A LOCAL EXISTENCE RESULT FOR A SYSTEM OF VISCOELASTICITY WITH PHYSICAL VISCOSITY

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Abstract. We prove the local in time existence of regular solutions to the system of equations of isothermal viscoelasticity with clamped boundary conditions. We deal with a general form of viscous stress tensor $\mathcal{Z}(F, \dot{F})$, assuming a Korn-type condition on its derivative $D_{\dot{F}}\mathcal{Z}(F, \dot{F})$. This condition is compatible with the balance of angular momentum, frame invariance and the Claussius-Duhem inequality. We give examples of linear and nonlinear (in $\dot{F}$) tensors $\mathcal{Z}$ satisfying these required conditions.

1. Introduction and the main results

In this paper, we are concerned with the local in time existence of the classical solutions to the system of equations of isothermal viscoelasticity. The system we study is given through the balance of linear momentum:

$$\xi_{tt} - \operatorname{div}\Big(DW(\nabla\xi) + \mathcal{Z}(\nabla\xi, \nabla\xi_t)\Big) = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1.1}$$

and it is subject to initial data:

$$\xi(0, \cdot) = \xi_0 \ \text{ and } \ \xi_t(0, \cdot) = \xi_1 \quad \text{in } \Omega, \tag{1.2}$$

the clamped boundary conditions:

$$\xi(\cdot, X) = X \qquad \forall X \in \partial\Omega, \tag{1.3}$$

and the non-interpenetration ansatz:

$$\det \nabla\xi > 0 \quad \text{in } \Omega. \tag{1.4}$$

Here, $\xi : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ denotes the deformation of a reference configuration $\Omega \subset \mathbb{R}^n$ which models a viscoelastic body with constant temperature and density. A typical point in Ω is denoted by X, and the deformation gradient, the velocity and velocity gradient are given as:

$$F = \nabla\xi \in \mathbb{R}^{n\times n}, \quad v = \xi_t \in \mathbb{R}^n, \quad Q = \nabla\xi_t = \nabla v = F_t \in \mathbb{R}^{n\times n}.$$

In (1.1) the operator *div* stands for the spacial divergence of an appropriate field. We use the convention that the divergence of a matrix field is taken row-wise. In what follows, we shall also use the matrix norm $|F| = (\operatorname{tr}(F^TF))^{1/2}$, which is induced by the inner product: $F_1 : F_2 = \operatorname{tr}(F_1^TF_2)$. To avoid notational confusion, we will often write $\langle F_1 : F_2\rangle$ instead of $F_1 : F_2$.

1.1. **The elastic energy density W.** The mapping $DW : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ in (1.1) is the Piola-Kirchhoff stress tensor which, in agreement with the second law of thermodynamics [9], is expressed as the derivative of an elastic energy density $W : \mathbb{R}^{n\times n} \longrightarrow \overline{\mathbb{R}}_+$.

The principles of material frame invariance, material consistency, and normalisation impose the following conditions on W, valid for all $F \in \mathbb{R}^{n\times n}$ and all proper rotations $R \in SO(n)$:

$$\begin{aligned} &\text{(i)}\ W(RF) = W(F),\\ &\text{(ii)}\ W(F) \to +\infty \quad \text{as } \det F \to 0,\\ &\text{(iii)}\ W(\mathrm{Id}) = 0. \end{aligned} \tag{1.5}$$

Examples of W satisfying the above conditions are:

$$\begin{aligned} W_1(F) &= |(F^TF)^{1/2} - \mathrm{Id}|^2 + |\log\det F|^q,\\ W_2(F) &= |(F^TF)^{1/2} - \mathrm{Id}|^2 + \left|\frac{1}{\det F} - 1\right|^q \ \text{ for } \det F > 0, \end{aligned}$$

where $q > 1$ and W is intended to be $+\infty$ if $\det F \leq 0$ [23]. Another case-study example, satisfying (i) and (iii) is: $W_0(F) = |F^TF - \mathrm{Id}|^2 = (|F^TF|^2 - 2|F|^2 + n)$.

We will assume that W is smooth in a neighborhood of $SO(n)$. Since $\mathrm{div}(DW(\nabla\xi))$ is a lower order term in (1.1), it follows that other properties of W play actually no role in the proof of our main Theorem 1.1 and 1.2. We hence remark that the same results are valid when $\mathrm{div}(DW(\nabla\xi))$ is replaced by $\mathrm{div}(DW((\nabla\xi)A(X)^{-1}))$. Such term corresponds to the so-called non-Euclidean elasticity, where the deformation ξ of the reference configuration strives to achieve a prescribed Riemannian metric $g = A^TA$ on Ω. This model pertains to the description of prestrained materials and morphogenesis of growing tissues [20, 19].

1.2. **The viscous stress tensor $\mathcal{Z}$.** The viscous stress tensor is given by the mapping $\mathcal{Z} : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$, depending on the deformation gradient F and the velocity gradient Q. It should be compatible with the following principles of continuum mechanics: balance of angular momentum, frame invariance, and the Claussius-Duhem inequality [9]. That is, for every $F, Q \in \mathbb{R}^{3\times 3}$ with $\det F > 0$, we require that:

(1.6)
- (i) $\mathrm{skew}\,(F^{-1}\mathcal{Z}(F,Q)) = 0$, i.e. $\mathcal{Z} = FS$ with S symmetric.
- (ii) $\mathcal{Z}(RF, R_tF + RQ) = R\mathcal{Z}(F,Q)$ for every path of rotations $R : \mathbb{R}_+ \longrightarrow SO(n)$, i.e. in view of (i): $S(RF, RKF + RQ) = S(F,Q)$ $\forall R \in SO(n)$ $\forall K \in \mathrm{skew}$.
- (iii) $\mathcal{Z}(F,Q) : Q \geq 0$, i.e. in view of (i): $S : \mathrm{sym}(F^TQ) \geq 0$.

Examples of $\mathcal{Z}$ satisfying the above are:

$$\begin{aligned} \mathcal{Z}_m(F,Q) &= [\mathrm{sym}(QF^{-1})]^{2m+1}F^{-1,T},\\ \mathcal{Z}_0'(F,Q) &= 2(\det F)\mathrm{sym}(QF^{-1})F^{-1,T},\\ \mathcal{Z}_0''(F,Q) &= 2F\mathrm{sym}(F^TQ). \end{aligned} \tag{1.7}$$

We note that in the case of $\mathcal{Z}_0'$, the related Cauchy stress tensor $T_0' = 2(\det F)^{-1}\mathcal{Z}_2 F^T = 2\mathrm{sym}(QF^{-1})$ is the Lagrangean version of the stress tensor $2\mathrm{sym}\nabla v$ written in the Eulerian coordinates. For incompressible fluids $2\mathrm{div}(\mathrm{sym}\nabla v) = \Delta v$, giving the usual parabolic viscous regularization of the fluid dynamics evolutionary system.

1.3. **The main results.** Our main assumption implying the dissipative properties of (1.1) will be expressed in terms of the following condition on a (constant coefficient) linear operator $M : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$:

$$\|\nabla\zeta\|^2_{L_2(\mathbb{R}^n)} \leq \gamma \int_{\mathbb{R}^n} (M\nabla\zeta) : \nabla\zeta \qquad \forall \zeta \in W^1_2(\mathbb{R}^n, \mathbb{R}^n). \tag{1.8}$$

Note that (1.8) is a Korn-type estimate, reducing to the classical Korn inequality for $M(F) = \mathrm{sym}F$ and $\gamma = 2$ [17]. Naturally, (1.8) is equivalent to (2.1) which is the same estimate, but required for all $\zeta \in W^2_1(U, \mathbb{R}^n)$ with $\zeta_{|\partial U} = 0$, on any fixed open bounded set $U \subset \mathbb{R}^n$. Further, it can be shown, via Fourier transform (see Lemma 2.2), that (1.8) is also equivalent to the strict positive definiteness of the operator M restricted to the space of rank-one matrices $Q = a \otimes b$:

$$\forall a, b \in \mathbb{R}^n \qquad |a|^2|b|^2 = |a \otimes b|^2 \leq \gamma\langle M(a \otimes b) : a \otimes b\rangle. \tag{1.9}$$

We point out that the above condition resembles, naturally, the local well-posedness criterion for the inviscid elasticity system [16], where the validity of (1.9) for $M = DW(\nabla\xi_0)$ is equivalent to the hyperbolicity of the first-order system (1.1) with $\mathcal{Z} = 0$.

The main result of this paper is the following:

Theorem 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and let:*

$$\xi_0 \in W^2_p(\Omega),\ \xi_1 \in W^{2-2/p}_p(\Omega) \quad \text{for some } \ p > n + 2, \tag{1.10}$$

satisfy:

$$\inf_{X\in\Omega} \det \nabla\xi_0(X) > 0, \qquad \xi_0(X) = X,\ \xi_1(X) = 0 \quad \forall X \in \partial\Omega.$$

Assume that the viscous tensor $\mathcal{Z}$ has the property that:

(1.11)

$\forall X \in \Omega$, *(1.8) holds with $M = D_Q\mathcal{Z}(\nabla\xi_0(X), \nabla\xi_1(X))$ and γ independent of X.*

Then there exists $T_{max} > 0$ such that the problem (1.1), (1.2), (1.3), (1.4) admits a unique regular solution $\xi \in W^{2,2}_p(\Omega \times (0,T)) \cap L_\infty((0,T), W^2_p(\Omega))$ with $\xi_t \in W^{2,1}_p(\Omega \times (0,T))$ for all $T < T_{max}$.

The proof of Theorem 1.1 will be given in sections 2, 3, 4. In section 5 we show that viscous stress tensors in (1.7) satisfy (1.11): for any initial data ξ_0, ξ_1 in case of the linear (in Q) tensors $\mathcal{Z}_0, \mathcal{Z}_0', \mathcal{Z}_0''$, and for initial data enjoying additionally: $\det\mathrm{sym}(\nabla\xi_1(\nabla\xi_0)^{-1}) \neq 0$ in case of nonlinear (in Q) tensors $\mathcal{Z}_1, \mathcal{Z}_2$; see Lemma 2.3. Thanks to this observation, Theorem 1.1 assures the mathematical well-posedness of a class of physically well-posed models.

With the same techniques of proof as in Theorem 1.1, one can show that:

Theorem 1.2. *Let $\mathcal{S}$ be the solution operator of the problem (1.1) - (1.4) as described in Theorem 1.1, given by:*

$$\mathcal{S}(\xi_0, \xi_1) = (\xi, \xi_t),$$

$$\mathcal{S} : W_p^2(\Omega) \times W_p^{2-2/p}(\Omega) \to \Big(W_p^{2,2}(\Omega \times (0,T)) \cap L_\infty((0,T), W_p^2(\Omega))\Big) \times W_p^{2,1}(\Omega \times (0,T)).$$

Then $\mathcal{S}$ is continuous.

We omit the proof of this result and refer instead to standard texts [1, 18, 21, 22], or to an application of the same methods in a more current context in Theorem 1.2 [10].

1.4. **Relation to previous works.** The dynamical viscoelasticity (1.1) has been the subject of vast studies in the last decades. For $\mathcal{Z}(F,Q) = Q$ conflicting with the frame invariance (1.6) (ii), various results on existence, asymptotics and stability have been obtained in [2, 26, 27, 13]. For dimension $n = 1$, existence of solutions to (1.1) has been shown in [8, 4] for $\mathcal{Z}$ depending nonlinearly on Q.

Existence and stability of viscoelastic shock profiles for a large class of models originating from (1.1) has been studied, among others, in [3, 5].

Existence of Young measure solutions to system (1.1) was shown in [11], without any additional assumptions on $\mathcal{Z}$, but with condition (1.6) (iii) strengthened to the uniform dissipativity i.e: $\mathcal{Z}(F,Q) \geq \gamma|Q|^2$. These measure-valued solutions were shown to be the unique classical weak solutions under the extra monotonicity assumption:

$$\langle \mathcal{Z}(F_1, Q_1) - \mathcal{Z}(F_2, Q_2) : Q_1 - Q_2 \rangle \geq \kappa |Q_1 - Q_2|^2 - l|F_1 - F_2|^2, \tag{1.12}$$

see also [28] for a treatment of slightly more general type of PDEs under the same condition. As noted in [11], (1.12) is incompatible with the balance of angular momentum (1.6) (i). In particular, (1.12) is not satisfied by any of the examples in (1.7), even $\mathcal{Z}_0$, $\mathcal{Z}_0'$, $\mathcal{Z}_0''$ which enjoy condition (1.11) for all invertible $F = \nabla\xi_0(X)$ and all $Q = \nabla\xi_1(X)$.

From the theory of PDEs viewpoint, our present result is a rather straightforward application of the theory of nonlinear (quasilinear) parabolic systems. Namely, we apply the maximal regularity estimates to control the nonlinearities of the system (1.1) by the dominating dissipative part. We choose the L_p-framework in order to avoid technical difficulties, but similar results to those of Theorems 1.1, 1.2 and estimates therein are expected in the Besov spaces framework [10]. In a sense, our result is hence a consequence of the classical theory of Ladyzhenskaya, Solonnikov and Uralceva [18], which has been further developed in [1, 12, 21], and which is a powerful tool in the study of the parabolic-elliptic systems [24, 25].

On a final note, observe that although we use the theory of quasilinear parabolic systems, the equations in (1.1) are not of parabolic type, and consequently the existence result in Theorem 1.1 does not follow directly from the general theory in [7, 12]. To see this, consider the simplest mono-dimensional linear reduction of (1.1):

$$u_{tt} - u_{xx} - u_{txx} = 0 \quad \text{in} \quad \mathbb{R} \times (0,T). \tag{1.13}$$

The solution has the following form, in Fourier variable k:

$$u(x,t) = \mathcal{F}^{-1}\left(\exp\Big(-t\ \frac{|k|^2 + |k|\sqrt{|k|^2-4}}{2}\Big)\ \hat{\alpha}\right) + \mathcal{F}^{-1}\left(\exp\Big(-t\ \frac{|k|^2 - |k|\sqrt{|k|^2-4}}{2}\Big)\ \hat{\beta}\right)$$

where $\mathcal{F}^{-1}$ denotes the inverse of the Fourier transform, and α, β represent the suitably chosen initial data. For large k we see that:

$$u(\cdot,t) \approx \exp\left(-\frac{1}{4}t\right)\beta \qquad \text{for } t \in [0,T),$$

and in particular, we see that there is no smoothing effect. Hence (1.13), which is a prototypical example of the system (1.1) should not be viewed as a parabolic problem.

1.5. **Notation.** By $L_p(\Omega)$ we denote the space of functions integrable with respect to the Lebesgue measure, with p-th power. By $W_p^{k,l}(\Omega \times (0,T))$ for $k, l \in \mathbb{N}$ we denote the anisotropic Sobolev space defined by the norm :

$$\|u\|_{W_p^{k,l}(\Omega\times(0,T))} = \|u\|_{L_p(\Omega\times(0,T))} + \|\nabla^k u, \partial_t^l u\|_{L_p(\Omega\times(0,T))},$$

where ∇^k is the k-th space derivative and ∂_t is the time derivative. The isotropic version is given by:

$$\|u\|_{W_p^k(\Omega)} = \|u\|_{L_p(\Omega)} + \|\nabla^k u\|_{L_p(\Omega)}.$$

The space $W_p^{2-2/p}(\Omega)$ is the trace (in time) space of $W_p^{2,1}(\Omega \times (0,T))$. For further details we refer to [6].

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2. The constant coefficient problem

The following auxiliary result will be needed in the proof of Theorem 1.1:

Lemma 2.1. *Assume that $M : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is a linear map satisfying the following Korn-type inequality:*

$$\|\nabla\zeta\|^2_{L_2(\Omega)} \leq \gamma \int_\Omega (M\nabla\zeta) : \nabla\zeta \qquad \forall \zeta \in W^1_2(\Omega, \mathbb{R}^n) \text{ with } \zeta_{|\partial\Omega} = 0 \tag{2.1}$$

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with the boundary of class $\mathcal{C}^2$. Then the solution to:

$$\begin{cases} \zeta_t - \operatorname{div}(M\nabla\zeta) = f & \text{in } \Omega\times(0,T), \\ \zeta = 0 & \text{on } \partial\Omega\times(0,T), \\ \zeta(0,\cdot) = \zeta_0 & \text{in } \Omega \end{cases} \tag{2.2}$$

admits the maximal regularity estimate:

$$\|\zeta_t, \nabla^2\zeta\|_{L_p(\Omega\times(0,T))} \leq C_{\gamma,p}(\|f\|_{L_p(\Omega\times(0,T))} + \|\zeta_0\|_{W^{2-2/p}_p(\Omega)}). \tag{2.3}$$

Towards a proof of Lemma 2.1, note first that for $M = \mathrm{Id}$, i.e. when (2.1) holds trivially, (2.3) is a classical maximal regularity parabolic estimate for the heat equation. When $M(F) = \mathrm{sym}F$, i.e. when (2.1) reduces to Korn's inequality, the proof of (2.3) is also immediate. For, take *div* of the equation in (2.2), and note that $\operatorname{div}^T\operatorname{div}(\mathrm{sym}\nabla\zeta) = \operatorname{div}\left(\frac{1}{2}\Delta\zeta + \frac{1}{2}\nabla\operatorname{div}\zeta\right) = \frac{1}{2}(\operatorname{div}\Delta\zeta + \frac{1}{2}\Delta\operatorname{div}\zeta) = \Delta\operatorname{div}\zeta$ so that:

$$(\operatorname{div}\zeta)_t - \Delta(\operatorname{div}\zeta) = \operatorname{div} f \quad \text{ in } \Omega\times(0,T).$$

By the maximal regularity estimate for the heat equation:

$$\|\nabla\operatorname{div}\zeta\|_{L_p(\Omega\times(0,T))} \leq C_{p,\Omega}\left(\|f\|_{L_p(\Omega\times(0,T))} + \|\operatorname{div}\zeta_0\|_{W^{1-1/p}_p(\Omega)}\right). \tag{2.4}$$

Now, (2.2) can be written as:

$$\zeta_t - \frac{1}{2}\Delta\zeta = f + \frac{1}{2}\nabla\operatorname{div}\zeta \quad \text{ in } \Omega\times(0,T).$$

We hence obtain:

$$\|\zeta_t, \nabla^2\zeta\|_{L_p(\Omega\times(0,T))} \leq C_{p,\Omega}\left(\|f + \frac{1}{2}\nabla\operatorname{div}\zeta\|_{L_p(\Omega\times(0,T))} + \|\operatorname{div}\zeta_0\|_{W^{2-1/p}_p(\Omega)}\right),$$

which combined with (2.4) yields (2.3).

In the general case, Lemma 2.1 follows from the maximal regularity theory developed for parabolic initial-boundary value problems in [12]. Under the ellipticity condition (b) on page 98 in there (see also Definition 5.1), the estimate (2.3) is a consequence of Theorem 7.11. We now prove that condition (2.1) implies that the constant coefficient operator $-\operatorname{div}(M\nabla\zeta)$ has its spectrum contained in the proper sector of the complex plane, which immediately gives ellipticity in the sense of [12].

Lemma 2.2. *Conditions (1.8), (2.1) and (1.9) are equivalent. Moreover, under any of these conditions the operator* $-\mathrm{div}\, M\nabla(\cdot)$ *is elliptic, i.e:*

$$\mathrm{spec}\big(-\mathrm{div}\, M\nabla(\cdot)\big) \subset \Big\{z \in \mathbb{C} : \mathrm{Re}\, z > 0, \ \textit{and} \ \arg z \in [\alpha_*, \alpha^*] \ \textit{with} \ -\frac{\pi}{2} < \alpha_* < \alpha^* < \frac{\pi}{2}\Big\}. \tag{2.5}$$

Proof. **1.** Conditions (1.8) and (2.1) are obviously equivalent, in view of the density of $C_c^\infty(\mathbb{R}^n)$ in $W_1^2(\mathbb{R}^n)$. To include condition (1.9), we use linearity of the Fourier transform and Plancherel's identity:

$$\|\nabla\zeta\|_{L_2(\mathbb{R}^n)})^2 = \|(\nabla\zeta)^\wedge\|_{L_2(\mathbb{R}^n)})^2 = \int_{\mathbb{R}^n} |\hat{\zeta}(k)\otimes k|^2 \ \mathrm{d}k$$

$$\int_{\mathbb{R}^n} \langle M(\nabla\zeta) : \nabla\zeta\rangle = \int_{\mathbb{R}^n} \langle M((\nabla\zeta)^\wedge) : \overline{(\nabla\zeta)^\wedge}\rangle = \int_{\mathbb{R}^n} \langle M(\hat{\zeta}(k)\otimes k) : \overline{(\hat{\zeta}(k)\otimes k)}\rangle \ \mathrm{d}k.$$

Hence, (1.8) may be rewritten as:

$$\forall \zeta \in W_1^2(\mathbb{R}^n) \qquad \|\hat{\zeta}\otimes k\|^2_{L_2(\mathbb{R}^n)} \leq \gamma \int_{\mathbb{R}^n} \Big\langle M(\hat{\zeta}\otimes k) : \overline{(\hat{\zeta}\otimes k)} \ \Big\rangle \ \mathrm{d}k. \tag{2.6}$$

It is therefore clear that (1.9) implies (2.6). On the other hand, given $k_0, a_0 \in \mathbb{R}^n$, consider: $\hat{\zeta}_m(k) = \Big(\rho_m^{1/2}(k-k_0) + \rho_m^{1/2}(k+k_0)\Big)\, a_0$, where ρ_m is the standard radially symmetric mollifier supported in the ball $B(0,1/m)$. Applying (2.6) to $\zeta_m(X) = (\hat{\zeta}_m)^\wedge(-X) \in W_1^2(\mathbb{R}^n,\mathbb{R}^n)$ and passing to the limit with $m\to\infty$, yield (1.9) for the rank-one matrix $Q = a_0\otimes k_0$.

2. To prove (2.5), consider the eigenvalue problem:

$$\lambda\zeta - \mathrm{div}(M(\nabla\zeta)) = 0 \qquad \text{in } \mathbb{R}^n,$$

which after passing to the Fourier variable $k\in\mathbb{R}^n$ becomes:

$$\lambda\hat{\zeta}(k) = M(\hat{\zeta}(k)\otimes k)k. \tag{2.7}$$

Upon writing $\lambda = \sigma|k|^2$, the problem (2.7) is equivalent to locating the eigenvalues σ of the family of linear operators $\{M_k\}_{|k|=1}$, $M_k : \mathbb{R}^n \to \mathbb{R}^n$ given by: $M_k(a) = M(a\otimes k)k$. Recalling (1.9), we see that each M_k is strictly positive definite:

$$M_k(a)\cdot a = \big\langle M(a\otimes k) : (a\otimes k)\big\rangle \geq \frac{|k|^2}{\gamma}|a|^2.$$

Consequently, spectrum of every M_k which consists of the eigenvalues σ, satisfies: $\mathrm{Re}\,\sigma > 0$. The inclusion (2.5) now easily follows by continuity with respect to the parameter k which varies in the compact set $|k| = 1$. ∎

Finally, we have the following:

Lemma 2.3. *The viscous stress tensors $\mathcal{Z}$ in (1.7) satisfy (2.1) with $M = D_Q\mathcal{Z}(F_0, Q_0)$, for every F_0, Q_0 with $\det F_0 > 0$, in the following manner:*

(i) $\mathcal{Z}_0''$ *with* $\gamma = |F_0^{-1,T}|^2$.
(ii) $\mathcal{Z}_0'$ *with* $\gamma = |F_0|^2(\det F_0)^{-1}$.
(iii) $\mathcal{Z}_0$ *with* $\gamma = \frac{1}{2}|F_0|^2$.

If we additionally assume that $\det \mathrm{sym}(Q_0 F_0^{-1}) \neq 0$ *then we also have:*

(iv) $\mathcal{Z}_1$ *with* $\gamma = 2|F_0|^2|\mathrm{sym}(Q_0F_0^{-1})^{-1}|^2$.
(v) $\mathcal{Z}_2$ *with* $\gamma = 2|F_0|^2|\mathrm{sym}(Q_0F_0^{-1})^{-1}|^4$.

The proof of Lemma 2.3 will be given in section 5. We now remark that in the proof of the main Theorem 1.1, Lemma 2.1 will be used to the operators $M = M_X = D_Q\mathcal{Z}(F_0, Q_0)$, at finitely many spacial points $X \in \Omega$, where $F_0 = \nabla\xi_0(X)$ and $Q_0 = \nabla\xi_1(X)$. It is clear that when the initial data ξ_0, ξ_1 with regularity (1.10) satisfy $\det \nabla\xi > 0$ (or the two conditions $\det\nabla\xi > 0$ and $\det \mathrm{sym}(\nabla\xi_1(\nabla\xi_0)^{-1}) \neq 0$ whenever required) then the constants γ in Lemma 2.3 have a common upper and lower bounds, independent of X. Therefore, Lemma 2.1 and the estimate (2.3) may be used with a uniform constant $C_{p,U}$, also independent of X.

3. The main a-priori estimate

Given ξ_0, ξ_1 as in Theorem 1.1, let $\bar{\xi}_1 \in W_p^{2,1}(\Omega\times\mathbb{R}_+)$ be the solution to:

$$\begin{cases} (\bar{\xi}_1)_t - \Delta\bar{\xi}_1 = 0 & \text{in } \Omega\times\mathbb{R}_+, \\ \bar{\xi}_1 = 0 & \text{on } \partial\Omega\times\mathbb{R}_+, \\ \bar{\xi}_1(0,\cdot) = \xi_1 & \text{in } \Omega, \end{cases}$$

Define the extension $\bar{\xi}$ of ξ_0, so that $\partial_t\bar{\xi} = \bar{\xi}_1$:

$$\bar{\xi}(t,x) = \xi_0(x) + \int_0^t \bar{\xi}_1(s,x)\,\mathrm{d}s. \tag{3.1}$$

By continuity, it is clear that: $\inf_{\Omega\times(0,T)} \det\nabla\xi > 0$ for T sufficiently small. We define:

$$D = D(T) = \|\bar{\xi}_{tt}, \nabla^2\bar{\xi}_t\|_{L_p(\Omega\times(0,T))}$$

and note that:

$$\lim_{T\to 0} D(T) = 0. \tag{3.2}$$

Lemma 3.1. *Let ξ_0, ξ_1 be as in Theorem 1.1 and assume that:*

$$\textit{for every } X\in\Omega \textit{ (1.8) holds with } M = D_Q\mathcal{Z}(\nabla\xi_0(X), \nabla\xi_1(X)).$$

Let $\xi \in W_p^{2,2}(\Omega\times(0,T_0))$ with $\xi_t \in W_p^{2,1}(\Omega\times(0,T_0))$ be a solution to the problem (1.1), (1.2), (1.3), (1.4), and denote:

$$\Theta = \Theta(T) = \|(\xi-\bar{\xi})_{tt}, \nabla^2(\xi-\bar{\xi})_t\|_{L_p(\Omega\times(0,T))},$$

where $\bar\xi$ is as in (3.1). Then, there exists $T_{00} < T_0$ and a constant C, both depending only on ξ_0 and ξ_1 (and, naturally, on Ω and p), such that for every $T < T_{00}$ we have:

$$\Theta \leq C(T^{1/p} + D + (T^{1/p} + D)\Theta + \Theta^2 + \Theta^4). \tag{3.3}$$

In particular:

$$\Theta(T) \leq C \qquad \forall T < T_{00}. \tag{3.4}$$

Before we give the proof of the lemma, we gather below some standard inequalities that will be frequently used for different functions: u defined on $\Omega \times (0,T)$, and w defined on Ω. We always assume that $T < 1$.

$$\|w\|_{W^2_p(\Omega)} \leq C_{p,\Omega}\|\Delta w\|_{L_p(\Omega)} \quad \text{when } w_{|\partial\Omega} = 0, \tag{3.5}$$

$$\sup_{t\in(0,T)} \|u(t,\cdot)\|_{W_p^{2-2/p}(\Omega)} \leq C_{p,\Omega}\left(\|u_t - \Delta u\|_{L_p(\Omega\times(0,T))} + \|u(0,\cdot)\|_{W_p^{2-2/p}(\Omega)}\right) \quad \text{when } u_{|\partial\Omega\times(0,T)} = 0, \tag{3.6}$$

$$\|w\|_{L_\infty(\Omega)} \leq C_{p,\Omega}\|w\|_{W_p^{1-1/p}(\Omega)}, \qquad \text{in fact: } \|w\|_{\mathcal{C}^{0,\alpha}(\Omega)} \leq C_{\alpha,p,\Omega}\|w\|_{W_p^{1-1/p}(\Omega)}, \tag{3.7}$$

$$\|\nabla u\|_{\mathcal{C}^{\alpha,\alpha/2}} \leq C_{\alpha,p,\Omega}\|u\|_{W_p^{2,1}(\Omega\times(0,T))}. \tag{3.8}$$

The inequality (3.5) is the usual elliptic estimate [15], and (3.6) is the parabolic estimate from [6]. The Morrey embedding gives (3.7) for $p > n+2$ [15], while (3.8) follows from the embedding $\nabla W_p^{2,1}(\Omega\times(0,T)) \subset L_\infty(\Omega\times(0,T))$, also valid for $p > n+2$ [18]. We stress that the constants C in all the above bounds are universal, i.e. they are independent of T.

We further remark the following simple bound:

$$\begin{aligned}
\|\nabla^2 u\|_{L_p(\Omega\times(0,T))} &= \left(\int_\Omega\int_0^T \left|\int_0^t \nabla^2 u_t \,\mathrm{d}s + \nabla^2 u(0,\cdot)\right|^p \,\mathrm{d}t\,\mathrm{d}X\right)^{1/p} \\
&\leq T^{1/p}\left(\int_\Omega T^{p/p'}\int_0^T |\nabla^2 u_t|^p \,\mathrm{d}t\,\mathrm{d}X\right)^{1/p} + T^{1/p}\|\nabla^2 u(0,\cdot)\|_{L_p(\Omega)} \\
&= T^{1/p}\left(\|\nabla^2 u_t\|_{L_p(\Omega\times(0,T))} + \|\nabla^2 u(0,\cdot)\|_{L_p(\Omega)}\right).
\end{aligned} \tag{3.9}$$

Let now ξ and $\bar\xi$ be as in Lemma 3.1. Using (3.6) to $(\xi-\bar\xi)_t$ we obtain:

$$\begin{aligned}
&\sup_{t\in(0,T)} \|(\xi-\bar\xi)_t(t,\cdot)\|_{L_p(\Omega)} + \sup_{t\in(0,T)} \|\nabla(\xi-\bar\xi)_t(t,\cdot)\|_{L_p(\Omega)} \\
&\qquad \leq C\left(\|(\xi-\bar\xi)_{tt}\|_{L_p(\Omega\times(0,T))} + \|\nabla^2(\xi-\bar\xi)_t\|_{L_p(\Omega\times(0,T))}\right) \leq C\Theta,
\end{aligned} \tag{3.10}$$

and consequently:

$$\|(\xi-\bar\xi)_t\|_{L_p(\Omega\times(0,T))} + \|\nabla(\xi-\bar\xi)_t\|_{L_p(\Omega\times(0,T))} \leq CT^{1/p}\Theta. \tag{3.11}$$

By (3.7), (3.6) used to $\xi - \bar{\xi}$, and (3.9), (3.11) we get:

$$\begin{aligned} \|\nabla(\xi - \bar{\xi})\|_{L_\infty(\Omega\times(0,T))} &= \sup_{t\in(0,T)} \|\nabla(\xi - \bar{\xi})(t,\cdot)\|_{L_\infty(\Omega)} \\ &\leq \sup_{t\in(0,T)} \|\nabla(\xi - \bar{\xi})(t,\cdot)\|_{W_p^{1-1/p}(\Omega)} \\ &\leq C\left(\|(\xi-\bar{\xi})_t\|_{L_p(\Omega\times(0,T))} + \|\nabla^2(\xi - \bar{\xi})\|_{L_p(\Omega\times(0,T))}\right) \leq CT^{1/p}\Theta. \end{aligned} \tag{3.12}$$

Likewise, using (3.7) and (3.6) to $(\xi - \bar{\xi})_t$, we directly obtain:

$$\|(\xi - \bar{\xi})_t\|_{L_\infty(\Omega\times(0,T))} + \|\nabla(\xi - \bar{\xi})_t\|_{L_\infty(\Omega\times(0,T))} \leq C\Theta. \tag{3.13}$$

In all the above inequalities (3.10) – (3.13), we write $\Theta = \Theta(T)$. The constant C depends only on the initial data of the problem ξ_0, ξ_1 (in addition to its dependence on Ω and p).

Proof of Lemma 3.1.
We will always assume that $T < 1$. Note that for $T < T_0$ sufficiently small, the constraint (1.4) is a consequence of the same constraint on the initial data ξ_0, by continuity. Likewise:

$$\|D\mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t), D^2\mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t), D^3\mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t)\|_{L_\infty(\Omega\times(0,T))} \leq C. \tag{3.14}$$

1. The system (1.1) can be rewritten as:

$$(\xi - \bar{\xi})_{tt} - \operatorname{div}\left(\mathcal{Z}(\nabla\xi, \nabla\xi_t) - \mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t)\right) = \operatorname{div}\left(DW(\nabla\xi)\right) + \operatorname{div}\left(\mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t)\right) - \bar{\xi}_{tt}$$

and further, it has the form:

$$(\xi - \bar{\xi})_{tt} - \operatorname{div}\left(D_Q\mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t)\nabla(\xi - \bar{\xi})_t\right) = F[\xi, \bar{\xi}], \tag{3.15}$$

where:

(3.16)

$$\begin{aligned} F[\xi,\bar{\xi}] &= \operatorname{div}\left(DW(\nabla\xi)\right) + \operatorname{div}\left(\mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t)\right) - \bar{\xi}_{tt} \\ &\qquad + \operatorname{div}\left(\mathcal{Z}(\nabla\xi, \nabla\xi_t) - \mathcal{Z}(\nabla\bar{\xi}, \nabla\xi_t)\right) \\ &\qquad + \operatorname{div}\left(\mathcal{Z}(\nabla\bar{\xi}, \nabla\xi_t) - \mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t) - D_Q\mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t)\nabla(\xi - \bar{\xi})_t\right) \\ &= \operatorname{div}\left(DW(\nabla\xi)\right) + \operatorname{div}\left(\mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t)\right) - \bar{\xi}_{tt} \\ &\quad + \operatorname{div}\left(D_F\mathcal{Z}(\nabla\bar{\xi}, \nabla\xi_t)\nabla(\xi - \bar{\xi})\right) \\ &\quad + \operatorname{div}\left(\int_0^1 (1-s)D^2_{FF}\mathcal{Z}(s\nabla\xi + (1-s)\nabla\bar{\xi}, \nabla\xi_t)(\nabla(\xi-\bar{\xi})\otimes\nabla(\xi-\bar{\xi}))\,\mathrm{d}s\right) \\ &\quad + \operatorname{div}\left(\int_0^1 (1-s)D^2_{QQ}\mathcal{Z}(\nabla\bar{\xi}, s\nabla\xi_t + (1-s)\nabla\bar{\xi}_t)(\nabla(\xi-\bar{\xi})_t\otimes\nabla(\xi-\bar{\xi})_t)\,\mathrm{d}s\right). \end{aligned}$$

We shall now prove the bound:

$$\|F[\xi,\bar{\xi}]\|_{L_p(\Omega\times(0,T)}\leq C\left(T^{1/p}+D+(T^{1/p}+D)\Theta+\Theta^2+\Theta^4\right). \tag{3.17}$$

By (3.9) and (3.12) it follows that:

$$\begin{aligned}
&\|\mathrm{div}\,(DW(\nabla\xi))\,\|_{L_p(\Omega\times(0,T))}\leq\|D^2W(\nabla\xi)\|_{L_\infty(\Omega\times(0,T))}\|\nabla^2\xi\|_{L_p(\Omega\times(0,T))}\\
&\qquad\leq\left(\|D^2W(\nabla\bar{\xi})\|_{L_\infty}+C\|\nabla(\xi-\bar{\xi})\|_{L_\infty}\right)T^{1/p}\cdot\\
&\qquad\qquad\cdot\left(\|\nabla^2(\xi-\bar{\xi})_t\|_{L_p}+\|\nabla^2\bar{\xi}_t\|_{L_p}+\|\nabla^2\xi_0\|_{L_p(\Omega)}\right)\\
&\qquad\leq C(1+T^{1/p}\Theta)T^{1/p}(1+\Theta+D)\leq CT^{1/p}(1+\Theta)(1+\Theta+D).
\end{aligned}\tag{3.18}$$

Using (3.14) and (3.9) to $\bar{\xi}$, we obtain:

$$\begin{aligned}
&\|\mathrm{div}\left(\mathcal{Z}(\nabla\bar{\xi},\nabla\bar{\xi}_t)\right)\|_{L_p(\Omega\times(0,T))}\\
&\qquad\leq\|D\mathcal{Z}(\nabla\bar{\xi},\nabla\bar{\xi}_t)\|_{L_\infty(\Omega\times(0,T))}\left(\|\nabla^2\bar{\xi}\|_{L_p(\Omega\times(0,T))}+\|\nabla^2\bar{\xi}_t\|_{L_p(\Omega\times(0,T))}\right)\\
&\qquad\leq C\left(T^{1/p}\|\nabla^2\bar{\xi}_t\|_{L_p}+T^{1/p}\|\nabla^2\xi_0\|_{L_p(\Omega)}+\|\nabla^2\bar{\xi}_t\|_{L_p}\right)\leq C(T^{1/p}+D).
\end{aligned}\tag{3.19}$$

By (3.14), (3.13), (3.9), (3.12) we get:

$$\begin{aligned}
&\|\mathrm{div}\left(D_F\mathcal{Z}(\nabla\bar{\xi},\nabla\xi_t)\nabla(\xi-\bar{\xi})\right)\|_{L_p(\Omega\times(0,T))}\\
&\leq\|D^2_{FF}\mathcal{Z}(\nabla\bar{\xi},\nabla\xi_t)\|_{L_\infty}\left(\|\nabla^2\bar{\xi}\|_{L_p(\Omega\times(0,T))}+\|\nabla^2\bar{\xi}_t\|_{L_p(\Omega\times(0,T))}\right)\|\nabla(\xi-\bar{\xi})\|_{L_\infty}\\
&\qquad\qquad+\|D_F\mathcal{Z}(\nabla\bar{\xi},\nabla\xi_t)\|_{L_\infty}\|\nabla^2(\xi-\bar{\xi})\|_{L_p(\Omega\times(0,T))}\\
&\leq C\left(1+\|\nabla(\xi-\bar{\xi})_t\|_{L_\infty}\right)\left(\|\nabla^2\bar{\xi}\|_{L_p}+\|\nabla^2\bar{\xi}_t\|_{L_p}+\|\nabla^2(\xi-\bar{\xi})_t\|_{L_p}\right)\|\nabla(\xi-\bar{\xi})\|_{L_\infty}\\
&\qquad\qquad+C\left(1+\|\nabla(\xi-\bar{\xi})_t\|_{L_\infty}\right)\|\nabla^2(\xi-\bar{\xi})\|_{L_p}\\
&\leq C(1+\Theta)(T^{1/p}+\Theta+D)T^{1/p}\Theta+C(1+\Theta)T^{1/p}\Theta\\
&\leq CT^{1/p}(1+\Theta+D)\Theta(1+\Theta).
\end{aligned}\tag{3.20}$$

and:

(3.21)

$$\begin{aligned}
&\|\mathrm{div}\left(\int_0^1 (1-s)D^2_{FF}\mathcal{Z}(s\nabla\xi + (1-s)\nabla\bar\xi, \nabla\xi_t)(\nabla(\xi-\bar\xi)\otimes\nabla(\xi-\bar\xi))\ \mathrm{d}s\right)\|_{L_p(\Omega\times(0,T))}\\
&\le \sup_{s\in[0,1]} \|\mathrm{div}\left(D^2_{FF}\mathcal{Z}(s\nabla\xi + (1-s)\nabla\bar\xi, \nabla\xi_t)(\nabla(\xi-\bar\xi)\otimes\nabla(\xi-\bar\xi))\right)\|_{L_p(\Omega\times(0,T))}\\
&\le \sup_{s\in[0,1]} \Bigg[\|D^3\mathcal{Z}(s\nabla\xi + (1-s)\nabla\bar\xi, \nabla\xi_t)\|_{L_\infty}\left(\|\nabla^2\bar\xi\|_{L_p} + \|\nabla^2(\xi-\bar\xi)\|_{L_p} + \|\nabla^2\xi_t\|_{L_p}\right)\|\nabla(\xi-\bar\xi)\|^2_{L_\infty}\\
&\qquad\qquad + \|D^2\mathcal{Z}(s\nabla\xi + (1-s)\nabla\bar\xi, \nabla\xi_t)\|_{L_\infty}\|\nabla^2(\xi-\bar\xi)\|_{L_p}\|\nabla(\xi-\bar\xi)\|_{L_\infty}\Bigg]\\
&\le C\left(1 + \|\nabla(\xi-\bar\xi)\|_{L_\infty} + \|\nabla(\xi-\bar\xi)_t\|_{L_\infty}\right)(T^{1/p} + \Theta + D)T^{2/p}\Theta^2\\
&\qquad\qquad + C\left(1 + \|\nabla(\xi-\bar\xi)\|_{L_\infty} + \|\nabla(\xi-\bar\xi)_t\|_{L_\infty}\right)T^{2/p}\Theta^2\\
&\le CT^{1/p}(1+\Theta+D)\Theta^2(1+\Theta).
\end{aligned}$$

In the same manner, we see that:

(3.22)

$$\begin{aligned}
&\|\mathrm{div}\left(\int_0^1 (1-s)D^2_{QQ}\mathcal{Z}(\nabla\bar\xi, s\nabla\xi_t + (1-s)\nabla\bar\xi_t)(\nabla(\xi-\bar\xi)_t\otimes\nabla(\xi-\bar\xi)_t)\ \mathrm{d}s\right)\|_{L_p(\Omega\times(0,T))}\\
&\quad\le C\left(1 + \|\nabla(\xi-\bar\xi)_t\|_{L_\infty}\right)\left(\|\nabla^2\bar\xi\|_{L_p} + \|\nabla^2(\xi-\bar\xi)_t\|_{L_p} + \|\nabla^2\xi_t\|_{L_p}\right)\|\nabla(\xi-\bar\xi)_t\|^2_{L_\infty}\\
&\qquad\qquad + C\left(1 + \|\nabla(\xi-\bar\xi)_t\|_{L_\infty}\right)\|\nabla^2(\xi-\bar\xi)_t\|_{L_p}\|\nabla(\xi-\bar\xi)_t\|_{L_\infty}\\
&\quad\le C(1+\Theta)(T^{1/p}+\Theta+D)\Theta^2 + C(1+\Theta)\Theta\\
&\quad\le C(T^{1/p}+\Theta+D)\Theta(1+\Theta)^2.
\end{aligned}$$

Combining (3.18) – (3.22), the bound (3.17) follows if only $T<1$, ensuring $D(T)<1$ by (3.2).

2. We will now work with the localizations of the system (3.15). Let $\{B_k\}_{k=1}^N$ be a covering of Ω by a finite number $N=N(r)$ of balls $B_k = B(X_k, r)$ with centers $X_k\in\Omega$ and radius $r<1$. This family of coverings (parametrized by r) should be such that all sets $2B_k\cap\Omega$ are uniformly bilipschitz homeomorphic to each other after appropriate dilations and that the covering numbers of $\{2B_k\cap\Omega\}_k$ are independent of r.

Let $\pi_k:\mathbb{R}^n\to[0,1]$ be smooth cut-off functions satisfying: $\pi_k=1$ on B_k, and $\pi_k=0$ on $\mathbb{R}^n\setminus 2B_k$ where $2B_k = B(X_k,2r)$, and $\|\nabla^\alpha\pi_k\|_{L_\infty}\le Cr^{-|\alpha|}$. After multiplying (3.15) by π_k, we obtain:

(3.23)

$$\left(\pi_k(\xi-\bar\xi)\right)_{tt} - \mathrm{div}\Big(D_Q\mathcal{Z}(\nabla\xi_0(X_k),\nabla\xi_1(X_k))\nabla\left(\pi_k(\xi-\bar\xi)_t\right)\Big) = \pi_k F[\xi,\bar\xi] + G_k[\xi,\bar\xi],$$

where:

$$\begin{aligned}G_k[\xi,\bar\xi] =& \pi_k \ \mathrm{div}\Big([D_Q\mathcal{Z}(\nabla\bar\xi,\nabla\bar\xi_t) - D_Q\mathcal{Z}(\nabla\xi_0(X_k),\nabla\xi_1(X_k))]\nabla(\xi-\bar\xi)_t\Big)\\ &- \Big(D_Q\mathcal{Z}(\nabla\xi_0(X_k),\nabla\xi_1(X_k))\nabla(\xi-\bar\xi)_t\Big)\nabla\pi_k\\ &- \mathrm{div}\left(D_Q\mathcal{Z}(\nabla\xi_0(X_k),\nabla\xi_1(X_k))((\xi-\bar\xi)_t\otimes\nabla\pi_k)\right).\end{aligned}$$

We shall now prove the bound:

$$\begin{aligned}\|G_k[\xi,\bar\xi]\|_{L_p(2B_k\times(0,T))} \leq& C(r^\alpha + T^{\alpha/2})\|\pi_k\nabla^2(\xi-\bar\xi)_t\|_{L_p(2B_k\times(0,T))}\\ &+ C(1+\frac{1}{r^2})(T^{1/p}+D)\Theta(1+\Theta).\end{aligned} \tag{3.24}$$

Using (3.11), we obtain:

$$\begin{aligned}&\|\left(D_Q\mathcal{Z}(\nabla\xi_0(X_k),\nabla\xi_1(X_k))\nabla(\xi-\bar\xi)_t\right)\nabla\pi_k\|_{L_p(2B_k\times(0,T))}\\ &\qquad\leq \frac{C}{r}\|\nabla(\xi-\bar\xi)_t\|_{L_p} \leq \frac{C}{r}T^{1/p}\Theta^2.\end{aligned} \tag{3.25}$$

Likewise:

$$\begin{aligned}&\|\mathrm{div}\left(D_Q\mathcal{Z}(\nabla\xi_0(X_k),\nabla\xi_1(X_k))((\xi-\bar\xi)_t\otimes\nabla\pi_k)\right)\|_{L_p(2B_k\times(0,T))}\\ &\qquad \frac{C}{r}\|\nabla(\xi-\bar\xi)_t\|_{L_p} + \frac{C}{r^2}\|(\xi-\bar\xi)_t\|_{L_p} \leq \frac{C}{r^2}T^{1/p}\Theta.\end{aligned} \tag{3.26}$$

Finally, by (3.14), (3.9), (3.13) and (3.8) we have:

$$\begin{aligned}&\|\pi_k \ \mathrm{div}\Big([D_Q\mathcal{Z}(\nabla\bar\xi,\nabla\bar\xi_t) - D_Q\mathcal{Z}(\nabla\xi_0(X_k),\nabla\xi_1(X_k))]\nabla(\xi-\bar\xi)_t\Big)\|_{L_p(2B_k\times(0,T))}\\ &\leq C\left(\|\nabla^2\bar\xi\|_{L_p} + \|\nabla^2\bar\xi_t\|_{L_p}\right)\|\nabla(\xi-\bar\xi)_t\|_{L_\infty}\\ &\quad + \|D_Q\mathcal{Z}(\nabla\bar\xi,\nabla\bar\xi_t) - D_Q\mathcal{Z}(\nabla\xi_0(X_k),\nabla\xi_1(X_k))\|_{L_\infty(2B_k\times(0,T))}\|\pi_k\nabla^2(\xi-\bar\xi)_t\|_{L_p(2B_k\times(0,T))}\\ &\leq C(T^{1/p}+D)\Theta\\ &\quad + C\left(\|\nabla\bar\xi - \nabla\xi_0(X_k)\|_{L_\infty(2B_k\times(0,T))} + \|\nabla\bar\xi_t - \nabla\xi_1(X_k)\|_{L_\infty(2B_k\times(0,T))}\right)\cdot\\ &\qquad\qquad\cdot\|\pi_k\nabla^2(\xi-\bar\xi)_t\|_{L_p(2B_k\times(0,T))}\\ &\leq C(T^{1/p}+D)\Theta + C(r^\alpha+T^{\alpha/2})\Big(\|\bar\xi\|_{W_p^{2,1}(\Omega\times(0,T))} + \|\bar\xi_1\|_{W_p^{2,1}}\Big)\|\pi_k\nabla^2(\xi-\bar\xi)_t\|_{L_p(2B_k\times(0,T))}.\end{aligned} \tag{3.27}$$

Combining (3.26) – (3.27) and noting that $\|\bar\xi,\bar\xi_1\|_{W_p^{2,1}(\Omega\times(0,T))} \leq C$ where as usual C depends only on ξ_0 and ξ_1, we conclude (3.24) in view of (3.2).

We now use Lemma 2.1 to the problem (3.23) in the domain Ω, i.e. we set $\zeta = \pi_k(\xi-\bar\xi)_t$, $M = D_Q\mathcal{Z}(\nabla\xi_0(X_k),\nabla\xi_1(X_k))$, where γ is the uniform constant from the assumption (1.11). By (2.3) we now obtain:

$$\|\pi_k(\xi-\bar\xi)_{tt}, \nabla^2(\pi_k(\xi-\bar\xi)_t)\|_{L_p(\Omega\times(0,T))} \leq C\|\pi_k F[\xi,\bar\xi], G_k[\xi,\bar\xi]\|_{L_p(\Omega\times(0,T))}.$$

which implies, in view of the localization π_k, present also in G_k:

$$\|\pi_k(\xi-\bar{\xi})_{tt}, \nabla^2(\pi_k(\xi-\bar{\xi})_t)\|_{L_p(2B_k\times(0,T))} \leq C\|\pi_k F[\xi,\bar{\xi}], G_k[\xi,\bar{\xi}]\|_{L_p(2B_k\times(0,T))}.$$

Summing over finitely many $k : 1\dots N$, we get by (3.24):

$$\begin{aligned}\|(\xi-\bar{\xi})_{tt}, \nabla^2(\xi-\bar{\xi})_t\|_{L_p(\Omega\times(0,T))} &\leq C(r^\alpha + T^{\alpha/2})\|\nabla^2(\xi-\bar{\xi})_t\|_{L_p(\Omega\times(0,T))} \\ &\quad + CN^{1/p}\Big((1+1/r)(T^{1/p}+D)\Theta(1+\Theta) + \|F[\xi,\bar{\xi}]\|_{L_p(\Omega\times(0,T))}\Big),\end{aligned}$$

where, again, C depends only on the covering number of $\{B_k\}_{k=1}^N$, on $\bar{\xi}$, p and Ω, but not on r, N, T or Θ. Consequently, for r and T sufficiently small, we arrive at:

$$\Theta \leq CN^{1/p}\left(1+\frac{1}{r}\right)\left(T^{1/p} + D + (T^{1/p}+D)\Theta + \Theta^2 + \Theta^4\right)$$

in virtue of (3.17). This concludes the proof of (3.3).

3. To prove (3.4), consider the functions:

$$g(\Theta) = \Theta \quad \text{and} \quad g_\epsilon(\Theta) = C(\epsilon + \epsilon\Theta + \Theta^2 + \Theta^4),$$

where C is a given constant and $\epsilon > 0$ is a small parameter.

Clearly, $g(0) < g_\epsilon(0)$ for every ϵ. Take now:

$$\epsilon < \min\{\frac{1}{16C^2}, \frac{1}{4C}, 1\} \tag{3.28}$$

and let $\Theta_0 \in (4C\epsilon, \frac{1}{4C})$ with $\Theta_0 < 1$. Then: $\max\{C\epsilon, C\epsilon^2\Theta_0, C\Theta_0^2, C\Theta_0^4\} < \frac{\Theta_0}{4}$ and hence $g(\Theta_0) > g_\epsilon(\Theta_0)$.

Taking now T_{00} so small that, in addition to other requirements imposed in the course of the proof, $\epsilon = T^{1/p} + D$ satisfies (3.28), we obtain that for every $T \in [0, T_{00})$ the quantity $\Theta(T)$ must stay below Θ_0, in virtue of continuity of the function $T \mapsto \Theta(T)$ and $\Theta(0) = 0$. This ends the proof of (3.4) and of Lemma 3.1. ■

4. A proof of Theorem 1.1

We only outline the proof of Theorem 1.1, which is standard, and we point to its most important steps. Let $\bar{\xi}$ be as in (3.1). Recall that the system (1.1) can be rewritten as:

$$(\xi-\bar{\xi})_{tt} - \operatorname{div}\left(D_Q\mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t)\nabla(\xi-\bar{\xi})_t\right) = F[\xi,\bar{\xi}], \tag{4.1}$$

where the right hand side $F[\xi,\bar{\xi}]$ is given in (3.16). We shall seek a solution ξ as the fixed point of the operator:

$$\mathcal{T}(\tilde{\xi}-\bar{\xi}) = \xi - \bar{\xi}, \qquad \xi \text{ is a solution to:}$$

$$(\xi-\bar{\xi})_{tt} - \operatorname{div}\left(D_Q\mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t)\nabla(\xi-\bar{\xi})_t\right) = \tilde{F}[\tilde{\xi},\bar{\xi}], \tag{4.2}$$

in the Banach space:

$$E_{\Omega,T} = \Big\{u \in L_p(\Omega \times (0,T));\ u(0,\cdot) = 0,\ u_t(0,\cdot) = 0,\ u_{|\partial\Omega\times(0,T)} = 0, \qquad u_{tt}, \nabla^2 u_t \in L_p(\Omega \times (0,T))\Big\}, \tag{4.3}$$

equipped with the norm:

$$\|u\|_{E_{\Omega,T}} = \Theta[u](T) = \|u_{tt}, \nabla^2 u_t\|_{L^p(\Omega\times(0,T))}.$$

1. Following calculations as in the proof of Lemma 3.1, it results that:

$$\forall \tilde\xi - \bar\xi \in E_{\Omega,T} \qquad F[\tilde\xi, \bar\xi] \in L^p(\Omega \times (0,T)).$$

2. Integrating (4.2) against $(\xi - \bar\xi)_t$ on $\Omega \times (0,T)$ and using the estimate (4.5) in Lemma 4.1 below with $\zeta = (\xi - \bar\xi)_t$, we obtain:

$$\begin{aligned}\sup_{t\in(0,T)} \|(\xi-\bar\xi)_t(t,\cdot)\|^2_{L_2(\Omega)} &+ \|\nabla(\xi-\bar\xi)_t\|^2_{L_2(\Omega\times(0,T))} \\ &\leq C\int_0^T\int_\Omega F^2[\tilde\xi,\bar\xi]\ \mathrm{d}x\ \mathrm{d}t + C\|(\xi-\bar\xi)_t\|^2_{L_2(\Omega\times(0,T))} \\ &\leq C\int_0^T\int_\Omega F^2[\tilde\xi,\bar\xi]\ \mathrm{d}x\ \mathrm{d}t + CT\sup_{t\in(0,T)}\|(\xi-\bar\xi)_t(t,\cdot)\|^2_{L_2(\Omega)},\end{aligned}$$

which implies the following energy estimate, for T small:

$$\sup_{t\in(0,T)} \|(\xi-\bar\xi)_t(t,\cdot)\|^2_{L_2(\Omega)} + \|\nabla(\xi-\bar\xi)_t\|^2_{L_2(\Omega\times(0,T))} \leq C\int_0^T\int_\Omega F^2[\tilde\xi,\bar\xi]\ \mathrm{d}x\ \mathrm{d}t. \tag{4.4}$$

In virtue of (4.4), the Galerkin construction of the approximants:

$$(\xi_N - \bar\xi)_t = \sum_{k=1}^N a_N^k(t) w_l(x),$$

where $\{w_l\}_{l=1}^\infty$ is an orthonormal base of $W_2^1(\Omega)$, yields existence of a weak solution $\xi - \bar\xi = \lim_{N\to\infty}(\xi_N - \bar\xi)$ of the problem (4.2), with: $(\xi-\bar\xi)_t \in L_\infty((0,T), L_2(\Omega))$ and $\nabla(\xi-\bar\xi)_t \in L_2(\Omega\times(0,T))$.

3. A modification of arguments in section 3 implies that the weak solution ξ is actually regular in the class determined by (4.3), i.e:

$$\xi - \bar\xi \in E_{\Omega,T}.$$

Moreover, for every small $\epsilon > 0$:

$$\text{if } \Theta[\tilde\xi - \bar\xi](T) \leq \epsilon \quad \text{then} \quad \Theta[\xi - \bar\xi](T) \leq \epsilon.$$

4. In now suffices to show that the map $\mathcal{T}$ is a contraction in some ball $\bar{B}_\epsilon \subset E_{\Omega,T}$. This is done by applying methods of (3) to the system:

$$(\xi_1 - \xi_2)_{tt} - \operatorname{div}\left(D_Q\mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t)\nabla(\xi_1 - \xi_2)_t\right) = F[\tilde{\xi}_1, \bar{\xi}] - F[\tilde{\xi}_2, \bar{\xi}],$$

where $\mathcal{T}(\tilde{\xi}_i - \bar{\xi}) = \xi_i - \bar{\xi}$. For $\epsilon > 0$ sufficiently small it follows that:

$$\Theta[\xi_1 - \xi_2](T) \leq \frac{1}{2}\Theta[\tilde{\xi}_1 - \tilde{\xi}_2](T),$$

which completes the proof. ■

The key role above was played by the following estimate:

Lemma 4.1. *Let $T < T_0$ be sufficiently small and assume that $\mathcal{Z}$ satisfies (1.11). Then for every $\zeta \in W_2^{2,1}(\Omega \times (0,T))$ such that $\zeta(0,\cdot) = 0$ and $\zeta_{|\partial\Omega\times(0,T)} = 0$, there holds:*

$$\|\nabla\zeta\|^2_{L_2(\Omega\times(0,T))} \leq 4\gamma \int_0^T \int_\Omega \langle D_Q\mathcal{Z}(\nabla\bar{\xi}, \nabla\bar{\xi}_t)\nabla\zeta : \nabla\zeta\rangle \ \mathrm{d}x \ \mathrm{d}t + C\|\zeta\|^2_{L_2(\Omega\times(0,T))}, \tag{4.5}$$

with a constant C independent of ζ.

Proof. Consider a covering $\{B_k\}_{k=1}^N$ of Ω by a finite number $N = N(r)$ of balls $B_k = B(X_k, r)$ with centers $X_k \in \Omega$ and radius $r > 0$. This family of coverings (parametrized by r) should be such that their covering numbers are uniform in r. Let $\{\pi_k\}_{k=1}^N$ be a partition of unity subject to $\{B_k\}_k$.

With a slight abuse of notation, we shall still write $\zeta = \zeta(t,\cdot) \in W_2^1(\Omega)$, for a fixed $t \in (0,T)$. By (1.11) it follows that:

$$\begin{aligned}
&\int_\Omega \langle D_Q\mathcal{Z}(\nabla\xi_0, \nabla\xi_1)\nabla\zeta : \nabla\zeta\rangle \\
&= \sum_{k=1}^N \int_{B_k} \langle D_Q\mathcal{Z}(\nabla\xi_0(X_k), \nabla\xi_1(X_k))\nabla(\pi_k^{1/2}\zeta) : \nabla(\pi_k^{1/2}\zeta)\rangle \ \mathrm{d}x + \sum \int_{B_k} E_k[\xi, \bar{\xi}] \\
&\geq \frac{1}{\gamma}\sum_{k=1}^N \|\nabla(\pi_k^{1/2}\zeta)\|^2_{L_2(B_k)} + \sum \int_{B_k} E_k[\xi, \bar{\xi}],
\end{aligned} \tag{4.6}$$

where we accumulated the error terms in:

$$\begin{aligned}
E_k[\xi, \bar{\xi}] =& \langle D_Q\mathcal{Z}(\nabla\xi_0, \nabla\xi_1)\pi_k^{1/2}\nabla\zeta : \pi_k^{1/2}\nabla\zeta\rangle \\
&- \langle D_Q\mathcal{Z}(\nabla\xi_0, \nabla\xi_1)\nabla(\pi_k^{1/2}\zeta) : \nabla(\pi_k^{1/2}\zeta)\rangle \\
&+ \langle [D_Q\mathcal{Z}(\nabla\xi_0, \nabla\xi_1) - D_Q\mathcal{Z}(\nabla\xi_0(X_k), \nabla\xi_1(X_k))]\nabla(\pi_k^{1/2}\zeta) : \nabla(\pi_k^{1/2}\zeta)\rangle.
\end{aligned}$$

Hence:

$$
\begin{aligned}
|\int_{B_k} E_k[\xi,\bar\xi]\ \mathrm{d}x| \leq |\int_{B_k} &\langle D_Q\mathcal{Z}(\nabla\xi_0,\nabla\xi_1)\pi_k^{1/2}\nabla\zeta : (\zeta\otimes\nabla\pi_k^{1/2})\rangle| \\
&+ |\int_{B_k}\langle D_Q\mathcal{Z}(\nabla\xi_0,\nabla\xi_1)(\zeta\otimes\nabla\pi_k^{1/2}) : \nabla(\pi_k^{1/2}\zeta)\rangle| + Cr\|\nabla(\pi_k^{1/2}\zeta)\|^2_{L_2(B_k)} \\
&\leq C_r\|\zeta\|_{W_2^1(B_k)}\|\zeta\|_{L_2(B_k)} + Cr\|\nabla(\pi_k^{1/2}\zeta)\|^2_{L_2(B_k)},
\end{aligned}
$$

where C is a universal constant depending only on the initial data and $\mathcal{Z}$, while the constant C_r depends on the covering $\{B_k\}_k$. Taking r small, so that $Cr < 1/(2\gamma)$, by (4.6) we now arrive at:

$$
\begin{aligned}
\int_\Omega &\langle D_Q\mathcal{Z}(\nabla\xi_0,\nabla\xi_1)\nabla\zeta : \nabla\zeta\rangle \\
&\geq \frac{1}{2\gamma}\sum_{k=1}^N \|\nabla(\pi_k^{1/2}\zeta)\|^2_{L_2(B_k)} - C_r\|\zeta\|_{W_2^1(\Omega)}\|\zeta\|_{L_2(\Omega)} \\
&\geq \frac{1}{2\gamma}\sum_{k=1}^N \|\pi_k^{1/2}\nabla\zeta\|^2_{L_2(B_k)} - C_r\|\zeta\|_{W_2^1(\Omega)}\|\zeta\|_{L_2(\Omega)} \\
&\geq \frac{1}{2\gamma}\|\nabla\zeta\|^2_{L_2(\Omega)} - C_r\left(\epsilon\|\nabla\zeta\|^2_{L_2(\Omega)} + \frac{1}{\epsilon}\|\nabla\zeta\|^2_{L_2(\Omega)}\right),
\end{aligned}
$$

where the last estimate follows through Young's inequality. With ϵ sufficiently small, it yields:

$$
\|\nabla\zeta\|^2_{L_2(\Omega)} \leq 3\gamma\int_\Omega\langle D_Q\mathcal{Z}(\nabla\xi_0,\nabla\xi_1)\nabla\zeta : \nabla\zeta\rangle + C\|\zeta\|^2_{L_2(\Omega)}. \tag{4.7}
$$

Integrating in t, we eventually arrive at:

$$
\begin{aligned}
\|\nabla\zeta\|^2_{L_2(\Omega\times(0,T))} &\leq 3\gamma\int_0^T\int_\Omega\langle D_Q\mathcal{Z}(\nabla\xi_0,\nabla\xi_1)\nabla\zeta : \nabla\zeta\rangle\ \mathrm{d}x\ \mathrm{d}t + C\|\zeta\|^2_{L_2(\Omega\times(0,T))} \\
&\leq 3\gamma\int_0^T\int_\Omega\langle D_Q\mathcal{Z}(\nabla\bar\xi,\nabla\bar\xi_t)\nabla\zeta : \nabla\zeta\rangle\ \mathrm{d}x\ \mathrm{d}t \\
&\qquad + CT\|\nabla\zeta\|^2_{L_2(\Omega\times(0,T))} + C\|\zeta\|^2_{L_2(\Omega\times(0,T))},
\end{aligned}
$$

which for T small enough implies (4.5). ■

5. A proof of Lemma 2.3

1. To prove (i), note that $D_Q\mathcal{Z}_0''(F_0,Q_0)Q = 2F_0\mathrm{sym}(F_0^TQ)$ so that:

$$
\forall Q\in\mathbb{R}^{n\times n} \qquad \langle D_Q\mathcal{Z}_0''(F_0,Q_0)Q : Q\rangle = 2\langle\mathrm{sym}(F_0^TQ_0) : F_0^TQ\rangle = |\mathrm{sym}(F_0^TQ)|^2.
$$

Take $\zeta \in W^1_2(\Omega, \mathbb{R}^n)$ with trace 0 on the boundary $\partial\Omega$. We have:

$$\begin{aligned}\int_\Omega |\nabla\zeta|^2 &\leq |F_0^{-1,T}|^2 \int_\Omega |\nabla(F_0^T\zeta)|^2 \leq 2|F_0^{-1,T}|^2 \int_\Omega |\mathrm{sym}\nabla(F_0^T\zeta)|^2 \\ &= |F_0^{-1,T}|^2 \int_\Omega \langle D_Q\mathcal{Z}_0''(F_0,Q_0)\nabla\zeta : \nabla\zeta\rangle,\end{aligned}$$

where we applied Korn's inequality to the map $x \mapsto F_0^T\zeta(x)$.

2. To prove (ii), observe that $D_Q\mathcal{Z}_0'(F_0,Q_0)Q = 2(\det F_0)\mathrm{sym}(QF_0^{-1})F_0^{-1,T}$ so that:

$$\langle D_Q\mathcal{Z}_0'(F_0,Q_0)Q : Q\rangle = 2(\det F_0)|\mathrm{sym}(QF_0^{-1})|^2.$$

Then, for any test function ζ as above, we have:

$$\begin{aligned}\int_\Omega |\nabla\zeta|^2 &\leq |F_0|^2 \int_\Omega |(\nabla\zeta)F_0^{-1}|^2 \,\mathrm{d}x = |F_0|^2 \int_{F_0\Omega} |\nabla(\zeta\circ(F_0^{-1}y))|^2(\det F_0^{-1})\,\mathrm{d}y \\ &\leq 2|F_0|^2(\det F_0^{-1}) \int_{F_0\Omega} |\mathrm{sym}((\nabla\zeta)\circ F_0^{-1})F_0^{-1})|^2\,\mathrm{d}y \\ &= 2|F_0|^2(\det F_0^{-1})(\det F_0) \int_\Omega |\mathrm{sym}((\nabla\zeta)F_0^{-1})|^2\,\mathrm{d}x \\ &= |F_0|^2(\det F_0)^{-1} \int_\Omega \langle D_Q\mathcal{Z}_0'(F_0,Q_0)\nabla\zeta : \nabla\zeta\rangle,\end{aligned} \tag{5.1}$$

where we applied Korn's inequality to the map $y \mapsto \zeta(F_0^{-1}y)$ on the open domain $F_0\Omega$.

3. To prove (iii) – (v), observe that:

$$\begin{aligned}\langle D_Q\mathcal{Z}_m(F_0,Q_0)Q : Q\rangle &= \Big\langle \sum_{j=0}^{2m} (\mathrm{sym}(Q_0F_0^{-1})^j\mathrm{sym}(QF_0^{-1})(\mathrm{sym}(Q_0F_0^{-1})^{2m-j} : QF_0^{-1}\Big\rangle \\ &= \Big\langle \sum_{j=0}^{2m} A^jBA^{2m-j} : QF_0^{-1}\Big\rangle,\end{aligned}$$

where we denoted:

$$A = \mathrm{sym}(Q_0F_0^{-1}), \qquad B = \mathrm{sym}(QF_0^{-1}).$$

Since the matrix $\sum_{j=0}^{2m} A^jBA^{2m-j}$ is symmetric, it follows that:

$$\langle D_Q\mathcal{Z}_m(F_0,Q_0)Q : Q\rangle = \Big\langle \sum_{j=0}^{2m} A^jBA^{2m-j} : B\Big\rangle$$

Let ζ be a test function as in Lemma 2.1. By calculations similar to (5.1) we get:

$$\int_\Omega |\nabla\zeta|^2 \leq \frac{1}{2}|F_0|^2 \int_\Omega \langle D_Q\mathcal{Z}_0(F_0,Q_0)\nabla\zeta : \nabla\zeta\rangle,$$

proving (iii). To prove (iv), we compute:

$$\begin{aligned}\langle D_Q\mathcal{Z}_1(F_0,Q_0)Q : Q\rangle &= \langle A^2B : B\rangle + \langle ABA : B\rangle + \langle BA^2 : B\rangle \\ &= \langle AB : AB\rangle\langle BA : AB\rangle + |AB|^2 = 2\langle \mathrm{sym}(AB) : AB\rangle + |AB|^2 \\ &= 2|\mathrm{sym}(AB)|^2 + |AB|^2 \geq |AB|^2.\end{aligned}$$

Therefore, by calculations similar to (5.1):

$$\begin{aligned}\int_\Omega |\nabla\zeta|^2 &\leq 2|F_0|^2 \int_\Omega |\mathrm{sym}((\nabla\zeta)F_0^{-1})|^2 \leq 2|F_0|^2|A^{-1}|^2 \int_\Omega |AB|^2 \\ &\leq 2|F_0|^2|\mathrm{sym}(Q_0F_0^{-1})^{-1}|^2 \int_\Omega \langle D_Q\mathcal{Z}_1(F_0,Q_0)\nabla\zeta : \nabla\zeta\rangle.\end{aligned} \tag{5.2}$$

Finally, in order to prove (v) we derive:

$$\begin{aligned}\langle D_Q\mathcal{Z}_2(F_0,Q_0)Q : Q\rangle &= \langle A^4B + A^3BA + A^2BA^2 + ABA^3 + BA^4 : B\rangle \\ &= |A^2B + ABA|^2 + |A^2B|^2 \geq |A^2B|^2,\end{aligned}$$

which, in the same manner as in (5.2) yields:

$$\begin{aligned}\int_\Omega |\nabla\zeta|^2 &\leq 2|F_0|^2|A^{-2}|^2 \int_\Omega |A^2B|^2 \\ &\leq 2|F_0|^2|\mathrm{sym}(Q_0F_0^{-1})^{-1}|^4 \int_\Omega \langle D_Q\mathcal{Z}_2(F_0,Q_0)\nabla\zeta : \nabla\zeta\rangle.\end{aligned}$$

The proof of Lemma 2.3 is done.

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