

# THE MONGE-AMPÈRE CONSTRAINT: MATCHING OF ISOMETRIES, DENSITY AND REGULARITY, AND ELASTIC THEORIES OF SHALLOW SHELLS

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ABSTRACT. The main analytical ingredients of the first part of this paper are two independent results: a theorem on approximation of  $W^{2,2}$  solutions of the Monge-Ampère equation by smooth solutions, and a theorem on the matching (in other words, continuation) of second order isometries to exact isometric embeddings of 2d surface in  $\mathbb{R}^3$ .

In the second part, we rigorously derive the  $\Gamma$ -limit of 3-dimensional nonlinear elastic energy of a shallow shell of thickness  $h$ , where the depth of the shell scales like  $h^\alpha$  and the applied forces scale like  $h^{\alpha+2}$ , in the limit when  $h \rightarrow 0$ . We offer a full analysis of the problem in the parameter range  $\alpha \in (1/2, 1)$ . We also complete the analysis in some specific cases for the full range  $\alpha \in (0, 1)$ , applying the results of the first part of the paper.

**Résumé. Les contraintes de type Monge-Ampère : Prolongement des isometries, densité et régularité, et les Modèles variationnels pour les coques minces.** On démontre d'abord deux résultats sur la densité des fonctions régulières dans l'ensemble des solutions de l'équation de Monge-Ampère et sur le prolongement des isometries infinitésimales de l'ordre 2 des surfaces bidimensionnelles aux isometries exactes.

On dérive ensuite un modèle nouveau pour les coques minces peu profondes d'épaisseur  $h$  et profondeur de l'ordre de  $h^\alpha$  départant de la théorie trois-dimensionnelle d'élasticité nonlinéaire. Le modèle limite obtenu par la  $\Gamma$ -convergence consiste en minimisant une énergie biharmonique sous un contraint de type Monge-Ampère. Ce résultat s'applique au cas où les forces sont de l'ordre de  $h^{\alpha+2}$  et  $1/2 < \alpha < 1$ . On peut l'étendre pour  $\alpha \in (0, 1)$  dans certains cas spécifiques, utilisant les résultats de la première partie de l'article.

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## 1. INTRODUCTION

The mathematical theory of elastic shells must account for the deformation of thin elastic surfaces of non-zero curvature and the associated energetics. The subject thus brings together the differential geometry of surfaces with the theory of elasticity appropriately modified to account for the small aspect ratio of these slender elastic structures. From a practical engineering perspective, many approximate theories have been proposed for the mechanical behavior of elastic shells over the last 150 years [4]. The mathematical foundations for these theories has lagged behind but has recently been the focus of much attention [6, 5, 10] from two perspectives: (i) as a means of understanding the rigorous derivation of these theories teaches to understand the limits of their practical applicability, and (ii) as a way to sheds light on the smoothness of allowable deformations with implications for the properties of elliptic operators that arise naturally in differential geometry. Both these questions can be couched in terms of the behavior of the elastic energy of the shell as a function of its aspect ratio, i.e. the ratio of its thickness to its curvature and/or lateral extent. For a flat sheet with no intrinsic curvature, the stretching energy per unit area of a thin sheet is proportional to its thickness, while the bending energy per unit area is proportional to the cube of its thickness. Thus, as the aspect ratio diminishes, the bending energy vanishes faster than the stretching or shearing energy. This leads to approximately isometric deformations of the sheet when it is subject to external loads, with a concomitant theory known as the Föppl-von Kármán theory [16] of elastic plates. For shallow shells with an intrinsic curvature, there is a new small parameter corresponding to the product of the intrinsic curvature and the thickness so that there are various possible distinguished limits associated with how small or large this parameter is, independent of the aspect ratio of the shell. This leads to various theories that penalize non-isometric deformations more or less depending on the relative magnitude of external loads that cause the shell to deform.

Over the past few years, we have begun to get a good understanding of the limiting models and the types of isometries involved in various similar contexts (e.g. see the review and a conjecture in [23] for thin shells), but each separate case enjoys its own peculiarities related to the the geometry of the shell (hyperbolic, elliptic, degenerate or of mixed type), the order of the relevant approximate isometries on the shell, the linearity or it lack in the PDEs governing them and their regularity. For shells of thickness  $h$  and depth  $h^\alpha$ , subject to applied forces that scale with the shell depth as  $h^{\alpha+2}$ , as the thickness of the shell  $h \rightarrow 0$ , depending on the choice of  $\alpha$ , various limiting theories arise. The regimes  $\alpha = 1$  and  $\alpha > 1$  can be treated in a similar manner as discussed in another context in [20] and are not of interest to us in this paper.

Here, we consider the case when the shell is shallow and the forces weak, but not too weak, i.e. when  $0 < \alpha < 1$ , so that nearly isometric deformations might be expected, but implemented via nonlinear constraints, rather than linear ones as appearing in the  $\alpha \geq 1$  regimes. In the vanishing thickness limit, using the basic methods of  $\Gamma$ -convergence which

were developed in this context in [9, 10], we derive a new thin film model (see also previous results in [19, 20]) corresponding to a situation where the second order infinitesimal isometries on a shallow shell reference configuration are given by the out-of-plane displacement  $h^\alpha v_0$ . In analogy with the results in [10] for plates with flat geometry  $v_0 = 0$  in the energy scaling regime  $2 < \beta < 4$ , we recover a Monge-Ampère constraint  $\det \nabla^2 v = \det \nabla^2 v_0$  as a second order isometry constraint on the limiting displacements of Sobolev regularity  $W^{2,2}$ . This emergence of a “linearized curvature” constraint is natural from a mechanical point of view since the Laplacian of the strain (which is proportional to the Laplacian of the stress for linear stress-strain relations) characterizes local area changes and the tensorial isometry condition is softened to a scalar constraint of local area-preservation.

In this paper, we offer a full analysis of the problem in the parameter range  $\alpha \in (1/2, 1)$  and obtain the required  $\Gamma$ -convergence result which identifies the proper limit model. However, similarly to the previously studied cases [10, 22], in order to complete the analysis for the full range  $(0, 1)$  we need a result on the matching property, i.e. the continuation of second order isometries, on shallow shells, to exact isometries (under suitable regularity assumptions), and another result on approximating the  $W^{2,2}$  solutions of the Monge-Ampère equation by smooth solutions. Contrary to [22, 13], the constraint here is a fully nonlinear one, and one cannot take advantage of the full degeneracy of  $\det \nabla^2 v = 0$  in [10, Theorems 7 and 10] [31]. Nevertheless, we prove the applicable versions of these results in the case  $\det \nabla^2 v_0 \equiv c > 0$ .

The outlay of the paper is as follows. In Sections 2 and 4, we present our main results, namely the matching property and density results, and the corresponding asymptotic behavior of the elastic energies when  $h \rightarrow 0$ . In Section 3 we discuss the various possible choices of  $\alpha$  and the known corresponding limiting theories. In Sections 5-7 we present the details of our proofs. Section 6 is dedicated to the study of not necessarily convex  $W^{2,2}$  solutions of the Monge-Ampère equation, using some key observations by Šverák from his unpublished manuscript [34]. These results are a necessary ingredient of the density result in Section 6. Sections 9 and 8 contain the construction of the recovery sequences for the upper bound in the corresponding  $\Gamma$ -limit results.

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## 2. THE MAIN RESULTS I: THE MATCHING AND DENSITY PROPERTIES

Throughout the paper  $\Omega \subset \mathbb{R}^2$  will be a domain, i.e. an open, bounded and simply connected set. Our first main result regards the matching of isometries on convex weakly shallow shells.

**Theorem 2.1.** *Assume that  $\Omega$  is simply connected and let  $v_0 \in \mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R})$  with  $\det \nabla^2 v_0 > c_0 > 0$ . Let  $v \in \mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R})$  satisfy:*

$$(2.1) \quad \det \nabla^2 v = \det \nabla^2 v_0 \quad \text{in } \Omega.$$

Then there exists a sequence  $w_h \in \mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R}^3)$  such that:

$$(2.2) \quad \forall h > 0 \quad \nabla(\text{id} + hve_3 + h^2w_h)^T \nabla(\text{id} + hve_3 + h^2w_h) = \nabla(\text{id} + hv_0e_3)^T \nabla(\text{id} + hv_0e_3)$$

and  $\sup \|w_h\|_{\mathcal{C}^{2,\beta}} < +\infty$ .

The condition (2.2) means that each deformation  $u^h : S_h \rightarrow \mathbb{R}^3$  of a surface  $S_h = \{x + hv_0(x)e_3; x \in \Omega\}$ , given by  $u^h(x + hv_0(x)e_3) = x + hv(x)e_3 + h^2w_h(x)$  is an isometry of  $S_h$ . In other words, the pull-back metrics from the Euclidean metric of  $S_h$  and of  $u^h(S_h) = \{x + hv(x)e_3 + h^2w_h(x); x \in \Omega\}$  coincide. Hence Theorem 2.1 asserts that if two convex out-of-plane displacements of first order have the same determinants of Hessians, then they can be matched by a family of equibounded higher order displacements (the fields  $w_h$ ) to be isometrically equivalent. For other results concerning matching of isometries see [10, Theorem 7], [22, Theorem 1.1], [13, Theorem 3.1] (which is comparable with [33, Lemma 3.3] and the remark which follows therein) and [12, Theorem 4.1].

We will put our analysis in a broader context in Remark 2.4. The proof of Theorem 2.1 will be given in Section 5.

Our next main result concerns the density of regular solutions to the elliptic 2d Monge-Ampère equation.

**Theorem 2.2.** *Let  $\Omega$  be open, bounded, connected and star-shaped with respect to an interior ball  $B \subset \Omega$ . For a fixed constant  $c_0 > 0$ , define:*

$$\mathcal{A} := \{u \in W^{2,2}(\Omega); \det \nabla^2 u = c_0 \text{ a.e. in } \Omega\}.$$

Then  $\mathcal{A} \cap C^\infty(\bar{\Omega})$  is dense in  $\mathcal{A}$  with respect to the strong  $W^{2,2}$  norm.

The main difficulty to overcome is the absence of convexity assumptions on the  $W^{2,2}$  solutions of the Monge-Ampère equation in our context. We first establish the convexity of elements of  $\mathcal{A}$  in a broader sense, combining some key observations by Šverák from his unpublished manuscript [34], with a theorem due to Iwaniec and Šverák in [15] (Theorem 6.5) on deformations with integrable dilatation in dimension 2. Namely, we deduce Theorem 2.2 by showing first the interior regularity of solutions:

**Theorem 2.3.** *Let  $u \in W^{2,2}(\Omega)$  be such that:*

$$(2.3) \quad \det \nabla^2 u = f \quad \text{in } \Omega, \quad \text{where } f : \Omega \rightarrow \mathbb{R}, \quad f(x) \geq c_0 > 0 \quad \forall \text{a.e. } x \in \Omega.$$

Then  $u \in \mathcal{C}^1(\Omega)$  and, modulo a global sign change,  $u$  is locally convex in  $\Omega$ .

Some of the results in [34] are known within the community, but for completeness we write the proofs of Theorem 2.3 in full detail in Sections 6 and 7.

**Remark 2.4.** In [23], the authors put forward a conjecture regarding existence of infinitely many small slope shell theories (with no prestrain) each valid for a corresponding range of energy scalings. This conjecture is based on formal asymptotic expansions and it is in accordance with the previously obtained results for plates and shells. It predicts the form of the 2-dimensional limit energy functional, and identifies the space of admissible deformations as infinitesimal isometries of a given integer order  $N > 0$  determined by the magnitude of the elastic energy. Hence, the influence of shells geometry on its qualitative response to

an external force, i.e. the shells rigidity, is reflected in a hierarchy of functional spaces of isometries (and infinitesimal isometries) arising as constraints of the derived theories.

In certain cases, a given  $N$ th order infinitesimal isometry can be modified by higher order corrections to yield an infinitesimal isometry of order  $M > N$ , a property to which we refer to by “matching property of infinitesimal isometries”. This feature, combined with certain density results for spaces of isometries, cause the theories corresponding to orders of infinitesimal isometries between  $N$  and  $M$ , to collapse all into one and the same theory. The examples of such behavior are observed for plates [10], where any second order infinitesimal isometry can be matched to an exact isometry ( $M = \infty$ ), for convex shells [22], where any first order infinitesimal isometry satisfies the same property, and for non-flat developable surfaces [33, 13] where first order isometries can be matched to higher order isometries (see also [12]). The effects of these geometric properties on the elasticity of thin films are drastic. A plate whose boundary is at least partially free possesses three types of small-slope theories: the linear theory, the von Kármán theory and the linearized Kirchhoff theory, whereas the only small slope theory for a convex shell with free boundary is the linear theory [22]: a convex shell transitions directly from the linear regime to the fully nonlinear bending one if the applied forces are adequately increased. In other words, while the von Kármán theory describes the buckling of thin plates at a body force magnitude of order thickness-cubed, the equivalent, variationally correct theory for buckling of elliptic shells is the purely nonlinear bending theory which comes only into effect when the body forces reach to a magnitude of order thickness-squared.

Our remaining results concern the variational limits of the elastic energies. We explain the set-up and present our findings in the next two sections.

### 3. ELASTIC SHELLS OF LOW CURVATURE. THE SET-UP AND DISCUSSION OF PAST RESULTS

For a given out-of-plate displacement  $v_0 \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R})$ , and for a given exponent  $\alpha > 0$ , consider a sequence of surfaces:

$$S_h = \phi_h(\Omega) \quad \text{where } \phi_h(x) = (x, h^\alpha v_0(x)) \quad \forall x = (x_1, x_2) \in \Omega,$$

and the family of thin plates  $\Omega^h = \Omega \times (-h/2, h/2)$  and thin shallow shells  $(S_h)^h$ :

$$(3.1) \quad (S_h)^h = \{\tilde{\phi}_h(x, x_3); x \in \Omega, x_3 \in (-h/2, h/2)\} \quad \forall 0 < h \ll 1.$$

Above, the Kirchhoff-Love extension  $\tilde{\phi}_h : \Omega^h \rightarrow \mathbb{R}^3$  of the parametrization  $\phi_h$ , is given by the formula:

$$(3.2) \quad \tilde{\phi}_h(x, x_3) = \phi_h(x) + x_3 \vec{n}^h(x) \quad \forall (x, x_3) \in \Omega^h,$$

while the vector  $\vec{n}^h(x)$  is the unit normal to  $S_h$  at  $\phi_h(x)$ :

$$\vec{n}^h(x) = \frac{\partial_1 \phi_h(x) \times \partial_2 \phi_h(x)}{|\partial_1 \phi_h(x) \times \partial_2 \phi_h(x)|} = \frac{1}{\sqrt{1 + h^{2\alpha} |\nabla v_0|^2}} (-h^\alpha \partial_1 v_0(x), -h^\alpha \partial_2 v_0(x), 1) \quad \forall x \in \Omega.$$

The thickness averaged elastic energy of a deformation  $u^h$  of  $(S_h)^h$  is now given by:

$$(3.3) \quad I^h(u^h) = \frac{1}{h} \int_{(S_h)^h} W(\nabla u^h) \quad \forall u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3).$$

The energy density  $W : \mathbb{R}^{3 \times 3} \rightarrow \bar{\mathbb{R}}_+$  above, in addition to being  $\mathcal{C}^2$  regular in a neighborhood of  $SO(3)$ , is assumed to satisfy the normalization, frame indifference and nondegeneracy conditions:

$$(3.4) \quad \begin{aligned} \exists c > 0 \quad \forall F \in \mathbb{R}^{3 \times 3} \quad \forall R \in SO(3) \quad & W(R) = 0, \quad W(RF) = W(F), \\ & W(F) \geq c \operatorname{dist}^2(F, SO(3)). \end{aligned}$$

where  $F = \nabla u$  is typically the deformation gradient associated with a mapping  $u$ . The following quadratic forms, generated by  $W$ , will be relevant in the subsequent analysis:

$$(3.5) \quad \mathcal{Q}_3(F) = D^2W(\operatorname{Id})(F, F), \quad \mathcal{Q}_2(F_{tan}) = \min\{\mathcal{Q}_3(\tilde{F}); \tilde{F} \in \mathbb{R}^{3 \times 3}, (\tilde{F} - F)_{tan} = 0\}.$$

The form  $\mathcal{Q}_3$  is defined for all  $F \in \mathbb{R}^{3 \times 3}$ , while  $\mathcal{Q}_2$  is defined on the  $2 \times 2$  principal minors  $F_{tan}$  of such matrices. By (3.4), both forms  $\mathcal{Q}_3$  and all  $\mathcal{Q}_2$  are nonnegative definite and depend only on the symmetric parts of their arguments.

Let  $f^h \in L^2((S_h)^h, \mathbb{R}^3)$  be a family of loads applied to the elastic shells under consideration. The total energy is then:

$$(3.6) \quad J^h(u^h) = \frac{1}{h} \int_{(S_h)^h} W(\nabla u^h) - \frac{1}{h} \int_{(S_h)^h} f^h u^h \quad \forall u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3).$$

In what follows, we will make the simplifying assumptions:

$$(3.7) \quad f^h = (0, 0, h^{\alpha'} f \circ \tilde{\phi}_h^{-1})^T$$

for some  $f \in L^2(\Omega)$ , normalizes so that:

$$(3.8) \quad \int_{\Omega} f = 0 \quad \text{and} \quad \int_{\Omega} x f(x) \, dx = 0.$$

Heuristically, stronger forces ( $\alpha' < \alpha + 2$ ) deform the shallow shell beyond the reference shape, while weaker forces ( $\alpha' > \alpha + 2$ ) leave it undeformed, with an asymptotic behavior of displacements of lower order similar to that of a plate. The remaining case where the forces are tuned with the curvature of the mid-surface (shallowness), is given by the scaling regime  $\alpha' = \alpha + 2$ .

When  $\alpha \geq 1$ , by a simple change of variables, we see that:

$$J^h(u^h) = \frac{1}{h} \int_{\Omega^h} W\left((\nabla v^h)(b^h)^{-1}\right) \det b^h - \frac{1}{h} \int_{\Omega^h} h^{\alpha_i} f v_3^h \det b^h,$$

with  $v^h = u^h \circ \tilde{\phi}_h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$  and  $b^h = \nabla \tilde{\phi}_h$ . Note that by the polar decomposition of positive definite matrices, there holds:  $b^h = R(x, x_3) a^h$  for some  $R(x, x_3) \in SO(3)$  and the symmetric tensor  $a^h = \sqrt{(b^h)^T b^h}$ . Therefore, for the isotropic energy  $W$  i.e. when:

$$(3.9) \quad W(FR) = W(F) \quad \forall F \in \mathbb{R}^{3 \times 3} \quad \forall R \in SO(3),$$

one obtains:

$$(3.10) \quad \begin{aligned} I^h(u^h) &= \frac{1}{h} \int_{\Omega^h} W\left((\nabla v^h)(a^h)^{-1} R(x, x_3)^{-1}\right) \det b^h \, d(x, x_3) \\ &= \frac{1}{h} \int_{\Omega^h} W\left((\nabla v^h)(a^h)^{-1}\right) (1 + \mathcal{O}(h)) \, d(x, x_3), \end{aligned}$$

which reduces the problem to studying deformations of the flat plate  $\Omega^h$  relative to the prestrain tensor  $a^h$ , see [19, 20] for a discussion of this topic.

When  $\alpha = 1$ , we derived in [20] the  $\Gamma$ -limit  $\mathcal{J}_{v_0}$  of the scaled energies  $\frac{1}{h^4}I^h$ . Namely, we showed that the energy of the almost minimizing deformations scales like:

$$\inf J^h \sim h^4,$$

and that the  $\Gamma$ -limit (in the general, possibly non-isotropic case) is given by:

$$(3.11) \quad \mathcal{J}_{v_0}(w, v) = \frac{1}{2} \int_{\Omega} \mathcal{Q}_2 \left( \text{sym} \nabla w + \frac{1}{2} \nabla v \otimes \nabla v - \frac{1}{2} \nabla v_0 \otimes \nabla v_0 \right) + \frac{1}{24} \int_{\Omega} \mathcal{Q}_2 (\nabla^2 v - \nabla^2 v_0) - \int_{\Omega} f v,$$

which is the von Kármán-type functional defined for all out-of-plane displacements  $v \in W^{2,2}(\Omega, \mathbb{R})$  and the in-plane displacements  $w \in W^{1,2}(\Omega, \mathbb{R}^2)$ . In analogy with the theory for flat plates with [10], due to the choice of scaling in (3.7), the limit energy is composed of two terms, corresponding to stretching and bending.

In the isotropic case (3.9), the Euler-Lagrange equations of  $\mathcal{J}_{v_0}$  are:

$$(3.12) \quad \begin{cases} \Delta^2 \Phi = -S(\det \nabla^2 v - \det \nabla^2 v_0) \\ B(\Delta^2 v - \Delta^2 v_0) = [v, \Phi] + f \end{cases}$$

where  $S$  is the Young modulus,  $B$  the bending stiffness,  $\nu$  the Poisson ratio (given in terms of the Lamé constants  $\mu$  and  $\lambda$ ). A more involved version of the system (3.12) incorporating prestrain was first introduced in [25] using a thermoelastic analogy to growth, as a mathematical model of blooming activated by differential lateral growth from an initial non-zero transverse displacement field  $v_0$  (See also [24, 8]). See [20] for the rigorous derivation of that model.

When  $\alpha > 1$ , the bending energy takes over the stretching and hence the limiting theory is a variant of the linear elasticity as discussed in [10], yielding the Euler-Lagrange equations:

$$B(\Delta^2 v - \Delta^2 v_0) = f.$$

We now turn to the case of interest treated in this paper, namely  $0 < \alpha < 1$ .

#### 4. THE MAIN RESULTS II: ELASTIC SHALLOW SHELLS

We first state the following lemma, whose proof is similar to [10, Theorem 2-i] and [21, Theorem 2.5], and hence it is omitted for brevity of presentation.

**Lemma 4.1.** *Assume (3.7) and (3.8). Then:*

(i) *For every small  $h > 0$  one has:*

$$0 \geq \inf \left\{ \frac{1}{h^{2\alpha+2}} J^h(u^h); \quad u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3) \right\} \geq -C.$$

(ii) *If  $u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3)$  is a minimizing sequence of  $\frac{1}{h^{2\alpha+2}} J^h$ , that is:*

$$(4.1) \quad \lim_{h \rightarrow 0} \left( \frac{1}{h^{2\alpha+2}} J^h(y^h) - \inf \frac{1}{h^{2\alpha+2}} J^h \right) = 0,$$

then  $\frac{1}{h^{2\alpha+2}}I^h(u^h)$  is bounded.

We now note, by a straightforward calculation, that  $a^h$  in (3.10), pertaining to the isotropic case (3.9), becomes:

$$a^h = \text{Id} + \frac{1}{2}h^{2\alpha}(\nabla v_0 \otimes \nabla v_0)^* - h^\alpha x_3(\nabla^2 v_0)^* + o(h^{2\alpha}) + x_3 o(h^\alpha)$$

where the uniform quantities in  $o(h^{2\alpha})$ ,  $o(h^\alpha)$  are independent of  $x_3$ . Following the proof of Theorem 1.3 in [19], we actually obtain in the general (possibly non-isotropic) case:

**Theorem 4.2.** *Assume that  $u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3)$  satisfies  $I^h(u^h) \leq Ch^{2\alpha+2}$ , where  $I^h$  is given in (3.3) and  $0 < \alpha < 1$ . Then there exists rotations  $\bar{R}^h \in SO(3)$  and translations  $c^h \in \mathbb{R}^3$  such that for the normalized deformations:*

$$(4.2) \quad y^h(x, t) = (\bar{R}^h)^T(u^h \circ \tilde{\phi}_h)(x, ht) - c^h : \Omega^1 \longrightarrow \mathbb{R}^3$$

defined by means of (3.2) on the common domain  $\Omega^1 = \Omega \times (-1/2, 1/2)$  the following holds:

(i)  $y^h(x, t)$  converge in  $W^{1,2}(\Omega^1, \mathbb{R}^3)$  to  $x$ .

(ii) the scaled displacements  $V^h(x) = h^{-\alpha} \int_{-1/2}^{1/2} y^h(x, t) - x \, dt$  converge (up to a subsequence) in  $W^{1,2}(\Omega, \mathbb{R}^3)$  to  $(0, 0, v)^T$  where  $v \in W^{2,2}(\Omega, \mathbb{R})$  and:

$$(4.3) \quad \det \nabla^2 v = \det \nabla^2 v_0.$$

(iii) Moreover:  $\liminf_{h \rightarrow 0} h^{-(2\alpha+2)} J^h(u^h) \geq \mathcal{J}_{v_0}(v)$  where:

$$(4.4) \quad \mathcal{J}_{v_0}(v) = \frac{1}{24} \int_{\Omega} \mathcal{Q}_2(\nabla^2 v - \nabla^2 v_0) - \int_{\Omega} f v.$$

The constraint (4.3) in the assertion (ii) follows by observing that  $h^{-\alpha} \text{sym} \nabla V^h$  converges in  $L^2(\Omega)$  to  $F = \frac{1}{2}(\nabla v_0 \otimes \nabla v_0 - \nabla v \otimes \nabla v)$ , and hence  $F = \text{sym} \nabla w$  for some  $w : \Omega \rightarrow \mathbb{R}^2$ . Consequently  $\text{curl}^T \text{curl} F = 0$ , which implies (4.3).

We now have the following:

**Theorem 4.3.** *Fix  $\alpha \in (1/2, 1)$ . Then, for every  $v \in W^{2,2}(\Omega, \mathbb{R})$  with  $\det \nabla^2 v = \det \nabla^2 v_0$ , there exists a sequence of deformations  $u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3)$  such that:*

(i) *The rescaled sequence  $y^h(x, t) = u^h(x + h^\alpha v_0(x) e_3 + ht \bar{n}^h(x))$  converges in  $W^{1,2}(\Omega^1, \mathbb{R}^3)$  to  $x$ .*

(ii) *The scaled displacements  $V^h$  as in (ii) Theorem 4.2 converge in  $W^{1,2}$  to  $(0, 0, v)$ .*

(iii)  $\lim_{h \rightarrow 0} h^{-(2\alpha+2)} J^h(u^h) = \mathcal{J}_{v_0}(v)$ .

The proof of Theorem 4.3 will be given in Section 9. Theorems 4.2 and 4.3 can be stated together, by identifying the linearized Kirchhoff-like energy (4.4) (4.3) as the  $\Gamma$ -limit of the rescaled energies  $h^{-(2\alpha+2)} J^h(u^h)$ , in the regime corresponding to  $\alpha \in (1/2, 1)$ . We conjecture that the same holds for all  $\alpha \in (0, 1)$ . Note that this result is established in [10] for the degenerate case  $v_0 \equiv 0$  and hence our model can be considered as a generalization of the linearized Kirchhoff model discussed in [10]. Also, note that when  $\Omega$  is not simply connected, one must replace (4.3) with a more general variant which states that  $\nabla v_0 \otimes \nabla v_0 - \nabla v \otimes \nabla v$  is a symmetric gradient.

We are able to prove our conjecture in the specific case when  $\det \nabla^2 v_0$  is constant. Its proof relies on Theorems 2.1 and 2.2 and it will be given in Section 8. Namely, we have:



**Theorem 4.4.** *Assume that  $\Omega$  is open, bounded and star-shaped with respect to an interior ball  $B \subset \Omega$ . Assume that:*

$$\det \nabla^2 v_0 \equiv c_0 > 0 \quad \text{in } \Omega.$$

*Fix  $\alpha \in (0, 1)$ . Then, for every  $v \in W^{2,2}(\Omega, \mathbb{R})$  with  $\det \nabla^2 v = \det \nabla^2 v_0$ , there exists a sequence of deformations  $u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3)$  such that the conclusions (i), (ii) and (iii) in Theorem 4.3 hold.*

**Remark 4.5.** We expect that the  $\mathcal{C}^{2,\beta}$  scalar fields  $v$  are dense in the set of the  $W^{2,2}$  fields with any prescribed, strictly positive but not necessarily constant  $\det \nabla^2$  of  $\mathcal{C}^{0,\beta}$  regularity. With such a result, it would follow that for all convex shells of sufficient regularity, the linearized Kirchhoff-type energy (4.4) is the rigorous variational limit on weakly shallow shells, in the same spirit as the matching and density of first order isometries on convex shells [22] resulted in that the only small slope theory for an elastic convex shell is the linear theory. The latter problems, when posed for surfaces of arbitrary geometry, are more difficult. One could hope to prove similar results for strictly hyperbolic surfaces  $S$ . In the general case, however, such problems reduce to the study of nonlinear PDEs of mixed types for which not so many suitable methods are at hand.

## 5. THE MATCHING PROPERTY ON CONVEX SHALLOW SHELLS: PROOF OF THEOREM 2.1

**Remark 5.1.** Writing  $w_h = w_{h,tan} + w_h^3 e_3$  where  $w_{h,tan}(x) \in \mathbb{R}^2$  and  $w_h^3(x) \in \mathbb{R}$ , equation (2.2) becomes:

$$(5.1) \quad \begin{aligned} & \text{Id} + h^2(2\text{sym}\nabla w_{h,tan} + \nabla v \otimes \nabla v) + 2h^3\text{sym}(\nabla v \otimes \nabla w_h^3) \\ & + h^4((\nabla w_{h,tan})^T \nabla w_{h,tan} + \nabla w_h^3 \otimes \nabla w_h^3) = \text{Id} + h^2 \nabla v_0 \otimes \nabla v_0. \end{aligned}$$

Recall that (2.1) is equivalent to  $\text{curl}^T \text{curl}(\nabla v \otimes \nabla v - \nabla v_0 \otimes \nabla v_0) = 0$ , and hence to:  $\nabla v \otimes \nabla v - \nabla v_0 \otimes \nabla v_0 = \text{sym}\nabla w$  for some  $w \in \mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R}^2)$  since  $\Omega$  is simply connected. Hence the constraint (2.1) is necessary and sufficient for matching the lowest order ( $h^2$ ) terms in (5.1).

Our result states that actually it is possible to perturb  $w$  by an equibounded 3d displacement  $w_h - w$  so that the full equality (5.1) holds. A natural way for proving this is by implicit function theorem. Indeed, this is how we proceed, and the ellipticity assumption  $\det \nabla^2 v_0 > 0$  is precisely a sufficient condition for the invertibility of the implicit derivative  $\mathcal{L}(p) : \mathcal{C}_0^{2,\beta}(\bar{\Omega}, \mathbb{R}) \rightarrow \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R})$ ,  $\mathcal{L}(p) = -\text{cof}\nabla^2 v : \nabla^2 p$  where  $p$  is the variation in  $w_h^3$ . An extra argument for the uniform boundedness of  $w_{h,tan}$  in  $\mathcal{C}^{2,\beta}$  concludes the proof.

### Proof of Theorem 2.1

1. By a direct calculation, (2.2) is equivalent to:

$$(5.2) \quad \nabla(\text{id} + h^2 w_{h,tan})^T \nabla(\text{id} + h^2 w_{h,tan}) = \text{Id} + h^2 \nabla v_0 \otimes \nabla v_0 - h^2(\nabla v + \nabla z_h) \otimes (\nabla v + \nabla z_h),$$

where  $w_{h,tan} \in \mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R}^2)$  and  $z_h = h w_h^3 \in \mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R})$  so that  $w_h = w_{h,tan} + w_h^3 e_3$  is the required correction in (2.2).

We shall first find the formula for the Gaussian curvature of the 2d metric in the right hand side of (5.2):

$$(5.3) \quad g_h(z_h) = \text{Id} + h^2 \nabla v_0 \otimes \nabla v_0 - h^2(\nabla v + \nabla z_h) \otimes (\nabla v + \nabla z_h).$$

**Lemma 5.2.** *Let  $v_0, v \in \mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R})$  and consider the  $\mathcal{C}^{1,\beta}$  regular metrics on  $\Omega$  of the type:*

$$g = [g_{ij}]_{i,j=1,2} = \text{Id} + h^2(\nabla v_0 \otimes \nabla v_0 - \nabla v_1 \otimes \nabla v_1).$$

*Then, for any  $h > 0$  small, the Gaussian curvature  $\kappa(g)$  of  $g$  is  $\mathcal{C}^{0,\beta}$  regular and it is given by the formula:*

$$(5.4) \quad \kappa(g) = h^2 \left[ \frac{\det(\nabla^2 v_0 - [\Gamma_{ij}^k \partial_k v_0]_{ij})}{(1 - h^2(g^{ij} \partial_i v_0 \partial_j v_0))^2 \det g} - \frac{(1 - h^2(g^{ij} \partial_i v_0 \partial_j v_0))^2}{(1 - h^2|\nabla v_1|^2)^2} \det \nabla^2 v_1 \right],$$

*where the Christoffel symbols of  $g$ , the inverse of  $g$ , and its determinant are:*

$$(5.5) \quad \Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_j g_{ij}),$$

$$(5.6) \quad g^{-1} = [g^{ij}] = \frac{1}{\det[g_{ij}]} \text{cof}[g_{ij}],$$

$$\det g = 1 - h^4 |(\nabla v_0)^\perp \cdot \nabla v_1|^2 + h^2 (|\nabla v_0|^2 - |\nabla v_1|^2).$$

*Proof.* Assume first that  $v_0$  and  $v_1$  are in fact smooth. By Lemma 2.1.2 in [11], we have:

$$\begin{aligned} \kappa(\text{Id} - h^2 \nabla v_1 \otimes \nabla v_1) &= -h^2 \frac{\det \nabla^2 v_1}{(1 - h^2 |\nabla v_1|^2)^2} \\ \kappa(g - h^2 \nabla v_0 \otimes \nabla v_0) &= \frac{1}{(1 - h^2(g^{ij} \partial_i v_0 \partial_j v_0))^2} \left[ \kappa(g) - \frac{h^2 \det(\nabla^2 v_0 - [\Gamma_{ij}^k \partial_k v_0]_{ij})}{(1 - h^2(g^{ij} \partial_i v_0 \partial_j v_0))^2 \det g} \right]. \end{aligned}$$

Since the two metrics above are equal, the formula (5.4) follows directly. The formula for  $\det g$  is obtained by a direct calculation, via  $\det(A + B) = \det A + \text{cof} A : B + \det B$ , valid for  $2 \times 2$  matrices  $A, B$ .

In the general case when  $v_0, v_1$  are only  $\mathcal{C}^{2,\beta}$  regular, one may approximate them by smooth sequences  $v_0^n, v_1^n$ . Then, each  $\kappa_n = \kappa(\text{Id} + h^2(\nabla v_0^n \otimes \nabla v_0^n - \nabla v_1^n \otimes \nabla v_1^n))$  is given by the formula in (5.4), and the sequence  $\kappa_n$  converges in  $\mathcal{C}^{0,\beta}$  to the right hand side in (5.4). On the other hand,  $\kappa_n$  converges in  $\mathcal{D}'(\Omega)$  to  $\kappa(g)$ , which follows from the definition of Gauss curvature  $\kappa = R_{1212}/\det g$ . Hence the lemma is proven.  $\blacksquare$

**2.** Applying Lemma 5.2 to  $v_1 = v + z_h$ , we now see that for small  $h$ , the Gauss curvature of metric  $g_h(z_h)$  vanishes:

$$(5.7) \quad \kappa(g_h(z_h)) = 0$$

if and only if:

$$(5.8) \quad \Phi(h, z_h) = 0,$$

where:

$$\begin{aligned} \Phi(h, z) &= (1 - h^2 |\nabla v + \nabla z|^2)^2 \det(\nabla^2 v_0 - [\Gamma_{ij}^k \partial_k v_0]_{ij}) \\ &\quad - (1 - h^2(g^{ij} \partial_i v_0 \partial_j v_0))^4 d(h, z) \det(\nabla^2 v + \nabla^2 z). \end{aligned}$$

Here:

$$d(h, z) = 1 - h^4 |(\nabla v_0)^\perp \cdot \nabla(v + z)|^2 + h^2 (|\nabla v_0|^2 - |\nabla v + \nabla z|^2)$$

and  $\Gamma_{ij}^k$  and  $g^{ij}$  are given by (5.5) and (5.6) for the metric  $g = \text{Id} + h^2 \nabla v_0 \otimes \nabla v_0 - h^2 (\nabla v + \nabla z) \otimes (\nabla v + \nabla z)$ . We shall consider:

$$\Phi : (-\epsilon, \epsilon) \times \mathcal{C}_0^{2,\beta}(\bar{\Omega}, \mathbb{R}) \longrightarrow \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R})$$

and seek for solutions  $z_h \in \mathcal{C}_0^{2,\beta}(\bar{\Omega}, \mathbb{R})$  of (5.8) with zero boundary data. It is elementary to check that  $\Phi$  is continuously Frechet differentiable at  $(0, 0)$  and that

$$\Phi(0, 0) = \det \nabla^2 v_0 - \det \nabla^2 v = 0.$$

Moreover, the partial Frechet derivative  $\mathcal{L} = \partial \Phi / \partial z(0, 0) : \mathcal{C}_0^{2,\beta}(\bar{\Omega}, \mathbb{R}) \longrightarrow \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R})$  is a linear continuous operator of the form:

$$\begin{aligned} \forall z \in \mathcal{C}_0^{2,\beta} \quad \mathcal{L}(z) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \Phi(0, \epsilon z) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\det \nabla^2 v_0 - \det(\nabla^2 v + \epsilon \nabla^2 z)) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (-\epsilon^2 \det \nabla^2 z - \epsilon \text{cof} \nabla^2 v : \nabla^2 z) = -\text{cof} \nabla^2 v : \nabla^2 z. \end{aligned}$$

Clearly,  $\mathcal{L}$  is invertible to a continuous linear operator, because of the uniform ellipticity of the matrix field  $\nabla^2 v$  which follows from the convexity assumption of  $\det \nabla^2 v = \det \nabla^2 v_0$  being strictly positive. Thus, invoking the implicit function theorem we obtain the solution operator:

$$\mathcal{Z} : (-\epsilon, \epsilon) \longrightarrow \mathcal{C}_0^{2,\beta}(\bar{\Omega}, \mathbb{R})$$

such that  $z_h = \mathcal{Z}(h)$  satisfies (5.8). Moreover,  $\mathcal{Z}$  is differentiable at  $h = 0$  and:

$$\mathcal{Z}'(0) = \mathcal{L}^{-1} \circ \left( \frac{\partial \Phi}{\partial h}(0, 0) \right) = 0,$$

because:

$$\frac{\partial \Phi}{\partial h}(0, 0) = (\text{cof} \nabla^2 v_0) : \left[ \left( \frac{\partial}{\partial h} \Gamma_{ij}^k \right) \partial_k v_0 \right]_{ij} + \frac{\partial}{\partial h} \det[\Gamma_{ij}^k \partial_k v_0]_{ij} - \left( \frac{\partial}{\partial h} d(0, 0) \right) \det \nabla^2 v = 0.$$

Consequently:

$$(5.9) \quad \|w_h^3\|_{\mathcal{C}^{2,\beta}} = \frac{1}{h} \|z_h\|_{\mathcal{C}^{2,\beta}} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

**3.** In conclusion, we have so far obtained a uniformly bounded sequence of  $\mathcal{C}_0^{2,\beta}$  out-of-plane displacements  $w_h^3 = z_h/h$  such that the Gauss curvature (5.7) of the metric  $g_h(z_h)$  in the right hand side of (5.2) is 0. By the result in [29] it follows that for each small  $h$  there exists exactly one (up to fixed rotations) orientation preserving isometric immersion  $\phi_h \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$  of  $g_h(z_h)$ :

$$(5.10) \quad \nabla \phi_h^T \nabla \phi_h = g_h(z_h) \quad \text{and} \quad \det \nabla \phi_h > 0.$$

What remains to be proven is that, in fact,  $\phi_h = \text{id} + h^2 w_{h,tan}$  with some  $w_{h,tan}$  uniformly bounded in  $\mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R}^2)$ .

It is a well known calculation (see [6, 29]) that (5.10) implies (is actually equivalent to):

$$(5.11) \quad \nabla^2 \phi_h - [\Gamma_{ij}^k \partial_k \phi_h]_{ij} = 0,$$

where  $\Gamma_{ij}^k$  are the Christoffel symbols (5.5) of the metric  $g = g_h(z_h)$  in (5.3). By (5.10)  $\|\nabla\phi_h\|_{L^\infty} \leq C$ , and by (5.11)  $\|\nabla^2\phi_h\|_{L^\infty} \leq C$ , hence  $\|\phi_h\|_{\mathcal{C}^{2,\beta}} \leq C$ . But  $\Gamma_{ij}^k$  are uniformly bounded (with respect to small  $h$ ) in  $\mathcal{C}^{0,\beta}$  so by (5.11)  $\|\nabla^2\phi_h\|_{\mathcal{C}^{0,\beta}} \leq C$  and thus:

$$\|\phi_h\|_{\mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R}^2)} \leq C.$$

Note now that  $\|\Gamma_{ij}^k\|_{\mathcal{C}^{0,\beta}} \leq Ch^2$  in view of the particular structure of the metrics  $g_h(z_h)$ . Hence, by (5.11):

$$(5.12) \quad \|\nabla^2\phi_h\|_{\mathcal{C}^{0,\beta}} \leq Ch^2.$$

Therefore, for some  $A_h \in \mathbb{R}^{2 \times 2}$  we have:

$$(5.13) \quad \|\nabla\phi_h - A_h\|_{\mathcal{C}^{1,\beta}} \leq Ch^2.$$

We now prove that the matrix  $A_h$  in the inequality above can be chosen as a rotation and hence, without loss of generality,  $A_h = \text{Id}$ . For each  $x \in \Omega$  there holds:

$$(5.14) \quad \text{dist}(A_h, SO(3)) \leq |A_h - \nabla\phi_h(x)| + \text{dist}(\nabla\phi_h(x), SO(3)).$$

To evaluate the last term above, write:  $\sqrt{\nabla\phi_h^T(x)\nabla\phi_h(x)} = QDQ^T$  for some  $Q \in SO(3)$  and  $D = \text{diag}(\lambda_1, \lambda_2)$  with  $\lambda_1, \lambda_2 > 0$ . Since  $\det \nabla\phi_h > 0$ , it follows by polar decomposition theorem that:

$$\begin{aligned} \text{dist}(\nabla\phi_h(x), SO(3)) &= |\sqrt{\nabla\phi_h^T(x)\nabla\phi_h(x)} - \text{Id}| \leq C|D - \text{Id}| \\ &= C \max_i \{|\lambda_i - 1|\} \leq C \max_i \{|\lambda_i^2 - 1|\} \leq C|D^2 - \text{Id}| \\ &= C|Q^T \nabla\phi_h^T(x)\nabla\phi_h(x)Q - \text{Id}| \leq C|\nabla\phi_h^T \nabla\phi_h(x) - \text{Id}| \leq Ch^2. \end{aligned}$$

By the above and (5.14), (5.13) we see that  $\text{dist}(A_h, SO(3)) \leq Ch^2$ . Hence, without loss of generality,  $\|\nabla\phi_h - \text{Id}\|_{\mathcal{C}^{1,\beta}} \leq Ch^2$  and:

$$\|\phi_h - \text{id}\|_{\mathcal{C}^{2,\beta}} \leq Ch^2.$$

Consequently,  $\phi_h = \text{id} + h^2 w_{h,tan}$  with  $\|w_{h,tan}\|_{\mathcal{C}^{2,\beta}} \leq C$ . This concludes the proof of Theorem 2.1, in view of (5.2) which is equivalent to (2.2).  $\blacksquare$

**Remark 5.3.** The above proof is somewhat similar to [11, Theorem 4.1.1]. In analogy, note the similarity between the proof of the matching property in [22] and the Weyl problem by Nirenberg in [32].

## 6. ŠVERÁK'S ARGUMENTS : PROOF OF THEOREM 2.3

In this section we provide a self-contained proof of Theorem 2.3. Observe first the following:

**Example 6.1.** Let  $B_1$  be the unit disk in  $\mathbb{R}^2$  and let  $u \in \mathcal{C}^1(B_1)$  be given by:

$$u(x, y) = \begin{cases} x^2 e^{y^2/2} & \text{if } x \geq 0 \\ -x^2 e^{y^2/2} & \text{otherwise.} \end{cases}$$

Note that  $u(0, y) = 0$  and  $\nabla u(0, y) = 0$  for all  $y \in (-1, 1)$ . Indeed, we have  $u_x = \pm 2x e^{y^2/2}$ ,  $u_y = \pm y x^2 e^{y^2/2}$ ,  $u_{xx} = \pm 2 e^{y^2/2}$ ,  $u_{xy} = u_{yx} = \pm 2xy e^{y^2/2}$ ,  $u_{yy} = \pm (x^2 e^{y^2/2} + y^2 x^2 e^{y^2/2})$  and  $\Delta u = \pm e^{y^2/2}(2 + x^2 + y^2 x^2)$ , respectively for  $x > 0$  and  $x < 0$ .

As a consequence  $u \in W^{2,\infty}(B_1)$ ,  $u$  is strictly convex in  $\{(x, y) \in B_1; x > 0\}$  and strictly concave in  $\{(x, y) \in B_1; x < 0\}$ . On the other hand  $\det \nabla^2 u = 2x^2 e^{y^2} (1 - y^2) \in C^\infty(B_1)$  and it is positive if  $x \neq 0$  and  $y^2 < 1$ . We right away note that  $u \notin C^2(B_1)$  although it solves the Monge-Ampère equation with smooth non-negative right hand side, a.e. in its domain.

Finally, our example shows that the assumption of strict positivity in Theorem 2.3 (and also 6.5 and Theorem 7.1) cannot be relaxed to  $\det \nabla^2 u > 0$  a.e., even assuming a better  $W^{2,\infty}$  regularity for  $u$ .

**Definition 6.2.** *We say that a mapping  $v \in C^0(\Omega, \mathbb{R}^2)$  is connectedly locally one-to-one iff it is locally one-to-one outside of a closed set  $S \subset \Omega$  of measure zero, for which  $\Omega \setminus S$  is connected.*

**Definition 6.3.** *Let  $v \in W^{1,2}(\Omega, \mathbb{R}^2)$  and let  $\det \nabla v \geq 0$  a.e. in  $\Omega$ . We say that  $v$  has integrable dilatation iff:*

$$\forall a.e. x \in \Omega \quad |\nabla v|^2(x) \leq K(x) \det \nabla v(x)$$

with some function  $K \in L^1(\Omega)$ .

For the proof of Theorem 2.3, the first result we propose is essentially a combination of arguments in Šverák’s unpublished paper [34]. In this section, we will gather all the details of its proof.

**Theorem 6.4** (Šverák). *If  $u \in W^{2,2}(\Omega)$  satisfies:*

$$(6.1) \quad \det \nabla^2 u(x) > 0 \quad \forall a.e. x \in \Omega,$$

*then  $u \in C^1(\Omega)$ . If additionally  $v = \nabla u$  is connectedly locally one-to-one, then modulo a global sign change,  $u$  is locally convex in  $\Omega$ . In particular, when  $\Omega$  is convex then  $u$  is either convex or concave in the whole  $\Omega$ .*

We quote now the result, which will be crucial for the proof of Theorem 6.4:

**Theorem 6.5** (Iwaniec and Šverák [15]). *Let  $v \in W^{1,2}(\Omega, \mathbb{R}^2)$  be as in Definition 6.3. Then there exists a homeomorphism  $h \in W^{1,2}(\Omega', \Omega)$  and a holomorphic function  $\varphi \in W^{1,2}(\Omega', \mathbb{R}^2 = \mathbb{C})$  such that:*

$$v = \varphi \circ h^{-1}.$$

*In particular,  $v$  is either constant or connectedly locally one-to-one, and in the latter case the singular set  $S = h((\nabla \varphi)^{-1}\{0\})$  is at most countable and it is finite on every subset compactly contained in  $\Omega$ .*

Without having Theorem 6.5 at hand, Šverák proved in [34] that if  $u \in W^{2,2}(\Omega)$  satisfies  $\det \nabla^2 u > 0$  a.e. in  $\Omega$ , then there exists a closed set  $S \subset \Omega$  of measure zero such that on each component of  $\Omega \setminus S$ ,  $u$  is either locally convex or locally concave. In fact, the main step in the proof is to show that any such map is locally one-to-one outside a set of measure zero, which Šverák has achieved by using consequences of a version of Lemma 6.10 below and the classical degree theory.

Combining Theorem 6.4 with Theorem 6.5 one directly obtains:

**Corollary 6.6.** *Let  $u \in W^{2,2}(\Omega)$  satisfy (6.1) and be such that  $\nabla u$  has integrable dilatation. Then  $u \in \mathcal{C}^1(\Omega)$  and modulo a global sign change,  $u$  is locally convex in  $\Omega$ .*

Theorem 2.3 is then, obviously, a particular case of the above corollary, where the displacement  $u$  trivially satisfies its assumptions. In the remaining part of this section, we will prove Theorem 6.4. We first remind a key result on the modulus of continuity of 2d deformations in  $W^{1,2}$  with positive Jacobian:

**Theorem 6.7** (Vodopyanov and Goldstein [38]). *Assume that  $v \in W^{1,2}(\Omega, \mathbb{R}^2)$  and that  $\det \nabla v > 0$  a.e. in  $\Omega$ . Then  $v$  is continuous in  $\Omega$ , and for any  $B(x, \delta) \subset B(x, R) \subset \Omega$  we have:*

$$(6.2) \quad \text{osc}_{B(x, \delta)} v \leq \sqrt{2\pi} \left( \ln \frac{R}{\delta} \right)^{-1/2} \|\nabla v\|_{L^2(B(x, R))}.$$

*Proof.* By a result of Vodopyanov and Goldstein [38]  $v$  is continuous (see also [27, 35]). In fact, a key ingredient of this result is to show that  $\phi$  is a monotone map, i.e. for  $B_\rho = B(x, \rho)$ :

$$(6.3) \quad \text{osc}_{B_\rho} v = \text{osc}_{\partial B_\rho} v,$$

and hence  $v$  has the asserted modulus of continuity by [30, Theorem 4.3.4] (see also [27]). We sketch the last part of the proof for the convenience of the reader. By Fubini's theorem  $v$  belongs to  $W^{1,2}(\partial B_\rho)$  for almost every  $\rho \in (\delta, R)$ . Hence the Morrey's theorem of embedding of  $W^{1,2}$  into  $\mathcal{C}^0$  for the one-dimensional set  $\partial B_\rho$  yields:

$$\forall a.e. \rho \in (\delta, R) \quad \text{osc}_{B_\delta} v \leq \text{osc}_{B_\rho} v = \text{osc}_{\partial B_\rho} v \leq \sqrt{2\pi\rho} \left( \int_{\partial B_\rho} |\nabla v|^2 \right)^{1/2}.$$

To conclude, one squares both sides of the above inequality, divides by  $\rho$  and integrates from  $\delta$  to  $R$ , in order to deduce (6.2).  $\blacksquare$

**Corollary 6.8.** *Assume that  $v_n \in W^{1,2}(\Omega, \mathbb{R}^2)$  is a bounded sequence such that  $\det \nabla v_n > 0$  a.e. in  $\Omega$ . Then, up to a subsequence,  $v_n$  converges locally uniformly and also weakly in  $W^{1,2}$  to a continuous mapping  $v \in W^{1,2}(\Omega, \mathbb{R}^2)$  satisfying  $\det \nabla v \geq 0$  a.e. in  $\Omega$ .*

*Proof.* The uniform convergence of a subsequence follows by Ascoli-Arzelá theorem in view of Theorem 6.7. Noting that  $\det \nabla v = -\nabla v_1 \cdot \nabla^\perp v_2$ , the Div-Curl Lemma implies then that the desired inequality is satisfied for the limit mapping  $v$ .  $\blacksquare$

**Corollary 6.9.** *Let  $u_n \in W^{2,2}(\Omega)$  be a bounded sequence such that  $\det \nabla^2 u_n \geq c_0 \in \mathbb{R}$  a.e. in  $\Omega$ . Then, up to a subsequence,  $u_n$  converges weakly in  $W^{2,2}$ , as well as it converges locally uniformly together with its gradients, to a  $\mathcal{C}^1$  function  $u \in W^{2,2}(\Omega, \mathbb{R}^2)$  satisfying  $\det \nabla^2 u \geq c_0$  a.e. in  $\Omega$ .*

*Proof.* Let  $v_n(x) = \nabla u(x) + (|c_0| + 1)^{1/2} x^\perp$ , where  $x^\perp = (x_1, x_2)^\perp = (-x_1, x_2)$ . Clearly,  $v_n \in W^{1,2}(\Omega, \mathbb{R}^2)$  and, since  $\nabla^2 u_n$  is a symmetric matrix, we get:

$$\det \nabla v_n(x) = \det \nabla^2 u_n(x) + |c_0| + 1 > 0 \quad \forall a.e. x \in \Omega.$$

The convergence assertion follows by Corollary 6.8. Again, the Div-Curl Lemma applied to sequence  $\nabla u_n$  implies the desired inequality for the limit function  $u$ .  $\blacksquare$

A consequence of Theorem 6.7 is the following assertion about  $W^{2,2}$  functions whose Hessian determinants are uniformly controlled from below:

**Lemma 6.10.** *Assume that  $u \in W^{2,2}(\Omega)$  satisfies:*

$$(6.4) \quad \det \nabla^2 u(x) \geq c_0 \in \mathbb{R} \quad \forall a.e. \ x \in \Omega.$$

*Then  $u \in \mathcal{C}^1(\Omega)$ . Moreover, if  $x_0 \in \Omega$  is a Lebesgue point for  $\nabla^2 u$ , i.e. for some  $A \in \mathbb{R}_{\text{sym}}^{2 \times 2}$ :*

$$(6.5) \quad \omega(r) := \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |\nabla^2 u - A|^2 dx \rightarrow 0 \quad \text{as } r \rightarrow 0^+,$$

*then for all  $\epsilon > 0$  there exists  $r_0 > 0$  such that:*

$$(6.6) \quad \begin{aligned} \forall r < r_0 \quad \forall a \in D_r = \overline{B(x_0, r)} \quad & \|\nabla u(x) - \nabla u(a) - A(x - a)\|_{C^0(D_r)} \leq \frac{1}{2}\epsilon r, \\ & \|u(x) - u(a) - \nabla u(a) \cdot (x - a) - \frac{1}{2}(x - a) \cdot A(x - a)\|_{C^0(D_r)} \leq \epsilon r^2. \end{aligned}$$

*Proof.* Following Kirchheim [17] we set  $v = \nabla u$  and we write  $\phi(x) = v(x) + (|c_0| + 1)^{1/2} x^\perp$ . Trivially  $\phi \in W^{1,2}(\Omega, \mathbb{R}^2)$  and, as before:

$$\det \nabla \phi = \det \nabla^2 u + |c_0| + 1 > 0 \quad \forall a.e. \ x \in \Omega.$$

Applying Theorem 6.7 to  $\phi$  shows that  $v$  is continuous and so  $u \in \mathcal{C}^1(\Omega)$ .

In what follows, we assume without loss of generality that  $x_0 = 0$ ,  $u(0) = 0$  and  $v(0) = \nabla u(0) = 0$  (otherwise it is sufficient to translate  $\Omega$  and to modify  $u$  by its tangent map at 0). For  $r$  sufficiently small and for all  $x \in B_2 = B(0, 2)$  we define:

$$v_r(x) := \frac{1}{r} v(rx), \quad \phi_r(x) := v_r(x) + (|c_0| + 1)^{1/2} x^\perp,$$

so that:

$$\begin{aligned} \forall x \in B_2 \quad \nabla v_r(x) = \nabla v(rx) = \nabla^2 u(rx), \quad \nabla \phi_r(x) = \nabla^2 u(rx) + (|c_0| + 1)^{1/2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \\ \det \nabla \phi_r(x) = \det \nabla v_r(x) + (|c_0| + 1) > 0. \end{aligned}$$

Since  $\phi_r \in W^{1,2}(B_2, \mathbb{R}^2)$ , we can apply (6.2) to  $x \in B_1$  and  $\delta < R = 1$ , to obtain for  $r < r_0$  small enough:

$$\begin{aligned} \text{osc}_{B(x, \delta)} \phi_r &\leq \sqrt{2\pi} \left( \ln \frac{1}{\delta} \right)^{-1/2} \left( \int_{B(x, 1)} |\nabla \phi_r|^2 \right)^{1/2} \\ &\leq \sqrt{2\pi} \left( \ln \frac{1}{\delta} \right)^{-1/2} \left( 2(|m| + 1)^{1/2} |B_1| + \left( \int_{B(x, 1)} |\nabla^2 u(ry)|^2 dy \right)^{1/2} \right) \\ &\leq C \ln \left( \frac{1}{\delta} \right)^{-1/2} \left( (|m| + 1)^{1/2} + \frac{1}{r} \|\nabla^2 u\|_{B(0, 2r)} \right) \\ &\leq C \ln \left( \frac{1}{\delta} \right)^{-1/2}, \end{aligned}$$

where  $C = C(m, |A|) > 0$ . Above we used the fact that  $B(x, 1) \subset B_2$  and that 0 is a Lebesgue point for  $\nabla^2 u$ . Now, given  $\epsilon > 0$  we choose  $\delta > 0$  such that:

$$\ln \left( \frac{1}{\delta} \right)^{-1/2} < \epsilon / C.$$

Consequently:

$$\forall x, y \in D_1 \quad \forall r < r_0 \quad |x - y| < \delta \implies |\phi_r(x) - \phi_r(y)| < \epsilon.$$

Since  $v_r - \phi_r$  is a given linear deformation, we conclude that the family:

$$\mathcal{F} = \{v_r : D_1 \rightarrow \mathbb{R}^2; \quad r < r_0\}$$

is equicontinuous. On the other hand:

$$\int_{D_1} |\nabla v_r - A|^2 = \pi\omega(r) \rightarrow 0 \quad \text{as } r \rightarrow 0.$$

Let  $\tilde{v}_r = v_r - \int_{B_1} v_r$  and apply the Poincaré inequality to obtain that

$$\tilde{v}_r \rightarrow Ax \quad \text{in } W^{1,2}(\Omega, \mathbb{R}^2) \quad \text{as } r \rightarrow 0.$$

Now, equicontinuity of  $\mathcal{F}$  and  $v_r(0) = 0$  yield, by Arzelà-Ascoli theorem, that a subsequence of  $v_r$  (which we do not relabel) converges uniformly to a continuous function  $V$  on  $D_1$ . Since  $v_r - \tilde{v}$  is constant, we deduce that  $V(x) - Ax = c$  is constant too. But then, evaluating at 0 gives  $c = 0$ . Hence,  $v_r$  uniformly converges to  $Ax$  on  $D_1$ .

Let us fix  $\epsilon > 0$  and choose  $r_0$  so that:

$$\forall r < r_0 \quad \|v_r(x) - Ax\|_{C^0(D_1)} \leq \frac{1}{4}\epsilon.$$

This implies:

$$\|\nabla u(x) - Ax\|_{C^0(D_r)} \leq \frac{1}{4}\epsilon r.$$

Fixing  $a \in D_r$  we get:

$$\|\nabla u(x) - \nabla u(a) - A(x - a)\|_{C^0(D_r)} \leq \frac{1}{2}\epsilon r,$$

giving the first estimate in (6.6). Since  $\text{diam } D_r = 2r$ , the second estimate follows.  $\blacksquare$

We now prove a simple useful lemma whose statement we quote from [34]:

**Lemma 6.11** ([34], Lemma 2). *Let  $u \in C^1(\bar{\Omega})$  and fix  $a \in \Omega$ . Suppose that:*

$$(6.7) \quad u(x) \geq u(a) + \nabla u(a) \cdot (x - a) \quad \forall x \in \partial\Omega$$

*and that  $\nabla u(x) \neq \nabla u(a)$  for all  $x \in \Omega \setminus \{a\}$ . Then  $u$  has a supporting hyperplane at  $a$ , i.e. (6.7) holds for all  $x \in \Omega$ . In particular, if  $\Omega$  is convex then:*

$$(Cu)(a) = u(a)$$

*where  $Cu$  denotes the convexification of the function  $u$  over  $\bar{\Omega}$ :*

$$(Cu)(a) = \sup \{T(a); \quad T : \Omega \rightarrow \mathbb{R} \text{ is affine and } T(x) \leq u(x) \quad \forall x \in \bar{\Omega}\}.$$

*Proof.* Consider the tangent map  $T(x) = u(a) + \nabla u(a) \cdot (x - a)$ . We now claim that  $T(x) \leq u(x)$  for all  $x \in \Omega$ . For otherwise, the continuous function  $g(x) = u(x) - T(x)$  would assume a negative minimum on  $\bar{\Omega}$  at some  $c \in \Omega \setminus \{a\}$ . Hence  $\nabla g(c) = 0$ , which is a contradiction with the second assumption as  $\nabla u(c) = \nabla T(c) = \nabla u(a)$ .  $\blacksquare$

We are ready to prove the key theorem of this section:



**Proof of Theorem 6.4.**

1. The  $\mathcal{C}^1$  regularity of  $u$  is an immediate consequence of Theorem 6.7. Recall the properties of the singular set  $S$  from Definition 6.2. Since  $\det \nabla^2 u > 0$  a.e. in  $\Omega$ , modulo a global change of sign for  $u$  we can choose  $x_0 \in \Omega \setminus S$  a Lebesgue point of  $\nabla^2 u$  as in (6.5), such that the matrix  $A$  is positive definite. Hence there exists  $\lambda > 0$  for which  $\xi \cdot A\xi \geq \lambda|\xi|^2$  for all  $\xi \in \mathbb{R}^2$ . By Lemma 6.10 for all  $r < r_0$  the estimate (6.6) holds true with  $\epsilon = \frac{1}{4}\lambda$ , and without loss of generality  $\nabla u$  is also one-to-one on  $D_r = \overline{B(x_0, r)} \subset \Omega \setminus S$ . By (6.6) it follows that:

$$\forall x \in \partial D_r \quad u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0) \geq \frac{1}{2}(x - x_0) \cdot A(x - x_0) - \epsilon r^2 \geq \frac{\lambda}{4}r^2 > 0.$$

In view of Lemma 6.11,  $u$  therefore admits a supporting hyperplane at  $x_0$  on  $D_r$ .

2. Our next claim is that  $u$  is locally convex in  $\Omega \setminus S$ . Since  $\Omega \setminus S$  is open and connected, it is also path-wise connected. Therefore, for a fixed  $x \in \Omega \setminus S$ , there exists a continuous path within  $\Omega \setminus S$  connecting  $x$  and  $x_0$ , which can be covered with a finite chain of open balls  $B_i \subset \Omega \setminus S, i = 1, 2 \dots n$ , such that  $x_0 \in B_1, B_i \cap B_{i+1} \neq \emptyset$  and  $x \in B_n$ . We now need the following strong theorem due to J. Ball:

**Theorem** ([1], Theorem 1). *Let  $\Omega \subset \mathbb{R}^n$  be open and convex, and let  $u \in \mathcal{C}^1(\Omega)$ . The necessary and sufficient condition for  $u$  to be strictly convex on  $\Omega$  is:*

- (i)  $\nabla u$  is locally one-to-one, and
- (ii) there exists a locally supporting hyperplane for  $u$  at some point of  $\Omega$ :

$$\exists x_0 \in \Omega \quad \forall \rho > 0 \quad \forall x \in B(x_0, \rho) \quad u(x) \geq u(x_0) + \nabla u(x_0) \cdot (x - x_0).$$

Applying this result consecutively to each ball  $B_i$ , we deduce that  $u$  is strictly convex on  $B_n$ , hence it is locally convex at  $x$ .

3. To finish the proof, we fix a direction in  $\mathbb{R}^2$  and consider the family of straight lines parallel to that direction. For almost every such line  $L$ , the 1-dimensional Lebesgue measure of  $L \cap S$  is zero and  $u \in W^{2,2}(L \cap \Omega)$ . Also,  $u$  is locally convex on  $(L \cap \Omega) \setminus S$  in view of the previously proven claim. We now state the following easy lemma to show that  $u$  is convex on connected components of  $L \cap \Omega$ :

**Lemma.** *Let  $I \subset \mathbb{R}$  be an open bounded interval,  $\phi \in W^{2,1}(I)$  and assume that  $\phi$  is locally convex on  $I \setminus S$ , where  $S$  is a set of measure 0. Then  $\phi$  is convex on  $I$ .*

The proof is elementary. Since  $\phi \in \mathcal{C}^1(I)$  and  $\phi$  is locally convex on a full measure open subset of  $I$ , we deduce that  $\phi'' \geq 0$  a.e. in  $I$ . But this immediately implies that  $\phi'$  is increasing in  $I$ , hence  $\phi$  is globally convex.

We have previously shown that  $u$  is convex on connected components of  $L \cap \Omega$ , for almost all straight lines  $L$  in any direction. By continuity of  $u$ , the same must hold, in fact, for all lines, by approaching any given line with a selected sequence of 'good' lines and passing to the limit in the convexity inequality. This implies that actually  $u$  is convex on any convex subset of  $\Omega$  and the proof is done. ■

For completeness, we now note another corollary of Lemma 6.10 and Lemma 6.11:

**Lemma 6.12.** *Let  $u \in W^{2,2}(\Omega)$  satisfy (6.4). Assume that  $x_0$  is a Lebesgue point for  $\nabla^2 u$  with  $A$  in (6.5) being positive definite. Assume that  $\nabla u$  is one-to-one in a neighborhood of  $x_0$ . Then  $u$  is locally convex at  $x_0$ .*

*Proof.* There exists  $\lambda > 0$  for which  $\xi \cdot A\xi \geq \lambda|\xi|^2$  for all  $\xi \in \mathbb{R}^2$ . By Lemma 6.10 for all  $r < r_0$  and all  $a \in D_r = \overline{B(x_0, r)}$ , estimate (6.6) holds true with  $\epsilon = \lambda/4$ . Without loss of generality,  $\nabla u$  is one-to-one on  $D_r$  and, by (6.6):

$$\forall a \in B(x_0, r/2) \quad \forall x \in \partial B(x_0, r/2)$$

$$u(x) - u(a) - \nabla u(a) \cdot (x - a) \geq \frac{1}{2}(x - a) \cdot A(x - a) - \epsilon r^2 \geq \frac{\lambda}{4}r^2 > 0.$$

The assumptions of Lemma 6.11 are satisfied and hence  $u(a) = (Cu)(a)$  for all  $a \in B(x_0, r/2)$ . The claim is proved.  $\blacksquare$

**Remark 6.13.** In proving Theorem 2.3 we only used the conclusion of Theorem 6.5 that  $v$  is locally one-to-one on a connected set of full measure. Therefore, the assumptions of Theorem 2.3 could potentially be relaxed (as in Theorem 6.4), but not to  $\det \nabla^2 u > 0$  a.e. Indeed, let  $v = \nabla u$  be as in Example 6.1. Then  $v$  is not of integrable dilatation because:

$$\frac{|\nabla v|^2}{\det \nabla v}(x, y) \geq \frac{2}{x^2(1 - y^2)},$$

and also the singular set  $S = \{(0, y); y \in (-1, 1)\}$  coincides with the vanishing set of  $v$  where  $v$  is obviously not locally one-to-one. On the other hand, Theorem 6.4 can be also applied to the cases where  $\det \nabla^2 u \in C^0(\Omega)$  is positive a.e. and  $\Omega \setminus f^{-1}(0)$  is connected.

## 7. DENSITY AND REGULARITY FOR ELLIPTIC 2-DIMENSIONAL MONGE-AMPÈRE EQUATION : PROOF OF THEOREM 2.2

As a consequence of Theorem 2.3 and of the monotonicity property by Vodopyanov and Goldstein which we quote in Theorem 6.7, we obtain:

**Theorem 7.1.** *Let  $f \in C^{k,\beta}(\Omega)$  be a positive function. Then any  $W^{2,2}(\Omega)$  solution of  $\det \nabla^2 u = f$ , is  $C^{k+2,\beta}$  regular, locally in  $\Omega$ .*

*Proof. 1.* We first note that  $u$  is a generalized Aleksandrov solution to (2.3). Since  $u$  is locally convex, it is twice differentiable in the classical sense a.e. in  $\Omega$ , and its gradient agrees with  $f$ . By Lemma 2.3 in [36], the regular part of the Monge-Ampère measure  $\mu_u$  equals  $(\det \nabla^2 u)dx = f dx$ . It suffices now to prove that there is no singular part of  $\mu_u$ , i.e. that  $\mu_u$  is absolutely continuous with respect to the Lebesgue measure  $dx$ .

Call  $v = \nabla u$ . By Theorem 2.3 we have  $u \in C^1(\Omega)$  and hence:

$$\mu_u(\omega) = |v(\omega)| \quad \text{for every Borel set } \omega \subset \Omega.$$

We thus need to show that  $v$  satisfies Luzin's condition (N):

$$|v(\omega)| = 0 \quad \forall \omega \subset \Omega; \quad |\omega| = 0.$$

The above claim follows directly from Theorem A in [26], in view of  $v \in W^{1,2}(\Omega, \mathbb{R}^2)$  and the monotonicity property (6.3) of  $v$  due to Vodopyanov and Goldstein.

**2.** Since  $f \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ , then Theorem 5.4 in [14] implies that  $u$  is locally  $\mathcal{C}^{2,\alpha}$ . We note that this statement is the well-known result due to Caffarelli [2]. Indeed, fix  $x_0 \in \Omega$ . By Remark 3.2 in [36] which gives an elementary proof of a result by Aleksandrov and Heinz, the displacement  $u$  as above must be strictly convex in some  $B(x_0, \epsilon)$ . By adding an affine function to  $u$ , we may without loss of generality assume that  $u = 0$  on the boundary of the convex set:

$$\Omega_0 = \{x \in \Omega; u(x) \leq u(x_0) + \delta\} \subset B(x_0, \epsilon),$$

for a sufficiently small  $\delta > 0$ . Therefore, the statement of Theorem 5.4 in [14] can be directly applied. Once the  $\mathcal{C}^{2,\beta}$  regularity is established, the  $\mathcal{C}^{k+2,\beta}$  regularity follows as in Proposition 9.1 in [3]. ■

**Proof of Theorem 2.2.** Without loss of generality we assume that  $\Omega$  is starshaped with respect to  $B = B(0, r) \subset \Omega$ . Let  $u \in \mathcal{A}$  and define  $u_\lambda(x) = \frac{1}{\lambda^2}u(\lambda x)$  for  $0 < \lambda < 1$ . Then:

$$\det \nabla^2 u_\lambda(x) = c_0 \quad \forall a.e. x \in \Omega$$

and:

$$\|u_\lambda\|_{L^2(\Omega)} = \lambda^{-3}\|u\|_{L^2(\lambda\Omega)}, \quad \|\nabla u_\lambda\|_{L^2(\Omega)} = \lambda^{-2}\|u\|_{L^2(\lambda\Omega)}, \quad \|\nabla^2 u_\lambda\|_{L^2(\Omega)} = \lambda^{-1}\|u\|_{L^2(\lambda\Omega)}.$$

As a consequence,  $u_\lambda \in \mathcal{A}$  for all  $\lambda \in (0, 1)$ , and  $u_\lambda \rightarrow u$  strongly in  $W^{2,2}(\Omega)$  as  $\lambda \rightarrow 1^-$ .

So far we have used only the fact that  $\Omega$  is starshaped with respect to the origin 0. Now, since  $\Omega$  is star-shaped with respect to an open ball  $B$ , we have  $\lambda\bar{\Omega} \subset \Omega$  for all  $0 < \lambda < 1$ . Hence, in view of Theorem 7.1,  $u_\lambda \in \mathcal{C}^\infty(\bar{\Omega}) \cap \mathcal{A}$ , which proves the claim. ■

## 8. RECOVERY SEQUENCE FOR THE RANGE $\alpha \in (0, 1)$ IN VIEW OF THE MATCHING PROPERTY: PROOF OF THEOREM 4.4

It is enough to prove Theorem 4.4 for  $v \in \mathcal{C}^{2,\beta}(\bar{\Omega})$  satisfying  $\det \nabla^2 v = \det \nabla^2 v_0$ . In the case of  $v \in W^{2,2}(\Omega)$  satisfying the same constraint, the result follows then by a diagonal argument in view of the density property established in Theorem 2.2.

**1.** We now recall the useful change of variable  $\tilde{\phi}_h \in \mathcal{C}^{1,\beta}(\Omega^h, \mathbb{R}^3)$  between thin plates  $\Omega^h = \Omega \times (-h/2, h/2)$  and thin shallow shells  $(S_h)^h$ :

$$(8.1) \quad \tilde{\phi}_h(x, x_3) = x + h^\alpha v_0(x)e_3 + x_3 \bar{n}^h(x) \quad \forall (x, x_3) \in \Omega,$$

where  $\bar{n}^h$  is the unit normal vector to the midsurface  $S_h$ , given as the image of the map  $\phi_h(x) = x + h^\alpha v_0(x)e_3$ :

$$\bar{n}^h(x) = \frac{\partial_1 \phi_h(x) \times \partial_2 \phi_h(x)}{|\partial_1 \phi_h(x) \times \partial_2 \phi_h(x)|} = \frac{1}{\sqrt{1 + h^{2\alpha} |\nabla v_0|^2}} (-h^\alpha (\nabla v_0)^* + e_3).$$

By Theorem 2.1, there exists an equibounded sequence  $w_h \in \mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R}^3)$  such that the deformations  $\xi_h(x) = x + h^\alpha v(x)e_3 + h^{2\alpha} w_h(x)$  are isometrically equivalent to  $id + h^\alpha v_0 e_3$ :

$$(8.2) \quad \forall 0 < h \ll 1 \quad (\nabla \xi_h)^T \nabla \xi_h = \nabla (id + h^\alpha v_0 e_3)^T \nabla (id + h^\alpha v_0 e_3).$$

Define now the recovery sequence  $u^h \in \mathcal{C}^{1,\beta}((S_h)^h, \mathbb{R}^3)$  by:

$$u^h(\tilde{\phi}_h(x, x_3)) = \bar{u}^h(x, x_3) = \xi_h(x) + x_3 \bar{N}^h(x) + \frac{x_3^2}{2} h^\alpha d^h(x),$$

where  $\vec{N}^h$  is the unit normal vector to the image surface  $\xi_h(\Omega)$ :

$$\vec{N}^h(x) = \frac{\partial_1 \xi_h(x) \times \partial_2 \xi_h(x)}{|\partial_1 \xi_h(x) \times \partial_2 \xi_h(x)|} = (-h^\alpha (\nabla v)^* + e_3) + \mathcal{O}(h^{2\alpha}),$$

while the 'warping' vector fields  $d^h \in \mathcal{C}^{1,\beta}(\bar{\Omega}, \mathbb{R}^3)$ , approximating the effective warping  $d \in \mathcal{C}^{0,\alpha}(\bar{\Omega}, \mathbb{R}^3)$  are defined so that:

$$(8.3) \quad \begin{aligned} h^\alpha \|d^h\|_{\mathcal{C}^{1,\beta}} &\leq C \quad \text{and} \quad \lim_{h \rightarrow 0} \|d^h - d\|_{L^\infty} = 0, \\ \mathcal{Q}_2(\nabla^2 v_0 - \nabla^2 v) &= \min \{ \mathcal{Q}_3(F); F \in \mathbb{R}^{3 \times 3}, F_{tan} = \nabla^2 v_0 - \nabla^2 v \} \\ &= \mathcal{Q}_3((\nabla^2 v_0 - \nabla^2 v)^* + \text{sym}(d \otimes e_3)). \end{aligned}$$

For  $F \in \mathbb{R}^{3 \times 3}$ , by  $F_{tan}$  we denote the principal  $2 \times 2$  minor of  $F$ . Recall also that the quadratic form  $\mathcal{Q}_3$  is given by  $\mathcal{Q}_3(F) = D^2 W(\text{Id})(F, F)$ .

**2.** Because of the first condition in (8.3), the statements in Theorem 4.4 (i), (ii) easily follow. In order to compute the energy limit in (iii), we write:

$$(8.4) \quad I^h(u^h) = \frac{1}{h} \int_{\Omega^h} W((\nabla \bar{u}^h)(b^h)^{-1}) \det \nabla \tilde{\phi}_h = \frac{1}{h} \int_{\Omega^h} W(\sqrt{K^h}) \det b^h,$$

where:

$$b^h = \nabla \tilde{\phi}^h$$

while the frame invariance of  $W$  justifies the second equality in (8.4) with:

$$K^h(x, x_3) = (b^h)^{-1,T} (\nabla \bar{u}^h)^T (\nabla \bar{u}^h) (b^h)^{-1}.$$

We will now compute the entries of the symmetric matrix field  $K^h$ , up to terms of order  $o(h^{\alpha+1})$ . In what follows we adopt the convention that all equalities hold modulo quantities which are uniformly  $o(h^{\alpha+1})$ . Call  $M^h = (\nabla \bar{u}^h)^T \nabla \bar{u}^h$ . Since:

$$\nabla_{tan} \bar{u}^h = \nabla \xi_h + x_3 \nabla \vec{N}^h + o(h^{\alpha+1}), \quad \partial_3 \bar{u}^h = \vec{N}^h + x_3 h^\alpha d^h,$$

we obtain, in view of (8.2):

$$\begin{aligned} M_{tan}^h &= \nabla(\text{id} + h^\alpha v_0 e_3)^T \nabla(\text{id} + h^\alpha v_0 e_3) + 2x_3 \text{sym} \left( (\nabla \xi_h)^T \nabla \vec{N}^h \right) \\ &= \text{Id}_2 + h^{2\alpha} \nabla v_0 \otimes \nabla v_0 - 2x_3 h^\alpha \nabla^2 v + o(h^{\alpha+1}), \\ M_{13,23}^h &= (M^h)_{13,23}^T = x_3 h^\alpha d_{tan}^h + o(h^{\alpha+1}), \\ M_{33}^h &= |\vec{N}^h + x_3 h^\alpha d^h|^2 = 1 + 2x_3 h^\alpha d_3^h + o(h^{\alpha+1}). \end{aligned}$$

Further, by a direct calculation, one obtains:

$$\begin{aligned} (b^h)_{tan} &= \text{Id}_2 - x_3 h^\alpha \nabla^2 v_0 + o(h^{\alpha+1}), \\ (b^h)_{13,23}^T &= h^\alpha \nabla v_0 + o(h^{\alpha+1}), \quad (b^h)_{13,23,33} = \vec{n}^h. \end{aligned}$$

and the inverse matrix  $(b^h)^{-1}$  has the following structure:

$$\begin{aligned} ((b^h)^{-1})_{tan} &= A + o(h^{\alpha+1}), \\ ((b^h)^{-1})_{13,23} &= h^\alpha A \nabla v_0 + o(h^{\alpha+1}), \quad ((b^h)^{-1})_{13,23,33}^T = \vec{n}^h. \end{aligned}$$

where the principal minor  $A(x) \in \mathbb{R}^{2 \times 2}$  of  $(b^h(x))^{-1}$  is the symmetric matrix:

$$(8.5) \quad A = (\text{Id}_2 + h^{2\alpha} \nabla v_0 \otimes \nabla v_0 - x_3 h^\alpha \nabla^2 v_0)^{-1}.$$

**3.** We now compute,

$$\begin{aligned} ((b^h)^{-1,T} M^h)_{tan} &= \text{Id} + x_3 h^\alpha A (\nabla^2 v_0 - 2\nabla^2 v) + x_3 h^\alpha \vec{n}_{tan}^h \otimes \vec{d}_{tan}^h + o(h^{\alpha+1}) \\ &= \text{Id}_2 + x_3 h^\alpha (\nabla^2 v_0 - 2\nabla^2 v) + o(h^{\alpha+1}) \\ ((b^h)^{-1,T} M^h)_{13,23}^T &= h^\alpha (\text{Id} + h^{2\alpha} \nabla v_0 \otimes \nabla v_0) A \nabla v_0 + x_3 h^\alpha d_{tan}^h + o(h^{\alpha+1}) \\ &= h^\alpha \nabla v_0 + x_3 h^\alpha d_{tan}^h + o(h^{\alpha+1}) \\ ((b^h)^{-1,T} M^h)_{13,23} &= x_3 h^\alpha d_{tan}^h + \vec{n}_{tan}^h + o(h^{\alpha+1}) \\ ((b^h)^{-1,T} M^h)_{33} &= \vec{n}_3^h + 2x_3 h^\alpha d_3^h + o(h^{\alpha+1}), \end{aligned}$$

where we used that  $A(\text{Id} + h^{2\alpha} \nabla v_0 \otimes \nabla v_0 - 2x_3 h^\alpha \nabla^2 v) = \text{Id} + x_3 h^\alpha A (\nabla^2 v_0 - 2\nabla^2 v)$  and that  $h^\alpha (\text{Id} + h^{2\alpha} \nabla v_0 \otimes \nabla v_0) A = h^\alpha (\text{Id} + x_3 h^\alpha \nabla^2 v_0 A) = h^\alpha \text{Id} + o(h^{\alpha+1})$ . Consequently:

$$\begin{aligned} K_{tan}^h &= A + x_3 h^\alpha (\nabla^2 v_0 - 2\nabla^2 v) + \vec{n}_{tan}^h \otimes \vec{n}_{tan}^h + o(h^{\alpha+1}) \\ &= A(\text{Id} + x_3 h^\alpha (\nabla^2 v_0 - 2\nabla^2 v) + \vec{n}_{tan}^h \otimes \vec{n}_{tan}^h + \frac{h^{4\alpha} |\nabla v_0|^2}{1 + h^{2\alpha} |\nabla v_0|^2} \nabla v_0 \otimes \nabla v_0) + o(h^{\alpha+1}) \\ &= A(\text{Id} + x_3 h^\alpha \nabla^2 v_0 - 2x_3 h^\alpha \nabla^2 v + h^{2\alpha} \nabla v_0 \otimes \nabla v_0) + o(h^{\alpha+1}) \\ &= \text{Id}_2 + 2x_3 h^\alpha A (\nabla^2 v_0 - \nabla^2 v) + o(h^{\alpha+1}) \\ &= \text{Id}_2 + 2x_3 h^\alpha (\nabla^2 v_0 - \nabla^2 v) + o(h^{\alpha+1}) \\ K_{13,23}^h &= h^\alpha A \nabla v_0 + x_3 h^\alpha d_{tan}^h + \vec{n}_3^h \vec{n}_{tan}^h + o(h^{\alpha+1}) \\ &= h^\alpha A \nabla v_0 + x_3 h^\alpha d_{tan}^h - \frac{h^\alpha}{1 + h^{2\alpha} |\nabla v_0|^2} \nabla v_0 + o(h^{\alpha+1}) \\ K_{33}^h &= h^{2\alpha} \langle A \nabla v_0, \nabla v_0 \rangle + 2x_3 h^\alpha d_3^h + |\vec{n}_3^h|^2 + o(h^{\alpha+1}) \\ &= 1 + h^{2\alpha} \langle A \nabla v_0, \nabla v_0 \rangle + 2x_3 h^\alpha d_3^h - \frac{h^{2\alpha} |\nabla v_0|^2}{1 + h^{2\alpha} |\nabla v_0|^2} + o(h^{\alpha+1}), \end{aligned}$$

where we used that  $\vec{n}_{tan}^h \otimes \vec{n}_{tan}^h = \frac{h^{2\alpha}}{1 + h^{2\alpha} |\nabla v_0|^2} \nabla v_0 \otimes \nabla v_0$ . Observe that:

$$\begin{aligned} h^\alpha A \nabla v_0 - \frac{h^\alpha}{1 + h^{2\alpha} |\nabla v_0|^2} \nabla v_0 &= \\ &= \frac{h^\alpha}{1 + h^{2\alpha} |\nabla v_0|^2} A ((1 + h^{2\alpha} |\nabla v_0|^2) \nabla v_0 - \nabla v_0 - h^{2\alpha} |\nabla v_0|^2 \nabla v_0) + o(h^{\alpha+1}) \\ &= o(h^{\alpha+1}). \end{aligned}$$

Therefore, in fact:

$$K_{13,23}^h = x_3 h^\alpha d_{tan}^h + o(h^{\alpha+1}), \quad K_{33}^h = 1 + 2x_3 h^\alpha d_3^h + o(h^{\alpha+1}).$$

Concluding, we get:

$$K^h = \text{Id}_3 + 2x_3 h^\alpha ((\nabla^2 v_0 - \nabla^2 v)^* + \text{sym}(d^h \otimes e_3)) + o(h^{\alpha+1}).$$

4. Taylor expanding  $W$  at  $\text{Id}_3$  and using (8.3) we now see that:

$$\begin{aligned} W(\sqrt{K^h}) &= W(\text{Id}_3 + x_3 h^\alpha ((\nabla^2 v_0 - \nabla^2 v)^* + \text{sym}(d^h \otimes e_3)) + o(h^{\alpha+1})) \\ &= \frac{1}{2} x_3^2 h^{2\alpha} \mathcal{Q}_3((\nabla^2 v_0 - \nabla^2 v)^* + \text{sym}(d^h \otimes e_3)) + o(h^{2\alpha+2}). \end{aligned}$$

Note that  $\nabla \tilde{\phi}_h = 1 + \mathcal{O}(h^\alpha)$ . By this fact, recalling (8.4) and the convergence in (8.3), it follows that:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h^{2\alpha+2}} I^h(u^h) &= \lim_{h \rightarrow 0} \frac{1}{h^{2\alpha+2}} \frac{1}{h} \int_{\Omega^h} W(\sqrt{K^h})(1 + \mathcal{O}(h^\alpha)) \\ &= \lim_{h \rightarrow 0} \frac{1}{2h^3} \int_{\Omega^h} x_3^2 \mathcal{Q}_3((\nabla^2 v_0 - \nabla^2 v)^* + \text{sym}(d \otimes e_3)) \\ &= \lim_{h \rightarrow 0} \frac{1}{2h^3} \left( \int_{-h/2}^{h/2} x_3^2 dx_3 \right) \int_{\Omega} \mathcal{Q}_2(\nabla^2 v_0 - \nabla^2 v) dx \\ &= \frac{1}{24} \int_{\Omega} \mathcal{Q}_2(\nabla^2 v_0 - \nabla^2 v) dx. \end{aligned}$$

Since, clearly  $\bar{u}_3^h = h^\alpha v + \mathcal{O}(h^{2\alpha})$ , we obtain:

$$\lim_{h \rightarrow 0} \frac{1}{h^{2\alpha+2}} \frac{1}{h} \int_{\Omega^h} h^{\alpha+2} f v_3^h \det b^h = \lim_{h \rightarrow 0} \frac{1}{h^{2\alpha+2}} \frac{1}{h} \int_{\Omega^h} h^{\alpha+2} f(h^\alpha v + \mathcal{O}(h^{2\alpha})) = \int_{\Omega} f v dx.$$

The proof of Theorem 4.4 is complete. ■

## 9. RECOVERY SEQUENCE IN THE GENERAL CASE FOR THE RANGE $\alpha \in (1/2, 1)$ : PROOF OF THEOREM 4.3

Let  $d(F) \in \mathbb{R}^3$  be the unique vector so that:

$$\mathcal{Q}_2(F) = \mathcal{Q}_3\left(F^* + \text{sym}(d \otimes e_3)\right).$$

The mapping  $d : \mathbb{R}_{sym}^{2 \times 2} \rightarrow \mathbb{R}^3$  is well-defined and linear.

1. Let the given out-of-plane displacement  $v$  be as in Theorem 4.3. The Monge-Ampère constraint on  $v$  can be rewritten as:

$$\text{curl}^T \text{curl}(\nabla v \otimes \nabla v) = \text{curl}^T \text{curl}(\nabla v_0 \otimes \nabla v_0).$$

Recall that a matrix field  $B \in L^2(\Omega, \mathbb{R}_{sym}^{2 \times 2})$  is in the kernel of the linear operator  $\text{curl}^T \text{curl}$  if and only if  $B = \text{sym} \nabla w$  for some  $w \in W^{1,2}(\Omega, \mathbb{R}^2)$ . Hence, we conclude that:

$$\text{sym} \nabla w = -\frac{1}{2} \nabla v \otimes \nabla v + \frac{1}{2} \nabla v_0 \otimes \nabla v_0.$$

By the Sobolev embedding theorem in the two-dimensional domain  $\Omega$ ,  $v \in W^{2,2}(\Omega)$  implies that:  $\nabla v \in W^{1,q}(\Omega, \mathbb{R}^2)$  for all  $q < \infty$ . Consequently:

$$\text{sym} \nabla w \in W^{1,p}(\Omega, \mathbb{R}^{3 \times 3}) \quad \forall 1 \leq p < 2.$$

Fix  $1 < p < 2$  such that:  $\gamma > 2/p$  and that  $W^{1,p}(\Omega)$  embeds in  $L^8(\Omega)$ . This is possible since  $\gamma < 2$  and so  $p$  can be chosen as close to 2 as we wish. Using Korn's inequality and through a possible modification of  $w$  by an affine mapping, we can assume that:

$$w \in W^{2,p} \cap W^{1,8}(\Omega, \mathbb{R}^2).$$

Call  $\lambda = 1/p$  and observe that:

$$(9.1) \quad \frac{2 - \gamma}{2(p - 1)} < \lambda < \frac{\gamma}{2}.$$

Following [10, Proposition 2], by partition of unity and a truncation argument, as a special case of the Lusin-type result for Sobolev functions, there exist sequences  $v^h \in W^{2,\infty}(\Omega)$  and  $w^h \in W^{2,\infty}(\Omega, \mathbb{R}^2)$  such that:

$$(9.2) \quad \begin{aligned} & \lim_{h \rightarrow 0} \|v^h - v\|_{W^{2,2}(\Omega)} + \|w^h - w\|_{W^{2,p}(\Omega, \mathbb{R}^2)} = 0, \\ & \|v^h\|_{W^{2,\infty}(\Omega)} + \|w^h\|_{W^{2,\infty}(\Omega, \mathbb{R}^2)} \leq Ch^{-\lambda}, \\ & \lim_{h \rightarrow 0} h^{-2\lambda} |\{x \in \Omega; v^h(x) \neq v(x)\}| + h^{-p\lambda} |\{x \in \Omega; w^h(x) \neq w(x)\}| = 0. \end{aligned}$$

Hence,  $\Omega$  is partitioned into a disjoint union  $\Omega = \mathcal{U}_h \cup O_h$ , where:

$$(9.3) \quad \begin{aligned} \mathcal{U}_h &= \{x \in \Omega; v^h(x) = v(x)\} \cap \{x \in \Omega; w^h(x) = w(x)\}, \\ |O_h| &= o(h^{p\lambda}) + o(h^{2\lambda}) = o(h^{p\lambda}). \end{aligned}$$

We observe that the second order stretching  $s(v^h, w^h)$  satisfies:

$$s(v^h, w^h) = \text{sym} \nabla w^h + \frac{1}{2} \nabla v^h \otimes \nabla v^h - \frac{1}{2} \nabla v_0 \otimes \nabla v_0 \quad \text{in } \mathcal{U}_h.$$

Now, a similar argument as in [21, Lemma 6.1] yields:

$$(9.4) \quad \|s(v^h, w^h)\|_{L^\infty(\Omega)} = o(h^{\lambda(p/2-1)}) \quad \text{and} \quad \|s(v^h, w^h)\|_{L^2(\Omega)}^2 = o(h^{2\lambda(p-1)}).$$

**2.** We now define the recovery sequence, using notation and formulas in (8.1) (8.4):

$$(9.5) \quad \begin{aligned} \forall (x, x_3) \in \Omega^h \quad u^h(\tilde{\phi}^h(x, x_3)) &= \bar{u}^h(x, x_3) = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} h^{2\alpha} w^h(x) \\ h^\alpha v^h(x) \end{bmatrix} \\ &+ x_3 \begin{bmatrix} -h^\alpha \nabla v^h(x) \\ 1 \end{bmatrix} + h^{2\alpha} x_3 d^{0,h}(x) + \frac{1}{2} h^\alpha x_3^2 d^{1,h}(x), \end{aligned}$$

where the Lipschitz continuous fields  $d^{0,h} \in W^{1,\infty}(\Omega, \mathbb{R}^3)$  are given by:

$$d^{0,h} = \left(1 - \frac{1}{2} |\nabla v^h|^2\right) e_3 + d \left( \text{sym} \nabla w^h + \frac{1}{2} \nabla v^h \otimes \nabla v^h - \frac{1}{2} \nabla v_0 \otimes \nabla v_0 \right),$$

while the smooth fields  $d^{1,h}$  obey:

$$(9.6) \quad \lim_{h \rightarrow 0} \sqrt{h} \|d^{1,h}\|_{W^{1,\infty}(\Omega)} = 0,$$

$$(9.7) \quad \lim_{h \rightarrow 0} d^{1,h} = e_3 + d \left( -\nabla^2 v + \nabla^2 v_0 \right) \quad \text{in } L^2(\Omega).$$

The convergence statements (i) and (ii) are now verified by a straightforward calculation. In order to establish (iii), we calculate the deformation gradient of  $\bar{u}^h$ :

$$\nabla \bar{u}^h = \text{Id} + h^{2\alpha} (\nabla w^h)^* + h^\alpha D^h - h^\alpha x_3 (\nabla^2 v^h)^* + h^{2\alpha} \begin{bmatrix} x_3 \nabla d^{0,h} & d^{0,h} \end{bmatrix} + h^\alpha \begin{bmatrix} \frac{1}{2} x_3^2 \nabla d^{1,h} & x_3 d^{1,h} \end{bmatrix},$$

where the skew-symmetric matrix field  $D^h$  is given as:

$$D^h = \begin{bmatrix} 0 & -(\nabla v^h)^T \\ \nabla v^h & 0 \end{bmatrix}.$$

Call  $S_g = \frac{1}{2} (\nabla v_0 \otimes \nabla v_0)^* + e_3 \otimes e_3$  and  $B_g = -(\nabla^2 v_0)^* + e_3 \otimes e_3$ . Write:  $A^h = \text{Id} + h^{2\alpha} S_g + h^\alpha x_3 B_g$ . We hence obtain:

$$(\nabla \bar{u}^h)(A^h)^{-1} = \text{Id}_3 + F^h$$

where, using  $\lambda < \gamma/2 < 1$ :

$$\begin{aligned} (9.8) \quad F^h &= h^{2\alpha} ((\nabla w^h)^* - S_g) + h^\alpha D^h - h^\alpha x_3 ((\nabla^2 v^h)^* + B_g) + h^{2\alpha} \begin{bmatrix} x_3 \nabla d^{0,h} & d^{0,h} \end{bmatrix} \\ &\quad + h^\alpha \begin{bmatrix} \frac{1}{2} x_3^2 \nabla d^{1,h} & x_3 d^{1,h} \end{bmatrix} - h^{2\alpha} S_g - h^\alpha x_3 B_g \\ &\quad + \mathcal{O}(h^{4\alpha})(|\nabla w^h| + |d^{0,h}|) + \mathcal{O}(h^{3\alpha})|D^h| + \mathcal{O}(h^{1+2\alpha}) \\ &= o(1). \end{aligned}$$

Hence:

$$(9.9) \quad (A^h)^{-1,T} (\nabla \bar{u}^h)^T (\nabla \bar{u}^h) (A^h)^{-1} = \text{Id}_3 + 2\text{sym} F^h + (F^h)^T F^h = \text{Id} + K^h + q^h,$$

where:

$$K^h = 2h^{2\alpha} \text{sym} \left( (\nabla w^h)^* - \frac{1}{2} (D^h)^2 - S_g + d^{0,h} \otimes e_3 \right) + 2h^\alpha x_3 \text{sym} \left( -(\nabla^2 v^h)^* - B_g + d^{1,h} \otimes e_3 \right),$$

and:

$$\begin{aligned} q^h &= \mathcal{O}(h^\alpha)(|\nabla w^h| + |\nabla w^h|^2 |d^{0,h}|) + \mathcal{O}(h^{3\alpha})|D^h|(1 + |\nabla w^h| + |D^h| + |d^{0,h}|) \\ &\quad + \mathcal{O}(h^{1+2\alpha-\lambda})(1 + |\nabla w^h|^2 + |D^h|^2 + |d^{0,h}|^2) + \mathcal{O}(h^{(2\alpha+3)/2}) \\ &= o(1). \end{aligned}$$

Note that  $(D^h)^2 = -(\nabla v^h \otimes \nabla v^h)^* - |\nabla v^h|^2 (e_3 \otimes e_3)$ . Therefore:

$$\begin{aligned} &\text{sym} \left( (\nabla w^h)^* - \frac{1}{2} (D^h)^2 - S_g + d^{0,h} \otimes e_3 \right) \\ &= \left( \text{sym} \nabla w^h + \frac{1}{2} \nabla v^h \otimes \nabla v^h - (S_g)_{2 \times 2} \right)^* + \text{sym} \left( (d^{0,h} - e_3) + \frac{1}{2} |\nabla v^h|^2 e_3 \right) \otimes e_3 \\ &= s(v^h, w^h)^* + \text{sym} \left( d(s(v^h, w^h)) \otimes e_3 \right). \end{aligned}$$

Call:

$$\begin{aligned} b(v^h) &= \text{sym} \left( -(\nabla^2 v^h)^* - B_g + d^{1,h} \otimes e_3 \right) \\ &= \left( -\nabla^2 v^h - (B_g)_{2 \times 2} \right)^* + \text{sym} \left( (d^{1,h} - e_3) \right) \otimes e_3. \end{aligned}$$

We therefore obtain:

$$K^h = 2h^\alpha x_3 b(v^h) + \mathcal{O}(h^{2\alpha})|s(v^h, w^h)| = o(1).$$



Note also that:

$$(9.10) \quad \lim_{h \rightarrow 0} b(v^h) = (-\nabla^2 v - (B_g)_{2 \times 2})^* + \text{sym} \left( d(-\nabla^2 v - (B_g)_{2 \times 2}) \otimes e_3 \right) \quad \text{in } L^2(\Omega).$$

**3.** We now observe the following convergence rates:

**Lemma 9.1.** *We have:*

$$(i) \quad h^{-1} \|q^h\|_{L^2(\mathcal{U}_h \times (-\frac{h}{2}, \frac{h}{2}))}^2 = o(h^{2\alpha+2}),$$

$$(ii) \quad h^{-1} \| |q^h| |K^h| \|_{L^1(\mathcal{U}_h \times (-\frac{h}{2}, \frac{h}{2}))} = o(h^{2\alpha+2}).$$

*Proof.* Recall that  $v^h$  and  $w^h$  are uniformly bounded in  $W^{1,8}(\Omega)$ . To prove (i) observe that:

$$\frac{1}{h} \|q^h\|_{L^2(\mathcal{U}_h \times (-\frac{h}{2}, \frac{h}{2}))}^2 \leq \|C^h\|_{L^1(\Omega)} \mathcal{O}(h^{8\alpha} + h^{6\alpha} + h^{2(1+2\alpha-\lambda)} + h^{2\alpha+3}) = o(h^{2\alpha+2}),$$

where we collected all the terms involving  $|D^h|$ ,  $|\nabla w^h|$  and  $|d^{0,h}| \leq C(1 + |\nabla w^h| + |D^h|^2)$  in the quantity  $C^h$ , which can be shown to be uniformly bounded in  $L^1(\Omega)$ .

To see (ii), we estimate:

$$\begin{aligned} \frac{1}{h} \| |q^h| |K^h| \|_{L^1(\mathcal{U}_h \times (-\frac{h}{2}, \frac{h}{2}))} &\leq h^{-1/2} \|q^h\|_{L^2} \left( h^{(2\alpha+2)/2} \|b(v^h)\|_{L^2(\Omega)} + h^{2\alpha} \|s(v^h, w^h)\|_{L^2(\Omega)} \right) \\ &= o(h^{(2\alpha+2)/2}) \left[ h^{\alpha+1} + o(h^{2\alpha+\lambda(p-1)}) \right] \\ &= o(h^{2\alpha+2}) + o(h^{3\alpha+\lambda p-\lambda+1}) = o(h^{2\alpha+2}), \end{aligned}$$

where we used (i), (9.4) and (9.10). ■

Now we observe that, since  $F^h = o(1)$  in (9.8), the matrix field  $\text{Id}_3 + F^h$  is uniformly close to  $SO(3)$  for appropriately small  $h$ , and hence it has a positive determinant. By (9.9) and in view of the polar decomposition theorem, there exists an  $SO(3)$  valued field  $R^h : \Omega^h \rightarrow \mathbb{R}^{3 \times 3}$  such that:

$$\text{Id}_3 + F^h = R^h \sqrt{\text{Id} + K^h + q^h} \quad \text{in } \Omega^h.$$

We hence obtain, by Taylor expanding the square root operator around  $\text{Id}_3$ , and using frame invariance:

$$W\left(\nabla \bar{u}^h (A^h)^{-1}\right) = W\left(R^h(\sqrt{\text{Id}_3 + K^h + q^h})\right) = W\left(\text{Id}_3 + \frac{1}{2}(K^h + q^h) + \mathcal{O}(|K^h + q^h|^2)\right).$$

Note that, used the fact that  $|K^h| + |q^h| = o(1)$ , one gets:

$$W\left(\nabla \bar{u}^h (A^h)^{-1}\right) \leq \mathcal{Q}_3 \left( \frac{1}{2} K^h \right) + \mathcal{O}(|K^h| |q^h| + |q^h|^2) + o(1) |K^h|^2,$$

Now let  $b^h = \nabla \tilde{\phi}^h$  as in the previous section. We have:

$$((b^h)^T b^h)^{1/2} = A^h + o(h^{\alpha+1}),$$

and we conclude from above calculations that:

$$(b^h)^{-1,T} (\nabla \bar{u}^h)^T (\nabla \bar{u}^h) (b^h)^{-1} = (A^h)^{-1,T} (\nabla \bar{u}^h)^T (\nabla \bar{u}^h) (A^h)^{-1} + o(h^{\alpha+1}).$$

In view of (8.4), the energy  $I^h$  can now be estimated exactly as in Step 4 of of the proof of Theorem 4.4 in Section 8, This proves the desired limit (iii) in Theorem 4.3. ■

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