{\Omega^1} W\big((Q^h)^T(\nabla u^h)g^{-1/2}(x,0) + \mathcal{O}(h\epsilon^{\alpha-1})\big)\;\mathrm{d}(x,z) \\
&= C\int_{\Omega^1} W\big((\mathrm{Id}_d + \mathcal{O}(\epsilon^{2\alpha}))^{1/2} + \mathcal{O}(h\epsilon^{\alpha-1})\big) = C\int_{\Omega^1} W\big(\mathrm{Id}_d + \mathcal{O}(\epsilon^{2\alpha}) + \mathcal{O}(h\epsilon^{\alpha-1})\big) \\
&\leq C\big(\epsilon^{2\alpha} + h\epsilon^{\alpha-1}\big)^2 = C\big(h^{2t\alpha} + h^{1+t(\alpha-1)}\big)^2.
\end{aligned}$$

Therefore, choosing $t = \frac{1}{\alpha+1}$ so that $2t\alpha = 1 + t(\alpha-1)$ which in particular validates the requirement (4.10), the above formula yields:

$$\inf \mathcal{E}_g^h \leq \mathcal{E}_g^h(u^h) \leq Ch^{2t\alpha} = Ch^{\frac{4\alpha}{\alpha+1}},$$

completing the proof of Theorem 1.2. ■

5. Appendix A: A proof of Lemma 2.3

In this section, we prove the key identity (2.3) stated in Lemma 2.3. Observe that the oscillatory perturbation is introduced in a single chose normal direction E_u and is accompanied by a matching tangential correction, following the construction inspired by [25]. A related formula was derived for the Monge-Ampère system in [26], and should be contrasted with the constructions in [5, Lemma 3.1] and [33] which contain additional terms of the form $\frac{\Gamma}{\lambda}a\nabla a\otimes\eta$. Such terms would prevent the implementation of the stage induction construction leading to estimates in Theorem 1.3. Naturally, the error terms in (2.3) are now considerably more intricate, additionally featuring the five orders of residual terms $\mathcal{R}_0\ldots\mathcal{R}_4$.

Proof of Lemma 2.5

We start by calculating the gradient of the modified immersion:

$$\begin{aligned}
\nabla\tilde u = {} & \nabla u + \Gamma'(\lambda t)aE_u\otimes\eta + \bar\Gamma'(\lambda t)a^2T_u\eta\otimes\eta \\
& + \frac{\Gamma(\lambda t)}{\lambda}\nabla(aE_u) + \frac{\bar\Gamma(\lambda t)}{\lambda}\nabla(a^2T_u\eta) + \frac{\bar{\bar\Gamma}'(\lambda t)}{\lambda}aT_u\nabla a\otimes\eta + \frac{\bar{\bar\Gamma}(\lambda t)}{\lambda^2}\nabla\big(aT_u\nabla a\big).
\end{aligned}$$

Taking into account that $(\nabla u)^T T_u = \mathrm{Id}_d$ we obtain the following formula, where we suppress the argument λt in Γ and $\bar\Gamma$, and order the terms according to powers of λ in the denominators:

$$
\begin{aligned}
&(\nabla\tilde u)^T\nabla\tilde u - (\nabla u)^T\nabla u \\
&= \Big\{2\bar\Gamma' a^2\eta\otimes\eta + (\Gamma')^2a^2\eta\otimes\eta + (\bar\Gamma')^2a^4|T_u\eta|^2\eta\otimes\eta\Big\} \\
&\quad + \Big\{\frac{2\Gamma}{\lambda}\mathrm{sym}\big((\nabla u)^T\nabla(aE_u)\big) + \frac{2\bar\Gamma}{\lambda}\mathrm{sym}\big((\nabla u)^T\nabla(a^2T_u\eta)\big) + \frac{2\bar{\bar\Gamma}'}{\lambda}a\ \mathrm{sym}\big(\nabla a\otimes\eta\big) \\
&\qquad + \frac{2\Gamma\Gamma'}{\lambda}a\ \mathrm{sym}\big(\nabla(aE_u)^T(E_u\otimes\eta)\big) + \frac{2\bar\Gamma\Gamma'}{\lambda}a\ \mathrm{sym}\big(\nabla(a^2T_u\eta)^T(E_u\otimes\eta)\big) \\
&\qquad + \frac{2\Gamma\bar\Gamma'}{\lambda}a^2\ \mathrm{sym}\big(\nabla(aE_u)^TT_u(\eta\otimes\eta)\big) + \frac{2\bar\Gamma\bar\Gamma'}{\lambda}a^2\ \mathrm{sym}\big(\nabla(a^2T_u\eta)^TT_u(\eta\otimes\eta)\big) \\
&\qquad + \frac{2\bar\Gamma'\bar{\bar\Gamma}'}{\lambda}a^3\ \big\langle T_u\eta, T_u\nabla a\big\rangle\eta\otimes\eta\Big\} \\
&\quad + \Big\{\frac{2\bar{\bar\Gamma}}{\lambda^2}\mathrm{sym}\big((\nabla u)^T\nabla(aT_u\nabla a)\big) + \frac{2\bar{\bar\Gamma}\Gamma'}{\lambda^2}a\ \mathrm{sym}\big(\nabla(aT_u\nabla a)^T(E_u\otimes\eta)\big) \\
&\qquad + \frac{2\bar\Gamma'\bar{\bar\Gamma}}{\lambda^2}a^2\ \mathrm{sym}\big(\nabla(aT_u\nabla a)^TT_u(\eta\otimes\eta)\big) + \frac{\Gamma^2}{\lambda^2}\nabla(aE_u)^T\nabla(aE_u) \\
&\qquad + \frac{2\Gamma\bar\Gamma}{\lambda^2}\mathrm{sym}\big(\nabla(aE_u)^T\nabla(a^2T_u\eta)\big) + \frac{2\Gamma\bar{\bar\Gamma}'}{\lambda^2}a\ \mathrm{sym}\big(\nabla(aE_u)^TT_u(\nabla a\otimes\eta)\big) \\
&\qquad + \frac{\bar\Gamma^2}{\lambda^2}\nabla(a^2T_u\eta)^T\nabla(a^2T_u\eta) + \frac{2\bar\Gamma\bar{\bar\Gamma}'}{\lambda^2}a\ \mathrm{sym}\big(\nabla(a^2T_u\eta)^TT_u(\nabla a\otimes\eta)\big) \\
&\qquad + \frac{(\bar{\bar\Gamma}')^2}{\lambda^2}a^2\ |T_u\nabla a|^2\eta\otimes\eta\Big\} \\
&\quad + \Big\{\frac{2\bar\Gamma\bar{\bar\Gamma}}{\lambda^3}\mathrm{sym}\big(\nabla(a^2T_u\eta)^T\nabla(aT_u\nabla a)\big) + \frac{2\Gamma\bar{\bar\Gamma}}{\lambda^3}\mathrm{sym}\big(\nabla(aE_u)^T\nabla(aT_u\nabla a)\big) \\
&\qquad + \frac{2\bar{\bar\Gamma}\bar{\bar\Gamma}'}{\lambda^3}a\ \mathrm{sym}\big(\nabla(aT_u\nabla a)^TT_u(\nabla a\otimes\eta)\big)\Big\} \\
&\quad + \frac{\bar{\bar\Gamma}^2}{\lambda^4}\nabla(aT_u\nabla a)^T\nabla(aT_u\nabla a).
\end{aligned}
\tag{5.1}
$$

We now rewrite the first, second and fourth terms in the second parentheses, in the form:

$$
\begin{aligned}
&\frac{2\Gamma}{\lambda}\mathrm{sym}\big((\nabla u)^T\nabla(aE_u)\big) = \frac{2\Gamma}{\lambda}a\ \mathrm{sym}\big((\nabla u)^T\nabla E_u\big) = -\frac{2\Gamma}{\lambda}a\ \big[\langle\partial^2_{ij}u, E_u\rangle\big]_{i,j=1\dots d}, \\
&\frac{2\bar\Gamma}{\lambda}\mathrm{sym}\big((\nabla u)^T\nabla(a^2T_u\eta)\big) = \frac{2\bar\Gamma}{\lambda}\Big(\mathrm{sym}\big(\nabla(a^2)\otimes\eta\big) + a^2\ \mathrm{sym}\big((\nabla u)^T\nabla(T_u\eta)\big)\Big) \\
&\qquad\qquad = \frac{4\bar\Gamma}{\lambda}a\ \mathrm{sym}\big(\nabla a\otimes\eta\big) - \frac{2\bar\Gamma}{\lambda}a^2\ \big[\langle\partial^2_{ij}u, T_u\eta\rangle\big]_{i,j=1\dots d}, \\
&\frac{2\Gamma\Gamma'}{\lambda}a\ \mathrm{sym}\big(\nabla(aE_u)^T(E_u\otimes\eta)\big) = \frac{2\Gamma\Gamma'}{\lambda}a\ \mathrm{sym}\big(\nabla a\otimes\eta\big),
\end{aligned}
$$

where we have used that $(\nabla u)^T E_u = 0$, $(\nabla u)^T T_u = \mathrm{Id}_d$ and $(\nabla E_u)^T E_u = 0$. Similarly, we rewrite the first and the fourth term in the third parentheses of (5.1), as follows:

$$\begin{aligned}
&\frac{2\bar{\bar{\Gamma}}}{\lambda^2}\mathrm{sym}\big((\nabla u)^T\nabla(aT_u\nabla a)\big) = \frac{2\bar{\bar{\Gamma}}}{\lambda^2}\Big(\nabla a\otimes\nabla a + a\nabla^2 a - a\,\big[\langle\partial^2_{ij}u, T_u\nabla a\rangle\big]_{i,j=1\dots d}\Big),\\
&\frac{\Gamma^2}{\lambda^2}\nabla(aE_u)^T\nabla(aE_u) = \frac{\Gamma^2}{\lambda^2}\Big(a^2\,(\nabla E_u)^T\nabla E_u + \nabla a\otimes\nabla a\Big).
\end{aligned}$$

In conclusion and recalling the error term $\mathcal{R}$, the formula (5.1) becomes:

$$\begin{aligned}
&(\nabla\tilde u)^T\nabla\tilde u - (\nabla u)^T\nabla u\\
&\quad = \big(2\bar\Gamma' + (\Gamma')^2\big)a^2\eta\otimes\eta - \frac{2\Gamma}{\lambda}a\,\big[\langle\partial^2_{ij}u, E_u\rangle\big]_{i,j=1\dots d} + \frac{4\bar\Gamma + 2\Gamma\Gamma' + 2\bar{\bar{\Gamma}}'}{\lambda}a\,\mathrm{sym}\big(\nabla a\otimes\eta\big)\\
&\qquad + \frac{2\bar{\bar{\Gamma}} + \Gamma^2}{\lambda^2}\nabla a\otimes\nabla a + \frac{2\bar{\bar{\Gamma}}}{\lambda^2}a\nabla^2 a + \mathcal{R},
\end{aligned}$$

whereas we conclude (2.3) in view of the identities:

$$2\bar\Gamma' + (\Gamma')^2 = 1, \qquad 4\bar\Gamma + 2\Gamma\Gamma' + 2\bar{\bar{\Gamma}}' = 0, \qquad 2\bar{\bar{\Gamma}} + \Gamma^2 = \bar{\bar{\bar{\Gamma}}}.$$

The proof is done. ■

6. Appendix B: proofs of propagation of normal vectors Lemmas 2.4, 2.5 and 2.7

In this section, we collect the proofs of the lemmas on the propagation of normal vectors stated in the preparatory section 2. The arguments follow the outline of [29, Section 3] now lifted to the general setting of arbitrary d and $k \geq 2$.

Proof of Lemma 2.4

The first assertion holds as $|\partial_i u(x)|^2 = \langle\nabla u(x)^T(\nabla u(x))e_i, e_i\rangle \leq \gamma$ for $i = 1\dots d$. For the second assertion, note that all eigenvalues of the symmetric matrix $(\nabla u)^T\nabla u$ lie within the interval $[1/\gamma, \gamma]$ for all $x \in \bar\omega$, by (2.4). This implies the stated determinant bound. ■

Proof of Lemma 2.5

Writing $T_u = \frac{1}{\det((\nabla u)^T\nabla u)}(\nabla u)\,\mathrm{cof}\big((\nabla u)^T\nabla u\big)$, we easily conclude that Lemma 2.4 implies the first assertion, with constant C depending on γ and d. For the second assertion, we estimate:

$$\begin{aligned}
\|\nabla^{(m)}\det((\nabla u)^T\nabla u)\|_0 &\leq C\sum_{p_1+\dots p_{2d}=m}\prod_{i=1}^{2d}\|\nabla^{(p_i+1)}u\|_0\\
&\leq C\mu^m A \qquad \text{for all } m = 1\dots n+1,
\end{aligned} \tag{6.1}$$

The first inequality above follows by observing that $\det((\nabla u)^T\nabla u)$ is a linear combination of products of $2d$ vectors chosen among $\{\partial_i u\}_{i=1}^d$. For the second inequality, at least one of the exponents $\{p_i\}_{i=1}^{2d}$ in there must be positive, say $p_1 \geq 1$ so that the assumption may be applied to estimate $\|\nabla^{(p_1+1)}u\|_0$ by $C\mu^{p_1}A$, whereas other terms in the product are estimated by $C\mu^{p_i}$

in view of Lemma 2.4 and $A < 1$. An application of Faà di Bruno's formula now yields:

$$\begin{aligned}
&\big\|\nabla^{(m)}\Big(\frac{1}{\det((\nabla u)^T\nabla u))}\Big)\big\|_0 \\
&\leq C \sum_{p_1+2p_2+\ldots mp_m=m} \big\|\det((\nabla u)^T\nabla u)^{-1-(p_1+\ldots+p_m)} \prod_{i=1}^m |\nabla^{(i)}\det((\nabla u)^T\nabla u)|^{p_i}\big\|_0 \\
&\leq C\mu^m A \quad \text{ for all } m = 1\ldots n+1.
\end{aligned}$$

In conclusion, and by a similar argument as in (6.1), we arrive at:

$$\begin{aligned}
&\|\nabla^{(m)}T_u\|_0 \leq C \sum_{q+s+p_1+\ldots p_{2(d-1)}=m} \big\|\nabla^{(q)}\Big(\frac{1}{\det((\nabla u)^T\nabla u))}\Big)\big\|_0 \|\nabla^{(s+1)}u\|_0 \prod_{i=1}^{2(d-1)} \|\nabla^{(p_i+1)}u\|_0 \\
&\leq C\mu^m A \quad \text{ for all } m = 1\ldots n+1.
\end{aligned}$$

This completes the proof. ■

Concerning existence of the normal frame in Lemma 2.6, we remark that the argument in [7] starts with a local construction on $\omega = B_1$, by first fixing an arbitrary orthonormal frame $\{\xi^i \in \mathbb{R}^{d+k}\}_{i=1}^k$ to $\nabla u(0)$ and then defining the smooth fields:

$$\nu_u^i(x) = \big(\mathrm{Id}_{d+k} - (\nabla u(x))(\nabla u(x)^T(\nabla u(x)))^{-1}\nabla u(x)^T\big)\xi^i \quad \text{ for } i = 1\ldots k.$$

These fields form a basis of the orthogonal complement of $\mathrm{span}\{\partial_i u(x)\}_{i=1}^k$ and can be Gramm-Schmidt orthonormalized to $\{E_u^i\}_{i=1}^k$ satisfying (2.5), (2.4) in a sufficiently small neighbourhood $\bar{B}_r$ of 0. Then, the key ingredient in the proof is to show that such local frame can be extended on $\bar{B}_{r+\delta}$ and obey the same bounds, with $\delta > 0$ that depends only on γ, d, k, N. The construction is explicit, via a partition of unity argument.

Proof of Lemma 2.7

1. Observe that, by the first bound in Lemma 2.4, we get:

$$\|(\nabla v)^T\nabla v - (\nabla u)^T\nabla u\|_0 \leq \|\nabla v - \nabla u\|_0(\|\nabla u\|_0 + \|\nabla v\|_0) \leq \rho(2(d\gamma)^{1/2}+1),$$

which implies for ρ small (in function of d, γ), that:

$$\frac{1}{2\gamma}\mathrm{Id}_d \leq (\nabla v)^T\nabla v \leq 2\gamma\mathrm{Id}_d \quad \text{ in } \bar{\omega}. \tag{6.2}$$

In particular, T_v is well defined, and since $(\nabla v)^T T_v = \mathrm{Id}_d$, we obtain:

$$(\nabla v)^T\nu_v^i = \big((\nabla v)^T - (\nabla v - \nabla u)^T\big)E_u^i = (\nabla u)^T E_u^i = 0 \quad \text{ for } i = 1\ldots k. \tag{6.3}$$

Further, applying Lemma 2.5 to bound $\|T_v\|_0$ by a constant that only depends on γ, d, we get:

$$\begin{aligned}
&\||\nu_v^i| - 1\|_0 \leq \|\nu_v^i - E_u^i\|_0 \leq \|T_v\|_0\|\nabla v - \nabla u\|_0 \leq C\rho \leq \epsilon \\
&\|\langle\nu_v^i, \nu_v^j\rangle - \delta_{ij}\|_0 \leq 2\|T_v\|_0\|\nabla v - \nabla u\|_0 + \|T_v\|_0^2\|\nabla v - \nabla u\|_0^2 \leq \epsilon \quad \text{ for } i,j = 1\ldots k.
\end{aligned} \tag{6.4}$$

provided that ρ sufficiently small, in function of the small parameter $\epsilon > 0$ that will be specified below. It is already clear that the fields $\{\nu_v^i\}_{i=1}^k$ can be Gramm-Schmidt orthonormalised to the normal frame $\{E_v^i\}_{i=1}^k$ satisfying conditions in (i). Nevertheless, we will require specific estimates to prove (ii), so we present the details of that construction in steps 2-4 below.

2. To show that $\{E_v^i\}_{i=1}^k$ in (2.7) are well defined, we now check that:

$$\||\tilde{\nu}_v^i| - 1\|_0 \leq \frac{1}{2} \quad \text{ where } \quad \tilde{\nu}_v^i = \nu_v^i - \sum_{j<i}\langle\nu_v^i, E_v^j\rangle E_v^j \quad \text{ for all } i = 1\ldots k. \tag{6.5}$$

For $i=1$, we have $\tilde\nu_v^1=\nu_v^1$, so (6.5) holds by (6.4) if $\epsilon\leq 1/2$. For $i=2\dots k$, we will show:

$$\|\langle \nu_v^i, E_v^j\rangle\|_0\leq \frac{1}{2^{2+3(k-i)}k^{1+(k-i)}}\quad \text{ for all }\ j<i. \tag{6.6}$$

Then, (6.5) will be implied by (6.6) and by taking $\epsilon\leq 1/4$ in (6.4), because:

$$\||\tilde\nu_v^i|-1\|_0\leq \||\nu_v^i|-1\|_0 + \sum_{j<i}\|\langle \nu_v^i, E_v^j\rangle\|_0\leq \epsilon + \frac{1}{2^{2+3(k-i)}k^{k-i}}\leq \epsilon+\frac{1}{4}\leq\frac{1}{2}.$$

We prove (6.6) by induction on i. When $i=2$, then:

$$|\langle \nu_v^2, E_v^1\rangle| = \frac{|\langle \nu_v^2,\nu_v^1\rangle|}{|\nu_v^1|}\leq 2\epsilon\leq \frac{1}{2^{2+3(k-2)}k^{k-1}}\quad \text{ in } \bar\omega,$$

in virtue of (6.4), provided that $\epsilon\leq 1/2$ is small enough (in function of the codimension k). For the induction step, assume that (6.6) holds up to the counter $i\geq 2$ and fix $j<i+1$. Then:

$$\begin{split} |\langle \nu_v^{i+1}, E_v^j\rangle| & = \frac{|\langle \nu_v^{i+1},\tilde\nu_v^j\rangle|}{|\tilde\nu_v^j|}\leq 2\Big(|\langle\nu_v^{i+1},\nu_v^j\rangle| + k\max_{s<j}|\langle \nu_v^j, E_v^s\rangle|\cdot|\langle \nu_v^{i+1}, E_v^s\rangle|\Big) \\ & \leq 2\epsilon + \frac{4k}{2^{2+3(k-j)}k^{1+(k-j)}}\leq \frac{1}{2^{2+3(k-i-1)}k^{1+(k-i-1)}}\quad \text{ in } \bar\omega, \end{split}$$

where we used that $|\tilde\nu_v^j|\geq 1/2$ in $\bar\omega$ from induction assumption and that $\|\langle\nu_v^{i+1}, E_v^s\rangle\|_0\leq \|\nu_v^{i+1}\|_0\leq 2$. The last bound above follows for ϵ sufficiently small, since $j\leq i$ in:

$$\frac{4k}{2^{2+3(k-j)}k^{1+(k-j)}} = \frac{1/2}{2^{2+3(k-j-1)}k^{1+(k-j-1)}}\leq \frac{1/2}{2^{2+3(k-i-1)}k^{1+(k-i-1)}}.$$

This ends the proof of (6.6) and thus of (6.5).

3. We now check that for every $x\in\bar\omega$, the k-tuple of unit vectors $\{E_v^i(x)\}_{i=1}^k$ is orthogonal, by inductively showing that:

$$\langle E_v^i, E_v^j\rangle = 0 \quad \text{ in } \bar\omega \quad \text{ for all } j<i. \tag{6.7}$$

Indeed, for $i=2$ there holds: $\langle E_v^2, E_v^1\rangle = \frac{1}{|\tilde\nu_v^2|}\langle \nu_v^2 - \langle \nu_v^2, E_v^1\rangle E_v^1, E_v^1\rangle = 0$. For the induction step, assume that (6.7) is satisfied up to some counter $i\geq 2$. Fix $j<i+1$ and compute:

$$\begin{split} \langle E_v^{i+1}, E_v^j\rangle & = \frac{1}{|\tilde\nu_v^{i+1}|}\Big\langle \nu_v^{i+1} - \sum_{s<i+1}\langle \nu_v^{i+1}, E_v^s\rangle E_v^s, E_v^j\Big\rangle \\ & = \frac{1}{|\tilde\nu_v^{i+1}|}\Big(\langle \nu_v^{i+1}, E_v^j\rangle - \sum_{s<i+1}\langle \nu_v^{i+1}, E_v^s\rangle\langle E_v^s, E_v^j\rangle\Big) = \frac{1}{|\tilde\nu_v^{i+1}|}\big(\langle \nu_v^{i+1}, E_v^j\rangle - \langle \nu_v^{i+1}, E_v^j\rangle\big) = 0, \end{split}$$

as the only nonzero term among $\langle E_v^s, E_v^j\rangle$ is that corresponding to $s=j$. Indeed, if either $s<j$ or $j<s$, we can use the induction assumption by $s,j<i+1$. This ends the proof of (6.7).

4. Finally, we show that $(\nabla v)^T E_v^i = 0$ in $\bar\omega$, for all $i=1\dots k$. For $i=1$ we directly use (6.3) in: $(\nabla v)^T E_v^1 = \frac{1}{|\nu_v^1|}(\nabla v)^T\nu_v^1 = 0$. For the induction step from i to $i+1$, we likewise conclude the claim by (6.3) and the induction assumption, writing:

$$(\nabla v)^T E_v^{i+1} = \frac{1}{|\tilde\nu_v^{i+1}|}\Big((\nabla v)^T\nu_v^{i+1} - \sum_{j<i+1}\langle \nu_v^i, E_v^j\rangle (\nabla v)^T E_v^j\Big) = 0.$$

This ends the proof of part (i) of Lemma.

5. In the next three steps we prove the assertion (ii). Since v obeys (6.2) and since from the assumptions we get: $\|\nabla^{(m)}\nabla^{(2)}v\|_0 \leq 2\bar{C}\mu^{m+1}$ for all $m = 0\dots n$, Lemma 2.5 implies that:

$$\|\nabla^{(m)}T_v\|_0 \leq C\mu^m \qquad \text{for all } m = 0\dots n+1. \tag{6.8}$$

Consequently, in view of the assumption in (ii):

$$\begin{aligned}
&\|\nabla^{(m)}(\nu_v^i - E_u^i)\|_0 \leq C\sum_{p+q+t=m}\|\nabla^{(p)}T_v\|_0\|\nabla^{(q)}(\nabla u - \nabla v)\|_0\|\nabla^{(t)}E_u^i\|_0 \leq C\mu^m A,\\
&\|\nabla^{(m)}\nu_v^i\|_0 \leq C\mu^m \qquad \text{for all } m = 0\dots n+1,\ i = 1\dots k.
\end{aligned} \tag{6.9}$$

We will prove (2.8) by induction on $i = 1\dots k$, along with the additional bound:

$$\|\nabla^{(m)}\langle\nu_v^i, E_v^j\rangle\|_0 \leq C\mu^m A, \qquad \text{for all } m = 0\dots n+1,\ j < i. \tag{6.10}$$

At the induction base $i = 1$, assertion (6.10) is void, whereas towards (2.8) we write:

$$E_v^1 - E_u^1 = f(\nu_v^1) - f(E_u^1) = \Big(\int_0^1 \nabla f(t\nu_v^1 + (1-t)E_u^1)\ \mathrm{d}t\Big)(\nu_v^1 - E_u^1)$$

$$\text{where } f(z) = \frac{z}{|z|} \quad \text{with} \quad \partial_i f^j(z) = \frac{\delta_{ij}|z|^2 - z_i z_j}{|z|^3} \qquad \text{for all } z \in \mathbb{R}^{d+k}\setminus\{0\},\ i,j = 1\dots d+k.$$

For each fixed $t \in (0,1)$, we now bound $\|\nabla_x^{(m)}\partial_i f^j(t(\nu_v^1 - E_u^1) + E_u^1)\|_0$. Firstly, by (6.9):

$$\|\nabla_x^{(m)}(t(\nu_v^1 - E_u^1) + E_u^1)\|_0 \leq C\mu^m \qquad \text{for all } m = 0\dots n+1,$$

and also the same estimate holds for the derivatives of $|t(\nu_v^1 - E_u^1) + E_u^1|^2$. Secondly, since $1/2 \leq |t(\nu_v^1 - E_u^1) + E_u^1| \leq 3/2$ by (6.4) upon taking $\epsilon \leq 1/2$, the Faà di Bruno formula yields:

$$\begin{aligned}
&\big\|\nabla_x^{(m)}\big(\frac{1}{|t(\nu_v^1 - E_u^1) + E_u^1|^3}\big)\big\|_0 = \big\|\nabla_x^{(m)}\big(|t(\nu_v^1 - E_u^1) + E_u^1|^2\big)^{-3/2}\big\|_0\\
&\leq C\sum_{p_1+2p_2+\dots mp_m = m}\Big\||t(\nu_v^1 - E_u^1) + E_u^1|^{2(-3/2-(p_1+\dots+p_m))}\prod_{j=1}^m\Big|\nabla_x^{(j)}|t(\nu_v^1 - E_u^1) + E_u^1|^2\Big|^{p_j}\Big\|_0\\
&\leq C\mu^m \qquad \text{for all } m = 1\dots n+1.
\end{aligned}$$

It hence follows that for all $m = 0\dots n+1$, independently of $t \in (0,1)$:

$$\begin{aligned}
&\|\nabla_x^{(m)}\nabla f\big(t(\nu_v^1 - E_u^1) + E_u^1\big)\|_0\\
&\leq C\sum_{p+q=m}\|\nabla_x^{(p)}|t(\nu_v^1 - E_u^1) + E_u^1|^2\|_0\big\|\nabla_x^q\big(\frac{1}{|t(\nu_v^1 - E_u^1) + E_u^1|^3}\big)\big\|_0 \leq C\mu^m,
\end{aligned}$$

which implies, through (6.9), the claimed bound (2.8):

$$\begin{aligned}
&\|\nabla^{(m)}(E_v^1 - E_u^1)\| \leq C\sum_{p+q=m}\Big\|\int_0^1 \nabla_x^{(p)}\nabla f\big(t(\nu_v^1 - E_u^1) + E_u^1\big)\ \mathrm{d}t\Big\|_0\|\nabla^{(q)}(\nu_v^1 - E_u^1)\|_0\\
&\leq C\mu^m A \qquad \text{for all } m = 0\dots n+1.
\end{aligned} \tag{6.11}$$

6. Assume (2.8) and (6.10) up to some $i \geq 1$. In the induction step, we aim to establish both estimates at $i+1$. For (6.10), fix $j < i+1$ and write:

$$\langle\nu_v^{i+1}, E_v^j\rangle = \frac{1}{|\tilde{\nu}_v^j|}P_v^{i+1,j} \qquad \text{where} \qquad P_v^{i+1,j} = \langle\nu_v^{i+1}, \nu_v^j\rangle - \sum_{s<j}\langle\nu_v^{i+1}, E_v^s\rangle\langle\nu_v^j, E_v^s\rangle. \tag{6.12}$$

We first analyze the components of $P_v^{i+1,j}$ defined above. Since:

$$\langle \nu_v^{i+1}, \nu_v^j\rangle = -2\big\langle \mathrm{sym}(T_v(\nabla v - \nabla u)^T) E_u^{i+1}, E_u^j\big\rangle + \big\langle T_v(\nabla v - \nabla u)^T E_u^{i+1}, T_v(\nabla v - \nabla u)^T E_u^j\big\rangle,$$

Lemma 2.5 and the assumption in (ii) imply:

$$\|\nabla^{(m)}\langle \nu_v^{i+1}, \nu_v^j\rangle\|_0 \leq C\mu^m A \quad \text{ for all } m = 0 \ldots n+1.$$

On the other hand, by (6.9) and the induction asssumption, we get:

$$\begin{aligned}\|\nabla^{(m)}\big(\langle \nu_v^{i+1}, E_v^s\rangle\langle \nu_v^j, E_v^s\rangle\big)\|_0 &\leq C \sum_{p+q+t=m} \|\nabla^{(p)}\nu_v^{i+1}\|_0 \|\nabla^{(q)}E_v^s\|_0 \|\nabla^{(t)}\langle \nu_v^j, E_v^s\rangle\|_0 \\ &\leq C\mu^m A \quad \text{ for all } m = 0 \ldots n+1, \ s<j.\end{aligned}$$

Consequently, it follows that:

$$\|\nabla^{(m)} P_v^{i+1,j}\|_0 \leq C\mu^m A \quad \text{ for all } m = 0 \ldots n+1. \tag{6.13}$$

Similarly, there holds by (6.9) and the induction assumption:

$$\begin{aligned}\|\nabla^{(m)}\tilde{\nu}_v^j\|_0 &\leq C\Big(\|\nabla^{(m)}\nu_v^j\|_0 + \sum_{s<j}\sum_{p+q=m}\|\nabla^{(p)}\langle \nu_v^j, E_v^s\rangle\|_0\|\nabla^{(q)}E_v^s\|_0\Big) \\ &\leq C\mu^m \quad \text{ for all } m = 0 \ldots n+1,\end{aligned}$$

which yields $\|\nabla^{(m)}|\tilde{\nu}_v^j|^2\|_0 \leq C\mu^m$ and further, using Faa di Bruno's inequality in view of (6.5):

$$\|\nabla^{(m)}\frac{1}{|\tilde{\nu}_v^j|}\|_0 \leq C\mu^m \quad \text{ for all } m = 0 \ldots n+1.$$

In conclusion, (6.13), (6.12) and the above estimate imply (6.10) at $i+1$.

7. We now prove (2.8) at $i+1$. First, by (6.9), the assumption in (ii) and the already established bound (6.10) at $i+1$, we observe that:

$$\begin{aligned}\|\nabla^{(m)}(\tilde{\nu}_v^{i+1} - E_u^{i+1})\|_0 &\leq C\Big(\|\nabla^{(m)}(\nu_v^{i+1} - E_u^{i+1})\|_0 \\ &\qquad + \sum_{j<i+1}\sum_{p+q=m}\|\nabla^{(p)}\big(\langle \nu_v^{i+1}, E_v^j\rangle\|_0\|\nabla^{(q)}E_v^j\|_0\Big) \\ &\leq C\mu^m A \quad \text{ for all } m = 0 \ldots n+1.\end{aligned}$$

We now proceed exactly as in step 5, by writing:

$$E_v^{i+1} - E_u^{i+1} = f(\tilde{\nu}_v^{i+1}) - f(E_u^{i+1}) = \Big(\int_0^1 \nabla f(t(\tilde{\nu}_v^{i+1} - E_v^{i+1}) + E_u^{i+1})\,\mathrm{d}t\Big)(\tilde{\nu}^{i+1} - E_u^{i+1}),$$

and conclude (2.8) by (6.10). This ends the proof of Lemma 2.7. ■

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