MODELS FOR ELASTIC SHELLS WITH INCOMPATIBLE STRAINS

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Abstract. The three-dimensional shapes of thin lamina such as leaves, flowers, feathers, wings etc, are driven by the differential strain induced by the relative growth. The growth takes place through variations in the Riemannian metric, given on the thin sheet as a function of location in the central plane and also across its thickness. The shape is then a consequence of elastic energy minimization on the frustrated geometrical object. Here we provide a rigorous derivation of the asymptotic theories for shapes of residually strained thin lamina with nontrivial curvatures, i.e. growing elastic shells in both the weakly and strongly curved regimes, generalizing earlier results for the growth of nominally flat plates. The different theories are distinguished by the scaling of the mid-surface curvature relative to the inverse thickness and growth strain, and also allow us to generalize the classical Föppl-von Kármán energy to theories of prestrained shallow shells.

1. Introduction

The physical basis for morphogenesis is now classical and elegantly presented in D'arcy Thompson's opus "On growth and form" as follows *"An organism is so complex a thing, and growth so complex a phenomenon, that for growth to be so uniform and constant in all the parts as to keep the whole shape unchanged would indeed be an unlikely and an unusual circumstance. Rates vary, proportions change, and the whole configuration alters accordingly."* From a mathematical and mechanical perspective, this reduces to a simple principle: differential growth in a body leads to residual strains that will generically result in changes in the shape of a tissue, organ or body. Eventually, the growth patterns are expected to themselves be regulated by these strains, so that this principle might well be the basis for the physical self-organization of biological tissues. Recent interest in characterizing the morphogenesis of low-dimensional structures such as filaments, laminae and their assemblies, is driven by the twin motivations of understanding the origin of shape in biological systems and the promise of mimicking them in artificial mimics [12, 13, 5]. The results lie at the interface of biology, physics and engineering, but they also have a deeply geometric character. Indeed, the basic question may be characterized in terms of a variation on a classical theme in differential geometry - that of embedding a shape with a given metric in a space of possibly different dimension [23, 24]. However, the goal now is not only to state the conditions when it might be done (or not), but also to constructively determine the resulting shapes in terms of an appropriate mechanical theory.

While these issues arise in three-dimensional tissues, the combination of the separation of scales that arises naturally in slender structures and the constraints associated with the

1991 *Mathematics Subject Classification.* 74K20, 74B20.

Key words and phrases. non-Euclidean plates, nonlinear elasticity, Gamma convergence, calculus of variations.

prescription of growth laws that are functions of space (and time) leads to the expectation that the resulting theories ought to be variants of classical elastic plate and shell theories such as the Föppl-von Kármán or the Donnell-Mushtari-Vlasov theories [1]. That this is the case, has been shown for bodies that are initially flat and thin i.e. elastic plates with no initial curvature, using analogies to thermoelasticity [22, 20], perturbation analysis [4, 5], and rigorous asymptotic analysis [14] that follows a program similar to the derivation of the equations for the nonlinear elasticity of thin plates and shells [7, 6, 16, 17, 18] and a linearized theory [19] for residually strained Kirchhoff plates [11]. However, most laminae are naturally curved in their strain-free configurations, as a consequence of slow relaxation, perhaps following a previous growth history. Since even infinitesimal deformations of a curved shell will potentially violate isometry relative to its rest state, one expects that differential growth of such an object will likely lead to a variety of possible low dimensional theories depending on the relative size of the metric changes imposed on the system. This potential multiplicity of asymptotic theories is of course presaged by a similar state of affairs for the derivation of a nonlinear theory of elastic shells [8, 18].

Here, we continue and extend the discussion in [20, 21, 14] and present a rigorous derivation of a series of asymptotic theories for the shape of *residually strained* thin lamina with nontrivial curvatures, i.e. growing elastic shells. As our starting point for a similar theory for growing curved shells, we use the observation that it is possible to change the shape of a lamina such as a blooming lily petal by driving it via excess growth of the margins relative to the interior, rather than via midrib deformations [27]. Here, a thermoelastic analogy [22] suggests a natural generalization of the Donnell-Mushtari-Vlasov shell theory [1] to growing shells [21], proposed as a mathematical model for blooming, activated from the initial (transverse) out-of-plane displacement v_0 of a petal's mid-surface. When $v_0 = 0$ the equations (6.5) reduce to the prestrained von Kármán equations (6.3) proposed in [20] and rigorously derived in [14] from *non-Euclidean elasticity*, where the imposed 3d prestrain is given via a Riemannian metric, whose components display the appropriate target (linear) stretching tensor ϵ_g (of order 2 in shell's thickness h), and the bending tensor κ_g (of order 1 in h, see (3.1)). Therefore, we focus on a particular regime of scaling for the prestrain tensor (2.6) which corresponds, a posteriori, in all different regimes of shallowness studied here, to von-Kàrmán type theories.

We start with a few comments regarding this particular choice of the scaling regime. From a mathematical point of view, the von-Kàrmàn regime, where the nonlinear elastic energy per unit thickness scales like the fourth power of thickness h, usually corresponds to sub-linear theories, i.e. the first nonlinear theories which arise (in a given context) when the magnitude of forces or of prestrain includes the elastic lamina to pass the threshold of linear behavior and to manifest phenomena such as buckling. For the same reason, the sublinear theories are also the least complicated among the nonlinear theories of plates and shells arising in the literature, and they are popular with engineers, physicists and applied mathematicians. More sophisticated technical challenges must be addressed when deriving lower order nonlinear theories by Γ-convergence. Thus it makes sense to begin the analysis of nonlinear shallow shell models, so far neglected in the literature, with the von Kármán regime. In a forthcoming paper [15], the authors address a shallow shell model arising in a force regime equivalent to the energy scaling h^β for $\beta < 4$, where, analogous to

[8], technical obstacles regarding properties of the Sobolev solutions to the Monge-Ampère equations must be addressed, before establishing the corresponding Γ-limit result.

In Section 2, we formulate our main results, in terms of a scaling analysis that leads to the hierarchy of limit models, in the corresponding prestrain and shallowness regimes. In Section 3 we argue that for non-flat mid-surface S (of arbitrary curvature or for the referential out-of-plane displacement $v_0 \neq 0$), the variationally correct 2d theory coincides with the extension of the classical *von Kármán energy* to shells, derived in [16]. In the special case $v_0 = 0$, this energy still reduces to the functional whose Euler-Lagrange equations are those derived growing elastic plates in [20]. In Section 4, we discuss a new shallow shell model, valid when the radius of curvature of the mid-surface is relatively large compared to the thickness. This limit is a prestrained plate model which inherits the geometric structure of the shallow shell. In Section 5, we consider the case where the radius of curvature and the thickness are comparable in magnitude, and appropriately compatible with the order of the prestrain tensor. We show that equations for a growing elastic shell can be formally derived by pulling back the tensors ϵ_g and κ_g, from *shallow shells* $(S_h)^h$ with reference mid-surface S_h given by the scaled out-of-plane displacement hv_0, onto a flat reference configuration. Furthermore, we argue that this theory for growing elastic shells is also the Euler-Lagrange equation of the variational limit for 3d nonlinear elastic energies on $(S_h)^h$. In Section 6 we proceed to discussing the models where the shallowness overcomes the given magnitude of the growth-induced prestrain. In this case the limit energy is impervious to the influence of shell's geometry, but the effects of growth may not be neglected, and indeed they lead to the the generalized von Kármán equations for a growing flat plate. In Section 7, we justify that under our prestrain or growth scaling assumptions, the derived models are the relevant ones when no boundary conditions or forces are present. Finally, in Section 8, we conclude with a discussion of the present results and prospects for the future. Since the proofs of the theorems consist of tedious yet minor (though necessary) modifications of the arguments detailed in [16, 14, 17], we refer the interested reader to the Appendix, where they are given for completeness.

2. Preliminaries and scaling limits

Let $v_0 \in \mathcal{C}^{1,1}(\bar\Omega)$ be an out-of-plate displacement on an open, bounded subset $\Omega \subset \mathbb{R}^2$, associated with a family of surfaces, parametrized by $\gamma \in [0,1]$:

$$S_\gamma = \phi_\gamma(\Omega), \text{ where } \phi_\gamma(x) = \big(x, \gamma v_0(x)\big) \qquad \forall x = (x_1, x_2) \in \Omega, \tag{2.1}$$

The unit normal vector to S_γ at $\phi_\gamma(x)$ is given by:

$$\vec n^\gamma(x) = \frac{\partial_1\phi_\gamma(x) \times \partial_2\phi_\gamma(x)}{|\partial_1\phi_\gamma(x) \times \partial_2\phi_\gamma(x)|} = \frac{1}{\sqrt{1+\gamma^2|\nabla v_0|^2}}\big(-\gamma\partial_1 v_0(x), -\gamma\partial_2 v_0(x), 1\big) \qquad \forall x \in \Omega.$$

For small $h > 0$, we now consider thin plates $\Omega^h = \Omega \times (-h/2, h/2)$ and 3d shells $(S_\gamma)^h$:

$$(S_\gamma)^h = \big\{\tilde\phi_\gamma(x, x_3);\ x \in \Omega,\ x_3 \in (-h/2, h/2)\big\}, \tag{2.2}$$

where the extension $\tilde\phi_\gamma : \Omega^h \to \mathbb{R}^3$ of ϕ_γ on Ω^h in (2.1) is given by the formula:

$$\tilde\phi_\gamma(x, x_3) = \phi_\gamma(x) + x_3\vec n^\gamma(x) \qquad \forall (x, x_3) \in \Omega^h. \tag{2.3}$$

For an elastic body with the reference configuration $(S_\gamma)^h$ we assume that its elastic energy density $W : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}_+$ is $\mathcal{C}^2$ regular in a neighborhood of $SO(3)$. Moreover, W satisfies the normalization, frame indifference and nondegeneracy conditions:

$$\begin{aligned} \exists c > 0 \quad \forall F \in \mathbb{R}^{3\times 3} \quad \forall R \in SO(3) \quad & W(R) = 0, \quad W(RF) = W(F), \\ & W(F) \geq c \,\mathrm{dist}^2(F, SO(3)). \end{aligned} \tag{2.4}$$

Above, F stands typically for the deformation gradient ∇u relative to the reference configuration $(S_\gamma)^h$. In the setting of the prestrained materials with the prestrain, given by the Riemannian metric:

$$p^h = (q^h)^T q^h \quad \text{on} \quad (S_\gamma)^h,$$

the tensor $F = \nabla u$ is replaced by $F = \nabla u (q^h)^{-1}$, so that the thickness averaged elastic energy is given by:

$$I^{\gamma,h}(u) = \frac{1}{h}\int_{(S_\gamma)^h} W(F)\,\mathrm{d}z = \frac{1}{h}\int_{(S_\gamma)^h} W(\nabla u (q^h)^{-1})\,\mathrm{d}z, \qquad \forall u \in W^{1,2}((S_\gamma)^h, \mathbb{R}^3). \tag{2.5}$$

Letting $\epsilon_g, \kappa_g : \bar\Omega \to \mathbb{R}^{3\times 3}$ be two given smooth tensors, for each small h we define the growth tensors q^h on $(S_\gamma)^h$ by:

$$q^h(\phi_\gamma(x) + x_3\vec n^\gamma(x)) = \mathrm{Id} + h^2\epsilon_g(x) + hx_3\kappa_g(x) \qquad \forall (x, x_3) \in \Omega^h. \tag{2.6}$$

The corresponding metric $p^h = (q^h)^T q^h$ on $(S_\gamma)^h$ is then:

$$p^h(\phi_\gamma(x) + x_3\vec n^\gamma(x)) = \mathrm{Id} + 2h^2 \mathrm{sym}\ \epsilon_g(x) + 2hx_3 \mathrm{sym}\ \kappa_g(x) + \mathcal{O}(h^3).$$

An important part of our study focuses on the asymptotic behavior in the limit of vanishing thickness $h \to 0$ of the variational models $I^{\gamma,h}$ in (2.5), when $\gamma = \gamma(h) = h^\alpha$ for a given exponent $0 \leq \alpha < +\infty$. The regime $\alpha > 0$ corresponds to the study of a *shallow* shell. However, we will identify *three distinct shallow shell limit models*, depending on the asymptotic behavior of the ratio γ/h, which in our setting depends only on the value of α. This allows us to rigorously derive the Γ-limits: $\Gamma\text{-}\lim_{h\to 0}\frac{1}{h^4}I^{h^\alpha,h}$, and show that under suitable incompatibility conditions on the strain tensors ϵ_g or κ_g, the infimum of energies $I^{h^\alpha,h}$ scales like h^4 irrespective of the value of α. This justifies our choice of the energy scaling and lends credibility to limiting models as physically relevant in the corresponding scaling regimes.

To get a sense of our results it is useful to summarize our analysis in terms of the $\Gamma-$limit of $\frac{1}{h^4}I^{h^\alpha,h}$, which can be identified as follows:

$$\Gamma\text{-}\lim_{h\to 0}\frac{1}{h^4}I^{h^\alpha,h} = \begin{cases} \mathcal{I}_4 & \text{if} \quad \alpha = 0 \\ \mathcal{I}_4^\infty & \text{if} \quad 0 < \alpha < 1 \\ \mathcal{I}_4^1 & \text{if} \quad \alpha = 1 \\ \mathcal{I}_4^0 & \text{if} \quad \alpha > 1. \end{cases} \tag{2.7}$$

The above four theories collapse into one and the same theory when $v_0 = 0$. Otherwise we must deal with four distinct potential limits depending on the choice of parameters, in the following order:

Case 1. $\alpha = 0$**.** This corresponds to $\gamma = 1$ where the 3d model is that of the prestrained non-linear elastic shell of arbitrarily large curvature (no shallowness involved). We will show that the Γ-limit in this case leads to a prestrained von Kármán model $\mathcal{I}_4$ for the 2d mid-surface S_1. This will be described in a more general framework in Section 3.

Case 2. $0 < \alpha < 1$**.** This corresponds to the flat limit $\gamma \to 0$ when the energy can be conceived as a limit of the von Kármán models $\mathcal{I}_4$ for shallow shells S_γ. In other words, this limiting model corresponds to the case when: $\lim_{h\to 0} \dfrac{\gamma(h)}{h} = \infty$, and it can be also identified as:

$$\mathcal{I}_4^\infty = \Gamma\text{-}\lim_{\gamma\to 0}\Big(\Gamma\text{-}\lim_{h\to 0} \frac{1}{h^4} I^{\gamma,h}\Big),$$

by choosing the distinguished sequence of limits, first as $h \to 0$ and then $\gamma \to 0$. In Section 4 we will see that $\mathcal{I}_4^\infty$ is formulated for displacements of a plate but it inherits certain geometric properties of shallow shells S_γ, such as the first-order infinitesimal isometry constraint.

Case 3. $\alpha = 1$**.** This corresponds to the case $\lim_{h\to 0} \gamma(h)/h = 1$. The limit model $\mathcal{I}_4^1$, derived in Section 5, is an unconstrained energy minimization, reflecting both the effect of shallowness and that of the prestrain. It corresponds to a simultaneous passing to the limit $(0, 0)$ of the pair (γ, h) in (2.5). The Euler-Lagrange equations (6.5) of $\mathcal{I}_4^1$ were suggested in [21] for the description of the deployment of petals during the blooming of a flower.

Case 4. $\alpha > 1$**.** Finally, the Γ-limit for all values of $\alpha > 1$, i.e. when $\lim_{h\to 0} \dfrac{\gamma(h)}{h} = 0$, coincides with the zero thickness limit of the degenerate case $\gamma = 0$, which is the prestrained plate von Kármán model, discussed in [14]. This limiting energy can be obtained by taking the consecutive limits:

$$\mathcal{I}_4^0 = \Gamma\text{-}\lim_{h\to 0}\Big(\Gamma\text{-}\lim_{\gamma\to 0} \frac{1}{h^4} I^{\gamma,h}\Big).$$

3. The prestrained von Kármán energy for shells of arbitrary curvature: $\alpha = 0$

When the parameter $\alpha = 0$, the 3d variational problem associated with (2.5) is reduced to the 3d nonlinear elastic energy on the thin shell S_1^h, where S_1 is the graph of v_0. It is useful to discuss this model in a more general framework. Let S be an arbitrary 2d surface embedded in $\mathbb{R}^3$, that is compact, connected, oriented, and of class $\mathcal{C}^{1,1}$. The boundary ∂S of S is assumed to be the union of finitely many (possibly none) Lipschitz continuous curves. We consider the family $\{S^h\}_{h>0}$ of thin shells of thickness h around S:

$$S^h = \Big\{z = x + t\vec{n}(x);\ x \in S,\ -\frac{h}{2} < t < \frac{h}{2}\Big\}, \qquad 0 < h < h_0 \ll 1$$

where we use the following notation: $\vec{n}(x)$ for the unit normal, T_xS for the tangent space, and $\Pi(x) = \nabla\vec{n}(x)$ for the shape operator on S, at a given $x \in S$. The projection onto S

along $\vec{n}$ is denoted by π, so that $\pi(z) = x$ for all $z = x + t\vec{n}(x) \in S^h$, and we assume that $h \ll 1$ is small enough to have π well defined on each S^h.

The instantaneous growth of S^h is described by smooth tensors: $\epsilon_g, \kappa_g : \overline{S} \longrightarrow \mathbb{R}^{3\times 3}$, by:

$$a^h = [a^h_{ij}] : \overline{S^h} \longrightarrow \mathbb{R}^{3\times 3}, \qquad a^h(x + t\vec{n}) = \mathrm{Id} + h^2\epsilon_g(x) + ht\kappa_g(x). \tag{3.1}$$

The growth tensor a^h is as in [20, 14], now in a general non-flat geometry setting. Given the elastic energy density $W : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}_+$ as in (2.4), the thickness averaged elastic energy induced by the prestrain a^h is given by:

$$I^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h (a^h)^{-1})\, \mathrm{d}z, \qquad \forall u^h \in W^{1,2}(S^h, \mathbb{R}^3). \tag{3.2}$$

Taking the asymptotic limit (the Γ-limit as $h \to 0$, see Theorem 3.1 and Theorem 3.2 below) of the energies I^h (note that $I^h = I^{1,h}$ in the notation of (2.5)) then leads to the variationally correct model for weakly prestrained shells. It corresponds to the following nonlinear energy functional $\mathcal{I}_4$ acting on the admissible limiting pairs (V, B):

$$\begin{aligned} \forall V \in \mathcal{V} \quad \forall B \in \mathcal{B} \qquad \mathcal{I}_4(V,B) = & \frac{1}{2}\int_S \mathcal{Q}_2\left(x, B - \frac{1}{2}(A^2)_{tan} - (\mathrm{sym}\ \epsilon_g)_{tan}\right) \\ & + \frac{1}{24}\int_S \mathcal{Q}_2\Big(x, (\nabla(A\vec{n}) - A\Pi)_{tan} - (\mathrm{sym}\ \kappa_g)_{tan}\Big). \end{aligned} \tag{3.3}$$

Here, the space $\mathcal{V}$ consists of *the first-order infinitesimal isometries* on S, defined by:

$$\mathcal{V} = \left\{V \in W^{2,2}(S, \mathbb{R}^3);\ \tau \cdot \partial_\tau V(x) = 0 \quad \forall \text{a.e. } x \in S \quad \forall \tau \in T_xS\right\}, \tag{3.4}$$

that is those $W^{2,2}$ regular displacements V for whom the change of metric on S due to the deformation $\mathrm{id} + \epsilon V$ is of order ϵ^2, as $\epsilon \to 0$. Furthermore, for a matrix field $A \in L^2(S, \mathbb{R}^{3\times 3})$, let $A_{tan}(x)$ denote the tangential minor of A at $x \in S$, that is $[(A(x)\tau)\eta]_{\tau,\eta \in T_xS}$. The skew-symmetric gradient of V as in (3.4) then uniquely determines a $W^{1,2}$ matrix field $A : S \longrightarrow so(3)$ so that: $\partial_\tau V(x) = A(x)\tau$ for all $\tau \in T_xS$. Hence, we equivalently write:

$$\begin{aligned} \mathcal{V} = \big\{V \in W^{2,2}(S, \mathbb{R}^3); \quad \exists A \in W^{1,2}(S, \mathbb{R}^{3\times 3}) \quad \forall \text{a.e. } x \in S \quad \forall \tau \in T_xS \\ \partial_\tau V(x) = A(x)\tau \text{ and } A(x)^T = -A(x)\big\}. \end{aligned}$$

For a plate, that is when $S \subset \mathbb{R}^2$, an equivalent analytic characterization for $V = (V^1, V^2, V^3) \in \mathcal{V}$ is given by: $(V^1, V^2) = (-\omega y, \omega x) + (b_1, b_2)$, while the out-of-plane displacement $V^3 \in W^{2,2}(S, \mathbb{R})$ remains unconstrained.

The space $\mathcal{B}$ in (3.3) consists of *finite strains*:

$$\mathcal{B} = \Big\{L^2 - \lim_{\epsilon \to 0} \mathrm{sym}\nabla w^\epsilon;\ w^\epsilon \in W^{1,2}(S, \mathbb{R}^3)\Big\}, \tag{3.5}$$

which are all limits of symmetrized gradients of sequences of displacements on S. By $\mathrm{sym}\nabla w(x)$ we mean here a bilinear form on T_xS given by: $(\mathrm{sym}\nabla w(x)\tau)\eta = \frac{1}{2}[(\partial_\tau w(x))\eta + (\partial_\eta w(x))\tau]$ for all $\tau, \eta \in T_xS$.

It follows (via Korn's inequality) that for a flat plate $S \subset \mathbb{R}^2$, the space $\mathcal{B}$ consists precisely of symmetrized gradients of all the in-plane displacements: $\mathcal{B} = \{\mathrm{sym}\nabla w;\ w \in W^{1,2}(S, \mathbb{R}^2)\}$. When S is strictly convex, rotationally symmetric, or developable without flat regions, it has been proven in [16, 26] that $\mathcal{B} = L^2(S, \mathbb{R}^{2\times 2}_{sym})$, i.e. it contains all symmetric matrix fields on S with square integrable entries.

Finally, in (3.3), the quadratic forms:

$$\mathcal{Q}_3(F) = D^2W(\mathrm{Id})(F,F), \quad \mathcal{Q}_2(x, F_{tan}) = \min\{\mathcal{Q}_3(\tilde F);\ \tilde F \in \mathbb{R}^{3\times 3},\ (\tilde F - F)_{tan} = 0\}. \tag{3.6}$$

where the form $\mathcal{Q}_3$ is defined for all $F \in \mathbb{R}^{3\times 3}$, while $\mathcal{Q}_2(x,\cdot)$ for a given $x \in S$ is defined on tangential minors F_{tan} of such matrices. Both forms $\mathcal{Q}_3$ and all $\mathcal{Q}_2(x,\cdot)$ are nonnegative definite and depend only on the symmetric parts of their arguments.

We now have the following results, stating in particular that the functional $\mathcal{I}_4$ is the Γ-limit [3] of the scaled energies $h^{-4}I^h$:

Theorem 3.1. *Let a sequence of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ satisfy $I^h(u^h) \le Ch^4$. Then there exists proper rotations $\bar R^h \in SO(3)$ and translations $c^h \in \mathbb{R}^3$ such that for the renormalized deformations:*

$$y^h(x + t\vec n(x)) = (\bar R^h)^T u^h(x + t\frac{h}{h_0}\vec n) - c^h : S^{h_0} \longrightarrow \mathbb{R}^3$$

defined on the common thin shell S^{h_0}, the following holds.

(i) *y^h converge in $W^{1,2}(S^{h_0}, \mathbb{R}^3)$ to π.*

(ii) *The scaled displacements:*

$$V^h(x) = h^{-1} \fint_{-h_0/2}^{h_0/2} y^h(x + t\vec n) - x \,\mathrm{d}t \tag{3.7}$$

converge (up to a subsequence) in $W^{1,2}(S, \mathbb{R}^3)$ to some $V \in \mathcal{V}$.

(iii) *The scaled averaged strains:*

$$B^h(x) = h^{-1}\mathrm{sym}\nabla V^h(x) \tag{3.8}$$

converge (up to a subsequence) weakly in $L^2(S, \mathbb{R}^{2\times 2})$ to a limit $B \in \mathcal{B}$.

(iv) *The lower bound holds:*

$$\liminf_{h\to 0} h^{-4} I^h(u^h) \ge \mathcal{I}_4(V, B).$$

Theorem 3.2. *For every couple $V \in \mathcal{V}$ and $B \in \mathcal{B}$, there exists a sequence of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ such that:*

(i) *The rescaled sequence $y^h(x + t\vec n) = u^h(x + t\frac{h}{h_0}\vec n)$ converges in $W^{1,2}(S^{h_0}, \mathbb{R}^3)$ to π.*

(ii) *The displacements V^h as in (3.7) converge in $W^{1,2}(S, \mathbb{R}^3)$ to V.*

(iii) *The strains B^h as in (3.8) converge in $W^{1,2}(S, \mathbb{R}^{2\times 2})$ to B.*

(iv) *There holds:*

$$\lim_{h\to 0} h^{-4} I^h(u^h) = \mathcal{I}_4(V, B).$$

The proofs follow through a combination of arguments in [14] and [16], which we do not repeat here but instead comment on the functional (3.3) and its relation with the prestrained von Kármán equations for plates.

Here, in analogy with the theory for flat plates $S \subset \mathbb{R}^2$ with incompatible strains [14], in (3.1) we have assumed that the target metric is 2nd order in thickness h for the in-plane stretching (sym ϵ_g), and 1st order in h for bending (sym κ_g). Due to this particular choice of scalings the limit energy $\mathcal{I}_4$ is composed of exactly two terms, corresponding to stretching and bending. The argument of the integrand in the first term, namely $B - \frac{1}{2}(A^2)_{tan} - (\mathrm{sym}\ \epsilon_g)_{tan}$, represents the difference of the second order stretching induced by the deformation $v^h = \mathrm{id} + hV + h^2 w^h$ from the target stretching (sym ϵ_g), with $V \in \mathcal{V}$ and $\mathrm{sym}\nabla w^h \to B$. The argument of the integrand in the second term $(\nabla(A\vec{n}) - A\Pi)_{tan} - (\mathrm{sym}\ \kappa_g)_{tan}$, represents the difference of the first order bending induced by v^h from the target bending (sym κ_g).

In general, the second order displacement w can be very oscillatory. Due to the non-trivial geometry of the mid-surface S, the finite strain space $\mathcal{B}$ is usually large and hence a bound on the L^2 norm of the symmetric gradients $\mathrm{sym}\nabla w^h$ implies only a very weak bound on w^h. The limiting tensor B can hence be written only as the symmetric gradient of a very weakly regular distribution (not a classical higher order displacement).

Remark 3.3. When the mid-surface S is elliptic, then for any first order isometry $V \in \mathcal{V}$ there exists $B \in \mathcal{B} = L^2(S, \mathbb{R}^{2\times 2}_{sym})$ such that $B - \frac{1}{2}(A^2)_{tan} - (\mathrm{sym}\ \epsilon_g)_{tan} = 0$ (see [17]). This implies that for any V there exists a higher order modification w^h for which in the limit, the second order target stretching is realized. Thus, the energy $\mathcal{I}_4$ reduces to:

$$\mathcal{I}_4(V) = \frac{1}{24}\int_S \mathcal{Q}_2\Big(x, (\nabla(A\vec{n}) - A\Pi)_{tan} - (\mathrm{sym}\ \kappa_g)_{tan}\Big)\ \mathrm{d}x,$$

i.e. the bending term which is to be minimized over the space $\mathcal{V}$. Note that this variational problem is convex (minimizing a convex integral over a linear space $\mathcal{V}$), and hence it admits only one solution (up to rigid motions). Following the analysis in [17], we see that for elliptic surfaces, all limiting theories for $h^{-\beta} I^h$ under the energy scaling $\beta > 2$, coincide with the linear theory $\mathcal{I}_4$ as above, while the sublinear theory, to be used in the description of buckling, is the Kirchhoff-like (nonlinear bending) theory corresponding to $\beta = 2$ and derived in [19].

4. The prestrained shallow shell with a first-order isometry constraint: $0 < \alpha < 1$

When the parameter $0 < \alpha < 1$, the highest order terms (of order $h^{2\alpha}$) in the prestrain metric p^h on $(S_\gamma)^h$ pulled back on the flat reference configuration Ω^h, turn out to be "compatible", i.e. entirely generated by the reference displacement $h^\alpha v_0$. In other words, the shallow shell will easily compensate for these terms by rigidly keeping its structure at the h^α order and only will make adjustments at higher orders to the prestrain induced by ϵ_g and κ_g. In the limit as $h \to 0$ we therefore expect that the effective energy functional on Ω will depend only on the out-of-plane and the in-plane displacements of respective orders h and h^2. Yet, as we shall see below, the residual curvature of mid-surfaces will appear in a two-fold manner: as a linearized first-order isometry constraint on the out-of-plate displacement (4.3), and also as a defining constraint on the space of admissible in-plane displacements. The mid-plate Ω will inherit the space of first order infinitesimal isometries (3.4) and the finite strain space (3.5), in the asymptotic limit of vanishing curvature shells.

The space of *finite strains* $\mathcal{B}_{v_0} \subset L^2(\Omega, \mathbb{R}^{2\times 2}_{sym})$ is defined as:

$$\mathcal{B}_{v_0} = \Big\{ L^2 - \lim_{\epsilon\to 0} \big(\mathrm{sym}\nabla w^\epsilon + \mathrm{sym}(\nabla v^\epsilon \otimes \nabla v_0)\big);\ w^\epsilon \in W^{1,2}(\Omega, \mathbb{R}^2),\ v^\epsilon \in W^{1,2}(\Omega, \mathbb{R}) \Big\}.$$

We now identify $\mathcal{B}_{v_0}$ with each of the finite strain spaces of the shallow surfaces S_γ:

Lemma 4.1. *Let the surfaces S_γ be as in (2.1). Then for all $\gamma \neq 0$, the finite strain spaces:*

$$\mathcal{B}^\gamma = \Big\{ L^2 - \lim_{\epsilon\to 0} \mathrm{sym}\nabla w^\epsilon;\ w^\epsilon \in W^{1,2}(S_\gamma, \mathbb{R}^3) \Big\},$$

are each isomorphic to $\mathcal{B}_{v_0}$ via the linear isomorphism:

$$\mathcal{T}^\gamma : L^2(S_\gamma, \mathcal{L}^2_{sym}(TS_\gamma, \mathbb{R})) \to L^2(\Omega, \mathbb{R}^{2\times 2}_{sym}).$$

Here, $L^2(S_\gamma, \mathcal{L}^2_{sym}(TS_\gamma, \mathbb{R}))$ is the space of all L^2-sections of the bundle of symmetric bilinear forms on S_γ, and $\mathcal{T}^\gamma$ is naturally defined by:

$$[\mathcal{T}^\gamma(\sigma)(x)]_{ij} = \sigma(\phi_\gamma(x))(\partial_i\phi_\gamma(x), \partial_j\phi_\gamma(x)) \quad \forall\, a.e.\ x\in\Omega \quad \forall \sigma \in L^2(S_\gamma, \mathcal{L}^2_{sym}(TS_\gamma, \mathbb{R})).$$

Proof. Let $w \in W^{1,2}(S_\gamma, \mathbb{R}^3)$ and write $\tilde w = (\tilde w_1, \tilde w_2, \tilde w_3) = w\circ\phi_\gamma \in W^{1,2}(\Omega, \mathbb{R}^3)$. Then, for $i, j = 1, 2$ we have:

$$(\mathrm{sym}\nabla w)(\partial_i\phi_\gamma, \partial_j\phi_\gamma) = \frac{1}{2}\left(\partial_i\tilde w \cdot \partial_j\phi_\gamma + \partial_j\tilde w\cdot\partial_i\phi_\gamma\right) = \Big[\mathrm{sym}\nabla(\tilde w_1, \tilde w_2) + \gamma\,\mathrm{sym}(\nabla\tilde w_3 \otimes \nabla v_0)\Big]_{ij}.$$

Take now a sequence $w^\epsilon \in W^{1,2}(S_\gamma, \mathbb{R}^3)$ such that $\lim_{\epsilon\to 0}\mathrm{sym}\nabla w^\epsilon = B_\gamma \in \mathcal{B}^\gamma$. Then:

$$\mathcal{T}^\gamma(B_\gamma) = \lim_{\epsilon\to 0}\mathcal{T}^\gamma(\mathrm{sym}\nabla w^\epsilon) = \lim_{\epsilon\to 0}\Big(\mathrm{sym}\nabla(\tilde w_1^\epsilon, \tilde w_2^\epsilon) + \mathrm{sym}(\nabla(\gamma\tilde w_3^\epsilon)\otimes\nabla v_0)\Big) \in \mathcal{B}_{v_0},$$

which proves the claim. ■

The following is a consequence of Lemma 4.1, [16, Lemma 5.6] and [26, Lemma 3.3]:

Corollary 4.2. *Assume that:*

(i) *either: $v_0 \in \mathcal{C}^{2,1}(\Omega)\cap\mathcal{C}^{1,1}(\bar\Omega)$ and $\det\nabla^2 v_0 \geq c > 0$ in Ω,*
(ii) *or: $v_0 \in \mathcal{C}^2(\bar\Omega)$ with $\det\nabla^2 v_0 = 0$ in Ω, and $\nabla^2 v_0$ does not vanish identically on any open region in Ω.*

Then:

$$\mathcal{B}_{v_0} = L^2(\Omega, \mathbb{R}^{2\times 2}_{sym}). \tag{4.1}$$

Indeed, in [17] we proved that for any strictly elliptic surface S, its finite strain space $\mathcal{B}$ equals $L^2(\Omega, \mathbb{R}^{2\times 2}_{sym})$. Since every S_γ is strictly elliptic under the assumption (i), the result follows by the equivalence of spaces $\mathcal{B}^\gamma$ and $\mathcal{B}_{v_0}$ in Lemma 4.1. The same observation can be derived directly, as follows. Given $B : \Omega \to \mathbb{R}^{2\times 2}_{sym}$ smooth enough, we first solve for v in:

$$\left\{\begin{array}{ll} \mathrm{cof}\,\nabla^2 v_0 : \nabla^2 v = -\mathrm{curl}^T\mathrm{curl}\,B & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{4.2}$$

Then we have:

$$\mathrm{curl}^T\mathrm{curl}\,B = -\mathrm{cof}\nabla^2 v : \nabla^2 v_0 = \mathrm{curl}^T\mathrm{curl}(\nabla v\otimes\nabla v_0) = \mathrm{curl}^T\mathrm{curl}\Big(\mathrm{sym}(\nabla v\otimes\nabla v_0)\Big)$$

(see also Remark 4.7), and therefore:

$$B = \mathrm{sym}\nabla(v_1, v_2) + \mathrm{sym}(\nabla v\otimes\nabla v_0),$$

for some in-plane displacement $(v_1, v_2) : \Omega \to \mathbb{R}^2$. The density of smooth fields B in the space $L^2(\Omega, \mathbb{R}^{2\times 2}_{sym})$ now yields the result.

Remark 4.3. We expect that the property (4.1) is satisfied for a generic v_0, whenever $\nabla^2 v_0$ does not vanish identically on any open region of Ω. The argument requires studying very weak solutions of the mixed-type equation (4.2). When this equation is degenerate ($v_0 \equiv 0$), $\mathcal{B}_{v_0}$ coincides with the space of all matrix fields in the kernel of the operator $\mathrm{curl}^T\mathrm{curl}$ and hence it is only a proper subset of $L^2(\Omega, \mathbb{R}^{2\times 2}_{sym})$, consisting of symmetric gradients.

We now present the main Γ-convergence result for the shallow shell regime $0 < \alpha < 1$. The proofs which consist of tedious modifications of the arguments in [16, 14], are outlined in the Appendix.

Theorem 4.4. *Let $0 < \alpha < 1$. Assume $u^h \in W^{1,2}((S_{h^\alpha})^h, \mathbb{R}^3)$ satisfies $I^{h^\alpha,h}(u^h) \le Ch^4$, where $I^{\gamma,h}$ is given as in (2.5). Then there exists $\bar R^h \in SO(3)$ and $c^h \in \mathbb{R}^3$ such that for the normalized deformations:*

$$y^h(x,t) = (\bar R^h)^T (u^h \circ \tilde\phi_{h^\alpha})(x, ht) - c^h : \Omega^1 \longrightarrow \mathbb{R}^3$$

with ϕ_γ and $\gamma = h^\alpha$ as in (2.1), we have:

(i) *$y^h(x,t)$ converge in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x.*

(ii) *The scaled displacements $V^h(x) = h^{-1} \fint_{-1/2}^{1/2} y^h(x,t) - x - h^\alpha v_0(x) e_3 \,\mathrm{d}t$ converge (up to a subsequence) in $W^{1,2}(\Omega, \mathbb{R}^3)$ to $(0,0,v)^T$ where $v \in W^{2,2}(\Omega, \mathbb{R})$ and:*

$$\mathrm{cof}\ \nabla^2 v_0 : \nabla^2 v = 0 \quad \textit{in } \Omega. \tag{4.3}$$

(iii) *The scaled strains:*

$$B_h = \frac{1}{h}\Big(\mathrm{sym}\nabla(V_1^h, V_2^h) + h^\alpha \mathrm{sym}(\nabla V_3^h \otimes \nabla v_0)\Big)$$

converge (up to a subsequence) weakly in L^2 to some $B \in \mathcal{B}_{v_0}$.

(iv) *Moreover: $\liminf_{h\to 0} h^{-4} I^{h^\alpha,h}(u^h) \ge \mathcal{I}_4^\infty(v, B)$, where:*

$$\mathcal{I}_4^\infty(v,B) = \int_\Omega \mathcal{Q}_2\left(B + \frac{1}{2}\nabla v \otimes \nabla v - (\mathrm{sym}\ \epsilon_g)_{tan}\right) + \frac{1}{24}\int_\Omega \mathcal{Q}_2\Big(\nabla^2 v + (\mathrm{sym}\ \kappa_g)_{tan}\Big), \tag{4.4}$$

with $\mathcal{Q}_2$ defined in (3.6).

Theorem 4.5. *Let $0 < \alpha < 1$. For every $v \in W^{2,2}(\Omega, \mathbb{R})$ satisfying (4.3) and every $B \in \mathcal{B}_{v_0}$, there exists a sequence of deformations $u^h \in W^{1,2}((S_{h^\alpha})^h, \mathbb{R}^3)$ such that:*

(i) *The sequence $y^h(x,t) = u^h(x + h^\alpha v_0(x) e_3 + ht\vec n^\gamma(x))$ converges in $W^{1,2}(\Omega^1)$ to x.*

(ii) *The scaled displacements V^h as in (ii) Theorem 4.4 converge in $W^{1,2}$ to $(0,0,v)$.*

(iii) *The scaled strains B^h as in (iii) Theorem 4.4 converge weakly in L^2 to B.*

(iv) $\lim_{h\to 0} h^{-4} I^{h^\alpha,h}(u^h) = \mathcal{I}_4^\infty(v, B)$.

In the special cases of Corollary 4.2, we have:

Theorem 4.6. *Assume additionally that v_0 is such that (4.1) holds. Then, for every $v \in W^{2,2}(\Omega, \mathbb{R})$ satisfying (4.3), there exists a sequence $u^h \in W^{1,2}((S_{h^\alpha})^h, \mathbb{R}^3)$ such that (i) and (ii) of Theorem 4.5 hold, and moreover:*

$$\lim_{h\to 0} h^{-4} I^{h^\alpha,h}(u^h) = \frac{1}{24}\int_\Omega \mathcal{Q}_2\Big(\nabla^2 v + (\mathrm{sym}\ \kappa_g)_{tan}\Big).$$

Remark 4.7. Comparing functionals (4.4) with (3.3), note that the space $\mathcal{V}(S_\gamma)$ of first-order infinitesimal isometries on S_γ is made of displacements $V : S_\gamma \to \mathbb{R}^3$ of the form:

$$\begin{aligned} & V(\phi_\gamma(x)) = (\gamma v_1(x), h^\alpha v_2(x), v_3) \qquad \forall x \in \Omega, \\ & \text{such that } \ (v_1, v_2, v_3) \in W^{2,2}(\Omega, \mathbb{R}^3) \ \text{ and } \ \mathrm{sym}\nabla(v_1, v_2) + \mathrm{sym}(\nabla v_3 \otimes \nabla v_0) = 0. \end{aligned} \tag{4.5}$$

Indeed, similarly as in the proof of Lemma 4.1, the condition $\mathrm{sym}\nabla V = 0$ on S_γ becomes:

$$0 = \frac{1}{2}\big(\partial_i (V \circ \phi_\gamma) \cdot \partial_j \phi_\gamma + \partial_j (V \circ \phi_\gamma) \cdot \partial_i \phi_\gamma\big) = \ \mathrm{sym}[\nabla(v_1, v_2) + \nabla v_3 \otimes \nabla v_0]_{ij}.$$

We also see that v_3 can be completed by (v_1, v_2) to $V \in \mathcal{V}_1(S_h)$ as in (4.5) only if:

$$\mathrm{cof}\nabla^2 v_0 : \nabla^2 v_3 = 0, \tag{4.6}$$

the latter being also a sufficient condition when Ω is simply connected. This follows from:

$$\begin{aligned} \mathrm{curl}^T \mathrm{curl}\Big(\mathrm{sym}(\nabla v_3 \otimes \nabla v_0)\Big) &= \mathrm{curl}^T \mathrm{curl}\Big(\nabla v_3 \otimes \nabla v_0\Big) \\ &= \partial_{22}(\partial_1 v_3 \cdot \partial_1 v_0) + \partial_{11}(\partial_2 v_3 \cdot \partial_2 v_0) - \partial_{12}(\partial_1 v_3 \cdot \partial_2 v_0 + \partial_2 v_3 \cdot \partial_1 v_0) \\ &= -\big(\partial_{11} v_3 \cdot \partial_{22} v_0 + \partial_{22} v_3 \cdot \partial_{11} v_0 - 2\partial_{12} v_3 \cdot \partial_{12} v_0\big) = -\mathrm{cof}\nabla^2 v_0 : \nabla^2 v_3. \end{aligned}$$

Hence, the admissible out-of-plane displacements v_3 relevant in (3.3), must obey for the least the constraint (4.6), which appears in the 2-scale limiting theory (4.4) as constraint (4.3). This is in contrast with the unconstrained 2-scale limiting theory (5.3) developed in the next section.

Remark 4.8. To put the last two results in another context, we draw the reader's attention to the forthcoming paper [15], where we analyze the Γ-limit of the shallow shell energies $\frac{1}{h^{2\alpha+2}} I^{h^\alpha, h}$ on shells with curvature of order h^α. This energy scaling is produced by forces of appropriate magnitude or by prestrains of a different order than those considered in the present paper. Our main result in [15] concerns the case $\alpha < 1$, where we can establish that in the special case $\det \nabla^2 v_0 \equiv c_0 > 0$, the Γ-limit is a linearized Kirchhoff model with a Monge-Ampère curvature constraint:

$$\det \nabla^2 v = \det \nabla^2 v_0 \tag{4.7}$$

on the admissible out-of-plane displacements $v \in W^{2,2}(\Omega)$. The constraint (4.3) can be interpreted as a linearization of (4.7), thereby highlighting the relationship between the two models for elliptic shallow shells.

5. The generalized Donnell-Mushtari-Vlasov model for a prestrained shallow shell: $\alpha = 1$

When the parameter $\alpha = 1$, i.e. the curvature of the mid-surface co-varies with the thickness, so that $\gamma = h$. For small h, the growth tensors on $(S_h)^h$ are then defined by (2.6) and the corresponding metric $p^h = (q^h)^T q^h$ is given by:

$$p^h(\phi_h(x) + x_3 \vec{n}^h(x)) = \mathrm{Id} + 2h^2 \mathrm{sym}\ \epsilon_g(x) + 2h x_3 \mathrm{sym}\ \kappa_g(x) + \mathcal{O}(h^3).$$

Let $v^h = u^h \circ \tilde\phi_h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$, via diffeomorphisms $\tilde\phi_h$ in (2.3). By this simple change of variables, we see that:

$$\begin{aligned} I^{h,h}(u^h) &= \frac{1}{h}\int_{(S_h)^h} W(\nabla u^h (q^h)^{-1}) = \frac{1}{h}\int_{\Omega^h} W\Big((\nabla v^h)(\nabla\tilde\phi_h)^{-1}(q^h\circ\tilde\phi_h)^{-1}\Big)\cdot \det\nabla\tilde\phi_h ~\mathrm{d}(x,x_3)\\ &= \frac{1}{h}\int_{\Omega^h} W\Big((\nabla v^h)(b^h)^{-1}\Big)\cdot\det\nabla\tilde\phi_h~\mathrm{d}(x,x_3), \end{aligned}$$

where:

$$b^h = (q^h\circ\tilde\phi_h)\nabla\tilde\phi_h.$$

In order to understand the structure of b^h we need the following result:

Lemma 5.1. *The pull-back of the metric p^h through $\tilde\phi_h$ satisfies:*

$$\begin{aligned} \forall (x,x_3)\in\Omega^h \qquad g^h(x,x_3) &= (\nabla\tilde\phi_h)^T(p^h\circ\tilde\phi_h)(\nabla\tilde\phi_h)\\ &= \mathrm{Id} + h^2\Big(2\mathrm{sym}~\epsilon_g(x) + (\nabla v_0(x)\otimes\nabla v_0(x))^*\Big)\\ &\qquad + 2hx_3\Big(\mathrm{sym}~\kappa_g(x) - (\nabla^2 v_0(x))^*\Big) + \mathcal{O}(h^3), \end{aligned}$$

where $F^*\in\mathbb{R}^{3\times3}$ denotes the matrix whose only non-zero entries are in its 2×2 principal minor given by $F\in\mathbb{R}^{2\times2}$.

Proof. By a direct calculation, we obtain:

$$\begin{aligned} \partial_1\tilde\phi_h &= \big(1 - x_3h\partial^2_{11}v_0, -x_3h\partial^2_{12}v_0, h\partial_1 v_0\big) + \mathcal{O}(h^3),\\ \partial_2\tilde\phi_h &= \big(-x_3h\partial^2_{12}v_0, 1 - x_3h\partial^2_{22}v_0, h\partial_2 v_0\big) + \mathcal{O}(h^3),\\ \partial_3\tilde\phi_h &= \vec n^h = \big(-h\partial_1 v_0, -h\partial_2 v_0, 1 - \frac{1}{2}h^2|\nabla v_0|^2\big) + \mathcal{O}(h^3). \end{aligned}$$

Hence:

$$\begin{aligned} &(\nabla\tilde\phi_h)^T(\nabla\tilde\phi_h) = \mathrm{Id}_3 - 2x_3h(\nabla^2 v_0)^* + h^2(\nabla v_0\otimes\nabla v_0)^* + \mathcal{O}(h^3)\\ &(\nabla\tilde\phi_h)^T\big(2h^2\mathrm{sym}~\epsilon_g + 2hx_3\mathrm{sym}~\kappa_g\big)(\nabla\tilde\phi_h) = 2h^2\mathrm{sym}~\epsilon_g + 2hx_3\mathrm{sym}~\kappa_g + \mathcal{O}(h^3), \end{aligned}$$

in view of $\nabla\tilde\phi_h = \mathrm{Id}_3 + \mathcal{O}(h)$, and the result follows. ■

Note that: $(b^h)^T b^h = g^h$ and therefore by the polar decomposition of matrices:

$$b^h = R(x,x_3)a^h \qquad \text{on } \Omega^h$$

for some $R(x,x_3)\in SO(3)$ and the symmetric growth tensor a^h given by:

$$a^h = \sqrt{g^h} = \mathrm{Id} + h^2\Big(\mathrm{sym}~\epsilon_g + \frac{1}{2}(\nabla v_0\otimes\nabla v_0)^*\Big) + hx_3\Big(\mathrm{sym}~\kappa_g - (\nabla^2 v_0)^*\Big) + \mathcal{O}(h^3). \tag{5.1}$$

For isotropic W it directly follows that:

$$\begin{aligned} I^{h,h}(u^h) &= \frac{1}{h}\int_{\Omega^h} W\Big((\nabla v^h)(a^h)^{-1}R(x)^{-1}\Big)\cdot\det\nabla\tilde\phi_h~\mathrm{d}(x,x_3)\\ &= \frac{1}{h}\int_{\Omega^h} W\Big((\nabla v^h)(a^h)^{-1}\Big)\cdot(1+\mathcal{O}(h))~\mathrm{d}(x,x_3). \end{aligned} \tag{5.2}$$

Heuristically, modulo the change of variable $\tilde{\phi}_h$ the problem reduces then to the study of deformations of the flat thin film Ω^h with the prestrain a^h. Indeed, by exactly the same analysis as in [14] Theorems 1.2 and 1.3, we obtain in the general (not necessarily isotropic) case, the following result:

Theorem 5.2. *Assume that $u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3)$ satisfies $I^{h,h}(u^h) \leq Ch^4$. Then there exists proper rotations $\bar{R}^h \in SO(3)$ and translations $c^h \in \mathbb{R}^3$ such that for the normalized deformations:*

$$y^h(x,t) = (\bar{R}^h)^T(u^h \circ \tilde{\phi}_h)(x, ht) - c^h : \Omega^1 \longrightarrow \mathbb{R}^3$$

defined by means of (2.3) on the common domain $\Omega^1 = \Omega \times (-1/2, 1/2)$ the following holds:

(i) *$y^h(x,t)$ converge in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x.*
(ii) *The scaled displacements $V^h(x) = h^{-1} \fint_{-1/2}^{1/2} y^h(x,t) - x \ \mathrm{d}t$ converge (up to a subsequence) in $W^{1,2}(\Omega, \mathbb{R}^3)$ to the vector field of the form $(0,0,v)^T$ and $v \in W^{2,2}(\Omega, \mathbb{R})$.*
(iii) *The scaled in-plane displacements $h^{-1}V^h_{tan}$ converge (up to a subsequence) weakly in $W^{1,2}$ to $w \in W^{1,2}(\Omega, \mathbb{R}^2)$.*
(iv) *Moreover: $\liminf_{h\to 0} h^{-4} I^{h,h}(u^h) \geq \mathcal{I}_4^1(w, v)$ where:*

$$\begin{aligned}\mathcal{I}_4^1(w,v) = \frac{1}{2}\int_\Omega \mathcal{Q}_2\left(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v \otimes \nabla v - \frac{1}{2}\nabla v_0 \otimes \nabla v_0 - (\mathrm{sym}\ \epsilon_g)_{tan}\right)\\ + \frac{1}{24}\int_\Omega \mathcal{Q}_2\Big(\nabla^2 v - \nabla^2 v_0 + (\mathrm{sym}\ \kappa_g)_{tan}\Big).\end{aligned} \tag{5.3}$$

In the same manner, applying the proof of Theorem 1.4 of [14] to (5.2), yields:

Theorem 5.3. *For every $v \in W^{2,2}(\Omega, \mathbb{R})$ and $w \in W^{1,2}(\Omega, \mathbb{R}^2)$, there exists a sequence of deformations $u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3)$ such that:*

(i) *The sequence $y^h(x,t) = u^h(x + hv_0(x)e_3 + ht\vec{n}^h(x))$ converges in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x.*
(ii) *The displacements V^h as in (ii) Theorem 5.2 converge in $W^{1,2}$ to $(0,0,v)$.*
(iii) *The in-plane displacements $h^{-1}V^h_{tan}$ converge in $W^{1,2}$ to w.*
(iv) $\lim_{h\to 0} h^{-4} I^{h,h}(u^h) = \mathcal{I}_{g,v_0}(w,v)$.

6. The prestrained plate model and the Euler-Lagrange equations: $\alpha > 1$

When the parameter $\alpha > 1$, we calculate the pull-back of the induced metric $p^h = (q^h)^T q^h$, to the flat plate Ω^h, via the change of variable $\tilde{\phi}_\gamma$ as in (2.3). Just as in Lemma 5.1, we obtain:

$$\begin{aligned} g^h = (\tilde{\phi}_{h^\alpha})^* p^h = \mathrm{Id}_3 + h^{2\alpha}(\nabla v_0 \otimes \nabla v_0)^* - 2h^\alpha x_3 (\nabla^2 v_0)^*\\ + 2h^2 \mathrm{sym}\ \epsilon_g + 2hx_3 \mathrm{sym}\ \kappa_g + \mathcal{O}(h^3).\end{aligned} \tag{6.1}$$

It is therefore clear that the prestrain terms (ϵ_g, κ_g) take over the effect of shallowness and hence the limiting theory in the scaling regime h^4 is that derived in [14], coinciding with results of Theorem 5.2 and Theorem 4.4 for the case $v_0 = 0$ and with the results of Theorem

3.1 for $S \subset \mathbb{R}^2$:

$$\forall v \in W^{2,2}(\Omega, \mathbb{R}) \quad \forall w \in W^{1,2}(\Omega, \mathbb{R}^2)$$

$$\mathcal{I}_4^0(w,v) = \frac{1}{2}\int_\Omega \mathcal{Q}_2\left(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v \otimes \nabla v - (\mathrm{sym}\ \epsilon_g)_{tan}\right) + \frac{1}{24}\int_\Omega \mathcal{Q}_2\Big(\nabla^2 v + (\mathrm{sym}\ \kappa_g)_{tan}\Big). \tag{6.2}$$

Indeed, consider the prestrained von Kármán shell model $\mathcal{I}_4$ discussed in Section 3 for a degenerate situation $S \subset \mathbb{R}^2$. The term $B - \frac{1}{2}(A^2)_{tan}$ reduces to: $\frac{1}{2}\left(\nabla w + (\nabla w)^T + \nabla v \otimes \nabla v\right)$, where w and $v = V^3$ are respectively the in-plane and the out-of-plane displacements of S. The term $(\nabla(A\vec{n}) - A\Pi)_{tan}$ reduces also to: $-\nabla^2 v$. Therefore, when $S \subset \mathbb{R}^2$, $\mathcal{I}_4$ coincides with the model $\mathcal{I}_4^0$ and with the models $\mathcal{I}_4^\infty$ and $\mathcal{I}_4^1$ in the degenerate case $v_0 = 0$.

Remark 6.1. We point out a qualitative difference between the out-of-plane displacements v in the argument of $\mathcal{I}_4^0$ and $\mathcal{I}_4^1$ and those appearing as the arguments of $\mathcal{I}_4^\infty$. The former are the net lowest order out of plane displacements of the limit deformations which are of order h, as suggested by Theorem 5.2 (ii), but, according to Theorem 4.4 (ii), when $\alpha < 1$, the latter are the second highest order term of the expansion of the deformation after $h^\alpha v_0$. Hence, one should replace v in (5.3) or (6.5) through a change of variables by $v + h^{\alpha-1}v_0$ in order to quantitatively compare this model with the variational model $\mathcal{I}_4^\infty$ in (4.4).

As shown in [14], under the assumption of W being isotropic, the Euler-Lagrange equations of $\mathcal{I}_4$ under this degeneracy condition (or equivalently the Euler-Lagrange equations of $\mathcal{I}_4^0$) can be then written in terms of the displacement v and the Airy stress potential Φ:

$$\begin{cases} \Delta^2\Phi = -Y(\det \nabla^2 v + \lambda_g) \\ Z\Delta^2 v = [v, \Phi] - Z\Omega_g\ , \end{cases} \tag{6.3}$$

where Y is the Young modulus, Z the bending stiffness, ν the Poisson ratio (given in terms of the Lamé constants μ and λ), and :

$$\lambda_g = \mathrm{curl}^T\mathrm{curl}\ (\epsilon_g)_{2\times 2} = \partial_{22}(\epsilon_g)_{11} + \partial_{11}(\epsilon_g)_{22} - \partial_{12}\Big((\epsilon_g)_{12} + (\epsilon_g)_{21}\Big),$$

$$\begin{aligned} \Omega_g &= \mathrm{div}^T\mathrm{div}\ \Big((\kappa_g)_{2\times 2} + \nu\ \mathrm{cof}\ (\kappa_g)_{2\times 2}\Big) \\ &= \partial_{11}\Big((\kappa_g)_{11} + \nu(\kappa_g)_{22}\Big) + \partial_{22}\Big((\kappa_g)_{22} + \nu(\kappa_g)_{11}\Big) + (1-\nu)\partial_{12}\Big((\kappa_g)_{12} + (\kappa_g)_{21}\Big). \end{aligned} \tag{6.4}$$

Equations (6.3) are based on a thermoelastic analogy to growth [22, 20] and can also be derived using a formal perturbation theory [4].

On the other hand, the following system was introduced in [21], as a mathematical model of blooming activated by differential lateral growth from an initial non-zero transverse displacement field v_0:

$$\begin{cases} \Delta^2\Phi = -Y(\det \nabla^2 v - \det \nabla^2 v_0 + \lambda_g) \\ Z(\Delta^2 v - \Delta^2 v_0) = [v, \Phi] - Z\Omega_g\ , \end{cases} \tag{6.5}$$

A similar calculation as in [14] then shows that (6.5) can be viewed as the Euler Lagrange equations corresponding to the energy functional $\mathcal{I}_4^1$. We will now show that (6.5) can be directly derived from the equations (6.3).

Proposition 6.2. *The system (6.5) can be derived from the equations (6.3) by pulling back the prestrain tensors ϵ_g and κ_g from a sequence of shallow shells $(S_h)^h$ generated by the vanishing out-of-plane displacements hv_0.*

Proof. By Lemma 5.1 we see that the growth tensor on Ω^h is given by (5.1). Applying (6.4) to the modified strain and curvature in a^h, to the leading order, we obtain:

$$\begin{aligned}\lambda_g(v_0) &= \mathrm{curl}^T\mathrm{curl}\,\Big((\mathrm{sym}\ \epsilon_g)_{tan} + \frac{1}{2}\nabla v_0 \otimes \nabla v_0\Big) = \lambda_g + \det \nabla^2 v_0 \\ \Omega_g(v_0) &= \mathrm{div}^T\mathrm{div}\,\Big(((\mathrm{sym}\ \kappa_g)_{tan} - \nabla^2 v_0) + \nu\ \mathrm{cof}\ ((\mathrm{sym}\ \kappa_g)_{tan} - \nabla^2 v_0)\Big) \\ &= \Omega_g - \Delta^2 v_0,\end{aligned}$$

where the last equality follows from div cof $\nabla^2 v_0 = 0$. Consequently, (6.3) for the growth tensor (5.1) becomes exactly (6.5). ■

7. The energy scaling

A straightforward consequence of our results is the following assertion about the scaling of the infimum elastic energies of the thin prestrained shallow shells in the von Kármán regime (2.6).

Theorem 7.1. *Let $\alpha > 0$ and let the sequence of thin shells $(S_\gamma)^h$ be given as in (2.2) with the elastic energies of deformations $I^{\gamma,h}$ as in (2.5). Assume that:*

$$\mathrm{curl}\,(\mathrm{sym}\ \kappa_g)_{tan} \not\equiv 0 \qquad in\ \Omega. \tag{7.1}$$

Then, there exists constants $c, C > 0$ for which:

$$\forall 0 < h \ll 1 \qquad c \leq \inf_{u\in W^{1,2}((S_{h^\alpha})^h,\mathbb{R}^3)} \frac{1}{h^4} I^{h^\alpha,h}(u) \leq C. \tag{7.2}$$

Indeed, the condition curl(sym $\kappa_g)_{tan} \equiv 0$ is equivalent to (sym $\kappa_g)_{tan} = \nabla^2 v$, for some $v : \Omega \to \mathbb{R}$. If not satisfied, the bending term in (4.4) is always positive, yielding the lower bound in (7.2). The existence of a recovery sequence in Theorem 4.5 and Theorem 5.3 and [14] implies the upper bound.

Remark 7.2. The incompatibility condition (7.1) can be relaxed depending on the specific value of α, and the assumed energy level, see e.g. [14] for a more involved scaling analysis when $\alpha > 1$. Heuristically, conditions of similar type imply that the Riemann curvature tensor of the induced metric p^h is non-zero and hence, in view of [19, Theorem 2.2], they guarantee the positivity of the infimum of $I^{\gamma,h}$. In a further step we observe that, when p^h is close to be flat, the scaling regime depends on the magnitude of the first non-zero term of the expansion of its curvature tensor. Note also that when $\alpha < 1$, the first two non-zero terms after identity in (6.1) have no bearing on the first non-zero terms in the expansion of the curvature. Analogously, the induced prestrains $\kappa_g' = \nabla^2 v_0$ and $\epsilon_g' = \frac{1}{2}(\nabla^2 v_0 \otimes \nabla^2 v_0)$ corresponding to the scalings h^α and $h^{2\alpha}$ do not satisfy neither conditions (1.13) nor (1.14) of [14]. Therefore the energy infimum must naturally fall below h^4, i.e. in the regime $h^{2\alpha+2}$.

8. Discussion

Our analysis has rigorously derived a general theory of shells with residual strain arising from relative growth, inhomogeneous swelling, plasticity etc. In fact, there are many such theories; each is a consequence of the scalings of shell's curvature relative to the magnitude of the strain incompatibility induced by curvature growth tensors. Indeed, for any exponent $\alpha \geq 0$ we considered the following energies of deformations on the weakly prestrained shallow shells:

$$I^h(u) = \frac{1}{h}\int_{(S_{h^\alpha})^h} W((\nabla u)(q^h)^{-1}) \qquad \forall u \in W^{1,2}((S_{h^\alpha})^h, \mathbb{R}^3),$$

with the growth tensor q^h given by (2.6), on thin shells of the form (2.2) around the mid-surface:

$$S_{h^\alpha} = \phi_{h^\alpha}(\Omega), \qquad \phi_{h^\alpha}(x) = (x, h^\alpha v_0(x)), \qquad v_0 \in \mathcal{C}^{1,1}(\bar{\Omega}, \mathbb{R}).$$

We have established that, disregarding the value of α, the scaling for the infimum of the energy is always determined by the prestrain and have to be of order h^4 under our current assumption (7.1). When $\alpha > 1$, the prestrain takes over the effect of shallowness and hence the limiting theory is the one derived in [14], coinciding with results of Theorem 5.2 for the case $v_0 = 0$ and yielding the Euler-Lagrange equations (6.3). When $\alpha = 1$ one recovers the recently postulated model [21], discussed in the present paper.

For the case $0 < \alpha < 1$, the limiting theory reduces to a new constrained theory. It can be viewed as a plate theory where the non-trivial geometric structure of the shallow shell is inherited by the plate, or equivalently it can be considered as the natural limit of the generalized von Kármán theories (3.3) on the shallow midsurface S_γ as $\gamma \to 0$. In contrast, a similar problem is considered by the authors in [15], where the Γ-limit is discussed under energy regime of order $h^{2\alpha+2}$. This order is compatible with the curvature, where the shallow shell is affected by the body forces or prestrains of large enough magnitude, and yet the choice of α has a bearing on the limit model. For simplicity of presentation, our analysis in the present paper and in [15] is limited to a subset of vast possible scenarios.

A natural generalization of our results would be to allow for different scaling regimes for the growth tensors in search of other possible limiting theories. Overall, there are three independent parameters: one associated with scaling of the shallowness, and two incompatible strains characterized in terms of their dependence on the thickness h in the form h^α. The resulting theories depend on the choice of scalings for these three parameters. Thus, there is no single *correct* model in general, but of course when dealing with a concrete situation, a choice of particular scalings for the relative magnitude of the thickness, the shallowness and the differential growth determines the effective theory.

Acknowledgments. This project is based upon work supported by, among others, the National Science Foundation. M.L. is partially supported by the NSF grants DMS-0707275 and DMS-0846996, and by the Polish MN grant N N201 547438. L.M. is supported by the MacArthur Foundation, M.R.P. is partially supported by the NSF grants DMS-0907844 and DMS-1210258.

Appendices

Here we provide a proof of Theorem 4.4 and Theorem 4.5. We first derive bounds on families of vector mappings $\{u^h\}_{h>0}$, defined on $(S_\gamma)^h$ as in (2.1) and (2.2), under the assumption of smallness on their energy and when $\gamma = h^\alpha$ for the scaling regime $0 < \alpha < 1$. In what follows, by C we denote an arbitrary positive constant, depending on v_0, but not on h or the vector mapping under consideration. In all proofs, the convergences are understood up to a subsequence, unless stated otherwise.

Appendix A. Proof of Theorem 4.4

Let a sequence of deformations $u^h \in W^{1,2}((S_\gamma)^h, \mathbb{R}^3)$ satisfy:

$$\frac{1}{h}\int_{(S_\gamma)^h} W((\nabla u^h)(q^h)^{-1})\,\mathrm{d}z \le Ch^4,$$

where, recalling the definition of $\tilde{\phi}_h : \Omega \times (-h/2, h/2) \to (S_\gamma)^h$ in (2.3), we have:

$$q^h \circ \tilde{\phi}_\gamma = \mathrm{Id} + h^2\epsilon_g + hx_3\kappa_g.$$

The following approximation results can be obtained by combining arguments in [16] and [14], in view of the seminal work [8].

Lemma A.1. *(Parallel to* [16, Lemma 3.1], [14, Theorem 1.6].*) There exists a matrix field* $R^h \in W^{1,2}(S_\gamma, \mathbb{R}^{3\times3})$ *with values in* $SO(3)$*, and a matrix* $Q^h \in SO(3)$*, such that:*

(i) $\frac{1}{h}\|\nabla u^h - R^h\|^2_{L^2(S^h_\gamma)} \le Ch^4$,

(ii) $\|\nabla R^h\|_{L^2(S_\gamma)} \le Ch$,

(iii) $\|(Q^h)^T R^h - \mathrm{Id}\|_{L^p(S_\gamma)} \le Ch$, *for all* $p \in [1, \infty)$.

Lemma A.2. *(Parallel to* [16, Lemma 3.2].*) Let* R^h, Q^h *be as in Lemma A.1. There holds, or all* $p \in [1, \infty)$*:*

(i) $\lim_{h\to0} (Q^h)^T R^h \circ \phi_\gamma = \mathrm{Id}$*, in* $W^{1,2}(\Omega)$ *and in* $L^p(\Omega)$.

Moreover, there exists a $W^{1,2}$ *skew-symmetric field* $A : S \longrightarrow so(3)$*, such that:*

(ii) $\lim_{h\to0} \frac{1}{h}\big((Q^h)^T R^h - \mathrm{Id}\big) \circ \phi_\gamma = A$*, weakly in* $W^{1,2}(\Omega)$ *and (strongly) in* $L^p(\Omega)$.

(iii) $\lim_{h\to0} \frac{1}{h^2}\mathrm{sym}\big((Q^h)^T R^h - \mathrm{Id}\big) \circ \phi_\gamma = \frac{1}{2}A^2$*, in* $L^p(\Omega)$.

Proof. The convergences in (i) follow from Lemma A.1. To prove (ii), notice that:

$$A^h = \frac{1}{h}\big((Q^h)^T R^h - \mathrm{Id}\big) \circ \phi_\gamma$$

is bounded in $W^{1,2}(\Omega)$ and so it has a weakly converging subsequence, as $h \to 0$. Consequently, convergence is strong in $L^p(\Omega)$. One has:

$$A^h + (A^h)^T = \frac{1}{h}\big((Q^h)^T R^h + (R^h)^T Q^h - 2\mathrm{Id}\big) = -h(A^h)^T A^h. \tag{A.1}$$

The latter converges to 0 in $L^p(\Omega)$, and therefore the limit matrix field A is skew-symmetric. The above equality proves as well that:

$$\lim_{h\to 0}\frac{1}{h}\,\mathrm{sym}\ A^h = \frac{1}{2}A^2$$

in $L^p(\Omega)$, which implies (iii). ∎

Consider (and compare with Theorem 4.4) the rescaling:

$$y^h(x+t\vec{n}^\gamma(x)) = u^h\left(x+th/h_0\vec{n}^\gamma(x)\right) \qquad \forall x\in S_\gamma \quad \forall t\in(-h_0/2,h_0/2),$$

so that $y^h\in W^{1,2}((S_\gamma)^{h_0},\mathbb{R}^3)$. Also, define: $\nabla_h y^h(x+t\vec{n}^\gamma(x)) = \nabla u^h\left(x+th/h_0\vec{n}^\gamma(x)\right)$. In what follows, $\Pi_\gamma = \nabla\vec{n}^\gamma$ denotes the second fundamental form of S_γ. By a straightforward calculation we obtain:

Proposition A.3. *(Parallel to* [16, Proposition 3.3]*.) For each $x\in S_\gamma$, $t\in(-h_0/2,h_0/2)$ and $\tau\in T_xS_\gamma$ there hold:*

$$\partial_\tau y^h(x+t\vec{n}^\gamma) = \nabla_h y^h\left(x+t\vec{n}^\gamma\right)\left(\mathrm{Id}+th/h_0\Pi_\gamma(x)\right)\left(\mathrm{Id}+t\Pi_\gamma(x)\right)^{-1}\tau$$

$$\partial_{\vec{n}^\gamma} y^h(x+t\vec{n}^\gamma) = \frac{h}{h_0}\nabla_h y^h\left(x+t\vec{n}^\gamma\right)\vec{n}^\gamma(x).$$

Moreover, for $I^h(y^h) = \frac{1}{h}\int_{(S_\gamma)^h} W((\nabla u^h)(q^h)^{-1})$ one has:

$$I^h(y^h) = \frac{1}{h_0}\int_{(S_\gamma)^{h_0}} W(\nabla_h y^h(x+t\vec{n}^\gamma)(q^h)^{-1})\cdot\det\left[\left(\mathrm{Id}+th/h_0\Pi_\gamma\right)\left(\mathrm{Id}+t\Pi_\gamma\right)^{-1}\right]$$

$$= \int_{S_\gamma}\fint_{-h_0/2}^{h_0/2} W(\nabla_h y^h(x+t\vec{n}^\gamma)(q^h)^{-1})\cdot\det\left[\mathrm{Id}+th/h_0\Pi_\gamma(x)\right]\ \mathrm{d}t\ \mathrm{d}x.$$

Directly from Lemma A.1 (i) and Lemma A.2 (ii) there follows:

Proposition A.4. *(Parallel to* [16, Proposition 3.4]*.)*

(i) $\|\nabla_h y^h - R^h\|_{L^2((S_\gamma)^{h_0})}\leq Ch^2$.

(ii) $\displaystyle\lim_{h\to 0}\frac{1}{h}\left((Q^h)^T\nabla_h y^h - \mathrm{Id}\right)\circ\tilde{\phi}_\gamma = A$, *in* $L^2(\Omega^{h_0})$.

We consider the corrected by rigid motions deformations $\tilde{y}^h\in W^{1,2}((S_\gamma)^{h_0},\mathbb{R}^3)$ and averaged displacements $V^h\in W^{1,2}(S_\gamma,\mathbb{R}^3)$:

$$\tilde{y}^h = (Q^h)^T y^h - c^h, \qquad V^h = V^h[\tilde{y}^h] = \frac{1}{h}\fint_{-h_0/2}^{h_0/2}\tilde{y}^h(x+t\vec{n}^\gamma) - x\ \mathrm{d}t,$$

where the constants c^h are chosen so that $\fint_\Omega V^h\circ\phi_\gamma = 0$.

Lemma A.5. *(Parallel to* [16, Lemma 3.5]*.)*

(i) $\lim_{h\to 0}(\tilde{y}^h\circ\tilde{\phi}_\gamma - \phi_\gamma) = 0$ *in* $W^{1,2}(\Omega^{h_0})$.

(ii) $\lim_{h\to 0}(V^h\circ\phi_\gamma) = V$ *in* $W^{1,2}(\Omega)$.

The vector field V in (ii) has regularity $W^{2,2}(\Omega,\mathbb{R}^3)$ and it satisfies $\partial_\tau V(x) = A(x)\tau$ for all $\tau\in\mathbb{R}^2S$. The $W^{1,2}$ skew-symmetric matrix field $A: S\longrightarrow so(3)$ is as in Lemma A.2.

Proof. **1.** In view of Proposition A.3 and Proposition A.4 we have:

$$\begin{aligned}
&\big\|\nabla_{tan}\tilde y^h - \big((Q^h)^T R^h\big)_{tan}\cdot(\mathrm{Id}+th/h_0\Pi^h_\gamma)(\mathrm{Id}+t\Pi_\gamma)^{-1}\big\|_{L^2(S^{h_0}_\gamma)} \le Ch^2\\
&\big\|\partial_{\vec n^\gamma}\tilde y^h\big\|_{L^2((S_\gamma)^{h_0})} \le Ch\|\nabla_h y^h\|_{L^2(S^{h_0}_\gamma)} \le Ch.
\end{aligned} \tag{A.2}$$

To prove convergence of $V^h\circ\phi_\gamma$, consider for $x\in S_\gamma$:

$$\begin{aligned}
\nabla V^h(x) &= \frac{1}{h}\fint_{-h_0/2}^{h_0/2} \nabla_{tan}\tilde y^h(x+t\vec n^\gamma)(\mathrm{Id}+t\Pi_\gamma) - \mathrm{Id}\ \mathrm{d}t\\
&= \frac{1}{h}\fint_{-h_0/2}^{h_0/2}\Big(\nabla_{tan}\tilde y^h - \big((Q^h)^T R^h\big)_{tan}(\mathrm{Id}+t\Pi_\gamma)^{-1}\Big)(\mathrm{Id}+t\Pi_\gamma)\ \mathrm{d}t\\
&\quad + \frac{1}{h}\Big((Q^h)^T R^h(x) - \mathrm{Id}\Big)_{tan} = A^h\circ(\phi_\gamma)^{-1} + \mathcal{O}(h).
\end{aligned} \tag{A.3}$$

We also have: $\nabla(V^h\circ\phi_\gamma) = (\nabla V^h\circ\phi_\gamma)(\nabla\phi_\gamma)$, hence:

$$\begin{aligned}
\nabla(V^h\circ\phi_\gamma) &= \Big[\big(\frac{1}{h}\fint_{-h_0/2}^{h_0/2}\nabla_{tan}\tilde y^h(\mathrm{Id}+t\Pi_\gamma) - \mathrm{Id}\ \mathrm{d}t\big)\circ\phi_\gamma\Big]\nabla\phi_\gamma\\
&= \Big[\big(\frac{1}{h}\fint_{-h_0/2}^{h_0/2}\Big(\nabla_{tan}\tilde y^h - \big((Q^h)^T R^h\big)_{tan}(\mathrm{Id}+t\Pi_\gamma)^{-1}\Big)(\mathrm{Id}+t\Pi_\gamma)\ \mathrm{d}t\big)\circ\phi_\gamma\Big]\nabla\phi_\gamma\\
&\quad + \Big[\big(\frac{1}{h}\Big((Q^h)^T R^h(x) - \mathrm{Id}\Big)_{tan}\big)\circ\phi_\gamma\Big]\nabla\phi_\gamma = A^h\nabla\phi_\gamma + \mathcal{O}(h).
\end{aligned} \tag{A.4}$$

Therefore, $\nabla(V^h\circ\phi_\gamma)$ converges to A_{tan} in $L^2(\Omega)$ and since $\fint_\Omega V^h\circ\phi_\gamma = 0$, we may use Poincaré inequality on Ω to deduce (ii).

2. To prove (i), notice that by (A.2) and Lemma A.2 we obtain the following convergences in $L^2(\Omega^{h_0})$:

$$\begin{aligned}
&\lim_{h\to 0}\nabla(\tilde y^h\circ\tilde\phi_\gamma) - \nabla\tilde\phi_\gamma = \lim_{h\to 0}\Big((\nabla\tilde y^h - \mathrm{Id})\circ\tilde\phi_\gamma\Big)\nabla\tilde\phi_\gamma = 0,\\
&\lim_{h\to 0}\partial_3(\tilde y^h\circ\tilde\phi_\gamma) = \lim_{h\to 0}(\partial_{\vec n^\gamma}\tilde y^h)\circ\tilde\phi_\gamma = 0.
\end{aligned}$$

Therefore $\nabla(\tilde y^h\circ\tilde\phi_\gamma) - \nabla\phi_\gamma$ converges to 0 in $L^2(\Omega^{h_0})$. Since the sequence $\{V^h\circ\phi_\gamma\}$ is bounded in $L^2(\Omega)$, it also follows that:

$$\lim_{h\to 0}\left\|\int_{-h_0/2}^{h_0/2}\tilde y^h(\tilde\phi_\gamma) - x\ \mathrm{d}t\right\|_{L^2(S_\gamma)} = 0. \tag{A.5}$$

Now, let $g(x+t\vec n^\gamma) = |\det\,(\mathrm{Id}+t\Pi_\gamma(x))|^{-1}$. Consider the two terms in the right hand side of:

$$\|\tilde y^h - \pi\|_{L^2((S_\gamma)^{h_0})} \le \left\|(\tilde y^h-\pi) - \int_{(S_\gamma)^{h_0}}(\tilde y^h-\pi)\cdot\frac{g}{\int_{(S_\gamma)^{h_0}} g}\right\|_{L^2((S_\gamma)^{h_0})} + \left|\int_{S^{h_0}_\gamma}(\tilde y^h-\pi)\cdot\frac{g}{\int_{(S_\gamma)^{h_0}} g}\right|.$$

The first term can be bounded by means of the weighted Poincaré inequality, by $\|\nabla(\tilde y^h - \pi)\|_{L^2(S^{h_0}_\gamma)}$ and therefore it converges to 0 as $h\to 0$. The second term converges to 0 as well, in view of (A.5) and:

$$\left|\int_{(S_\gamma)^{h_0}} (\tilde{y}^h - \pi)\cdot g\right| = \left|\int_{S_\gamma}\int_{-h_0/2}^{h_0/2} \tilde{y}^h - \pi \,\mathrm{d}t\,\mathrm{d}x\right| \leq C\left\|\int_{-h_0/2}^{h_0/2} \tilde{y}^h - \pi\,\mathrm{d}t\right\|_{L^2(S_\gamma)}.$$

■

Proposition A.6. *We have:*

(i) $\lim_{h\to 0}\frac{1}{h^\alpha}\Big((\tilde{y}^h\circ\tilde{\phi}_\gamma) - x\Big) = v_0 e_3$ *in* $W^{1,2}(\Omega,\mathbb{R}^3)$,

(ii) $\mathrm{cof}\ \nabla^2 v_0 : \nabla^2 V^3 = \mathrm{cof}\ \nabla^2 v_0 : \nabla^2 v = 0$ *in* Ω.

Proof. The statement (i) easily follows from Lemma A.5 (i). By (A.4) and (A.1) we calculate:

$$\begin{aligned}\forall i,j=1,2\quad 2\left\langle\partial_i\phi_\gamma, (\mathrm{sym}\nabla V^h)\partial_j\phi_\gamma\right\rangle &= \left\langle\partial_i(V^h\circ\phi_\gamma),\partial_j\phi_\gamma\right\rangle + \left\langle\partial_j(V^h\circ\phi_\gamma),\partial_i\phi_\gamma\right\rangle\\ &= \left\langle\partial_j\phi_\gamma, A^h\partial_i\phi_\gamma\right\rangle + \left\langle\partial_i\phi_\gamma, A^h\partial_j\phi_\gamma\right\rangle + \mathcal{O}(h) = \mathcal{O}(h),\end{aligned}$$

and hence, denoting: $V^h\circ\phi_\gamma = (v_1^h, v_2^h, v_3^h)$, we get:

$$\begin{aligned}\mathcal{O}(h) = 2\left\langle\partial_i\phi_\gamma, (\mathrm{sym}\nabla V^h)\partial_j\phi_\gamma\right\rangle &= \left\langle\partial_i(V^h\circ\phi_\gamma),\partial_j\phi_\gamma\right\rangle + \left\langle\partial_j(V^h\circ\phi_\gamma),\partial_i\phi_\gamma\right\rangle\\ &= 2\left\langle e_i, \big(\mathrm{sym}\nabla(v_1^h,v_2^h) + h^\alpha\mathrm{sym}(\nabla v_3^h\otimes\nabla v_0)\big)e_j\right\rangle.\end{aligned}$$

Dividing by h^α and passing to 0 in h we obtain:

$$\lim_{h\to 0}\Big(\mathrm{sym}(\nabla v_3^h\otimes\nabla v_0) + \frac{1}{h^\alpha}\mathrm{sym}\nabla(v_1^h,v_2^h)\Big) = 0.$$

On the other hand $V^h\circ\phi_\gamma$ converges to V in Ω and so $\mathrm{sym}\nabla V = 0$ imply that (v_1^h, v_2^h) converge to a constant, while v_3^h converges to $V^3 = v\in W^{2,2}(\Omega)$. Passing to the limit we obtain:

$$\mathrm{sym}\big(\nabla v\otimes\nabla v_0\big) = -\lim_{h\to 0}\frac{1}{h^\alpha}\mathrm{sym}\nabla(v_1^h,v_2^h) = -\mathrm{sym}\nabla\tilde{w},$$

for some $\tilde{w}\in W^{1,2}(\Omega)$ (we used Korn's inequality for deducing the existence of $\tilde{w}$). By applying the operator $\mathrm{curl}^T\mathrm{curl}$ on both sides, we conclude:

$$\mathrm{cof}\ \nabla^2 v_0 : \nabla^2 v = 0,$$

as claimed in (ii). ■

We now need to study the following sequence of matrix fields on $(S_\gamma)^{h_0}$:

$$G^h = \frac{1}{h}\Big((R^h)^T\nabla_h y^h - \mathrm{Id}\Big)\circ\tilde{\phi}_\gamma.$$

In view of Proposition A.4 (i), the tensor 2sym G^h is the h^2 order term in the expansion of the nonlinear strain $(\nabla u^h)^T\nabla u^h$, at Id.

Lemma A.7. *(Parallel to* [16, Lemma 3.6]*.) The sequence* $\{G^h\}$ *as above has a subsequence, converging weakly in* $L^2(\Omega^{h_0})$ *to a matrix field* G*. The tangential minor of* G *is affine in the* e_3 *direction. More precisely:*

$$G(x,t)_{2\times 2} = G_0(x)_{2\times 2} - \frac{t}{h_0}\nabla^2 v(x),\quad \textit{with}\ \ G_0(x) = \fint_{-h_0/2}^{h_0/2} G(x,t)\,\mathrm{d}t.$$

Proof. **1.** The sequence $\{G^h\}$ is bounded in $L^2(\Omega^{h_0})$ by Proposition A.4 (i). Therefore it has a subsequence (which we do not relabel) converging weakly to some G. For a fixed $s > 0$, consider now the sequence of vector fields $f^{s,h} \in W^{1,2}((S_\gamma)^{h_0}, \mathbb{R}^3)$:

$$f^{s,h}(x+t\vec{n}^\gamma) = \frac{1}{sh^2}\Big[\Big(h_0\tilde{y}^h(x+(t+s)\vec{n}^\gamma) - h(x+(t+s)\vec{n}^\gamma)\Big) - \Big(h_0\tilde{y}^h(x+t\vec{n}^\gamma) - h(x+t\vec{n}^\gamma)\Big)\Big]$$

We claim that $f^{s,h} \circ \tilde{\phi}_\gamma$ converges in $L^2(\Omega^{h_0})$ to Ae_3. Indeed, by Proposition A.3 one has:

$$\begin{aligned} f^{s,h}(x+t\vec{n}^\gamma) &= \frac{1}{h^2} \fint_t^{t+s} \Big(h_0 \partial_{\vec{n}^\gamma}\tilde{y}^h(x+\sigma\vec{n}^\gamma) - h\vec{n}^\gamma\Big)\, \mathrm{d}\sigma \\ &= \frac{1}{h} \fint_t^{t+s} \Big((Q^h)^T \nabla_h y^h(x+\sigma\vec{n}^\gamma) - \mathrm{Id}\Big)\vec{n}^\gamma\, \mathrm{d}\sigma, \end{aligned}$$

and the convergence follows by Proposition A.4 (ii).

2. We claim that this convergence is actually weak in $W^{1,2}(\Omega^{h_0})$. First, notice that the x_3 derivatives converge to 0 in $L^2(\Omega^{h_0})$ by Proposition A.4 (ii):

$$\partial_3(f^{s,h}\circ\tilde{\phi}_\gamma) = \frac{1}{sh}\,\Big[(Q^h)^T\Big(\nabla_h y^h(x+(t+s)\vec{n}^\gamma) - \nabla_h y^h(x+t\vec{n}^\gamma)\Big)\circ\tilde{\phi}_\gamma\Big]\vec{n}^\gamma(x).$$

We now find the weak limit of the tangential gradients of $f^{s,h}$. By Proposition A.3:

$$\begin{aligned} \partial_i(f^{s,h}\circ\tilde{\phi}_\gamma) &= \frac{1}{sh^2}\,\Big[\Big(h_0\nabla\tilde{y}^h(x+(t+s)\vec{n}^\gamma)(\mathrm{Id}+(t+s)\Pi_\gamma)(\mathrm{Id}+t\Pi_\gamma)^{-1} \\ &\qquad - h_0\nabla\tilde{y}^h(x+t\vec{n}^\gamma) - hs\Pi_\gamma(\mathrm{Id}+t\Pi_\gamma)^{-1}\Big)\Big]\circ\tilde{\phi}_\gamma\partial_i\phi_\gamma \\ &= \frac{h_0}{sh^2}\,\Big[(Q^h)^T\Big(\nabla_h y^h(x+(t+s)\vec{n}^\gamma) - \nabla_h y^h(x+t\vec{n}^\gamma)\Big)(\mathrm{Id}+th/h_0\Pi_\gamma)(\mathrm{Id}+t\Pi_\gamma)^{-1}\Big]\circ\tilde{\phi}_\gamma\partial_i\phi_\gamma \\ &\quad + \frac{1}{sh}\,\Big[\Big((Q^h)^T\nabla_h y^h(x+(t+s)\vec{n}^\gamma) - \mathrm{Id}\Big)s\Pi_\gamma(\mathrm{Id}+t\Pi_\gamma)^{-1}\Big]\circ\tilde{\phi}_\gamma\partial_i\phi_\gamma. \end{aligned}$$

By Proposition A.4 (ii), the following expression in the right hand side above:

$$\frac{1}{h}\,\Big[\Big((Q^h)^T\nabla_h y^h(x+(t+s)\vec{n}^\gamma) - \mathrm{Id}\Big)\Pi_\gamma(\mathrm{Id}+t\Pi_\gamma)^{-1}]\Big]\circ\tilde{\phi}_\gamma$$

converges in $L^2(\Omega^{h_0})$ to 0. On the other hand, the first term in this expression converges weakly in $L^2(\Omega^{h_0})$ to:

$$\frac{h_0}{s}(G(x,t+s)) - G(x,t)),$$

by Lemma A.2 (i). This establishes the (weak) convergence of $f^{s,h}$ in $W^{1,2}(\Omega^{h_0})$.

3. Equating the weak limits of tangential derivatives, we obtain:

$$\partial_i(Ae_3)(x) = \frac{h_0}{s}\Big(G(x,(t+s)) - G(x,t)\Big)e_i.$$

This proves the lemma. ■

Finally, we have the following bound for convergence of the scaled energies I^h.

Lemma A.8. *(Parallel to* [16, Lemma 3.7] *and* [14, Theorem 1.3]*.)*

$$\liminf_{h\to 0} \frac{1}{h^4} I^h(y^h) \geq \frac{1}{2}\int_S \mathcal{Q}_2\left((\text{sym }(G_0)_{2\times 2} - (\epsilon_g)_{2\times 2}\right) + \frac{1}{24}\int_S \mathcal{Q}_2\left(\nabla^2 v + (\kappa_g)_{2\times 2}\right).$$

In view of Lemma A.5 and Lemma A.8, it remains to understand the structure of G_0.

Lemma A.9. *(Parallel to* [16, Lemma 3.7]*.) Let G_0 be as in Lemma A.7. Then we have the following convergence (up to a subsequence) weakly in $L^2(\Omega)$:*

$$\text{(A.6)}\qquad \lim_{h\to 0}\frac{1}{h}\left\langle \partial_j\phi_\gamma, ([\text{sym }\nabla V^h]\circ\phi_\gamma)\partial_i\phi_\gamma\right\rangle = \left\langle e_i, \left(\text{sym } G_0 + \frac{1}{2}A^2\right)_{2\times 2} e_j\right\rangle.$$

Proof. We use the formula (A.3) composed with ϕ_γ to calculate $\frac{1}{h}(\text{sym }\nabla V^h)\circ\phi_\gamma$. The last term in the right hand side gives:

$$\frac{1}{h^2}\text{sym}\left((Q^h)^T R^h - \text{Id}\right)_{tan}\circ\phi_\gamma = \frac{1}{h^2}\text{sym}\left((Q^h)^T R^h - \text{Id}\right)_{tan}\circ\phi_\gamma,$$

which converges in $L^2(\Omega)$ to $1/2(A^2)_{tan}$ by Lemma A.2 (iii). To treat the first term in the right hand side of (A.3), notice that for every $\tau\in T_x S_\gamma$:

$$\begin{aligned}
&\langle\frac{1}{h^2}\left[\fint_{-h_0/2}^{h_0/2}\nabla\tilde{y}^h(x+t\vec{n}^\gamma)(\text{Id}+t\Pi_\gamma) - (Q^h)^T R^h(x)\ \text{d}t\right]\circ\phi_\gamma,\tau\rangle\\
&\quad=\frac{1}{h^2}\langle(Q^h)^T\left[\fint_{-h_0/2}^{h_0/2}\nabla_h y^h(x+t\vec{n}^\gamma) - R^h(x)\ \text{d}t + \fint_{-h_0/2}^{h_0/2} th/h_0\nabla_h y^h\Pi_\gamma\ \text{d}t\right]\circ\phi_\gamma,\tau\rangle\\
&\quad=\frac{1}{h^2}\langle(Q^h)^T R^h(x)\left[\fint_{-h_0/2}^{h_0/2}(R^h)^T\nabla_h y^h - \text{Id}\ \text{d}t\right]\circ\phi_\gamma,\tau\rangle\\
&\qquad\qquad+\frac{h_0}{h}\langle\Big((Q^h)^T\left[\fint_{-h_0/2}^{h_0/2} t\left(\nabla_h y^h - R^h\pi\right)\ \text{d}t\right]\Pi_\gamma(x)\Big)\circ\phi_\gamma,\tau\rangle,
\end{aligned}$$

where we used Proposition A.3. Now, the second term in the right hand side above converges in $L^2(\Omega)$ to 0, by Proposition A.4 (i). Further, the matrix in the first term equals to:

$$((Q^h)^T R^h(x)\fint_{-h_0/2}^{h_0/2} G^h(x+t\vec{n}^\gamma)\ \text{d}t)\circ\phi_\gamma,$$

and by Lemma A.2 (i) and Lemma A.7, it converges weakly in $L^2(\Omega)$ to G_0. This completes the proof, in view of the fact that $\nabla\phi_\gamma$ converges to (e_1, e_2). ■

Conclusion of the proof of Theorem 4.4

It now remains to identify B as a member of $\mathcal{B}_{v_0}$. We have:

$$\text{(A.7)}\quad \frac{1}{h}\langle\partial_j\phi_\gamma, ([\text{sym }\nabla(V^h)]\circ\phi_\gamma)\partial_i\phi_\gamma\rangle = \frac{1}{h}\langle e_i, \left(\text{sym}\nabla(v_1^h, v_2^h) + h^\alpha\text{sym}(\nabla v_3^h\otimes\nabla v_0)\right)e_j\rangle.$$

Therefore:

$$\text{sym}(G_0)_{2\times 2} = B - \frac{A^2}{2}$$

where B is given by the following limit:

$$B = \lim_{h\to 0} B^h = \lim_{h\to 0} \frac{1}{h}\Big[\mathrm{sym}\nabla(v_1^h, v_2^h) + h^\alpha \mathrm{sym}(\nabla v_3^h \otimes \nabla v_0)\Big] \in \mathcal{B}_{v_0},$$

whose existence is assured by Lemma A.9. ■

Appendix B. Proof of Theorem 4.5

Let v and B be given as in the statement of Theorem 4.5. Since v satisfies the constraint (4.3) on a simply-connected Ω, there exists a displacement field $\tilde{w} \in W^{2,2}(\Omega, \mathbb{R}^2)$ for which:

$$\mathrm{sym}\nabla\tilde{w} + \mathrm{sym}(\nabla v \otimes \nabla v_0) = 0.$$

Let $V_\gamma = (\gamma\tilde{w}, v) \circ (\phi_\gamma)^{-1}$. Then it is straightforward as in (A.7) to verify that V_γ is a first order isometry of class $W^{2,2}$ on S_γ, i.e. $\mathrm{sym}\nabla V_\gamma = 0$ on S_γ. Also, using the isomorphism $\mathcal{T}^\gamma$ in Lemma 4.1, we let $B_\gamma = (\mathcal{T}^\gamma)^{-1}(B) \in \mathcal{B}^\gamma$, where the latter space is identified in [16] as the finite strain space of S_γ. Given V_γ and B_γ as above, we proceed as in [16, Theorem 2.2], as follows. With a slight abuse of notation, we write:

$$\mathcal{Q}_2(x, F_{tan}) = \min\left\{\mathcal{Q}_3(F_{tan} + c\otimes e_3 + e_3 \otimes c);\ c \in \mathbb{R}^3\right\}. \tag{B.1}$$

The unique vector c, which attains the above minimum will be denoted $c(x, F_{tan})$. By uniqueness, the map c is linear in its second argument. Also, for all $F \in \mathbb{R}^{3\times 3}$, by $l(F)$ we denote the unique vector in $\mathbb{R}^3$, linearly depending on F, for which:

$$\mathrm{sym}\big(F - (F_{2\times 2})^*\big) = \mathrm{sym}\big(l(F)\otimes e_3\big).$$

Recall that $\gamma = h^\alpha$. Given $B_\gamma \in \mathcal{B}^\gamma$, there exists a sequence of vector fields $w^h \in W^{1,2}(S_\gamma, \mathbb{R}^3)$ such that $\|\mathrm{sym}\ \nabla w^h - B_\gamma\|_{L^2(S_\gamma)}$ converges to 0. Without loss of generality, we may assume that w^h are smooth, and (by possibly reparameterizing the sequence) that:

$$\lim_{h\to 0} \sqrt{h}\|w^h\|_{W^{2,\infty}(S_\gamma)} = 0. \tag{B.2}$$

In the same manner, we approximate V_γ by a sequence $v^h \in W^{2,\infty}(S_\gamma, \mathbb{R}^3)$ such that, for a sufficiently small, fixed $\epsilon_0 > 0$:

$$\begin{aligned} &\lim_{h\to 0}\|v^h - V_\gamma\|_{W^{2,2}(S_\gamma)} = 0, \qquad h\|v^h\|_{W^{2,\infty}(S_\gamma)} \le \epsilon_0,\\ &\lim_{h\to 0}\frac{1}{h^2}\ \mu\left\{x \in S;\ v^h(x) \neq V(x)\right\} = 0. \end{aligned} \tag{B.3}$$

The existence of such v^h follows by partition of unity and a truncation argument, as a special case of the Lusin-type result for Sobolev functions (see [8, Proposition 2]).

Finally, define the sequence of deformations $u^h \in W^{1,2}((S_\gamma)^h, \mathbb{R}^3)$ by:

$$\begin{aligned} u^h(x + t\vec{n}^\gamma) = {} & x + hv^h(x) + h^2 w^h(x)\\ & + t\vec{n}^\gamma(x) + th\Big(\Pi_\gamma v^h_{tan} - \nabla(v^h\vec{n}^\gamma)\Big)(x)\\ & - th^2(\vec{n}^\gamma)^T\nabla w^h + th^2 d^{0,h}(x) + \frac{1}{2}t^2 h d^{1,h}(x). \end{aligned} \tag{B.4}$$

The vector fields $d^{0,h}, d^{1,h} \in W^{1,\infty}(S_\gamma, \mathbb{R}^3)$ are defined so that:

$$\lim_{h\to 0} \sqrt{h}\left(\|d^{0,h}\|_{W^{1,\infty}(S_\gamma)} + \|d^{1,h}\|_{W^{1,\infty}(S_\gamma)}\right) = 0 \tag{B.5}$$

and that, in $L^2(\Omega)$:

$$\begin{aligned} &\lim_{h\to 0} d^{0,h}\circ\phi_\gamma = l(\epsilon_g) - \frac{1}{2}|\nabla v|^2 e_3 + c\Big(B - \frac{1}{2}\nabla v\otimes\nabla v - (\mathrm{sym}\ \epsilon_g)_{2\times 2}\Big),\\ &\lim_{h\to 0} d^{1,h}\circ\phi_\gamma = l(\kappa_g) + c\Big(-\nabla^2 v - (\mathrm{sym}\ \kappa_g)_{2\times 2}\Big). \end{aligned} \tag{B.6}$$

Now, the convergence statements of Theorem 4.5 are verified by straightforward calculations as in the proofs of [16, Theorem 2.2] and [14, Theorem 1.4].

Remark B.1. One may define the recovery sequence explicitly on Ω, without the diagonal argument presented in the proof above. Namely we proceed as follows.

We approximate $V = (\tilde{w}, v)$ by a sequence $V^h = (\tilde{w}^h, v^h) \in W^{2,\infty}(\Omega, \mathbb{R}^3)$ such that, for a sufficiently small, fixed $\epsilon_0 > 0$:

$$\begin{aligned} &\lim_{h\to 0} \|V^h - V\|_{W^{2,2}(\Omega)} = 0, \qquad h\|V^h\|_{W^{2,\infty}(\Omega)} \le \epsilon_0,\\ &\lim_{h\to 0} \frac{1}{h^2}\,\mu\left\{x\in\Omega;\ V^h(x)\neq V(x)\right\} = 0. \end{aligned} \tag{B.7}$$

Also, let $w^h : \Omega \to \mathbb{R}^3$ be such that

$$B = \lim_{h\to 0}\Big[\mathrm{sym}\nabla(w_1^h, w_2^h) + \mathrm{sym}(\nabla w_3^h\otimes\nabla v_0)\Big].$$

Without loss of generality, we may assume that w^h are smooth, and that:

$$\lim_{h\to 0}\sqrt{h}\|w^h\|_{W^{2,\infty}(\Omega)} = 0. \tag{B.8}$$

The recovery sequence is then given by:

$$\begin{aligned} y^h(x,t) &= u^h(x + h^\alpha v_0(x)e_3 + th\vec{n}^\gamma)\\ &= \begin{bmatrix} x \\ h^\alpha v_0(x)\end{bmatrix} + h\begin{bmatrix} h^\alpha\tilde{w}^h(x) \\ v^h(x)\end{bmatrix} + h^2\begin{bmatrix} w^h_{tan} \\ h^{-\alpha}w_3^h\end{bmatrix}\\ &\qquad + th\begin{bmatrix} -h^\alpha\nabla v_0(x) \\ 1\end{bmatrix} + th\begin{bmatrix} -h\nabla v^h(x) \\ 1\end{bmatrix}\\ &\qquad - th^3\begin{bmatrix} 0 \\ h^{-\alpha}\nabla w_3^h\end{bmatrix} + h^3 t d^{0,h}(x) + \frac{1}{2}h^3t^2 d^{1,h}(x), \end{aligned} \tag{B.9}$$

where the vector fields $d^{0,h}, d^{1,h} \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ are defined similarly as before.

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