

A STABILITY RESULT FOR THE STOKES-BOUSSINESQ EQUATIONS IN INFINITE 3D CHANNELS

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ABSTRACT. We consider the Stokes-Boussinesq (and the stationary Navier-Stokes-Boussinesq) equations in a slanted, i.e. not aligned with the gravity's direction, 3d channel and with an arbitrary Rayleigh number. For the front-like initial data and under the no-slip boundary condition for the flow and no-flux boundary condition for the reactant temperature, we derive uniform estimates on the burning rate and the flow velocity, which can be interpreted as stability results for the laminar front.

1. INTRODUCTION AND THE MAIN RESULTS

The Boussinesq system for a reactive flow is a model describing flame propagation in a gravitationally stratified medium [13]. It consists of the reaction-advection-diffusion equation for the reactant temperature θ (normalised so that $0 \leq \theta \leq 1$) and the Navier-Stokes equations for the evolution of the incompressible flow u . The variables u and θ are coupled through the advection velocity in the reaction equation, and through the force term in the fluid equation. After passing to nondimensional variables [2], the simplified Stokes-Boussinesq system takes the form:

$$(1.1) \quad \theta_t + u \cdot \nabla \theta - \Delta \theta = f(\theta),$$

$$(1.2) \quad u_t - \nu \Delta u + \nabla p = \theta \vec{\rho},$$

$$(1.3) \quad \operatorname{div} u = 0.$$

Here, $\nu > 0$ is the Prandtl number (inversely proportional to the Reynolds number), while the reaction rate is given by a nonnegative, nonlinear function $f(\theta)$ of ignition type. That is, f is Lipschitz continuous, and there is a threshold temperature $0 < \vartheta_0 < 1$ such that:

$$(1.4) \quad f(\theta) = 0 \text{ for } \theta \leq \vartheta_0 \text{ and } \theta \geq 1, \quad f(\theta) > 0 \text{ on } (\vartheta_0, 1), \quad f'(1) < 0.$$

The vector $\vec{\rho} = \rho \vec{g}$ corresponds to the non-dimensional gravity \vec{g} scaled by the Rayleigh number ρ , and we assume that $\vec{\rho}$ is non-parallel to the unbounded direction of the 3d channel D , which amounts to studying the system (1.1) - (1.3) in:

$$D = (-\infty, \infty) \times \Omega = \{(x, \tilde{x}); x \in \mathbb{R}, \tilde{x} \in \Omega\},$$

with:

$$\vec{\rho} \cdot e_3 \neq 0.$$

The crosssection Ω is a sufficiently regular, connected and bounded domain in \mathbb{R}^2 . For the solutions to (1.1) - (1.3), we impose the Neumann condition in θ , and the no-slip (Dirichlet) boundary condition in u :

$$(1.5) \quad \frac{\partial \theta}{\partial \vec{n}} = 0 \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial D.$$

The classical result by Kanel [7, 1] states that there exists the unique speed c_0 (necessarily positive) and a (necessarily decreasing) traveling wave profile $\Phi : \mathbb{R} \rightarrow [0, 1]$, satisfying:

$$-c_0 \Phi' - \Phi'' = f(\Phi), \quad \Phi(-\infty) = 1, \quad \Phi(+\infty) = 0.$$

We call c_0 and Φ the *laminar speed* and *laminar front*, and to fix the ideas, we let $\Phi(0) = \vartheta_0$. Then $\theta(x, t) = \Phi(x - c_0 t)$ is the unique (up to translations) traveling-wave solution to:

$$(1.6) \quad \theta_t - \theta_{xx} = f(\theta), \quad \theta(x, 0) = \theta_0(x)$$

joining the two equilibria 1 and 0. Moreover, if the initial data $\theta_0(x)$ differs from the Heaviside function $H(x)$ by a compactly supported error, then there exists a shift x_0 such that, for the solution θ to (1.6) there holds:

$$\|\theta(\cdot, t) - \Phi(\cdot + c_0 t + x_0)\|_{L^\infty(\mathbb{R})} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.$$

Our purpose is to reproduce this result for the problem (1.1) - (1.3), (1.5). Following [3] and [2], we define the *bulk burning rate* $\bar{B}(t)$ and the *Nusselt number* $\bar{N}(t)$ by:

$$(1.7) \quad \begin{aligned} B(t) &= \frac{1}{|\Omega|} \int_D f(\theta) \, dx \, d\tilde{x}, & \bar{B}(t) &= \frac{1}{t} \int_0^t B(s) \, ds \\ N(t) &= \frac{1}{|\Omega|} \int_D |\nabla \theta|^2 \, dx \, d\tilde{x}, & \bar{N}(t) &= \frac{1}{t} \int_0^t N(s) \, ds, \end{aligned}$$

and the *average flow* $\bar{U}(t)$ by

$$(1.8) \quad \bar{U}(t) = \frac{1}{t} \int_0^t \|u(\cdot, s)\|_\infty \, ds.$$

Theorem 1.1. *Assume that the initial temperature $\theta_0(x, \tilde{x}) \in [0, 1]$ is such that $\theta_0(x, \tilde{x}) - H(x)$ is compactly supported in D . Let $u_0 \in W^{1,2}(D)$ and let (θ, u, p) be a global solution of (1.1) - (1.3), (1.5) with: $\theta(\cdot, 0) = \theta_0$ and $u(\cdot, 0) = u_0$. Then, there exists a constant C_Ω , depending only on Ω , such*

that as $t \rightarrow +\infty$:

$$(1.9) \quad c_0 - C_\Omega \left(\frac{\rho}{\nu} + \frac{\rho^2}{\nu^2} \right) - o(1) \leq \bar{B}(t) \leq c_0 + C_\Omega \left(\frac{\rho}{\nu} + \frac{\rho^2}{\nu^2} \right) + o(1),$$

$$(1.10) \quad \bar{N}(t) \leq \left(C_\Omega \frac{\rho}{\nu} + \sqrt{\frac{c_0}{2} + C_\Omega \frac{\rho^2}{\nu^2}} \right)^2 + o(1),$$

$$(1.11) \quad \bar{U}(t) \leq C_\Omega \left(\frac{\rho}{\nu} + \frac{\rho^2}{\nu^2} \right) + o(1).$$

The above result shows that the solution of the initial-boundary problem for (1.1) - (1.3), with small ρ/ν propagates with finite speed close to the laminar front speed. Also, note that if we replace θ with the laminar front Φ , then a simple integration of (1.6) gives: $\bar{B}(t) = c_0$. This corresponds with the estimates in Theorem 1.1 when $\rho = 0$, i.e. the system (1.1) - (1.3) turns out to be a regular perturbation of the reaction-diffusion equation.

Our next result states that with front-like initial data, the solution to the studied system stays front-like:

Theorem 1.2. *With the same assumptions as in Theorem 1.1, we have:*

$$(1.12) \quad \begin{aligned} \Phi \left(x - c_0 t + x_0 + \bar{U}(t)t + C_{\Omega,0}\sqrt{t} \right) - \frac{C_{\Omega,0}}{\sqrt{t}} &\leq \theta(x, \tilde{x}, t) \\ &\leq \Phi \left(x - c_0 t - x_0 + \bar{U}(t)t + C_{\Omega,0}\sqrt{t} \right) + \frac{C_{\Omega,0}}{\sqrt{t}} \quad \forall t \gg 1, \end{aligned}$$

with appropriate $x_0 > 0$ and $C_{\Omega,0}$ depending on Ω , f and the initial data θ_0 .

Finally, we have:

Theorem 1.3. *The results in Theorem 1.1 and Theorem 1.2 remain valid in either of the following cases:*

- (i) *the channel $D = (-\infty, +\infty) \times [0, \lambda]$ is 2-dimensional*
- (ii) *the flow equation (1.2) is replaced by the stationary Navier-Stokes:*

$$-\nu \Delta u + u \cdot \nabla u + \nabla p = \theta \vec{\rho},$$

and the crosssection Ω is sufficiently "thin", i.e it satisfies the same type of condition as in [10]:

$$\frac{\sqrt{3}}{2} \frac{\rho}{\nu \sqrt{\pi \nu}} |\Omega|^{1/2} C_P \left(C_{PW} + \left(\int |\vec{g} \cdot (0, \tilde{x})|^2 d\tilde{x} \right)^{1/2} \right) < 1.$$

Existence of traveling fronts for the Navier-Stokes-Boussinesq system in infinite channels has been the subject of research during recent years [3, 2, 11, 5, 8, 10]. Of interest has been also an understanding of the regularizing and mixing effect of convection [6, 4]. In [3] and [2], the solutions of the system with front-like datum in a 2d strip have been considered and uniform estimates for the full Navier-Stokes-Boussinesq system have been

obtained for the *stress-free* boundary conditions on u . In this paper we generalize these results to dimension 3 and the more physically relevant *no-slip* boundary conditions. The analysis follows [2] closely; it first seeks bounds for \bar{N} and \bar{B} using the parabolic equation (1.1). Where our argument diverges from that of [2] is in finding an estimate for $\|u\|_\infty$. In [2], this has been done using Poincaré's inequality for vorticity, based on the assumption that the vorticity vanishes at the boundary, and using the rather simple form of the vorticity equation in dimension two. In the present case, we rely on the almost-uniform Xie bound in Lemma 2.2.

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2. AUXILIARY RESULTS

Our first result is a technical lemma, which extends Lemma 4.3 in [2], proved there for 2d channels (see also [4]). Here we prove a similar result in dimension 3, showing that diffusion in a tube behaves like 1d heat equation.

Lemma 2.1. *Let $u \in W^{1,2}(D, \mathbb{R}^3)$ with $\operatorname{div} u = 0$ be a given solenoidal flow satisfying $u = 0$ on ∂D . Let ϕ be the solution to the advection-diffusion equation:*

$$(2.1) \quad \begin{aligned} \phi_t + u \cdot \nabla \phi - \Delta \phi &= 0 \quad \text{for } (x, \tilde{x}) \in D \text{ and } t > 0 \\ \frac{\partial \phi}{\partial \vec{n}}(x, \tilde{x}, t) &= 0 \quad \text{for } (x, \tilde{x}) \in \partial D \\ \phi(x, \tilde{x}, 0) &= \phi_0(x, \tilde{x}). \end{aligned}$$

Then there exists a constant C_Ω depending only on Ω (in particular independent of u and ϕ_0) such that, for all sufficiently large t , there holds:

$$\|\phi(\cdot, t)\|_\infty \leq \frac{C_\Omega}{\sqrt{t}} \|\phi_0\|_{L^1(D)} \quad \forall t \gg 1.$$

Proof. 1. We first prove a Nash-type inequality, valid for solutions of (2.1):

$$(2.2) \quad \|\nabla \phi(\cdot, t)\|_{L^2(D)}^2 \geq C_\Omega \frac{\|\phi(\cdot, t)\|_{L^2}^6}{\|\phi(\cdot, t)\|_{L^1}^4 + \|\phi(\cdot, t)\|_{L^1} \|\phi(\cdot, t)\|_{L^2}^3} \quad \forall t.$$

To simplify the notation, in what follows we suppress the dependence on t and write ϕ instead of $\phi(\cdot, t)$, etc.

Define $\psi(x) = \frac{1}{|\Omega|} \int_\Omega \phi(x, \tilde{x}) \, d\tilde{x}$ and let $k(x, \tilde{x}) = \phi(x, \tilde{x}) - \psi(x)$, so that $\phi = \psi + k$ and $\int_\Omega k(x, \tilde{x}) \, d\tilde{x} = 0$ for every $x \in \mathbb{R}$. Notice that, by Cauchy-Schwarz inequality:

$$(2.3) \quad \|\psi\|_{L^2(D)}^2 = |\Omega| \int_{\mathbb{R}} \left(\int_\Omega \phi \, d\tilde{x} \right)^2 dx \leq \int_{\mathbb{R}} \int_\Omega |\phi|^2 = \|\phi\|_{L^2(D)}^2,$$

Similarly:

$$(2.4) \quad \begin{aligned} \|\psi'\|_{L^2(D)} &\leq \|\phi_x\|_{L^2(D)}, \quad \|\psi\|_{L^1(D)} \leq \|\phi\|_{L^1(D)}, \\ \|k\|_{L^2} &\leq 2\|\phi\|_{L^2}, \quad \|\nabla k\|_{L^2} \leq 2\|\nabla\phi\|_{L^2}, \quad \|k\|_{L^1} \leq 2\|\phi\|_{L^1}. \end{aligned}$$

Let $\hat{\psi} : \mathbb{R} \rightarrow \mathbb{C}$ be the Fourier transform of ψ , i.e. $\hat{\psi}(\omega) = \int_{\mathbb{R}} \psi(s)e^{-2\pi i\omega s} ds$. By Plancherel's identity, we have:

$$(2.5) \quad \|\psi\|_{L^2(\mathbb{R})}^2 = \|\hat{\psi}\|_{L^2(\mathbb{R})}^2, \quad \|\psi'\|_{L^2(\mathbb{R})}^2 = \|2\pi i\omega\hat{\psi}\|_{L^2(\mathbb{R})}^2.$$

For a given positive m , we now write:

$$\begin{aligned} \|\psi\|_{L^2(\mathbb{R})}^2 &= \int_{|\omega| \leq m} |\hat{\psi}(\omega)|^2 d\omega + \int_{|\omega| > m} |\hat{\psi}(\omega)|^2 d\omega \\ &\leq 2m\|\psi\|_{L^1(\mathbb{R})}^2 + \frac{1}{m^2} \int_{\mathbb{R}} \omega^2 |\hat{\psi}(\omega)|^2 d\omega \\ &\leq 2m\|\psi\|_{L^1(\mathbb{R})}^2 + \frac{1}{4\pi^2 m^2} \|\psi'\|_{L^2(\mathbb{R})}^2 \end{aligned}$$

where the estimate of the first term follows by: $|\hat{\psi}(\omega)| \leq \|\psi\|_{L^1(D)}$, while to estimate the second term we used (2.5).

Setting $m = \|\psi'\|_{L^2(\mathbb{R})}^{2/3} \|\psi\|_{L^1(\mathbb{R})}^{-2/3}$, we obtain:

$$\|\psi\|_{L^2(\mathbb{R})}^2 \leq \left(\frac{1}{4\pi^2} + 2\right) \|\psi\|_{L^1(\mathbb{R})}^{4/3} \|\psi'\|_{L^2(\mathbb{R})}^{2/3},$$

which implies:

$$\|\psi\|_{L^2(D)}^2 \leq \left(\frac{1}{4\pi^2} + 2\right) |\Omega|^{-2/3} \|\psi\|_{L^1(D)}^{4/3} \|\psi_x\|_{L^2(D)}^{2/3}.$$

Now, by Cauchy-Schwarz inequality, the Sobolev embedding $W^{1,2}(D) \hookrightarrow L^4(D)$, and the Poincare-Wirtinger inequality on Ω , it follows that:

$$\|k\|_{L^2(D)}^2 \leq \|k\|_{L^1(D)}^{2/3} \|k\|_{L^4(D)}^{4/3} \leq C_{\Omega} \|k\|_{L^1(D)}^{2/3} \|\nabla k\|_{L^2(D)}^{4/3}.$$

Therefore, by (2.3) and (2.4):

$$\begin{aligned} \|\phi\|_{L^2(D)}^2 &\leq 2(\|\psi\|_{L^2(D)}^2 + 2\|k\|_{L^2(D)}^2) \\ &\leq C_{\Omega} \left(\|\phi\|_{L^1(D)}^{4/3} \|\nabla\phi\|_{L^2(D)}^{2/3} + \|\phi\|_{L^1(D)}^{2/3} \|\nabla\phi\|_{L^2(D)}^{4/3} \right). \end{aligned}$$

We now argue as in [2]. Since $y := \|\nabla\phi\|_{L^2}^{2/3}$ satisfies: $ay^2 + by - c \geq 0$ with appropriate $a, b, c \geq 0$, then: $y \geq \frac{-b + \sqrt{b^2 + 4ac}}{2a} = \frac{2c}{b + \sqrt{b^2 + 4ac}} \geq \frac{c}{\sqrt{b^2 + 4ac}}$. Hence:

$$\|\nabla\phi\|_{L^2}^{2/3} \geq \|\phi\|_{L^2}^2 \left(\|\phi\|_{L^1}^{8/3} + \|\phi\|_{L^1}^{2/3} \|\phi\|_{L^2}^2 \right)^{-1/2},$$

which gives:

$$\|\nabla\phi\|_{L^2}^2 \geq C_{\Omega} \|\phi\|_{L^2}^6 \left(\|\phi\|_{L^1}^4 + \|\phi\|_{L^1} \|\phi\|_{L^2}^3 \right)^{-1},$$

yielding exactly (2.2).

2. Recall now that:

$$(2.6) \quad \|\phi\|_{L^1(D)} \leq \|\phi_0\|_{L^1(D)}.$$

Indeed, the L^1 norm of a solution to (2.1) is conserved when the initial data is positive. In the general case one can write $\phi_0 = \phi_0^+ - \phi_0^-$ where ϕ_0^+ and ϕ_0^- are positive, with disjoint supports. Solving (2.1) for each, one obtains the inequality (2.6). Further, integrating (2.1) against ϕ and using incompressibility of u and the boundary condition, it follows that:

$$(2.7) \quad \frac{d}{dt} \|\phi\|_{L^2}^2 = -2\|\nabla\phi\|_{L^2}^2.$$

In view of (2.2), (2.6), (2.7) we now obtain:

$$\frac{d\|\phi\|_{L^2}}{dt} \leq C_\Omega \frac{\|\phi\|_{L^2}^5}{\|\phi_0\|_{L^1}^4 + \|\phi_0\|_{L^1} \|\phi\|_{L^2}^3},$$

which, after integrating in time, gives:

$$(2.8) \quad t \leq C_\Omega \left(\frac{\|\phi_0\|_{L^1}^4}{\|\phi\|_{L^2}^4} + \frac{\|\phi_0\|_{L^1}}{\|\phi\|_{L^2}} \right).$$

Call $\alpha = \|\phi\|_{L^2}/\|\phi_0\|_{L^1}$. From (2.8) it follows that $\frac{\alpha^4}{\alpha^3+1} \leq \frac{C_\Omega}{t}$. The function $\alpha \mapsto \frac{\alpha^4}{\alpha^3+1}$ is increasing and it converges to 0 as $\alpha \rightarrow 0$. Let β be the unique solution to $\frac{\beta^4}{\beta^3+1} = \frac{C_\Omega}{t}$, so that $\alpha \leq \beta$. Now, for $t \rightarrow \infty$ clearly $\frac{C_\Omega}{t} \rightarrow 0$, hence also $\beta \rightarrow 0$ and $\frac{\beta^4}{2} \leq \frac{\beta^4}{\beta^3+1} = \frac{C_\Omega}{t}$. Consequently, $\beta \leq \frac{C_\Omega}{t^{1/4}}$ and we arrive at:

$$\|\phi\|_{L^2} \leq \frac{C_\Omega}{t^{1/2}} \|\phi_0\|_{L^1} \quad \forall t \gg 1.$$

3. We now argue as in [2]. Let \mathcal{P}_t be the solution operator for (2.1). Thus far, we have showed that:

$$\|\mathcal{P}_t\|_{L^1 \rightarrow L^2} \leq \frac{C_\Omega}{t^{1/4}} \quad \forall t \gg 1.$$

Now if \mathcal{P}_t^* is the adjoint operator, then \mathcal{P}_t^* is the solution operator to (2.1), with u replaced by $-u$. Therefore the above argument works again:

$$\|\mathcal{P}_t^*\|_{L^1 \rightarrow L^2} \leq \frac{C_\Omega}{t^{1/4}} \quad \forall t \gg 1.$$

Finally, we conclude the lemma:

$$\begin{aligned} \|\mathcal{P}_{2t}\|_{L^1 \rightarrow L^\infty} &\leq \|\mathcal{P}_t\|_{L^1 \rightarrow L^2} \|\mathcal{P}_t\|_{L^2 \rightarrow L^\infty} \\ &= \|\mathcal{P}_t\|_{L^1 \rightarrow L^2} \|\mathcal{P}_t^*\|_{L^1 \rightarrow L^2} \leq \frac{C_\Omega}{t^{1/2}} \quad \forall t \gg 1. \end{aligned}$$

■

We now present a lemma taken from [10]:

Lemma 2.2. *Let $g \in L^2(D)$. There exists a constant C_Ω , depending only on the crosssection Ω , such that for any solenoidal flow $u \in W^{2,2} \cap W_0^{1,2}(D)$ satisfying:*

$$-\nu \Delta u + \nabla p = g, \quad \operatorname{div} u = 0 \text{ in } D,$$

there holds:

$$(2.9) \quad \|u\|_\infty \leq \frac{\sqrt{2}}{\sqrt{\nu\pi}} \|\nabla u\|_{L^2(D)}^{1/2} \|g\|_{L^2(D)}^{1/2} + C_\Omega \|\nabla u\|_{L^2(D)}.$$

We remark that the proof of (2.9) relies on Xie's estimate [12]:

$$\|u\|_\infty \leq \frac{1}{\sqrt{2\pi}} \|\Delta u\|_{L^2(D)}^{1/2} \|\nabla u\|_{L^2(D)}^{1/2}$$

valid for $u \in W^{2,2}(D) \cap W_0^{1,2}(D)$, and on a commutator estimate [9]:

$$\|(\mathcal{P}\Delta - \Delta\mathcal{P})u\|_{L^2(D)}^2 \leq \left(\frac{1}{2} + \epsilon\right) \|\Delta u\|_{L^2(D)}^2 + C_{D,\epsilon} \|\nabla u\|_{L^2(D)}^2$$

where \mathcal{P} is the Helmholtz projection onto the space of solenoidal vector fields.

The interest in the inequality (2.9) lies in the independence of the constant $\frac{\sqrt{2}}{\sqrt{\pi\nu}}$ at the term involving g . Indeed, this was the key argument allowing to prove [10] existence of traveling waves for the full Navier-Stokes-Boussinesq system in 3d channels, satisfying appropriate ‘‘thinness’’ condition on the crosssection Ω . The same argument is needed to obtain the uniform bounds for the stationary Navier-Stokes-Boussinesq system in Theorem 1.3 (ii).

On the other hand, by elliptic estimates and the Sobolev imbedding, it follows directly that:

$$(2.10) \quad \|u\|_\infty \leq C_\Omega \|u\|_{W^{2,2}(D)} \leq C_\Omega \left(\|\nabla u\|_{L^2(D)} + \frac{1}{\nu} \|g\|_{L^2(D)} \right).$$

In fact, already this inequality is sufficient for the estimates in case of the Stokes-Boussinesq system.

We will also need the following result, in the line of Lemma 3.6 from [10]:

Lemma 2.3. *For each t , there exists a function $h \in W_{loc}^{1,2}(D)$, such that:*

$$\|\theta(\cdot, t)\vec{\rho} - \nabla h\|_{L^2(D)} \leq C_\Omega \rho \|\nabla \theta(\cdot, t)\|_{L^2(D)},$$

with C_Ω depending only on Ω . In fact:

$$(2.11) \quad C_\Omega = C_{PW} + \left(\int_\Omega |\vec{g} \cdot (0, \tilde{x})|^2 d\tilde{x} \right)^{1/2},$$

where C_{PW} stands for the Poincare-Wirtinger constant of Ω .

Proof. Let e_1, e_2, e_3 be the standard basis for \mathbb{R}^3 . Suppressing the time variable t , we define:

$$h(x, \tilde{x}) = \vec{\rho} \cdot e_1 \int_0^x \int_\Omega \theta(x, \tilde{x}) d\tilde{x} dx + \vec{\rho} \cdot (0, \tilde{x}) \int_\Omega \theta(x, \tilde{x}) d\tilde{x}.$$

Therefore, the following identity concludes the proof of lemma:

$$\theta \vec{\rho} - \nabla h = \vec{\rho} \left(\theta(x, \tilde{x}) - \int_{\Omega} \theta(x, \tilde{x}) \, d\tilde{x} \right) - \vec{\rho} \cdot (0, \tilde{x}) \int_{\Omega} \theta_x(x, \tilde{x}) \, d\tilde{x} \, e_1,$$

by the Poincare-Wirtinger inequality: $\|\theta - f\theta\|_{L^2(\Omega)} \leq C_{PW} \|\nabla \theta\|_{L^2(\Omega)}$. ■

3. PROOFS OF THE MAIN RESULT

The following lemma has been proven in [2] for the case of 2d channels and vorticity of the flow u vanishing at ∂D . Exactly the same proof, relying on the construction of super and sub-solutions to (1.1) is valid also in the present 3d case. Since the argument uses the estimate in Lemma 2.1, we partially reproduce it below for the sake of completeness.

Lemma 3.1. *There exists a constant $C_{\Omega,0}$ depending on Ω , f , and on the initial data θ_0 , such that:*

$$(3.1) \quad \bar{N}(t) \leq \frac{1}{2} \bar{B}(t) + \bar{U}(t) + C_{\Omega,0} \left(\frac{1}{t} + \frac{1}{\sqrt{t}} \right) \quad \forall t \gg 1.$$

Moreover, there exists $x_0 > 0$ and $q \in L^1(\mathbb{R})$, such that for all sufficiently large $t \gg 1$:

$$(3.2) \quad \begin{aligned} \Phi \left(x - c_0 t + x_0 + \bar{U}(t)t + C_{\Omega,0} \sqrt{t} \right) - Q(x, \tilde{x}, t) &\leq \theta(x, \tilde{x}, t) \\ &\leq \Phi \left(x - c_0 t - x_0 - \bar{U}(t)t - C_{\Omega,0} \sqrt{t} \right) + Q(x, \tilde{x}, t), \end{aligned}$$

where Q is the solution to:

$$(3.3) \quad \begin{aligned} Q_t + u \cdot \nabla Q - \Delta Q &= 0 \quad \text{in } D, \\ \frac{\partial Q}{\partial \vec{n}} &= 0 \quad \text{on } \partial D, \quad Q(x, \tilde{x}, 0) = q(x). \end{aligned}$$

Proof. We only prove (3.2), since it implies (3.1) as in [2], Lemma 4.2. Define:

$$\psi_l(x, \tilde{x}, t) = \Phi \left(x - c_0 t + x_0 + \bar{U}(t)t + C\sqrt{t} \right) - Q(x, \tilde{x}, t),$$

where $x_0, C > 0$ are to be determined later, and Q is as in (3.3) with q appropriately chosen.

To prove that ψ is a subsolution, we first need to show the non-positivity of the following expression:

$$(3.4) \quad \begin{aligned} &(\psi_l)_t + u \cdot \nabla \psi_l - \Delta \psi_l - f(\psi_l) \\ &= \left(\|u(\cdot, t)\|_{L^\infty(D)} + \frac{C}{2\sqrt{t}} + u_1(x, \tilde{x}, t) \right) \Phi' \left(x - c_0 t + x_0 + \bar{U}(t)t + C\sqrt{t} \right) \\ &\quad + f(\Phi) - f(\Phi - Q) \\ &\leq \frac{C}{2\sqrt{t}} \Phi' + f(\Phi) - f(\Phi - Q). \end{aligned}$$

Take $q \in L^1(\mathbb{R})$ such that $0 \leq q(x) \leq \alpha = \min(\vartheta_0/2, (1 - \vartheta_0)/4)$. By the maximum principle we have:

$$(3.5) \quad 0 \leq Q(x, \tilde{x}, t) \leq \alpha.$$

Let now $x_0 > 0$ be such that:

$$(3.6) \quad \theta_0(x, \tilde{x}, 0) \geq \Phi(x + x_0) - q(x).$$

Finally, let C be large enough so that when $\Phi \in (\vartheta_0, 1 - (1 - \vartheta_0)/4)$ then:

$$\begin{aligned} \frac{C}{2\sqrt{t}}\Phi' + f(\Phi) - f(\Phi - Q) &\leq \frac{C}{2\sqrt{t}}\Phi' + \|f'\|_{L^\infty}\|Q(\cdot, t)\|_{L^\infty(D)} \\ &\leq -\frac{C}{2\sqrt{t}} \min_{\Phi(s) \in (\vartheta_0, (3+\vartheta_0)/4)} |\Phi'(s)| + \frac{C_\Omega|\Omega|}{\sqrt{t}}\|f'\|_{L^\infty}\|q\|_{L^1(\mathbb{R})} \leq 0, \end{aligned}$$

where we used Lemma 2.1 to estimate $\|Q\|_\infty$. On the other hand, for $\Phi \in (0, \vartheta_0) \cup (1 - (1 - \vartheta_0)/4, 1)$ the nonpositivity of the right hand side in (3.4) follows directly, via (3.5). Concluding, (3.4) and (3.6) imply the lower bound in (3.2).

The upper bound follows by assuring that:

$$\psi_r(x, \tilde{x}, t) = \Phi\left(x - c_0t - x_0 - \bar{U}(t)t - C\sqrt{t}\right) + Q(x, \tilde{x}, t)$$

is a supersolution. Similarly as above, this follows by choosing x_0 such that:

$$\theta_0(x, \tilde{x}, 0) \leq \Phi(x - x_0) + q(x)$$

in addition to (3.6), and having C large enough so that:

$$\begin{aligned} (\psi_r)_t + u \cdot \nabla \psi_r - \Delta \psi_r - f(\psi_r) &\geq -\frac{C}{2\sqrt{t}}\Phi' - \|f'\|_{L^\infty}\|Q(\cdot, t)\|_{L^\infty(D)} \\ &\geq \frac{C}{2\sqrt{t}} \min_{\Phi(s) \in (\vartheta_0/2, (1+\vartheta_0)/2)} |\Phi'(s)| - \frac{C_\Omega|\Omega|}{\sqrt{t}}\|f'\|_{L^\infty}\|q\|_{L^1(\mathbb{R})} \geq 0, \end{aligned}$$

when $\Phi \in (\vartheta_0/2, 1 - (1 - \vartheta_0)/2)$. ■

Clearly, Theorem 1.2 follows from (3.2) and Lemma 2.1. We now have:

Lemma 3.2. *There exists a constant $C_{\Omega,0}$ depending on Ω , f , and on the initial data θ_0 , such that for all sufficiently large $t \gg 1$:*

$$(3.7) \quad \bar{B}(t) \leq c_0 + \bar{U}(t) + C_{\Omega,0} \left(\frac{1}{t} + \frac{1}{\sqrt{t}} \right),$$

and

$$(3.8) \quad \bar{B}(t) \geq c_0 - \bar{U}(t) - C_{\Omega,0} \left(\frac{1}{t} + \frac{1}{\sqrt{t}} \right).$$

Proof. Denoting $\xi(t) = \bar{U}(t)t + C_{\Omega,0}\sqrt{t}$ and using the bound (3.2), we obtain:

$$\begin{aligned}
(3.9) \quad \bar{B}(t) &= \frac{1}{|\Omega|t} \int_0^t \int_D f(\theta) = \frac{1}{|\Omega|t} \int_0^t \int_D \theta_t \, dx \, d\tilde{x} \\
&= \frac{1}{|\Omega|t} \int_D \theta(x, \tilde{x}, t) - \theta(x, \tilde{x}, 0) \, dx \, d\tilde{x} \\
&\leq \frac{1}{|\Omega|t} \int_D \Phi(x - c_0t - x_0 - \xi(t)) + Q(x, \tilde{x}, t) - \Phi(x + x_0) + Q(x, \tilde{x}, 0) \, dx \, d\tilde{x}.
\end{aligned}$$

By (2.6) and the construction of the corrector Q , it follows that:

$$(3.10) \quad \frac{1}{|\Omega|t} \int_D |Q(x, \tilde{x}, t) + Q(x, \tilde{x}, 0)| \, dx \, d\tilde{x} \leq \frac{2}{t} \|q\|_{L^1(\mathbb{R})} \leq \frac{C_{\Omega,0}}{t}.$$

Further:

$$\begin{aligned}
(3.11) \quad &\frac{1}{|\Omega|t} \int_D \Phi(x - c_0t - x_0 - \xi(t)) - \Phi(x + x_0) \\
&= \frac{1}{t} \left[\int_{-\infty}^{c_0t+x_0+\xi(t)} (\Phi(x - c_0t - x_0 - \xi(t)) - 1) \right. \\
&\quad + \int_{-\infty}^{-x_0} (1 - \Phi(x + x_0)) + \int_{-x_0}^{c_0t+x_0+\xi(t)} (1 - \Phi(x + x_0)) \\
&\quad + \int_{c_0t+x_0+\xi(t)}^{\infty} \Phi(x - c_0t - x_0 - \xi(t)) \\
&\quad \left. - \int_{c_0t+x_0+\xi(t)}^{-x_0} \Phi(x + x_0) - \int_{-x_0}^{\infty} \Phi(x + x_0) \right] \\
&= \frac{1}{t} \int_{-x_0}^{c_0t+x_0+\xi(t)} 1 \, ds = \frac{|c_0t + 2x_0 + \xi(t)|}{t} \\
&\leq c_0 + \bar{U}(t) + C_{\Omega,0} \left(\frac{1}{t} + \frac{1}{\sqrt{t}} \right),
\end{aligned}$$

which together with (3.10) proves (3.7). To prove (3.8), similarly as in (3.9) we note that:

$$\bar{B}(t) \geq \frac{1}{|\Omega|t} \int_D \Phi(x - c_0t + x_0 + \xi(t)) - Q(x, \tilde{x}, t) - \Phi(x - x_0) - Q(x, \tilde{x}, 0) \, dx \, d\tilde{x},$$

and as in (3.11) we obtain:

$$\begin{aligned}
&\frac{1}{|\Omega|t} \int_D \Phi(x - c_0t + x_0 + \xi(t)) - \Phi(x - x_0) = \frac{|c_0t - 2x_0 - \xi(t)|}{t} \\
&\geq c_0 - \bar{U}(t) - C_{\Omega,0} \left(\frac{1}{t} + \frac{1}{\sqrt{t}} \right).
\end{aligned}$$

Together with (3.10) the above implies (3.8). ■

Proof of Theorem 1.1.

Multiplying the fluid equation (1.2) by u and integrating over D , one obtains:

$$(3.12) \quad \frac{d}{dt} \|u\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 \leq C_\Omega \frac{\rho^2}{\nu} \|\nabla \theta\|_{L^2}^2.$$

Integrating (1.2) against u_t gives, in turn:

$$\|u_t\|_{L^2}^2 + \nu \frac{d}{dt} \|\nabla u\|_{L^2}^2 \leq C_\Omega \rho^2 \|\nabla \theta\|_{L^2}^2,$$

where in both inequalities above we used Lemma 2.3 to “replace” the term $\theta \bar{\rho}$ by $\rho \nabla \theta$, Taking averages in time, we get:

$$(3.13) \quad \begin{aligned} \frac{1}{t} \int_0^t \|\nabla u\|_{L^2}^2 dt &\leq C_\Omega \frac{\rho^2}{\nu^2} N(t) + \frac{1}{\nu t} \|\nabla u_0\|_{L^2}^2, \\ \frac{1}{t} \int_0^t \|u_t\|_{L^2}^2 dt &\leq C_\Omega \rho^2 N(t) + \frac{\nu}{t} \|\nabla u_0\|_{L^2}^2. \end{aligned}$$

By Lemma 2.2 or (2.10), and Lemma 2.3 it now follows that:

$$\|u\|_\infty \leq C_\Omega \left(\|\nabla u\|_{L^2} + \frac{1}{\nu} \|u_t\|_{L^2} + \frac{\rho}{\nu} \|\nabla \theta\|_{L^2} \right).$$

Using (3.13) we obtain:

$$(3.14) \quad \begin{aligned} \bar{U}(t) &\leq C_\Omega \left(\left(\frac{1}{t} \int_0^t \|\nabla u\|_{L^2}^2 \right)^{1/2} + \frac{1}{\nu} \left(\frac{1}{t} \int_0^t \|u_t\|_{L^2}^2 \right)^{1/2} + \frac{\rho}{\nu} \sqrt{N(t)} \right) \\ &\leq C_\Omega \left(\frac{\rho}{\nu} \sqrt{N(t)} + \frac{1}{\sqrt{\nu t}} \|\nabla u_0\|_{L^2} + \frac{1}{\sqrt{t}} \|\nabla u_0\|_{L^2} \right). \end{aligned}$$

By (3.1) and (3.7) we hence get, for large $t \gg 1$:

$$(3.15) \quad \begin{aligned} \bar{N}(t) &\leq \frac{1}{2} c_0 + \frac{3}{2} \bar{U}(t) + C_{\Omega,0} \left(\frac{1}{t} + \frac{1}{\sqrt{t}} \right) \\ &\leq \frac{1}{2} c_0 + C_\Omega \frac{\rho}{\nu} \sqrt{N(t)} + C_{all} \left(\frac{1}{t} + \frac{1}{\sqrt{t}} \right). \end{aligned}$$

where C_{all} depends on Ω, f, ν, θ_0 and u_0 . Consequently:

$$\begin{aligned} \bar{N}(t) &\leq \left(C_\Omega \frac{\rho}{\nu} + \sqrt{\frac{1}{2} c_0 + C_\Omega \frac{\rho^2}{\nu^2}} + C_{all} \sqrt{\frac{1}{t} + \frac{1}{\sqrt{t}}} \right)^2 \\ &\leq \left(C_\Omega \frac{\rho}{\nu} + \sqrt{\frac{1}{2} c_0 + C_\Omega \frac{\rho}{\nu}} \right)^2 + C_{all} \left(\frac{1}{t} + \frac{1}{\sqrt{t}} \right), \end{aligned}$$

and, returning to (3.14):

$$\bar{U}(t) \leq C_\Omega \left(\frac{\rho^2}{\nu^2} + \frac{\rho}{\nu} \sqrt{\frac{1}{2} c_0 + C_\Omega \frac{\rho}{\nu}} \right) + C_{all} \left(\frac{1}{t} + \frac{1}{\sqrt{t}} \right).$$

In view of Lemma 3.2, Theorem 1.1 is hence proven. ■

Proof of Theorem 1.3.

We only prove the assertion (ii), because the 2d case in (i) follows with exactly the same calculations as in Theorem 1.1 and Theorem 1.2 .

For the stationary Navier-Stokes-Boussinesq system, using the Poincaré inequality and (2.11), we obtain the following counterpart of (3.12):

$$(3.16) \quad \|\nabla u\|_{L^2(D)} \leq \frac{\rho}{\nu} C_P \left(C_{PW} + \left(\int |\vec{g} \cdot (0, \tilde{x})|^2 d\tilde{x} \right)^{1/2} \right) \|\nabla \theta\|_{L^2(D)},$$

where C_P and C_{PW} are, respectively, the Poincaré and the Poincaré-Wirtinger constants of Ω . By Lemma 2.1 and arguing as (3.14), we arrive at:

$$\bar{U}(t) \leq \frac{\rho^2 |\Omega|}{\nu^3 2\pi} C_P^2 \left(C_{PW} + \left(\int |\vec{g} \cdot (0, \tilde{x})|^2 d\tilde{x} \right)^{1/2} \right)^2 \bar{N}(t) + C_\Omega \frac{\rho}{\nu} \sqrt{\bar{N}(t)},$$

while the counterpart of (3.15) in the present case is:

$$\begin{aligned} \bar{N}(t) \leq & \frac{1}{2} c_0 + C_\Omega \frac{\rho}{\nu} \sqrt{\bar{N}(t)} + C_{\Omega,0} \left(\frac{1}{t} + \frac{1}{\sqrt{t}} \right) \\ & + \frac{3|\Omega| \rho^2}{4\pi \nu^3} C_P^2 \left(C_{PW} + \left(\int |\vec{g} \cdot (0, \tilde{x})|^2 d\tilde{x} \right)^{1/2} \right)^2 \bar{N}(t) \quad \forall t \gg 1. \end{aligned}$$

It is therefore clear that when the constant in front of $\bar{N}(t)$ in the last term of the right hand side above is smaller than 1, the results of Theorem 1.1 and Theorem 1.2 follow as in the case of the Stokes-Boussinesq system. ■

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