VARIATIONAL MODELS FOR PRESTRAINED PLATES WITH MONGE-AMPÈRE CONSTRAINT

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Abstract. We derive a new variational model in the description of prestrained elastic thin films. The model consists of minimizing a biharmonic energy of the out-of plane displacements $v \in W^{2,2}(\Omega,\mathbb{R})$, satisfying the Monge-Ampère constraint $\det \nabla^2 v = f$. Here, $f = -\mathrm{curl}^T\mathrm{curl}(S_g)_{2\times 2}$ is the linearized Gauss curvature of the incompatibility (prestrain) family of Riemannian metrics $G^h = \mathrm{Id}_3 + 2h^\gamma S_g + h.o.t.$ imposed on the referential configurations of the thin films with midplate Ω and small thickness h. We further discuss multiplicity properties of the minimizers of this model in some special cases.

1. Introduction of the problem and the main results

Materials which assume non-trivial rest configurations in the absence of exterior forces or boundary conditions arise in various contexts, such as: morphogenesis by growth, swelling or shrinkage, torn plastic sheets, engineered polymer gels, to mention a few. This paper is a continuation of the analysis initiated in [21, 16, 17, 1, 22] and it regards the derivation of the dimensionally-reduced models in the description of prestrained thin films, and of the related residual energy scaling laws. Below we first briefly remind the mathematical setting of the problem and then we present the main results of this paper. These are: (i) the derivation of the variational model for the linearized Kirchhoff-like energy subject to Monge-Ampère constraint, (ii) the derivation of the matching property for the continuation of infinitesimal isometries to exact isometries, valid for metrics with positive Gauss curvature, and (iii) a study of uniqueness/multiplicity of the minimizers to the derived model in the rotationally symmetric case.

1.1. **The set-up and the non-Euclidean elasticity model.** Let Ω be an open bounded subset of $\mathbb{R}^2$. Consider a family of 3d plates:

$$\Omega^h = \Omega \times (-h/2, h/2), \qquad 0 < h << 1,$$

viewed as the reference configurations of thin elastic tissues. A typical point in Ω^h is denoted by $x = (x', x_3)$ where $x' \in \Omega$ and $|x_3| < h/2$. Each Ω^h is assumed to undergo an activation process, whose instantaneous growth is described by a smooth, invertible tensor:

$$A^h = [A^h_{ij}] : \overline{\Omega^h} \to \mathbb{R}^{3\times 3} \quad \text{ with: } \det A^h(x) > 0.$$

The multiplicative decomposition model [30, 21, 4, 12] in the description of shape formation due to the prestrain, relies on the assumption that for a deformation $u^h : \Omega^h \to \mathbb{R}^3$, its elastic energy $I^h_W(u^h)$ is written in terms of the elastic tensor $F = \nabla u^h (A^h)^{-1}$ accounting for the reorganization of the body Ω^h in response to A^h. That is, we write:

$$\nabla u^h = F A^h,$$

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and define:

$$I_W^h(u^h) = \frac{1}{h}\int_{\Omega^h} W(F)\,\mathrm{d}x = \frac{1}{h}\int_{\Omega^h} W(\nabla u^h (A^h)^{-1})\,\mathrm{d}x \qquad \forall u^h \in W^{1,2}(\Omega^h,\mathbb{R}^3). \tag{1.1}$$

The elastic energy density $W : \mathbb{R}^{3\times 3} \to \mathbb{R}_+$ is assumed to satisfy the standard [2, 6] conditions of normalization, frame indifference (with respect to the special orthogonal group $SO(3)$ of proper rotations in $\mathbb{R}^3$), and second order nondegeneracy:

$$\begin{aligned} \forall F \in \mathbb{R}^{3\times 3} \quad \forall R \in SO(3) \qquad & W(R) = 0, \quad W(RF) = W(F) \\ & W(F) \geq c\ \mathrm{dist}^2(F, SO(3)), \end{aligned} \tag{1.2}$$

for a constant $c > 0$. We also assume that there exists a monotone nonnegative function $\omega : [0,+\infty] \to [0,+\infty]$ which converges to zero at 0, and a quadratic form $\mathcal{Q}_3$ on $\mathbb{R}^{3\times 3}$, with:

$$\forall F \in \mathbb{R}^{3\times 3} \qquad |W(\mathrm{Id} + F) - \mathcal{Q}_3(F)| \leq \omega(|F|)|F|^2. \tag{1.3}$$

This condition is satisfied in particular if W is $\mathcal{C}^2$ regular in a neighborhood of $SO(3)$, wheras $\mathcal{Q}_3 = \frac{1}{2}D^2W(\mathrm{Id})$. Also, note that (1.2) implies that $\mathcal{Q}_3$ is nonnegative, is positive definite on symmetric matrices and $\mathcal{Q}_3(F) = \mathcal{Q}_3(\mathrm{sym}\, F)$ for all $F \in \mathbb{R}^{3\times 3}$ (see Lemma 2.2 for a proof of these standard observations).

The model (1.1) has been extensively studied in [21, 16, 17, 24, 25, 26, 3, 13, 14]. Recall (which is quite easy to check) that $I_W^h(u^h) = 0$ is equivalent, via (1.2) and the polar decomposition theorem, to:

$$(\nabla u^h)^T \nabla u^h = (A^h)^T(A^h) \quad \text{and} \quad \det \nabla u^h > 0 \quad \text{in } \Omega^h. \tag{1.4}$$

The above can be interpreted in the following way: $I_W^h(u^h) = 0$ if and only if u^h is an isometric immersion of the Riemannian metric $G^h = (A^h)^T(A^h)$. Therefore, the quantity:

$$e_h = \inf\left\{ I_W^h(u^h);\ u^h \in W^{1,2}(\Omega^h,\mathbb{R}^3)\right\} \tag{1.5}$$

measures the residual energy at free equilibria of the configuration Ω^h that has been prestrained by G^h. This is consistent with [21, Theorem 2.2], which observes that $e_h > 0$ whenever G^h has no smooth isometric immersion in $\mathbb{R}^3$, i.e. when there is no u^h with (1.4) or, equivalently, when the Riemann curvature tensor of the metric G^h does not vanish identically on Ω^h.

1.2. **Growth tensors A^h considered in this paper.** Given now a sequence of growth tensors A^h, the main objective is to analyze the scaling of the residual energy in (1.5) in terms of the thickness h, and the asymptotic behavior of the minimizers of the energies I_W^h as $h \to 0$.

Note that when $A^h \equiv \mathrm{Id}_3$, the model (1.1) reduces to the classical nonlinear elasticity, and it is augmented by the applied force term $\int_{\Omega^h} f^h u^h$. In this context, questions of dimension reduction through Γ-convergence have been studied in the seminal papers [15, 5, 6] and led to the rigorous derivation of the hierarchy of elastic 2d models, differentiated by the scaling of f^h (see [23] for a more complete review of the literature). In this paper, we will be concerned with growth tensors A^h which bifurcate from the Euclidean case $A = \mathrm{Id}_3$, and are of the form:

$$A^h(x', x_3) = \mathrm{Id}_3 + h^\gamma S_g(x') + h^{\gamma/2} x_3 B_g(x'). \tag{1.6}$$

The "stretching" and "bending" tensors $S_g, B_g : \overline{\Omega} \to \mathbb{R}^{3\times 3}$ are two given smooth matrix fields, while the scaling exponent γ belongs to the range:

$$0 < \gamma < 2.$$

The critical cases $\gamma = 0, 2$ have been analyzed previously, and they led to the fully nonlinear bending models in [21, 1] for $\gamma = 0$, and to the von Kármán-like morphogenesis models [16, 17, 23] for $\gamma = 2$ (or $\gamma = 0$ under partial curvature vanishing conditions).

Observe now that A^h in (1.6) yields:

$$G^h(x', x_3) = (A^h)^T(A^h) = \mathrm{Id}_3 + 2h^\gamma\ \mathrm{sym}S_g(x') + 2h^{\gamma/2}x_3\mathrm{sym}B_g(x') + \text{ higher order terms.}$$

Interpreting the term $p_h = \mathrm{Id}_2 + 2h^\gamma(\mathrm{sym}S_g)_{2\times2}$ as the first fundamental form of the mid-plate Ω, and $h^{\gamma/2}(\mathrm{sym}B_g)_{2\times2}$ as its second fundamental form, the compatibility of these forms through the Gauss-Codazzi equations at the leading order terms in the expansion in h, is expressed by the following conditions:

$$\text{(1.7)} \qquad \mathrm{curl}\big((\mathrm{sym}\ B_g)_{2\times2}\big) \equiv 0 \quad \text{and} \quad \mathrm{curl}^T\mathrm{curl}\ (S_g)_{2\times2} + \det\big((\mathrm{sym}\ B_g)_{2\times2}\big) \equiv 0 \text{ in } \Omega,$$

Hence, if (1.7) is violated, then any isometric immersion $u_h : \Omega \to \mathbb{R}^3$ of p_h will have the second fundamental form: $h^{\gamma/2}\Pi \neq h^{\gamma/2}\mathrm{sym}B_g$. Expanding the energy of the deformation:

$$\text{(1.8)} \qquad u^h(x', x_3) = u_h(x') + x_3N^h(x'), \qquad N^h(x') = \frac{\partial_1 u_h \times \partial_2 u_h}{|\partial_1 u_h \times \partial_2 u_h|}$$

(which is the Kirchhoff-Love extension of u_h in the direction of the normal vector N^h to the surface $u_h(\Omega)$), and gathering the remaining terms after the cancellation of p_h, we obtain:

$$I^h_W(u^h) \approx \frac{1}{h}\int_{\Omega^h} |(\nabla u^h)^T(\nabla u^h) - G^h|^2\,\mathrm{d}x \approx \frac{1}{h}\int_{\Omega^h} |2h^{\gamma/2}x_3\big((\mathrm{sym}B_g(x'))_{2\times2} - \Pi(x')\big)|^2\,\mathrm{d}x \approx Ch^{\gamma+2}.$$

As we shall see, the scaling $h^{\gamma+2}$ above is sharp, and the residual 2d energy is indeed given in terms of the square of difference in the scaled second fundamental forms: $|(\mathrm{sym}B_g)_{2\times2} - \Pi|^2$. We state our main results in the next subsections.

1.3. The variational limit with Monge-Ampère constraint: case of $1 < \gamma < 2$.

The main result of this paper is the identification of the asymptotic behavior of the minimizers of I^h_W as $h \to 0$, through deriving the Γ-limit of the rescaled energies $h^{-(\gamma+2)}I^h_W$. This limit, given in the Theorem below, consists of minimizing the bending content, relative to the ideal bending $(\mathrm{sym}B_g(x'))_{2\times2}$, under the nonlinear constraint of the form $\det\nabla^2 v = f$. Our result, which concerns arbitrary functions f, is a generalization to the non-Euclidean setting of [6, Theorem 2], where the degenerate Monge-Ampère type constraint ($f \equiv 0$) was rigorously derived in the context of standard nonlinear elasticity.

Theorem 1.1. *Let A^h be given as in (1.6), with an arbitrary exponent γ in the range:*

$$0 < \gamma < 2.$$

Assume that a sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ satisfies:

$$\text{(1.9)} \qquad I^h_W(u^h) \leq Ch^{\gamma+2},$$

where W fulfills (1.2) and (1.3). Then there exist rotations $\bar{R}^h \in SO(3)$ and translations $c^h \in \mathbb{R}^3$ such that for the normalized deformations:

$$y^h \in W^{1,2}(\Omega^1, \mathbb{R}^3), \qquad y^h(x', x_3) = (\bar{R}^h)^T u^h(x', hx_3) - c^h,$$

the following holds (up to a subsequence that we do not relabel):

(i) *$y^h(x', x_3)$ converge in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x'.*

(ii) *The scaled displacements:* $V^h(x') = \frac{1}{h^{\gamma/2}} \int_{-1/2}^{1/2} y^h(x',t) - x' \,\mathrm{d}t$ *converge to a vector field* V *of the form* $V = (0,0,v)^T$. *This convergence is strong in* $W^{1,2}(\Omega, \mathbb{R}^3)$. *The only non-zero out-of-plane scalar component* v *of* V *satisfies:* $v \in W^{2,2}(\Omega, \mathbb{R})$ *and:*

$$\det \nabla^2 v = -\mathrm{curl}^T \mathrm{curl}\ (S_g)_{2\times 2} \quad \text{in } \Omega. \tag{1.10}$$

In other words: $v \in \mathcal{A}_f$, *where:*

$$\mathcal{A}_f = \Big\{ v \in W^{2,2}(\Omega);\ \det \nabla^2 v = f \Big\} \qquad \text{and} \qquad f = -\mathrm{curl}^T \mathrm{curl}\ (S_g)_{2\times 2}.$$

(iii) *Moreover:*

$$\liminf_{h\to 0} \frac{1}{h^{\gamma+2}} I_W^h(u^h) \geq \mathcal{I}_f(v), \tag{1.11}$$

where $\mathcal{I}_f : W^{2,2}(\Omega) \to \bar{\mathbb{R}}_+$ *is given by:*

$$\mathcal{I}_f(v) = \left\{ \begin{array}{ll} \frac{1}{12}\int_\Omega \mathcal{Q}_2\Big(\nabla^2 v + (\mathrm{sym}\ B_g)_{2\times 2}\Big), & \text{if } v \in \mathcal{A}_f, \\ +\infty & \text{if } v \notin \mathcal{A}_f \end{array} \right. \tag{1.12}$$

and the quadratic nondegenerate form $\mathcal{Q}_2$, *acting on matrices* $F \in \mathbb{R}^{2\times 2}$ *is:*

$$\mathcal{Q}_2(F) = \min \Big\{ \mathcal{Q}_3(\tilde{F});\ \tilde{F} \in \mathbb{R}^{3\times 3}, \tilde{F}_{2\times 2} = F \Big\}. \tag{1.13}$$

The result above can be interpreted as follows. The smallness of the energy scaling in (1.9) relative to the scaling in (1.6), induces the deformations $u_h(x') = u^h(x',0)$ of the mid-plate Ω to be perturbations of a rigid motion:

$$u_h(x') = x' + h^{\gamma/2} v(x') e_3 + \text{higher order terms}. \tag{1.14}$$

Moreover, the Gaussian curvatures κ of the metric $p_h = \mathrm{Id}_2 + 2h^\gamma (\mathrm{sym} S_g)_{2\times 2}$ and of the surface $u_h(\Omega)$ coincide at their highest order in the expansion in terms of h. This is precisely the meaning of the constraint (1.10), in view of the formulas:

$$\begin{aligned} &\kappa\big(\mathrm{Id}_2 + 2\epsilon^2 (\mathrm{sym}\ S_g)_{2\times 2}\big) = -\epsilon^2 \mathrm{curl}^T \mathrm{curl}\ (S_g)_{2\times 2} + \mathcal{O}(\epsilon^4) \\ &\kappa\big(\nabla(\mathrm{id}_2 + \epsilon v e_3)^T \nabla(\mathrm{id}_2 + \epsilon v e_3)\big) = \epsilon^2 \det \nabla^2 v + \mathcal{O}(\epsilon^4). \end{aligned} \tag{1.15}$$

All other curvatures, besides κ, contribute to the limiting energy $\mathcal{I}_f$. Indeed, $\mathcal{I}_f$ measures the L^2 difference between the full second fundamental forms: the form $h^{\gamma/2}(\mathrm{sym}\ B_g)_{2\times 2}$ deduced from A^h, and that of the surface $u_h(\Omega)$ given by:

$$(\nabla u_h)^T \nabla N^h = -h^{\gamma/2} \nabla^2 v + \text{ higher order terms.}$$

We now turn to the optimality of the energy bound in (1.11) and of the scaling (1.9).

Theorem 1.2. *Assume (1.6), (1.2) and (1.3). Moreover, assume that* Ω *is simply connected and:*

$$1 < \gamma < 2.$$

Then, for every $v \in \mathcal{A}_f$, *there exists a sequence of deformations* $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ *such that the following holds:*

(i) *The sequence* $y^h(x', x_3) = u^h(x', hx_3)$ *converges in* $W^{1,2}(\Omega^1, \mathbb{R}^3)$ *to* x'.

(ii) $V^h(x') = h^{-\gamma/2} \fint_{-h/2}^{h/2} (u^h(x',t) - x') \,\mathrm{d}t$ *converge in* $W^{1,2}(\Omega, \mathbb{R}^3)$ *to* $(0,0,v)^T$.

(iii) *One has:* $\lim_{h\to 0} \frac{1}{h^{\gamma+2}} I_W^h(u^h) = \mathcal{I}_f(v)$, *where* $\mathcal{I}_f$ *is as in (1.12).*

Theorem 1.3. *Assume (1.6), (1.2), (1.3). Let Ω be simply connected and let $1 < \gamma < 2$. Then:*

(i) *$\mathcal{A}_f \neq \emptyset$ if and only if there exists a uniform constant $C \geq 0$ such that:*

$$e_h = \inf I_W^h \leq Ch^{\gamma+2}.$$

Under this condition, for any minimizing sequence $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ for I_W^h, i.e. when:

$$\lim_{h\to 0} \frac{1}{h^{\gamma+2}} \Big(I_W^h(u^h) - \inf I_W^h \Big) = 0, \tag{1.16}$$

the convergences (i), (ii) of Theorem 1.1 hold up to a subsequence, and the limit v is a minimizer of the functional $\mathcal{I}_f$ defined as in (1.12).

Moreover, for any (global) minimizer v of $\mathcal{I}_f$, there exists a minimizing sequence u^h, satisfying (1.16) together with (i), (ii) and (iii) of Theorem 1.2.

(ii) *If (1.7) is violated, i.e. when:*

$$\operatorname{curl}\big((\operatorname{sym} B_g)_{2\times 2}\big) \not\equiv 0, \quad \textit{or} \quad \operatorname{curl}^T\operatorname{curl}\,(S_g)_{2\times 2} + \det\big((\operatorname{sym} B_g)_{2\times 2}\big) \not\equiv 0, \tag{1.17}$$

then:

$$\exists c > 0 \qquad \inf I_W^h \geq ch^{\gamma+2}.$$

The conditions in (1.17) guarantee that the highest order terms in the expansion of the Riemann curvature tensor components R_{1213}, R_{2321} and R_{1212} of $G^h = (A^h)^T A^h$ do not vanish. Also, vanishing of either of them implies that $\inf \mathcal{I}_f > 0$ (see Lemma 3.2), which combined with Theorem 1.1 yields the lower bound on $\inf I_W^h$. The mechanical significance of these components of the curvature tensor is not known to the authors, but it seems that certain components have a more important role in determining the energy scaling; compare with [1, Theorems 5.1, 5.3, 5.5].

The scaling analysis in Theorem 1.3 is new, and in particular it does not follow from our prior results in [16], valid for another family of growth tensors A^h than (1.6). In a sense, the scaling exponents γ, $\gamma/2$ and $\gamma + 2$ pertain to the critical case in [16, Theorem 1.1], and thus the results in Theorem 1.3 and Theorem 1.1 are also optimal from this point of view.

1.4. **The matching property: a full range case of $0 < \gamma < 2$.** It is clear from Theorem 1.1 that the recovery sequence u^h in Theorem 1.2 will have the form (1.8), with u_h as in (1.14). We can write this expansion with more precision, including a higher order correction $w_h : \Omega \to \mathbb{R}^3$:

$$u_h(x') = x' + h^{\gamma/2} v(x') e_3 + h^\gamma w_h + \text{higher order terms}. \tag{1.18}$$

In order to match the ideal metric $p_h = \mathrm{Id}_2 + 2h^\gamma (\operatorname{sym} S_g)_{2\times 2}$ with the metric induced by u_h:

$$(\nabla u_h)^T (\nabla u_h) = \mathrm{Id}_2 + 2h^\gamma \Big(\frac{1}{2} \nabla v \otimes \nabla v + \operatorname{sym}\nabla w_h \Big) + \mathcal{O}(h^{3\gamma/2}), \tag{1.19}$$

one hence needs that:

$$-\operatorname{sym}\nabla w_h = \frac{1}{2}\nabla v \otimes \nabla v - (\operatorname{sym} S_g)_{2\times 2}. \tag{1.20}$$

On a simply connected domain Ω, equation (1.20) is solvable in terms of w_h if and only if the tensor in its right hand side belongs to the kernel of the operator $\operatorname{curl}^T\operatorname{curl}$, which becomes:

$$0 = \operatorname{curl}^T\operatorname{curl}\Big(\frac{1}{2}\nabla v \otimes \nabla v - (\operatorname{sym} S_g)_{2\times 2} \Big) = -\det \nabla^2 v - \operatorname{curl}^T\operatorname{curl}(S_g)_{2\times 2},$$

and is readily satisfied in view of (1.10). It follows from careful calculations in the proof of Theorem 1.2 that the constraint (1.10) allows precisely for the existence of a correction w_h in (1.18) so that the discrepancy of the metrics in p_h and (1.19) does not exceed the residual energy bound (1.9), when γ is in the range $1 < \gamma < 2$. In order to cover a larger range of γ, one needs hence to "improve" the recovery sequence (1.18) towards matching the metrics in (1.19) and the metrics $G^h(\cdot, x_3 = 0) = p_h +$ higher order terms, with a better accuracy. This is the content of our next result (see [6, Theorem 7] for a parallel result valid in the degenerate case $S_g \equiv 0$).

Theorem 1.4. *Assume that Ω is simply connected and that $-\mathrm{curl}^T\mathrm{curl}(S_g)_{2\times 2} \geq c > 0$ in Ω. For $0 < \beta < 1$ let $v \in \mathcal{C}^{2,\beta}(\bar\Omega, \mathbb{R})$ satisfy:*

$$\det \nabla^2 v = -\mathrm{curl}^T\mathrm{curl}(S_g)_{2\times 2} \qquad in\ \Omega.$$

Let $s_\epsilon : \Omega \to \mathbb{R}^{2\times 2}_{sym}$ be a given sequence of smooth symmetric tensor fields, such that: $\sup \|s_\epsilon\|_{\mathcal{C}^{1,\beta}} < +\infty$*. Then there exists a sequence $w_\epsilon \in \mathcal{C}^{2,\beta}(\bar\Omega, \mathbb{R}^3)$, such that:*

$$(1.21) \qquad \forall \epsilon > 0 \quad \nabla(\mathrm{id}_2 + \epsilon v e_3 + \epsilon^2 w_\epsilon)^T \nabla(\mathrm{id}_2 + \epsilon v e_3 + \epsilon^2 w_\epsilon) = \mathrm{Id}_2 + 2\epsilon^2(\mathrm{sym}\ S_g)_{2\times 2} + \epsilon^3 s_\epsilon,$$

and: $\sup \|w_\epsilon\|_{\mathcal{C}^{2,\beta}} < +\infty$.

The applicability of Theorem 1.4 is limited by the strong assumption of Hölder regularity in $v \in \mathcal{C}^{2,\beta}(\bar\Omega, \mathbb{R})$. Clearly, it is too restrictive for constructing a recovery sequence when $v \in W^{2,2}(\Omega)$. However, when the $\mathcal{C}^{2,\beta}(\bar\Omega, \mathbb{R})$ solutions v of (1.10) are dense in the set of all $W^{2,2}(\Omega, \mathbb{R})$ solutions of the same equation, with respect to the $W^{2,2}$ topology, then one can use a diagonal argument. As shown in [18], the mentioned density property holds for star-shaped domains with a constant positive linearized curvature constraint, and consequently we obtain:

Theorem 1.5. *Assume (1.6), (1.2) and (1.3). Moreover, assume that Ω is star-shaped with respect to a ball, and that $f = -\mathrm{curl}^T\mathrm{curl}(S_g)_{2\times 2} \equiv c_0 > 0$ in Ω. Let:*

$$0 < \gamma < 2.$$

Then, for every $v \in \mathcal{A}_f$, there exists a sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ such that (i), (ii) and (iii) of Theorem 1.2 hold. Moreover, all the assertions of Theorem 1.3 hold as well.

1.5. **On the multiplicity of solutions to the limit model.** Our final set of results concerns the question of uniqueness of the minimizers to the model (1.12). We first observe that both uniqueness and existence of a one-parameter family of global minimizers are possible (see Example 5.1 and Example 5.2). Naturally, for the radial function $f = f(r) \geq 0$, uniqueness is tied to the radial symmetry of minimizers. One approach is to study the relaxed problem, and replace the constraint set $\mathcal{A}_f$ by $\mathcal{A}_f^* = \{v \in W^{2,2}(\Omega);\ \det \nabla^2 v \geq f\}$.

In particular, as a corollary to Theorem 6.4 and Corollary 6.2 we obtain the following result:

Theorem 1.6. *Assume that $f \in L^2(B(0,1))$ is radially symmetric i.e.: $f = f(r)$ and $\int_0^1 r f^2(r)\,\mathrm{d}r < \infty$. Assume further that $f \geq c > 0$, and that f is a.e. nonincreasing, i.e.:*

$$(1.22) \qquad \forall a.e.\ r \in [0,1] \quad \forall a.e.\ x \in [0,r] \qquad f(r) \leq f(x).$$

Then the functional $\mathcal{I}(v) = \int_{B(0,1)} |\nabla^2 v|^2$, restricted to the constraint set $\mathcal{A}_f$, has a unique (up to an affine map) minimizer, which is radially symmetric and given by v_f in (6.3).

It is unclear to the authors whether the above theorem holds for every positive f. However, we can establish that the radial solution to the constraint equation is always a critical point. More precisely, we have the following result:

Theorem 1.7. *Assume that $f \in \mathcal{C}^\infty(\bar B(0,1))$ is radially symmetric i.e. $f = f(r)$, and that $f \geq c > 0$. Then the radially symmetric $v = v(r) \in \mathcal{A}_f$ must be a critical point of the functional $\mathcal{I}(v) = \int_{B(0,1)} |\nabla^2 v|^2$, restricted to the constraint set $\mathcal{A}_f$.*

1.6. **Notation.** Throughout the paper, we use the following notational convention. For a matrix F, its $n\times m$ principal minor is denoted by $F_{n\times m}$. When $m = n$ then the symmetric part of a square matrix F is: sym $F = 1/2(F + F^T)$. The superscript ^{T} refers to the transpose of a matrix or an operator. The operator $\mathrm{curl}^T\mathrm{curl}$ acts on 2×2 square matrix fields F by taking first curl of each row (returning 2 scalars) and then taking curl of the resulting 2d vector, so that: $\mathrm{curl}^T\mathrm{curl}F = \partial^2_{11}F_{22} - \partial^2_{12}(F_{12} + F_{21}) + \partial^2_{22}F_{11}$. In particular, we see that: $\mathrm{curl}^T\mathrm{curl}\ F = \mathrm{curl}^T\mathrm{curl}(\mathrm{sym}\ F)$.

Further, for any $F \in \mathbb{R}^{2\times 2}$, by $F^* \in \mathbb{R}^{3\times 3}$ we denote the matrix for which $(F^*)_{2\times 2} = F$ and $(F^*)_{i3} = (F^*)_{3i} = 0$, $i = 1, \dots, 3$. By ∇_{tan} we denote taking derivatives ∂_1 and ∂_2 in the in-plate directions $e_1 = (1, 0, 0)^T$ and $e_2 = (0, 1, 0)^T$. The derivative ∂_3 is taken in the out-of-plate direction $e_3 = (0, 0, 1)^T$.

Finally, we will use the Landau symbols $\mathcal{O}(h^\alpha)$ and $o(h^\alpha)$ to denote quantities which are of the order of, or vanish faster than h^α, as $h \to 0$. By C we denote any universal constant, depending on Ω and W, but independent of other involved quantities, so that $C = \mathcal{O}(1)$.

1.7. **Discussion and relation to previous works.** We now comment on the "critical exponents" of γ, i.e. the boundary values of ranges in which our analysis is valid. To draw a parallel with the previous results, in particular the seminal paper [6] and the conjecture in [22] for the hierarchy of models for nonlinear elastic shells, we note the following heuristics.

Given an exponent $\gamma > 0$, we expect (in view of Theorem 1.1 and its proof) the residual energy to scale as $h^{\gamma+2}$, under suitable non-vanishing curvature conditions on the prestrain metric. Following [22], where the critical exponents for the energy were shown to be: $\{\beta_n = 2 + \frac{2}{n}\}_{n\in\mathbb{N}}$, we let $\gamma_n = \beta_n - 2 = \frac{2}{n}$, with $\gamma_0 = \infty$ and $\gamma_\infty = 0$. We say that $(V_1, \dots, V_n) : \Omega \to (\mathbb{R}^3)^n$ is an nth order isometry of the prestrained plate when the sequence of metrics induced by the one-parameter family of infinitesimal bendings $u_h = \mathrm{id}_2 + \sum_{k=1}^n h^{k\gamma/2}V_k$ differ from the prescribed metrics G^h by terms of order at most $\mathcal{O}(h^{(n+1)\gamma/2})$. If $n = 1$, any normal out-of-plane displacement $V_1 = (0, 0, v)^T$ is a 1st order isometry, while for $n = \infty$, the resulting bending u_h is formally an exact isometry.

In this framework, several regimes can be distinguished:

- (i) When $\gamma_n < \gamma < \gamma_{n-1}$, we expect the limiting energy to be a linearized bending model with the nth-order isometry constraint.
- (ii) At the critical values $\gamma = \gamma_n$, the isometry constraint of the limit model should be of order $n - 1$, but in addition to the bending energy term, the limiting energy will also contain the nth order stretching term.
- (iii) Whenever the structure of the pre-strain tensor S_g allows for it, any nth order isometry can be matched to a higher order isometry of some order $m > n$. In that case, the theories in the range $\gamma_m < \gamma < \gamma_n$ are expected to collapse to the same theory (with the nth order isometry constraint).

The results in this paper can be now interpreted as follows. In Theorems 1.2 and 1.3 we derived the correct model, with the second order isometry constraint (1.10), corresponding to the values of γ between $\gamma_2 = 1$ and $\gamma_1 = 2$. The constraint (1.10) is naturally derived for the full range $0 < \gamma < \gamma_1$ (Theorem 1.1), but this information is not enough for characterizing the limiting model when $\gamma \leq \gamma_2 = 1$. Theorem 1.4 and the corresponding density result provides the tools to let all the expected higher order constraints for the full range $0 = \gamma_\infty < \gamma \leq \gamma_2 = 1$ be derived from the second order constraint (1.10). This leads to Theorem 1.5. For other instances where such matching properties have been proved and exploited to a similar purpose see [19, 6, 11, 18]. The continuation of infinitesimal bendings has also been used in [9, 10] to derive the Euler-Lagrange equations of elastic shell models.

In the absence of better techniques to show a direct nth order to exact isometry continuation (when the assumptions of Theorem 1.4 do not hold), one could hope to improve the results of Theorem 1.3, say to the range $2/3 = \gamma_3 < \gamma \leq \gamma_2 = 1$, provided that a matching of 2nd order isometries to 3rd order isometries is at hand. Solving this problem involves analyzing a linear system of PDEs, rather than the full nonlinear isometry equation as in Theorem 1.4. In general, this strategy, which was adapted in [11] for developable surfaces (see also [9]) leads to matching of nth order isometries to $(n+1)$th order isometries, and hence it could potentially imply that Theorem 1.3 is indeed true for the full range $0 < \gamma < 2$. This is, however, still a technically difficult problem and beyond our current understanding.

The two extreme critical cases are: $\gamma_1 = 2$ which leads to the prestrained von Kármán model, whose rigorous derivation was given in [16], and $\gamma_\infty = 0$ which corresponds to the prestrained Kirchhoff model, that has been considered in [1, 21, 23]. The Monge-Ampère constrained model studied in this paper lies in between the Kirchhoff and von Kármán models and can be compared to either of them. It can also be seen as a natural generalization, to the prestrained case, of a similar model derived in [6] which involves the degenerate constraint $\det \nabla^2 v = 0$. Finally, the regime $\gamma > \gamma_1$ will lead to a simple linear bending model.

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2. Compactness and lower bound: A proof of Theorem 1.1

1. Recall that in [16] we dealt with the general growth tensor family A_h. The following quantities, which we compute for the present case scenario (1.6) play the role in the scaling analysis below:

$$\begin{aligned} &\|\nabla_{tan}(A^h_{|x_3=0})\|_{L^\infty(\Omega)} + \|\partial_3 A^h\|_{L^\infty(\Omega^h)} \leq Ch^{\gamma/2} \\ &\|A^h\|_{L^\infty(\Omega^h)} + \|(A^h)^{-1}\|_{L^\infty(\Omega^h)} \leq C. \end{aligned} \tag{2.1}$$

We now quote the following approximation result, which can be directly obtained from the geometric rigidity estimate [5], in view of the bounds (2.1):

Theorem 2.1. [16, Theorem 1.6] *Let $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ satisfy $\lim_{h\to 0} \frac{1}{h^2} I_W^h(u^h) = 0$ (which is in particular implied by (1.9). Then there exist matrix fields $R^h \in W^{1,2}(\Omega, \mathbb{R}^{3\times 3})$, such that $R^h(x') \in SO(3)$ for a.e. $x' \in \Omega$, and:*

$$\frac{1}{h}\int_{\Omega^h} |\nabla u^h(x) - R^h(x')A^h(x)|^2 \,\mathrm{d}x \leq Ch^{2+\gamma}, \qquad \int_\Omega |\nabla R^h|^2 \leq Ch^\gamma. \tag{2.2}$$

Towards the proof of compactness in Theorem 1.1, we now outline the argument in [16] which yields (i) and (ii). We only emphasize points that lead to the new constraint (1.10).

Assume (1.9) and let $R^h \in W^{1,2}(\Omega, SO(3))$ be the matrix fields as in Theorem 2.1. Define the averaged rotations: $\tilde{R}^h = \mathbb{P}_{SO(3)} \fint_\Omega R^h$, which satisfy:

$$\int_\Omega |R^h - \tilde{R}^h|^2 \leq C\Big(\int_\Omega |R^h - \fint R^h|^2 + \mathrm{dist}^2(\fint R^h, SO(3))\Big) \leq Ch^\gamma, \tag{2.3}$$

and also let:

$$\hat{R}^h = \mathbb{P}_{SO(3)} \fint_{\Omega^h} (\tilde{R}^h)^T \nabla u^h, \tag{2.4}$$

which is well defined in view of (2.2) and (2.3). Consequently:

$$|\hat{R}^h - \mathrm{Id}|^2 \leq C|\fint_{\Omega^h} (\tilde{R}^h)^T\nabla u^h - \mathrm{Id}|^2 \leq Ch^\gamma. \tag{2.5}$$

Defining: $\bar{R}^h = \tilde{R}^h\hat{R}^h$, or equivalently: $\bar{R}^h = \mathbb{P}_{SO(3)}\fint_{\Omega^h}\nabla u^h$, it follows by (2.3), (2.5) and (2.2):

$$\int_\Omega |R^h - \bar{R}^h|^2 \leq Ch^\gamma \quad \text{and} \quad \lim_{h\to 0}(\bar{R}^h)^T R^h = \mathrm{Id} \quad \text{in } W^{1,2}(\Omega, \mathbb{R}^{3\times 3}). \tag{2.6}$$

Consider the translation vectors $c^h \in \mathbb{R}^3$, such that:

$$\int_\Omega V^h = 0 \quad \text{and} \quad \mathrm{skew}\int_\Omega \nabla V^h = 0. \tag{2.7}$$

To prove Theorem 1.1 (i), we now use (2.5) in:

$$\begin{aligned}\|(\nabla y^h - \mathrm{Id})_{3\times 2}\|^2_{L^2(\Omega^1)} &\leq \frac{1}{h}\int_{\Omega^h} |(\bar{R}^h)^T\nabla u^h - \mathrm{Id}|^2 \\ &\leq C\left(\frac{1}{h}\int_{\Omega^h} |(\tilde{R}^h)^T\nabla u^h - \mathrm{Id}|^2 \,\mathrm{d}x + |\hat{R}^h - \mathrm{Id}|^2\right) \leq Ch^\gamma,\end{aligned} \tag{2.8}$$

and notice that by (2.2) one has: $\|\partial_3 y^h\|^2_{L^2(\Omega^1)} \leq Ch\int_{\Omega^h}|\nabla u^h|^2 \leq Ch^2$. This yields convergence of y^h by means of the Poincaré inequality and (2.7). We also remark that (2.8) implies the weak convergence of V^h (up to a subsequence) in $W^{1,2}(\Omega, \mathbb{R}^3)$.

2. Consider the matrix fields $D^h \in W^{1,2}(\Omega, \mathbb{R}^{3\times 3})$:

$$\begin{aligned} D^h(x') &= \frac{1}{h^{\gamma/2}}\fint_{-h/2}^{h/2} (\bar{R}^h)^T R^h(x')A^h(x',t) - \mathrm{Id}\,\mathrm{d}t \\ &= h^{\gamma/2}(\bar{R}^h)^T R^h(x')S_g(x') + \frac{1}{h^{\gamma/2}}\Big((\bar{R}^h)^T R^h(x') - \mathrm{Id}\Big).\end{aligned} \tag{2.9}$$

By (2.6) and (2.2), it clearly follows that: $\|D^h\|_{W^{1,2}(\Omega)} \leq C$. Hence, up to a subsequence:

$$\begin{gathered}\lim_{h\to 0} D^h = D \quad \text{and} \quad \lim_{h\to 0}\frac{1}{h^{\gamma/2}}\Big((\bar{R}^h)^T R^h - \mathrm{Id}\Big) = D \\ \text{weakly in } W^{1,2}(\Omega, \mathbb{R}^{3\times 3}) \text{ and (strongly) in } L^q(\Omega, \mathbb{R}^{3\times 3}) \quad \forall q \geq 1.\end{gathered} \tag{2.10}$$

Using (2.6), (2.2) and the identity $(R-\mathrm{Id})^T(R-\mathrm{Id}) = -2\mathrm{sym}(R-\mathrm{Id})$, valid for all $R \in SO(3)$, we obtain: $\|\mathrm{sym}((\bar{R}^h)^T R^h - \mathrm{Id})\|_{L^2(\Omega)} \leq Ch^\gamma$. Consequently, the limiting matrix field D has skew-symmetric values.

Further, by (2.6) and (2.10):

(2.11)

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h^{\gamma/2}}\mathrm{sym}D^h &= \lim_{h\to 0}\left(\mathrm{sym}\Big((\bar{R}^h)^T R^h S_g\Big) - \frac{1}{2}\frac{1}{h^\gamma}\Big((\bar{R}^h)^T R^h(x') - \mathrm{Id}\Big)^T\Big((\bar{R}^h)^T R^h(x') - \mathrm{Id}\Big)\right) \\ &= \mathrm{sym}\, S_g + \frac{1}{2}D^2 \quad \text{in } L^q(\Omega, \mathbb{R}^{3\times 3}) \quad \forall q \geq 1.\end{aligned}$$

Regarding convergence of V^h, we have:

$$\begin{aligned}\|\nabla V^h - D^h_{3\times 2}\|^2_{L^2(\Omega)} &\leq \frac{C}{h^\gamma}\int_\Omega \left| \fint_{-h/2}^{h/2} R^h(x')A^h_{3\times 2}(x',t) - \nabla_{tan}u^h(x',t)\ \mathrm{d}t\right|^2 \mathrm{d}x' \\ &\leq \frac{C}{h^{\gamma+1}}\int_{\Omega^h} |\nabla u^h(x) - R^h(x')A^h(x)|^2\ \mathrm{d}x \leq Ch^2,\end{aligned} \tag{2.12}$$

and hence by (2.10) ∇V^h converges in $L^2(\Omega, \mathbb{R}^{3\times 2})$ to D. Consequently, by (2.7):

$$\lim_{h\to 0} V^h = V \text{ in } W^{1,2}(\Omega,\mathbb{R}^3), \quad V \in W^{2,2}(\Omega,\mathbb{R}^3) \quad \text{ and } \ \nabla V = D_{3\times 2}. \tag{2.13}$$

By Korn's inequality, V_{tan} must be constant, hence 0 in view of (2.7). This ends the proof of the first claim in Theorem 1.1 (ii).

3. We now show (1.10). By (2.8) we have:

$$\|\mathrm{sym}\nabla V^h\|^2_{L^2(\Omega)} \leq \frac{1}{h}\int_{\Omega^h} |(\bar{R}^h)^T\nabla u^h - \mathrm{Id}|^2 \leq Ch^\gamma.$$

We conclude, using (2.12) and (2.11) that:

$$\lim_{h\to 0}\frac{1}{h^{\gamma/2}}\mathrm{sym}\,\nabla V^h_{tan} = \lim_{h\to 0}\Big(\frac{1}{h^{\gamma/2}}\mathrm{sym}(D^h)_{tan} + \mathcal{O}(h^{1-\gamma/2})\Big) = (\mathrm{sym}\ S_g + \frac{1}{2}D^2)_{tan},$$

weakly in $L^2(\Omega)$. As a consequence, Korn's inequality implies the existence of a displacement field $w \in W^{1,2}(\Omega,\mathbb{R}^2)$ for which

$$\mathrm{sym}\nabla w = (\mathrm{sym}\ S_g + \frac{1}{2}D^2)_{tan} = \mathrm{sym}\ (S_g)_{2\times 2} - \frac{1}{2}\nabla v\otimes\nabla v,$$

where we calculated D^2 through (2.13), knowing that $\mathrm{sym}\,D = 0$ and $V = (0,0,v)^T$. Applying the operator $\mathrm{curl}^T\mathrm{curl}$ to both sides of the above formula yields the required result.

4. To prove the lower bound in (ii), define the rescaled strains $P^h \in L^2(\Omega^1,\mathbb{R}^{3\times 3})$ by:

$$P^h(x',x_3) = \frac{1}{h^{\gamma/2+1}}\Big((R^h(x'))^T\nabla u^h(x',hx_3)A^h(x',hx_3)^{-1} - \mathrm{Id}\Big).$$

Clearly, by (2.2) $\|P^h\|_{L^2(\Omega^1)} \leq C$ and hence, up to a subsequence:

$$\lim_{h\to 0} P^h = P \qquad \text{weakly in } L^2(\Omega^1,\mathbb{R}^{3\times 3}). \tag{2.14}$$

Precisely the same arguments as in [19], yield:

$$P(x)_{3\times 2} = P_0(x')_{3\times 2} + x_3P_1(x')_{3\times 2}, \tag{2.15}$$

for some $P_0 \in L^2(\Omega,\mathbb{R}^{3\times 3})$ where:

$$P_1(x') = \nabla(D(x')e_3) - B_g(x'). \tag{2.16}$$

Before concluding the proof of the lower bound in Theorem 1.1 (iii), we need to gather a few simple consequences of (1.2) and (1.3).

Lemma 2.2. *Assume that W satisfies (1.2) and (1.3). Then the quadratic form Q_3 is nonnegative, is positive definite on symmetric matrices, and $Q_3(F) = Q_3(\mathrm{sym}F)$ for all $F \in \mathbb{R}^3$.*

Proof. Let $F \in \mathbb{R}^{3\times 3}$ and $A \in so(3)$. Since $e^{tA} \in SO(3)$, by the frame invariance of W we get:

$$\begin{aligned}\forall t \in \mathbb{R} \qquad W(\mathrm{Id}_3 + tF) &= W\big(e^{tA}(\mathrm{Id}_3 + tF)\big) = W\Big((\mathrm{Id}_3 + tA + \mathcal{O}(t^2))(\mathrm{Id}_3 + tF)\Big)\\ &= W\big(\mathrm{Id}_3 + t(F+A) + \mathcal{O}(t^2)\big).\end{aligned}$$

Applying (1.3) to both sides of the above equality, it follows that:

$$\begin{aligned}t^2|Q_3(F) - Q_3\big((F+A) + \mathcal{O}(t)\big)| &= |Q_3(tF) - Q_3(t(F+A) + \mathcal{O}(t^2))|\\ &\leq \omega\big(t|F|\big)t^2|F|^2 + \omega\big(t|F+A| + \mathcal{O}(t^2)\big)t^2||F+A| + \mathcal{O}(t)|^2.\end{aligned}$$

Dividing both sides by t^2 and passing to the limit with $t \to 0$, implies that $Q_3(F+A) = Q_3(F)$, where we also used the fact that ω converges to zero at 0. Consequently:

$$\forall F \in \mathbb{R}^{3\times 3} \qquad Q_3(F) = Q_3(\mathrm{sym} F).$$

It remains now to prove that Q_3 is strictly positive definite on symmetric matrices. Let $F \in \mathbb{R}^{3\times 3}_{sym}$. Then, for every t small enough, $\mathrm{dist}(\mathrm{Id}_3 + tF, SO(3)) = |(\mathrm{Id}_3 + tF) - \mathrm{Id}_3| = |tF|$. It now follows that:

$$\begin{aligned}Q_3(F) &= \frac{1}{t^2}Q_3(tF) \geq \frac{1}{t^2}\Big(W(\mathrm{Id}_3 + tF) - \omega(tF)t^2|F|^2\Big)\\ &\geq \frac{1}{t^2}\Big(c\ \mathrm{dist}^2(\mathrm{Id}_3 + tF, SO(3)) - \omega(tF)t^2|F|^2\Big) \geq \frac{c}{2}|F|^2,\end{aligned}$$

where again we used (1.3) and (1.2). ∎

We are now ready to conclude the proof of Theorem 1.1. Recalling (1.3), we obtain:

$$\begin{aligned}\frac{1}{h^{\gamma+2}}W\Big(\nabla u^h(x)A^h(x)^{-1}\Big) &= \frac{1}{h^{\gamma+2}}W\Big(R^h(x)^T\nabla u^h(x)A^h(x)^{-1}\Big)\\ &= \frac{1}{h^{\gamma+2}}W(\mathrm{Id} + h^{\gamma/2+1}P^h(x)) = \mathcal{Q}_3(P^h(x)) + \omega(h^{\gamma/2+1}|P^h|)\mathcal{O}(|P^h(x)|^2).\end{aligned}$$

Consider now sets $\mathcal{U}_h = \{x \in \Omega^1;\ h|P^h(x', x_3)| \leq 1\}$. Clearly $\chi_{\mathcal{U}_h}$ converges to 1 in $L^1(\Omega^1)$, with $h \to 0$, as hP^h converges to 0 pointwise a.e. by (2.2). Remembering that $\lim_{t\to 0}\omega(t) = 0$, we get:

$$\begin{aligned}\liminf_{h\to 0}\frac{1}{h^{\gamma+2}}I_W^h(u^h) &\geq \liminf_{h\to 0}\frac{1}{h^{\gamma+2}}\int_{\Omega^1}\chi_{\mathcal{U}_h}W\Big(\nabla u^h(x', hx_3)A^h(x', hx_3)^{-1}\Big)\ \mathrm{d}x\\ &= \liminf_{h\to 0}\Big(\int_{\Omega^1}\mathcal{Q}_3(\chi_{\mathcal{U}_h}P^h) + o(1)\int_{\Omega^1}|P^h|^2\Big)\\ &\geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_3\Big(\mathrm{sym}\ P(x)\Big)\ \mathrm{d}x,\end{aligned}\tag{2.17}$$

where the last inequality follows by (2.2) guaranteeing convergence to 0 of the term $o(1)\int|P^h|^2$, and by the fact that $\chi_{\mathcal{U}_h}P^h$ converges weakly to P in $L^2(\Omega^1, \mathbb{R}^{3\times 3})$ (see (2.14)) in view of the properties of $\mathcal{Q}_3$ in Lemma 2.2. Further, by (1.13) and (2.16):

$$\begin{aligned}\frac{1}{2}\int_{\Omega^1}\mathcal{Q}_3(\text{sym } P) &\geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_2(\text{sym } P_{2\times 2}(x))\ \mathrm{d}x\\ &= \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_2\Big(\text{sym } P_0(x')_{2\times 2} + x_3\text{sym } P_1(x')_{2\times 2}\Big)\ \mathrm{d}x\\ &= \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_2(\text{sym } P_0(x')_{2\times 2}) + \frac{1}{2}\int_{\Omega^1}x_3^2\mathcal{Q}_2(\text{sym } P_1(x')_{2\times 2})\\ &\geq \frac{1}{12}\int_{\Omega}\mathcal{Q}_2\Big(\text{sym } (\nabla De_3)_{2\times 2} - (\text{sym } B_g)_{2\times 2}\Big).\end{aligned} \tag{2.18}$$

Now, in view of Theorem 1.1 (ii) and (2.13) one easily sees that:

$$\Big(\nabla De_3\Big)_{2\times 2} = -\nabla v^2,$$

which yields the claim in Theorem 1.1 (iii), by (2.17) and (2.18). ■

3. Recovery sequence: Proofs of Theorem 1.2 and Theorem 1.3

Recalling (1.13), let $c(F)\in\mathbb{R}^3$ be the unique vector so that:

$$\mathcal{Q}_2(F) = \mathcal{Q}_3\Big(F^* + \text{sym}(c\otimes e_3)\Big).$$

The mapping $c:\mathbb{R}^{2\times 2}_{sym}\to\mathbb{R}^3$ is well-defined and linear, by the properties of Q_3 in Lemma 2.2. Also, for all $F\in\mathbb{R}^{3\times 3}$, by $l(F)$ we denote the unique vector in $\mathbb{R}^3$, linearly depending on F, for which:

$$\text{sym}\big(F - (F_{2\times 2})^*\big) = \text{sym}\big(l(F)\otimes e_3\big). \tag{3.1}$$

1. Let the given out-of-plane displacement $v\in\mathcal{A}_f$ be as in Theorem 1.2. The constraint (1.10) can be rewritten as:

$$-\frac{1}{2}\text{curl}^T\text{curl}(\nabla v\otimes\nabla v) = -\text{curl}^T\text{curl}(S_g)_{2\times 2} = -\text{curl}^T\text{curl}(\text{sym}\, S_g)_{2\times 2}.$$

Recall that a matrix field $B\in L^2(\Omega,\mathbb{R}^{2\times 2}_{sym})$ is in the kernel of the linear operator curl^Tcurl if and only if $B = \text{sym}\nabla w$ for some $w\in W^{1,2}(\Omega,\mathbb{R}^2)$. Hence, we conclude that:

$$\text{sym}\nabla w = -\frac{1}{2}\nabla v\otimes\nabla v + \text{sym}(S_g)_{2\times 2}.$$

By the Sobolev embedding theorem in the two-dimensional domain Ω, $v\in W^{2,2}(\Omega)$ implies that: $\nabla v\in W^{1,q}(\Omega,\mathbb{R}^2)$ for all $q<\infty$. Consequently:

$$\text{sym}\nabla w\in W^{1,p}(\Omega,\mathbb{R}^{3\times 3})\qquad\forall 1\leq p<2.$$

Fix $1<p<2$ such that: $\gamma>2/p$ and that $W^{1,p}(\Omega)$ embeds in $L^8(\Omega)$. This is possible since $\gamma<2$ and so p can be chosen as close to 2 as we wish. Using Korn's inequality and through a possible modification of w by an affine mapping, we can assume that:

$$w\in W^{2,p}\cap W^{1,8}(\Omega,\mathbb{R}^2).$$

Call $\lambda = 1/p$ and observe that:

$$\frac{2-\gamma}{2(p-1)} < \lambda < \frac{\gamma}{2}. \tag{3.2}$$

Following [6, Proposition 2], by partition of unity and a truncantion argument, as a special case of the Lusin-type result for Sobolev functions, there exist sequences $v^h \in W^{2,\infty}(\Omega)$ and $w^h \in W^{2,\infty}(\Omega,\mathbb{R}^2)$ such that:

$$\begin{aligned}
&\lim_{h\to 0} \|v^h - v\|_{W^{2,2}(\Omega)} + \|w^h - w\|_{W^{2,p}(\Omega,\mathbb{R}^2)} = 0,\\
&\|v^h\|_{W^{2,\infty}(\Omega)} + \|w^h\|_{W^{2,\infty}(\Omega,\mathbb{R}^2)} \leq Ch^{-\lambda},\\
&\lim_{h\to 0} h^{-2\lambda}\left|\Big\{x\in\Omega;\ v^h(x)\neq v(x)\Big\}\right| + h^{-p\lambda}\left|\Big\{x\in\Omega;\ w^h(x)\neq w(x)\Big\}\right| = 0.
\end{aligned} \tag{3.3}$$

Hence, Ω is partitioned into a disjoint union $\Omega = \mathcal{U}_h \cup O_h$, where:

$$\begin{aligned}
&\mathcal{U}_h = \Big\{x\in\Omega;\ v^h(x) = v(x)\Big\} \cap \Big\{x\in\Omega;\ w^h(x) = w(x)\Big\},\\
&|O_h| = o(h^{p\lambda}) + o(h^{2\lambda}) = o(h^{p\lambda}).
\end{aligned} \tag{3.4}$$

We observe that the second order stretching $s(v^h, w^h)$ satisfies:

$$s(v^h, w^h) = \mathrm{sym}\nabla w^h + \frac{1}{2}\nabla v^h \otimes \nabla v^h - \mathrm{sym}(S_g)_{2\times 2} = 0 \quad \text{ in } \mathcal{U}_h.$$

Now, a similar argument as in [19, Lemma 6.1] yields:

$$\|s(v^h, w^h)\|_{L^\infty(\Omega)} = o(h^{\lambda(p/2-1)}) \quad \text{and} \quad \|s(v^h, w^h)\|^2_{L^2(\Omega)} = o(h^{2\lambda(p-1)}). \tag{3.5}$$

2. Define the recovery sequence:

$$\begin{aligned}
\forall (x', x_3)\in\Omega^h \qquad u^h(x',x_3) = &\begin{bmatrix} x' \\ 0 \end{bmatrix} + \begin{bmatrix} h^\gamma w^h(x') \\ h^{\gamma/2} v^h(x') \end{bmatrix} + x_3 \begin{bmatrix} -h^{\gamma/2}\nabla v^h(x') \\ 1 \end{bmatrix}\\
&+ h^\gamma x_3 d^{0,h}(x') + \frac{1}{2} h^{\gamma/2} x_3^2 d^{1,h}(x),
\end{aligned} \tag{3.6}$$

where the Lipschitz continuous fields $d^{0,h} \in W^{1,\infty}(\Omega,\mathbb{R}^3)$ is given by:

$$d^{0,h} = l(S_g) - \frac{1}{2}|\nabla v^h|^2 e_3 + c\Big(\mathrm{sym}\nabla w^h + \frac{1}{2}\nabla v^h\otimes\nabla v^h - (\mathrm{sym}\ S_g)_{2\times 2}\Big),$$

while the smooth fields $d^{1,h}$ obey:

$$\lim_{h\to 0} \sqrt{h}\|d^{1,h}\|_{W^{1,\infty}(\Omega)} = 0, \tag{3.7}$$

$$\lim_{h\to 0} d^{1,h} = l(B_g) + c\Big(- \nabla^2 v - (\mathrm{sym}\ B_g)_{2\times 2}\Big) \quad \text{ in } L^2(\Omega). \tag{3.8}$$

The convergence statements in (i), (ii) of Theorem 1.2 are now verified by a straightforward calculation. In order to establish (iii) we will estimate the energy of the sequence u^h in (3.6). Calculating the deformation gradient we first obtain:

$$\nabla u^h = \mathrm{Id} + h^\gamma(\nabla w^h)^* + h^{\gamma/2} D^h - h^{\gamma/2} x_3 (\nabla^2 v^h)^* + h^\gamma \begin{bmatrix} x_3\nabla d^{0,h} & d^{0,h} \end{bmatrix} + h^{\gamma/2}\begin{bmatrix} \frac{1}{2}x_3^2 \nabla d^{1,h} & x_3 d^{1,h} \end{bmatrix},$$

where the skew-symmetric matrix field D^h is given as:

$$D^h = \begin{bmatrix} 0 & -(\nabla v^h)^T \\ \nabla v^h & 0 \end{bmatrix}.$$

Recall that: $(A^h)^{-1} = \mathrm{Id} - h^\gamma S_g - h^{\gamma/2} x_3 B_g + \mathcal{O}(h^{2\gamma})$. We hence obtain:

$$(\nabla u^h)(A^h)^{-1} = \mathrm{Id} + F^h$$

where, using $\lambda < \gamma/2 < 1$:

$$\begin{aligned}F^h &= h^\gamma((\nabla w^h)^* - S_g) + h^{\gamma/2}D^h - h^{\gamma/2}x_3((\nabla^2 v^h)^* + B_g) + h^\gamma \left[\ x_3\nabla d^{0,h} \quad d^{0,h}\ \right] \\ &\quad + h^{\gamma/2}\left[\ \tfrac{1}{2}x_3^2\nabla d^{1,h} \quad x_3 d^{1,h}\ \right] - h^\gamma S_g - h^{\gamma/2}x_3 B_g \\ &\quad + \mathcal{O}(h^{2\gamma})(|\nabla w^h| + |d^{0,h}|) + \mathcal{O}(h^{3\gamma/2})|D^h| + \mathcal{O}(h^{1+\gamma}) \\ &= o(1).\end{aligned} \tag{3.9}$$

Hence:

$$(A^h)^{-1,T}(\nabla u^h)^T(\nabla u^h)(A^h)^{-1} = \mathrm{Id}_3 + 2\mathrm{sym}\, F^h + (F^h)^T F^h = \mathrm{Id} + K^h + q^h, \tag{3.10}$$

where:

$$K^h = 2h^\gamma \mathrm{sym}\Big((\nabla w^h)^* - \frac{1}{2}(D^h)^2 - S_g + d^{0,h}\otimes e_3\Big) + 2h^{\gamma/2}x_3\ \mathrm{sym}\Big(- (\nabla^2 v^h)^* - B_g + d^{1,h}\otimes e_3\Big),$$

and:

$$\begin{aligned}q^h &= \mathcal{O}(h^{2\gamma})\big(|\nabla w^h| + |\nabla w^h|^2|d^{0,h}|\big) + \mathcal{O}(h^{3\gamma/2})|D^h|\big(1 + |\nabla w^h| + |D^h| + |d^{0,h}|\big) \\ &\quad + \mathcal{O}(h^{1+\gamma-\lambda})\big(1 + |\nabla w^h|^2 + |D^h|^2 + |d^{0,h}|^2\big) + \mathcal{O}(h^{(\gamma+3)/2}) \\ &= o(1).\end{aligned}$$

Note that $(D^h)^2 = -(\nabla v^h \otimes \nabla v^h)^* - |\nabla v^h|^2(e_3\otimes e_3)$. Therefore:

$$\begin{aligned}&\mathrm{sym}\left((\nabla w^h)^* - \frac{1}{2}(D^h)^2 - S_g + d^{0,h}\otimes e_3\right) \\ &\quad = \left(\mathrm{sym}\nabla w^h + \frac{1}{2}\nabla v^h\otimes\nabla v^h - (\mathrm{sym}\ S_g)_{2\times 2}\right)^* + \mathrm{sym}\left(\Big(d^{0,h} - l(S_g) + \frac{1}{2}|\nabla v^h|^2 e_3\Big)\otimes e_3\right) \\ &\quad = s(v^h, w^h)^* + \mathrm{sym}\Big(c(s(v^h,w^h))\otimes e_3\Big).\end{aligned}$$

Call:

$$\begin{aligned}b(v^h) &= \mathrm{sym}\left(-(\nabla^2 v^h)^* - B_g + d^{1,h}\otimes e_3\right) \\ &= \left(-\nabla^2 v^h - (\mathrm{sym}\ B_g)_{2\times 2}\right)^* + \mathrm{sym}\left((d^{1,h} - l(B_g))\otimes e_3\right).\end{aligned}$$

We therefore obtain:

$$K^h = 2h^{\gamma/2}x_3 b(v^h) + \mathcal{O}(h^\gamma)|s(v^h,w^h)| = o(1).$$

Note also that:

$$\lim_{h\to 0} b(v^h) = \big(-\nabla^2 v - (\mathrm{sym}\ B_g)_{2\times 2}\big)^* + \mathrm{sym}\left(c\left(-\nabla^2 v - (\mathrm{sym}\ B_g)_{2\times 2}\right)\otimes e_3\right) \quad \text{in } L^2(\Omega). \tag{3.11}$$

3. We now observe the following convergence rates:

Lemma 3.1. *We have:*

(i) $h^{-1}\|q^h\|^2_{L^2(\mathcal{U}_h\times(-\frac{h}{2},\frac{h}{2}))} = o(h^{\gamma+2})$,

(ii) $h^{-1}\||q^h||K^h|\|_{L^1(\mathcal{U}_h\times(-\frac{h}{2},\frac{h}{2}))} = o(h^{\gamma+2})$.

Proof. Recall that v^h and w^h are uniformly bounded in $W^{1,8}(\Omega)$. To prove (i) observe that:

$$\frac{1}{h}\|q^h\|^2_{L^2(\mathcal{U}_h\times(-\frac{h}{2},\frac{h}{2}))} \leq \|C^h\|_{L^1(\Omega)}\mathcal{O}(h^{4\gamma} + h^{3\gamma} + h^{2(1+\gamma-\lambda)} + h^{\gamma+3}) = o(h^{\gamma+2}),$$

where we collected all the terms involving $|D^h|, |\nabla w^h|$ and $|d^{0,h}| \leq C(1 + |\nabla w^h| + |D^h|^2)$ in the quantity C^h, which can be shown to be uniformly bounded in $L^1(\Omega)$.

To see (ii), we estimate:

$$\begin{aligned}\frac{1}{h}\||q^h||K^h|\|_{L^1(\mathcal{U}_h\times(-\frac{h}{2},\frac{h}{2}))} &\leq h^{-1/2}\|q^h\|_{L^2}\Big(h^{(\gamma+2)/2}\|b(v^h)\|_{L^2(\Omega)}+h^\gamma\|s(v^h,w^h)\|_{L^2(\Omega)}\Big)\\ &= o(h^{(\gamma+2)/2})\Big[h^{(\gamma+2)/2}+o(h^{\gamma+\lambda(p-1)})\Big]\\ &= o(h^{\gamma+2})+o(h^{3\gamma/2+\lambda p-\lambda+1})=o(h^{\gamma+2}),\end{aligned}$$

where we used (i), (3.5) and (3.11). ■

Now we observe that, since $F^h=o(1)$ in (3.9), the matrix field Id_3+F^h is uniformly close to $SO(3)$ for appropriately small h, and hence it has a positive determinant. By (3.10) and in view of the polar decomposition theorem, there exists an $SO(3)$ valued field $R^h:\Omega^h\to\mathbb{R}^{3\times 3}$ such that:

$$\mathrm{Id}_3+F^h=R^h\sqrt{\mathrm{Id}+K^h+q^h}\quad \text{ in } \Omega^h.$$

We hence obtain, by Taylor expanding the square root operator around Id_3, and using (1.2) :

$$W\Big(\nabla u^h(A^h)^{-1}\Big)=W\Big(R^h(\sqrt{\mathrm{Id}_3+K^h+q^h})\Big)=W\Big(\mathrm{Id}_3+\frac{1}{2}(K^h+q^h)+\mathcal{O}(|K^h+q^h|^2)\Big).$$

Recalling (1.3), we hence obtain:

$$\begin{aligned}W\Big(\nabla u^h(A^h)^{-1}\Big) &\leq \mathcal{Q}_3\left(\frac{1}{2}(K^h+q^h)+\mathcal{O}(|K^h+q^h|^2)\right)\\ &\qquad +\omega\Big(|K^h+q^h|+\mathcal{O}(|K^h+q^h|^2)\Big)\Big||K^h+q^h|+\mathcal{O}(|K^h+q^h|^2)\Big|^2\\ &\leq \mathcal{Q}_3\left(\frac{1}{2}K^h\right)+\mathcal{O}\left(|K^h||q^h|+|q^h|^2\right)+o(1)|K^h|^2,\end{aligned}$$

where we used the fact that $|K^h|+|q^h|=o(1)$ and $\omega(t)\to 0$ as $t\to 0$. We now estimate the energy I^h_W using the above inequality and Lemma 3.1:

$$\begin{aligned}I^h_W(u^h)=\frac{1}{h}\int_{\Omega^h}W(\nabla u^h(A^h)^{-1}) &= \frac{1}{h}\int_{\Omega^h}\mathcal{Q}_3\left(\frac{1}{2}K^h\right)+\mathcal{O}\left(|K^h||q^h|+|q^h|^2\right)+o(1)|K^h|^2\,\mathrm{d}x\\ &\leq \frac{1}{h}\int_{\Omega^h}\mathcal{Q}_3\Big(h^{\gamma/2}x_3 b(v^h)+\mathcal{O}(h^\gamma)|s(v^h,w^h)|\Big)\,\mathrm{d}x+o(h^{\gamma+2}).\end{aligned}$$

Integrating in the x_3 direction and applying the estimate (3.5) finally yields:

$$\begin{aligned}I^h_W(u^h) &\leq \frac{1}{12}\int_\Omega h^{\gamma+2}\mathcal{Q}_3\Big(b(v^h)\Big)\,\mathrm{d}x+\mathcal{O}(h^{2\gamma})\|s(v^h,w^h)\|^2_{L^2(\Omega)}+o(h^{\gamma+2})\\ &= \frac{1}{12}h^{\gamma+2}\int_\Omega\mathcal{Q}_3\Big(b(v^h)\Big)\,\mathrm{d}x+o(h^{2\lambda(p-1)+2\gamma})+o(h^{\gamma+2})\\ &= \frac{1}{12}h^{\gamma+2}\int_\Omega\mathcal{Q}_3\Big(b(v^h)\Big)\,\mathrm{d}x+o(h^{\gamma+2}),\end{aligned}$$

since by the choice of λ in (3.2), we have $2\lambda(p-1)+2\gamma>\gamma+2$. In view of (3.11) it follows that:

$$\limsup_{h\to 0}\frac{1}{h^{\gamma+2}}I^h_W(u^h)\leq \mathcal{I}_f(v), \tag{3.12}$$

which, combined with Theorem (1.1), proves the desired limit (iii) in Theorem 1.2. ■

Theorem 1.3 follows now from the next result:

Lemma 3.2. *When Ω is simply connected, the following are equivalent:*

(i) *There exists* $v \in W^{2,2}(\Omega)$ *such that* $\det(\nabla^2 v) = -\mathrm{curl}^T \mathrm{curl}(S_g)_{2\times 2}$ *and* $\mathcal{I}_f(v) = 0$,

(ii) $\mathrm{curl}\big((\mathrm{sym}\ B_g)_{2\times 2}\big) = 0$ *and* $\mathrm{curl}^T \mathrm{curl}\ (S_g)_{2\times 2} = -\det\big((\mathrm{sym}\ B_g)_{2\times 2}\big)$.

The two equations in (ii) are the linearized Gauss-Codazzi-Mainardi equations corresponding to the metric $\mathrm{Id} + 2h^\gamma(\mathrm{sym}\ S_g)_{2\times 2}$ *and the shape operator* $h^{\gamma/2}(\mathrm{sym}\ B_g)_{2\times 2}$ *on the mid-plate* Ω.

Proof. The proof is straightforward and equivalent to that of [16, Lemma 6.1]. ■

Remark 3.3. Another construction of the recovery sequence, following the general approach of [5], will appear in [29]. We briefly present this argument for the simplified case when $B_g = 0$. Define u^h as in (3.6), where instead of (3.7) and (3.8) we require the following of the Lipschitz warping coefficients $d^{0,h}$ and $d^{1,h}$:

$$
\begin{aligned}
& d^{0,h} = l(S_g) - \frac{1}{2}|\nabla v^h|^2 e_3, \\
& \lim_{h\to 0} \|d^{1,h} - c(-\nabla^2 v)\|_{L^2(\Omega)} = 0, \qquad \lim_{h\to 0} h^{\gamma/2}\|d^{1,h}\|_{W^{1,\infty}(\Omega)} = 0.
\end{aligned} \tag{3.13}
$$

The truncation sequences $v^h \in W^{2,\infty}(\Omega)$ and $w^h \in W^{1,\infty}(\Omega, \mathbb{R}^2)$ should satisfy the conditions below. Define the truncation scale and the truncation exponent:

$$
\lambda = 1 + \frac{\gamma}{2}, \qquad q = \frac{2+\gamma}{\gamma - 1} > 4,
$$

so that $w \in W^{1,q}(\Omega, \mathbb{R}^2)$. Then, given an appropriately small constant $\epsilon_0 > 0$, the result in [6, Proposition 2] allows for having:

$$
\begin{aligned}
& \lim_{h\to 0} \|v^h - v\|_{W^{2,2}(\Omega)} + \|w^h - w\|_{W^{1,q}(\Omega,\mathbb{R}^2)} = 0, \\
& \|v^h\|_{W^{2,\infty}(\Omega)} \le \epsilon_0 h^{-\lambda}, \qquad \|w^h\|_{W^{1,\infty}(\Omega,\mathbb{R}^2)} \le \epsilon_0 h^{-2\lambda/q}, \\
& \lim_{h\to 0} h^{-2\lambda}\left|\Big\{x\in\Omega;\ v^h(x) \neq v(x)\Big\}\right| + h^{-2\lambda}\left|\Big\{x\in\Omega;\ w^h(x)\neq w(x)\Big\}\right| = 0,
\end{aligned} \tag{3.14}
$$

where the constants C above depend only on Ω and γ, but are independent of h and ϵ_0. The main new observation follows now from the Brezis-Wainger inequality [31, Theorem 2.9.4], applied to the sequence $\nabla v^h \in W^{1,4}$, uniformly bounded in $W^{1,2}$, which yields:

$$
\|\nabla v^h\|_{L^\infty} \le C\Big(1 + \log^{1/2}\left(1 + \|\nabla v^h\|_{W^{1,4}}\right)\Big) \le C\Big(1 + \log^{1/2}\left(1 + S_0 h^{-\lambda}\right)\Big) \le C\log(1/h) \tag{3.15}
$$

for all h sufficiently small. In particular: $\|\nabla v^h\|_{L^\infty} \le Ch^{-\gamma/4}$ and as a result, we obtain the following bounds:

$$
\|D^h\|_{L^\infty} \le Ch^{-\gamma/4}, \quad \|d^{0,h}\|_{L^\infty} \le C(1 + h^{-\gamma/2}), \quad \|\nabla d^{0,h}\|_{L^\infty} \le C(1 + h^{-\lambda-\gamma/4}),
$$

which together with (3.13), (3.14) give:

$$
\|\nabla u^h - \mathrm{Id}_3\|_{L^\infty} \le C\epsilon_0.
$$

Consequently:

$$
\mathrm{dist}(\nabla u^h (A^h)^{-1}, SO(3)) \le \|\nabla u^h(A^h)^{-1} - \mathrm{Id}_3\|_{L^\infty} \le \|\nabla u^h - \mathrm{Id}_3\|_{L^\infty} + \|\nabla u^h\big((A^h)^{-1} - \mathrm{Id}_3\big)\|_{L^\infty} \le C\epsilon_0,
$$

for all h sufficiently small. Let the sets $\mathcal{U}_h$, O_h be as in (3.4). Then, in view of boundedness of W close to $SO(3)$ and (3.14) we have:

$$
\frac{1}{h^{2+\gamma}}\frac{1}{h}\int_{O_h\times(-\frac{h}{2},\frac{h}{2})} W(\nabla u^h (A^h)^{-1})\,\mathrm{d}x \le \frac{C}{h^{2+\gamma}}|O_h| = \frac{C}{h^{-2\lambda}}|O_h| \to 0 \quad \text{as } h\to 0,
$$

while on the "good set" $\mathcal{U}_h$, the estimates follow using the fact that $v^h = v$ and $w^h = w$, as in [5].

4. The matching property and an efficient recovery sequence: A proof of Theorem 1.4 and Theorem 1.5

1. We decompose the unknown vector field w_ϵ into its tangential and normal components:

$$w_\epsilon = w_{\epsilon,tan} + w_\epsilon^3 e_3,$$

where $w_{\epsilon,tan} \in \mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R}^2)$. Denoting: $z_\epsilon = \epsilon w_\epsilon^3 \in \mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R})$, the equation (1.21) is equivalent to:

$$\nabla(\mathrm{id}_2+\epsilon^2 w_{\epsilon,tan})^T \nabla(\mathrm{id}_2+\epsilon^2 w_{\epsilon,tan}) = \mathrm{Id}_2+2\epsilon^2(\mathrm{sym}\ S_g)_{2\times 2}-\epsilon^2(\nabla v+\nabla z_\epsilon)\otimes(\nabla v+\nabla z_\epsilon)+\epsilon^3 s_\epsilon. \tag{4.1}$$

We shall first find the formula for the Gaussian curvature of the 2d metric in the right hand side of (4.1), where we denote $v_1 = v + z_\epsilon$, and:

$$g_\epsilon(z_\epsilon) = \mathrm{Id}_2 + 2\epsilon^2(\mathrm{sym}\ S_g)_{2\times 2} - \epsilon^2 \nabla v_1 \otimes \nabla v_1 + \epsilon^3 s_\epsilon. \tag{4.2}$$

Call $P_\epsilon = [P_{ij}]_{i,j=1,2} = \mathrm{Id}_2 + 2\epsilon^2(\mathrm{sym}\ S_g)_{2\times 2} + \epsilon^3 s_\epsilon$. The Christoffel symbols, the inverse and the determinant of P_ϵ, satisfy:

$$\begin{aligned}
&\Gamma_{ij}^k = \frac{1}{2}P^{kl}\left(\partial_j P_{il} + \partial_i P_{jl} - \partial_j P_{ij}\right) = 1 + \mathcal{O}(\epsilon^2)\\
&(P_\epsilon)^{-1} = [P^{ij}] = \frac{1}{\det P_\epsilon}\mathrm{cof}[P_{ij}] = \mathrm{Id}_2 + \mathcal{O}(\epsilon^2)\\
&\det P_\epsilon = 1 + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{4.3}$$

By [7, Lemma 2.1.2], we have:

$$\kappa\big(P_\epsilon - \epsilon^2\nabla v_1 \otimes \nabla v_1\big) = \frac{\kappa(P_\epsilon)}{\big(1-\epsilon^2(P^{ij}\partial_i v_1 \partial_j v_1)\big)^2} - \frac{\epsilon^2 \det(\nabla^2 v_1 - [\Gamma_{ij}^k \partial_k v_1]_{ij})}{\big(1-\epsilon^2(P^{ij}\partial_i v_1\partial_j v_1)\big)^4 \det P_\epsilon}. \tag{4.4}$$

In fact, the formula above is obtained, by a direct calculation, for v_1 smooth. When $v_1 \in \mathcal{C}^{2,\beta}$, one approximates v_1 by smooth sequence v_1^n, and notes that each $\kappa_n = \kappa(\mathrm{Id}_2 + 2\epsilon^2(S_g)_{2\times 2} - \epsilon^2(\nabla v_1^n \otimes \nabla v_1^n)) + \epsilon^3 s_\epsilon$ is given by (4.4), while the sequence κ_n converges in $\mathcal{C}^{0,\beta}$ to the right hand side in (4.4). Since κ_n converges in distributions to $\kappa(P_\epsilon - \epsilon^2(\nabla v_1 \otimes \nabla v_1))$, as follows from the definition of Gauss curvature $\kappa = R_{1212}/\det g_\epsilon$, (4.4) holds for $v_1 \in \mathcal{C}^{2,\beta}$ as well.

2. We now see that $\kappa(g_\epsilon(z_\epsilon)) = 0$ if and only if $\Phi(\epsilon, z_\epsilon) = 0$, where:

$$\Phi(\epsilon, z) = \big(1-\epsilon^2 P^{ij}\partial_i(v+z)\partial_j(v+z)\big)^2\big(\det P_\epsilon\big)\frac{1}{\epsilon^2}\kappa(P_\epsilon) - \det\big(\nabla^2 v + \nabla^2 z - [\Gamma_{ij}^k \partial_k(v+z)]_{ij}\big).$$

Consider $\Phi : (-\epsilon_0, \epsilon_0)\times\mathcal{C}_0^{2,\beta}(\bar{\Omega}, \mathbb{R}) \to \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R})$ and look for $z_\epsilon \in \mathcal{C}_0^{2,\beta}(\bar{\Omega}, \mathbb{R})$ satisfying $\Phi(\epsilon, z_\epsilon) = 0$. By using (1.15) to approximate $\kappa(P_\epsilon)$ and recalling (4.3), we get:

$$\begin{aligned}
\Phi(\epsilon, z) = &-\big(1 + \mathcal{O}(\epsilon^2)|\nabla v + \nabla z|^2\big)^2(1+\mathcal{O}(\epsilon^2))\big(\mathrm{curl}^T\mathrm{curl}(S_g)_{2\times 2} + \mathcal{O}(\epsilon^2)\big)\\
&- \det\big(\nabla^2 v + \nabla^2 z + \mathcal{O}(\epsilon^2)|\nabla v + \nabla z|\big).
\end{aligned}$$

It easily follows that: $\Phi(0,0) = -\mathrm{curl}^T\mathrm{curl}(S_g)_{2\times 2} - \det\nabla^2 v = 0$, and that the partial derivative $\mathcal{L} = \partial\Phi/\partial z(0,0) : \mathcal{C}_0^{2,\beta}(\bar{\Omega}, \mathbb{R}) \to \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R})$ is a linear continuous operator of the form:

$$\forall z \in \mathcal{C}_0^{2,\beta} \qquad \mathcal{L}(z) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Phi(0, \epsilon z) = -\lim\frac{1}{\epsilon}\big(\det(\nabla^2 v + \epsilon\nabla^2 z) - \det\nabla^2 v\big) = -\mathrm{cof}\nabla^2 v : \nabla^2 z.$$

Clearly, $\mathcal{L}$ above is invertible to a continuous linear operator, because of the uniform ellipticity of $\nabla^2 v$, implied by $\det\nabla^2 v$ being strictly positive. By the implicit function theorem there exists hence the solution operator: $\mathcal{Z} : (-\epsilon_0, \epsilon_0) \to \mathcal{C}_0^{2,\beta}(\bar\Omega, \mathbb{R})$ such that $z_\epsilon = \mathcal{Z}(\epsilon)$ satisfies $\Phi(\epsilon, z_\epsilon) = 0$. Moreover:

$$\mathcal{Z}'(0) = \mathcal{L}^{-1} \circ \left(\frac{\partial \Phi}{\partial \epsilon}(0,0)\right) = 0, \qquad \text{because } \frac{\partial \Phi}{\partial \epsilon}(0,0) = 0.$$

Consequently, we also obtain: $\|w_\epsilon^3\|_{\mathcal{C}^{2,\beta}} = \frac{1}{\epsilon}\|z_\epsilon\|_{\mathcal{C}^{2,\beta}} \to 0$, as $\epsilon \to 0$.

3. By [27] it now follows that for each small ϵ there is exactly one (up to rotations) orientation preserving isometric immersion $\phi_\epsilon \in \mathcal{C}^2(\bar\Omega, \mathbb{R}^2)$ of $g_\epsilon(z_\epsilon)$:

$$\nabla \phi_\epsilon^T \nabla \phi_\epsilon = g_\epsilon(z_\epsilon) \quad \text{and} \quad \det \nabla \phi_\epsilon > 0. \tag{4.5}$$

We now sketch the argument that in fact: $\phi_\epsilon = \mathrm{id} + \epsilon^2 w_{\epsilon,tan}$ with some $w_{\epsilon,tan}$ uniformly bounded in $\mathcal{C}^{2,\beta}(\bar\Omega, \mathbb{R}^2)$. The proof proceeds as in [18, Theorem 4.1], where the reader may find many more details. Firstly, (4.5) is equivalent to: $\nabla^2 \phi_\epsilon - [\tilde\Gamma_{ij}^k \partial_k \phi_\epsilon]_{ij} = 0$, where $\tilde\Gamma_{ij}^k$ are the Christoffel symbols of the metric $g_\epsilon(z_\epsilon)$ in (4.2). By (4.5) and the boundedness of $\tilde\Gamma_{ij}^k$, it follows that: $\|\phi_\epsilon\|_{\mathcal{C}^{2,\beta}(\bar\Omega,\mathbb{R}^2)} \le C$. Further, $\|\tilde\Gamma_{ij}^k\|_{\mathcal{C}^{0,\beta}} = \mathcal{O}(\epsilon^2)$ and so:

$$\exists A_\epsilon \in \mathbb{R}^{2\times 2} \qquad \|\nabla \phi_\epsilon - A_\epsilon\|_{\mathcal{C}^{1,\beta}} \le C\epsilon^2. \tag{4.6}$$

In fact, $\mathrm{dist}(A_\epsilon, SO(3)) \le C\epsilon^2$, so without loss of generality: $\|\nabla\phi_\epsilon - \mathrm{Id}_3\|_{\mathcal{C}^{1,\beta}} \le C\epsilon^2$ and therefore: $\|\phi_\epsilon - \mathrm{id}\|_{\mathcal{C}^{2,\beta}} \le C\epsilon^2$. Consequently, $\phi_\epsilon = \mathrm{id}_2 + \epsilon^2 w_{\epsilon,tan}$ with $\|w_{\epsilon,tan}\|_{\mathcal{C}^{2,\beta}} \le C$. This ends the proof of Theorem 1.4. ■

4. We now sketch the proof of Theorem 1.5. The complete calculations are similar to [18, Theorem 3.5] and can be found in [29]. We recall first a result on density of regular solutions to the elliptic 2d Monge-Ampère equation:

Proposition 4.1. [18, Theorem 3.2] *Assume that Ω is star-shaped with respect to an interior ball $B \subset \Omega$. For a constant $c_0 > 0$, recall the definition:*

$$\mathcal{A}_{c_0} = \big\{u \in W^{2,2}(\Omega);\ \det \nabla^2 u = c_0 \ \textit{a.e. in } \Omega\big\}.$$

Then $\mathcal{A}_{c_0} \cap C^\infty(\bar\Omega)$ is dense in $\mathcal{A}_{c_0}$ with respect to the $W^{2,2}$ norm.

In view of the above, it is enough to prove Theorem 1.5 for $v \in \mathcal{C}^{2,\beta}(\bar\Omega)$ satisfying $\det \nabla^2 v = c_0$. In the general case of $v \in W^{2,2}(\Omega)$ satisfying the same constraint, the result follows by a diagonal argument.

By Theorem 1.4 used with $\epsilon = h^{\gamma/2}$ and $s_\epsilon = \epsilon(S_g^2)_{2\times 2}$, there exists an equibounded sequence $w_h \in \mathcal{C}^{2,\beta}(\bar\Omega, \mathbb{R}^3)$ such that the deformations $u_h(x') = x' + h^{\gamma/2} v(x') e_3 + h^\gamma w_h(x')$ are isometrically equivalent to the metric in:

$$\forall 0 < h \ll 1 \qquad (\nabla u_h)^T \nabla u_h = \mathrm{Id}_2 + 2h^\gamma(\mathrm{sym}\ S_g)_{2\times 2} + h^{2\gamma}(S_g^2)_{2\times 2}. \tag{4.7}$$

Define now the recovery sequence $u^h \in \mathcal{C}^{1,\beta}(\Omega^h, \mathbb{R}^3)$ by the formula:

$$u^h(x', x_3) = u_h(x') + x_3 b^h(x') + \frac{x_3^2}{2} h^{\gamma/2}\big(d^h(x') - l(B_g(x'))\big), \tag{4.8}$$

where $l(B_g)$ is defined as in (3.1), the "Cosserat" vector fields $b^h : \Omega \to \mathbb{R}^3$ are given by:

$$\begin{bmatrix} \partial_1 u_h & \partial_2 u_h & b^h \end{bmatrix}^T \begin{bmatrix} \partial_1 u_h & \partial_2 u_h & b^h \end{bmatrix} = G^h(\cdot, 0) \quad \text{in } \Omega,$$

and $d^h \in \mathcal{C}^{1,\beta}(\bar\Omega,\mathbb{R}^3)$ are the "warping" vector fields, approximating the effective warping $d \in \mathcal{C}^{0,\beta}(\bar\Omega,\mathbb{R}^3)$:

$$\begin{aligned} &h^{\gamma/2}\|d^h\|_{\mathcal{C}^{1,\beta}} \leq C \quad \text{and} \quad \lim_{h\to 0}\|d^h - d\|_{L^\infty} = 0,\\ &\mathcal{Q}_2\big(\nabla^2 v + \mathrm{sym}(B_g)_{2\times 2}\big) = \mathcal{Q}_3\big((\nabla^2 v + \mathrm{sym}(B_g)_{2\times 2})^* + \mathrm{sym}(d\otimes e_3)\big). \end{aligned} \tag{4.9}$$

Note that (4.8) is consistent with (1.8) at the highest order terms in the expansion in h. ■

5. On the uniqueness of minimizers to the Monge-Ampère constrained energy

In this section, we discuss the multiplicity of minimizers to the limiting problem (1.12). Given a bounded, simply connected $\Omega \subset \mathbb{R}^2$ and a function $f \in L^1(\Omega)$, we consider the functional:

$$\mathcal{I}(v) = \int_\Omega |\nabla^2 v|^2 \,\mathrm{d}x' \quad \text{subject to the constraint: } \mathcal{A}_f = \{v \in W^{2,2}(\Omega);\ \det\nabla^2 v = f\}. \tag{5.1}$$

Here, we assumed that $\mathcal{Q}_2(F_{2\times 2}) = |\mathrm{sym}(F_{2\times 2})|^2$ for every $F_{2\times 2} \in \mathbb{R}^{2\times 2}$, which is consistent with (1.13) and (1.3), when $W(F) = \frac12\mathrm{dist}^2(F, SO(3))$ for F close to $SO(3)$. Indeed, expanding $\mathrm{dist}^2(\mathrm{Id}+\epsilon A, SO(3)) = |\sqrt{(\mathrm{Id}+\epsilon A)^T(\mathrm{Id}+\epsilon A)} - \mathrm{Id}|^2 = \epsilon^2|\mathrm{sym}\ A|^2 + \mathcal{O}(\epsilon^3)$, we see that $\mathcal{Q}_3(A) = |\mathrm{sym}\ A|^2$, which implies the form of $\mathcal{Q}_2$. This scenario corresponds to the isotropic elastic energy density with the Lamé coefficients $\lambda = 0$, $\mu = \frac12$ (see [6] for more details).

We now observe that the minimization problem for (5.1) may have multiple or unique solutions, depending on the choice of a smooth constraint function f.

Example 5.1. (i) Let $\Omega = B(0,1) \subset \mathbb{R}^2$. Then for $f \equiv -1$ the problem (5.1) has a non-trivial one-parameter family of absolute minimizers: $v_\theta(x_1,x_2) = (\cos\theta)\dfrac{x_1^2 - x_2^2}{2} + (\sin\theta)(x_1x_2)$. Indeed, for $v \in \mathcal{A}_{f\equiv -1}$ the quantity $|\nabla^2 v|^2 = (\mathrm{tr}\ \nabla^2 v)^2 - 2\det\nabla^2 v = (\mathrm{tr}\ \nabla^2 v)^2 + 2$ is minimized when $\mathrm{tr}\nabla^2 v = \Delta v = 0$, that is readily satisfied with: $\nabla^2 v_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$.

(ii) On the other hand, for $f \equiv 1$, (5.1) has a unique minimizer: $v(x_1,x_2) = \dfrac{x_1^2 + x_2^2}{2}$. This is because for $v \in \mathcal{A}_{f\equiv 1}$ we have: $|\nabla^2 v|^2 = (\mathrm{tr}\ \nabla^2 v)^2 - 2 = (\lambda_1+\lambda_2)^2 - 2$, where λ_1, λ_2 are the eigenvalues of $\nabla^2 v$. This quantity achieves its minimum, under the constraint $\lambda_1\lambda_2 = 1$, precisely when $\lambda_1 = \lambda_2 = 1$. ■

Example 5.2. A similar argument as in Example 5.1 (i), allows for a construction of a one-parameter family of absolute minimizers v_θ to (5.1) when a smooth function $f : \bar\Omega \to \mathbb{R}$ satisfies:

$$f \leq c_0 < 0 \quad \text{and} \quad \Delta(\log|f|) = 0 \quad \text{in } \Omega. \tag{5.2}$$

Indeed, define $\lambda = \sqrt{|f|}$. Clearly, the function λ is positive, smooth and satisfies $\Delta(\log\lambda) = 0$ in $\bar\Omega$. Hence there exists $\phi \in \mathcal{C}^\infty(\bar\Omega)$ such that the function $(\log\lambda + i\phi)$ is holomorphic in $\Omega \subset \mathbb{C}$. Trivially, for every $\theta \in \mathbb{R}$, the function $(\log\lambda + i(\phi+\theta))$ is holomorphic, as is its exponential:

$$\exp(\log\lambda + i(\phi+\theta)) = \lambda\cos(\phi+\theta) + i\lambda\sin(\phi+\theta).$$

Writing the associated Cauchy-Riemann equations we note that they are precisely the vanishing of the *curl* of the symmetric matrix field in the left hand side of:

$$\begin{bmatrix} \lambda\cos(\phi+\theta) & -\lambda\sin(\phi+\theta) \\ -\lambda\sin(\phi+\theta) & -\lambda\cos(\phi+\theta) \end{bmatrix} = \nabla^2 v_\theta. \tag{5.3}$$

Consequently, since Ω is simply connected, for each θ there exists a smooth $v_\theta : \bar{\Omega} \to \mathbb{R}$ as in (5.3). We see that:

$$\Delta v_\theta = 0 \quad \text{and} \quad \det \nabla^2 v_\theta = -\lambda^2 = -|f| = f, \tag{5.4}$$

which proves the claim.

For completeness, we now prove that (5.2) is in fact equivalent to the existence of some v satisfying (5.4). Denote $\lambda = \sqrt{f}$ and let $r_1, r_2 : \Omega \to \mathbb{R}^3$ be the (unit-length) eigenvectors fields of $\nabla^2 v$ corresponding to the eigenvalues λ and $-\lambda$. Since $\langle r_1, r_2 \rangle = 0$, we may write: $[r_1, r_2] = R_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \in SO(2)$, for some smooth function $\phi : \Omega \to (0, 2\pi)$. The fact that the range of ϕ may be taken in $(0, 2\pi)$ follows from the simply-connectedness of Ω. We obtain:

$$\nabla^2 v = R_\phi \ \mathrm{diag}\{\lambda, -\lambda\} \ R_\phi^T = \begin{bmatrix} \lambda\cos(2\phi) & \lambda\sin(2\phi) \\ \lambda\sin(2\phi) & -\lambda\cos(2\phi) \end{bmatrix} = \begin{bmatrix} \lambda\cos(-2\phi) & -\lambda\sin(-2\phi) \\ -\lambda\sin(-2\phi) & -\lambda\cos(-2\phi) \end{bmatrix}.$$

Since *curl* of the matrix field in the right hand side above vanishes in Ω, we reason as in (5.3) and see that the (nonzero) function $\lambda\exp(-2i\phi)$ satisfy the Cauchy-Riemann equations, and hence it is holomorphic in $\Omega \subset \mathbb{C}$. Further, its logarithm: $(\log\lambda - 2i\phi)$ is well defined and holomorphic as well. Consequently: $\Delta(\log\lambda) = 0$, which concludes the proof of (5.2). ■

In what follows, we want to derive conditions for uniqueness of minimizers to (5.1). In this context, it is useful to consider the relaxed constraint:

$$\mathcal{A}_f^* = \{v \in W^{2,2}(\Omega);\ \det\nabla^2 v \geq f\}.$$

We will denote by $\mathcal{I}_f$ and $\mathcal{I}_f^*$ the restrictions of I to $\mathcal{A}_f$ and $\mathcal{A}_f^*$, respectively. Clearly:

$$\inf \mathcal{I}_f^* \leq \inf \mathcal{I}_f.$$

The following straightforward lemma has been observed in [8] as well:

Lemma 5.3. *Assume that $\mathcal{A}_f \neq \emptyset$ ($\mathcal{A}_f^* \neq \emptyset$). Then I_f (I_f^*) admits a minimizer. Moreover, there must be $f \in L^1\log L^1(\Omega)$, namely:*

$$\int_{\Omega'} |f\log(2+f)| < \infty,$$

for every subset Ω' compactly contained in Ω.

Proof. Take a minimizing sequence $v_n \in \mathcal{A}_f$; it satisfies: $\|\nabla^2 v_n\|_{L^2(\Omega)} \leq C$. By modifying v_n by $⨍ v$ and $(⨍ \nabla v)x$, in view of the Poincare inequality it follows that: $\|v_n\|_{W^{2,2}(\Omega)} \leq C$. Therefore $v_n \rightharpoonup v$ weakly in $W^{2,2}(\Omega)$ (up to a subsequence), which implies $\mathcal{I}(v) \leq \liminf \mathcal{I}(v_n)$. We hence see that v is a minimizer of $\mathcal{I}_f$ ($\mathcal{I}_f^*$) if only v satisfies the appropriate constraint.

Since $\nabla v_n \rightharpoonup \nabla v$ weakly in $W^{1,2}(\Omega)$, then the same convergence is also valid strongly in any $L^p(\Omega)$ for $p \in [1,\infty)$, and so $\nabla v_n \otimes \nabla v_n \to \nabla v \otimes \nabla v$ strongly in $L^2(\Omega)$. Applying $\mathrm{curl}^T\mathrm{curl}$, this yields the following convergence, in the sense of distributions:

$$\det\nabla^2 v_n = -\frac{1}{2}\mathrm{curl}^T\mathrm{curl}(\nabla v_n \otimes \nabla v_n) \to -\frac{1}{2}\mathrm{curl}^T\mathrm{curl}(\nabla v \otimes \nabla v) = \det\nabla^2 v.$$

Consequently, if $v_n \in \mathcal{A}_f$ then $v \in \mathcal{A}_f$ as well (likewise, if $v_n \in \mathcal{A}_f^*$ then $v \in \mathcal{A}_f^*$).

The final assertion follows from the celebrated result in [28]: If $v \in W^{1,2}(\Omega, \mathbb{R}^n)$ on $\Omega \subset \mathbb{R}^n$ satisfies $\det\nabla v \geq 0$ then $\det\nabla v \in L^1\log L^1(\Omega)$. ■

Lemma 5.4. *Assume that $f \geq c > 0$ in Ω. Let $v_1, v_2 \in \mathcal{A}_f^*$ be two minimizers of $\mathcal{I}_f^*$. Then $\nabla^2 v_1 = \nabla^2 v_2$, i.e. $v_1 - v_2$ is an affine function. In particular, the function:*

$$\psi[f] = \det \nabla^2(\text{argmin } \mathcal{I}_f^*) = \det \nabla^2 v_1$$

is well defined and it satisfies: $\psi[f] \geq f$ and $\psi[f] \in L^1 \log L^1(\Omega)$.

Proof. By [18, Theorem 6.1], without loss of generality (possibly replacing v_i by $-v_i$) we may assume that $\nabla^2 v_1$ and $\nabla^2 v_2$ are strictly positive definite a.e. in the domain. For $\lambda \in [0,1]$, consider $v_\lambda = \lambda v_1 + (1-\lambda)v_2$. We claim that $v_\lambda \in \mathcal{A}_f^*$. This follows by the Brunn-Minkowski inequality:

$$(\det \nabla^2 v_\lambda)^{1/2} \geq \lambda(\det \nabla^2 v_1)^{1/2} + (1-\lambda)(\det \nabla^2 v_2)^{1/2} \geq \lambda\sqrt{f} + (1-\lambda)\sqrt{f} = \sqrt{f}.$$

Also: $\mathcal{I}(v_\lambda) \leq \lambda \mathcal{I}(v_1) + (1-\lambda)\mathcal{I}(v_2) = \min \mathcal{I}_f^*$, and so this inequality is in fact an equality. Since the L^2 norm is a strictly convex function, we conclude that $\nabla^2 v_1 = \nabla^2 v_2$. ■

Remark 5.5. Consider the related functional $I_\Delta(v) = \int_\Omega |\Delta v|^2$, constrained to $\mathcal{A}_f$ or $\mathcal{A}_f^*$, which we respectively denote by $I_{\Delta,f}$ and $I_{\Delta,f}^*$. Since $|\nabla^2 v|^2 = |\Delta v|^2 - 2\det \nabla^2 v$, any minimizing sequence v_n of $I_{\Delta,f}$ or $I_{\Delta,f}^*$, satisfies $\|\nabla^2 v_n\|_{L^2(\Omega)} \leq C$. Arguing as in the proof of Lemma 5.3 we obtain existence of minimizers to both problems. On the other hand, there is no uniqueness as in Lemma 5.4, in the sense that two minimizers of $I_{\Delta,f}^*$ may differ by a non-affine harmonic function. We now observe that if $\min \mathcal{I}_f = \min \mathcal{I}_f^*$, then $\min I_{\Delta,f} = \min I_{\Delta,f}^*$. Indeed, let $v_0 \in \mathcal{A}_f$ be the common minimizer of $\mathcal{I}_f$ and $\mathcal{I}_f^*$. Then:

$$\forall v \in \mathcal{A}_f^* \quad \mathcal{I}_\Delta(v) = I(v) + 2\int_\Omega \det \nabla^2 v \geq \mathcal{I}(v_0) + 2\int_\Omega f = I_\Delta(v_0),$$

hence v_0 is also the common minimizer of $I_{\Delta,f}$ and $I_{\Delta,f}^*$.

6. On the uniqueness of minimizers: the radially symmetric case

In this section we assume that $\Omega = B(0,1) \subset \mathbb{R}^2$ and that:

$$f = f(r) \geq c > 0$$

is a radial function such that $f \in L^1(\Omega)$, i.e.: $\int_0^1 rf(r)\, \mathrm{d}r < \infty$.

Lemma 6.1. *If a radial function $v = v(r) \in W^{2,2}(\Omega)$ satisfies* $\det \nabla^2 v = f$, *then:*

$$|v'(r)|^2 = \int_0^r 2sf(s)\, \mathrm{d}s.$$

In particular, there exists at most one (up to a constant) radial function $v = v_f$ as above.

Proof. Let $v = v(r)$ be as in the statement of the Lemma. Recall that writing $\partial_r v = v'$, the gradient of v in polar coordinates has the form: $\nabla v(r,\theta) = (v'(r)\cos\theta, v'(r)\sin\theta)^T$. We now check directly that:

$$\det \nabla^2 v = \frac{1}{r}v'v'' = \frac{1}{2r}\left(|v'|^2\right)'.$$

Hence, there must be:

$$|v'(r)|^2 = \int_0^r 2sf(s)\, \mathrm{d}s + C, \tag{6.1}$$

for some $C \geq 0$. Since $v \in W^{2,2}(\Omega)$, we get: $\Delta v = v'' + \frac{1}{r}v' \in L^2(\Omega)$, or equivalently:

$$\int_\Omega |v''|^2 + \frac{1}{r^2}|v'|^2 + \frac{2}{r}v'v'' < \infty.$$

Note that the last term above equals $2f \in L^1(\Omega)$, and thus $\frac{1}{r^2}|v'|^2 \in L^1(\Omega)$. By (6.1) we conclude:

$$\int_0^1 \frac{2\pi C}{r} < 2\pi \int_0^1 \frac{1}{r}|v'(r)|^2 \, \mathrm{d}r = \int_\Omega \frac{1}{r^2}|v'|^2 < \infty,$$

and so there must be $C = 0$. ■

Corollary 6.2. *A necessary and sufficient condition for existence of a radial function $v = v(r) \in W^{2,2}(\Omega)$ solving $\det \nabla^2 v = f$ is:*

$$\int_0^1 r|\log r| f(r) \, \mathrm{d}r < \infty \quad \text{and} \quad \int_0^1 \frac{r^3 f(r)^2}{\int_0^r s f(s) ds} \, \mathrm{d}r < \infty. \tag{6.2}$$

The solution v_f is then given by (uniquely, up to a constant):

$$v_f(r) = \int_0^r \left(\int_0^s 2t f(t) \, \mathrm{d}t \right)^{1/2} \, \mathrm{d}s. \tag{6.3}$$

In particular, (6.2) is satisfied when $f \in L^2(\Omega)$, and consequently $\mathcal{A}_f \neq \emptyset$.

Proof. By Lemma 6.1 it follows that the solution v is given by v_f in (6.3). Clearly $\nabla v_f \in \mathcal{C}^1(\bar{\Omega})$, so it remains to check when $\nabla^2 v_f \in L^2(\Omega)$. We compute:

$$\begin{aligned} \int_\Omega |\nabla^2 v_f|^2 &= \int_\Omega |v_f''|^2 + \frac{1}{r^2}|v_f'|^2 = 2\pi \int_0^1 r|v_f''|^2 + \frac{|v_f'|^2}{r} \, \mathrm{d}r \\ &= 2\pi \int_0^1 \frac{r^3 f(r)^2}{\int_0^r 2s f(s) \mathrm{d}s} \, \mathrm{d}r + 2\pi \int_0^1 2r|\log r| f(r) \, \mathrm{d}r, \end{aligned} \tag{6.4}$$

proving the first claim. When $f \in L^2(\Omega)$, then $\int_0^1 r f^2(r) \, \mathrm{d}r < \infty$, and so:

$$\begin{aligned} &\int_0^1 r|\log r| f(r) \, \mathrm{d}r \leq \Big(\int_0^1 r|\log r|^2 \Big)^{1/2} \Big(\int_0^1 r f^2 \Big)^{1/2} < \infty \\ &\int_0^1 \frac{r^3 f(r)^2}{\int_0^r s f(s) \mathrm{d}s} \, \mathrm{d}r \leq \int_0^1 \frac{r^3 f(r)^2}{\int_0^r cs \mathrm{d}s} \, \mathrm{d}r \leq \int_0^1 r f^2 < \infty \end{aligned}$$

which concludes the proof. ■

Lemma 6.3. *(i) Assume that $\mathcal{A}_f^* \neq \emptyset$. Then the unique (up to an affine map) minimizer of $\mathcal{I}_f^*$ is radially symmetric, given by $v_{\psi[f]}$ where $\psi[f]$ satisfies (6.2).*

(ii) Assume that $\mathcal{I}_f$ has the unique (up to an affine map) minimizer. Then, it is radially symmetric and hence given by v_f in (6.3). Also, f satisfies conditions (6.2).

Proof. We will prove (ii). The proof of (i) relies on Lemma 5.3 and Lemma 5.4 and the same argument as below.

Let $v \in W^{2,2}(\Omega)$ be a minimizer of $\mathcal{I}_f$, which we modify (if needed) so that: $v(0) = 0$ and $\fint \nabla v = 0$. For any $\theta \in [0, 2\pi)$ let $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be the planar rotation by angle θ. Note that $\nabla^2(v \circ R_\theta) = R_\theta^T \big((\nabla^2 v) \circ R_\theta \big) R_\theta$, so $\det \nabla^2(v \circ R_\theta) = (\det \nabla^2 v) \circ R_\theta$. In view of radial symmetry of f, if follows that $v \circ R_\theta \in \mathcal{A}_f^*$ and $\mathcal{I}(v \circ R_\theta) = \mathcal{I}(v)$. Therefore, by uniqueness, $v = v \circ R_\theta$ is radially symmetric and so the result follows from Corollary 6.2. ■

Theorem 6.4. *Assume that $\mathcal{A}_f^* \neq \emptyset$, and that f is a.e. nonincreasing, i.e.:*

$$\forall a.e.\ r \in [0,1] \quad \forall a.e.\ x \in [0,r] \qquad f(r) \leq f(x). \tag{6.5}$$

Then both problems $\mathcal{I}_f$ and $\mathcal{I}_f^*$ have a unique (up to an affine map) minimizer. The minimizer is common to both problems, necessarily radially symmetric and given by v_f in (6.3).

Proof. By Lemma 6.3, the radial function $v_{\psi[f]}$ is the unique minimizer of $\mathcal{I}_f^*$. Consider v_f given by (6.3). We will prove that $\mathcal{I}(v_f) \leq \mathcal{I}(v_\psi)$. This will imply that $v_f \in W^{2,2}(\Omega)$ and hence, by uniqueness of minimizers there must be: $v_f = v_\psi$, as claimed in the Theorem.

Recall that $\psi \geq f$ and note that $\int_0^r 2sf(s)\,\mathrm{d}s \geq r^2 f(r)$ in view of (6.5). As in (6.4), we compute:

$$\begin{aligned}
\int_\Omega |\nabla^2 v_\psi|^2 - \int_\Omega |\nabla^2 v_f|^2 &= 2\pi \int_0^1 \frac{r^3\psi(r)^2}{\int_0^r 2s\psi(s)\mathrm{d}s} - \frac{r^3 f(r)^2}{\int_0^r 2sf(s)\mathrm{d}s}\,\mathrm{d}r + 2\pi\int_0^1 \frac{\int_0^r 2s(\psi-f)\mathrm{d}s}{r}\,\mathrm{d}r \\
&\geq -2\pi \int_0^1 \frac{r^3 f^2 \int_0^r 2s(\psi-f)\mathrm{d}s}{(\int_0^r 2s\psi(s)\mathrm{d}s)(\int_0^r 2sf(s)\mathrm{d}s)}\mathrm{d}r + 2\pi\int_0^1 \frac{\int_0^r 2s(\psi-f)\mathrm{d}s}{r}\,\mathrm{d}r \\
&\geq -2\pi \int_0^1 \frac{r^3 f^2 \int_0^r 2s(\psi-f)\mathrm{d}s}{(\int_0^r 2sf(s)\mathrm{d}s)^2}\mathrm{d}r + 2\pi\int_0^1 \frac{\int_0^r 2s(\psi-f)\mathrm{d}s}{r}\,\mathrm{d}r \\
&\geq -2\pi \int_0^1 \frac{r^3 f^2 \int_0^r 2s(\psi-f)\mathrm{d}s}{(r^2 f(r))^2}\mathrm{d}r + 2\pi\int_0^1 \frac{\int_0^r 2s(\psi-f)\mathrm{d}s}{r}\,\mathrm{d}r = 0.
\end{aligned}$$

The proof is now achieved in view of Corollary 6.2 and Lemma 6.3. ∎

Remark 6.5. Note that v_f in general, is not a minimizer of the relaxed problem $\mathcal{I}_f^*$. Consider $f_\epsilon(r) = \epsilon\chi_{(0,1/2]} + \chi_{(1/2,1]}$. Then $v_{f_\epsilon} \in W^{2,2}(\Omega)$ and, by (6.4):

$$\begin{aligned}
\int_\Omega |\nabla^2 v_{f_\epsilon}|^2 &\geq 2\pi\int_0^1 r|v''(r)|^2\,\mathrm{d}r \geq 2\pi\int_{1/2}^1 \frac{r^3}{\frac{\epsilon}{4} + (r^2 - \frac14)}\,\mathrm{d}r \geq C\int_{1/2}^1 \frac{1}{r^2 - (1-\epsilon)/4}\,\mathrm{d}r \\
&\geq C\left(\log(1 - \frac{\sqrt{1-\epsilon}}{2}) - \log(\frac{1-\sqrt{1-\epsilon}}{2}) - \log(1 + \frac{\sqrt{1-\epsilon}}{2}) + \log(\frac{1+\sqrt{1-\epsilon}}{2})\right) \\
&\to \infty \qquad \text{as } \epsilon \to 0.
\end{aligned}$$

On the other hand $f_\epsilon \leq \psi \equiv 1$ and we see that $\int_\Omega |\nabla^2 v_\psi|^2 = 2\pi$, where $v_\psi = \frac12 r^2$. Therefore $\mathcal{I}(v_\psi) < \mathcal{I}(v_{f_\epsilon})$ for all small ϵ. A standard approximation argument leads to similar counter-examples with smooth f.

7. Critical points of the Monge-Ampère constrained energy in the radial case: a proof of Theorem 1.7

The Euler-Lagrange equations for the problem (5.1) are complicated, which is due to the, in general, unknown structure of the tangent space to the constraint set $\mathcal{A}_f$. Consider instead the functional:

$$\Lambda(v,\lambda) = \int_\Omega |\nabla^2 v|^2 + \int_\Omega \lambda(\det\nabla^2 v - f), \qquad v \in W^{2,2}, \quad \lambda \in L^\infty.$$

The following result is to be compared with [8], where a converse statement is proved in a limited setting:

Lemma 7.1. *If (v,λ) is a critical point for Λ then v is a critical point for (5.1).*

Proof. Let w be a tangent vector to $\mathcal{A}_f$ at a given $v \in \mathcal{A}_f$, so that there exists a continuous curve $\phi : [0,1] \to \mathcal{A}_f$ with $\phi(0) = v$ such that $\phi'(0) = w$. Note that $\phi(\epsilon) = v + \epsilon w + o(\epsilon) \in \mathcal{A}_f$. Expanding det in the usual manner we obtain:

$$f = \det \nabla^2 \phi(\epsilon) = \det(\nabla^2 v + \epsilon \nabla^2 w + o(\epsilon)) = \det \nabla^2 v + \epsilon \mathrm{cof} \nabla^2 w : \nabla^2 v + o(\epsilon)$$

which implies that:

$$\mathrm{cof}\nabla^2 w : \nabla^2 v = 0 \qquad \text{a.e. in } \Omega. \tag{7.1}$$

To prove (i), let (v, λ) be a critical point of Λ. Taking variation μ in λ we get: $\int \mu(\det \nabla^2 v - f) = 0$, thus $v \in \mathcal{A}_f$. Taking now a variation w in v we obtain:

$$2\int \nabla^2 v : \nabla^2 w + \int \lambda \,\mathrm{cof}\nabla^2 v : \nabla^2 w = 0 \qquad \forall w \in W^{2,2}. \tag{7.2}$$

In particular, for every w satisfying (7.1) the above reduces to $\int \nabla^2 v : \nabla^2 w = 0$ which is the variation of pure bending functional $\mathcal{I}$. Hence v must indeed be a critical point of (5.1). ■

Lemma 7.2. *The Euler-Lagrange equations of Λ and the natural boundary conditions are:*

$$\begin{aligned} &2\Delta^2 v + \mathrm{cof}\nabla^2 v : \nabla^2 \lambda = 0 \qquad \textit{in } \Omega, \\ &\det\nabla^2 v = f \qquad \textit{in } \Omega, \end{aligned} \tag{7.3}$$

$$\begin{aligned} &\partial_\tau \Big[\Big(2\nabla^2 v + \lambda \mathrm{cof}\nabla^2 v \Big) : (\tau \otimes \vec{n}) \Big] + \Big(2\nabla \Delta v + (\mathrm{cof}\nabla^2 v)\nabla\lambda \Big)\vec{n} = 0 \qquad \textit{on } \partial\Omega, \\ &\Big(2\nabla^2 v + \lambda \mathrm{cof}\nabla^2 v \Big) : (\vec{n} \otimes \vec{n}) = 0 \qquad \textit{on } \partial\Omega. \end{aligned} \tag{7.4}$$

Proof. Assuming enough regularity on v, λ, integration by parts gives:

$$2\int_\Omega \nabla^2 v : \nabla^2 w = 2\int_\Omega w \Delta^2 v + 2\int_{\partial\Omega} \Big[(\nabla^2 v \nabla w)\vec{n} - w(\nabla \Delta v)\vec{n} \Big],$$

$$\int_\Omega \lambda \,\mathrm{cof}\nabla^2 v : \nabla^2 w = \int_\Omega w \,\mathrm{cof}\nabla^2 v : \nabla^2 \lambda + \int_{\partial\Omega} \Big[\lambda((\mathrm{cof}\nabla^2 v)\nabla w)\vec{n} - w((\mathrm{cof}\nabla^2 v)\nabla\lambda)\vec{n} \Big]$$

In view of (7.2) the above calculations yield (7.3) and:

$$\int_{\partial\Omega} \Big[\Big((2\nabla^2 v + \lambda \mathrm{cof}\nabla^2 v)\nabla w \Big)\vec{n} - w\Big(2\nabla\Delta v + (\mathrm{cof}\nabla^2 v)\nabla\lambda \Big)\vec{n} \Big] = 0 \qquad \forall w \in W^{2,2}.$$

Writing now $\nabla w = (\partial_\tau w)\tau + (\partial_{\vec{n}} w)\vec{n}$, where τ is the unit vector tangent to $\partial\Omega$ we get:

$$\begin{aligned} \int_{\partial\Omega} \Big[(\partial_\tau w)\Big(2\nabla^2 v + \lambda \mathrm{cof}\nabla^2 v \Big) : (\tau \otimes \vec{n}) - w\Big(2\nabla\Delta v + (\mathrm{cof}\nabla^2 v)\nabla\lambda \Big)\vec{n} \Big] & \\ + \int_{\partial\Omega} (\partial_{\vec{n}} w)\Big(2\nabla^2 v + \lambda \mathrm{cof}\nabla^2 v \Big) : (\vec{n} \otimes \vec{n}) = 0 \qquad \forall w \in W^{2,2}. \end{aligned}$$

Integrating by parts on the boundary in the first integral above, we deduce (7.4). ■

The proof of Theorem 1.7 follows now directly from the result below.

Proposition 7.3. *Assume that $f \in \mathcal{C}^\infty(\bar{B}(0,1))$ is radially symmetric i.e. $f = f(r)$, and that $f \geq c > 0$. Let $v = v(r) \in \mathcal{A}_f$ be a radial solution to the constraint: $\det \nabla^2 v = f$ in $B(0,1)$. Then there is a radial function $\lambda = \lambda(r) \in \mathcal{C}^\infty(\bar{B}(0,1))$ such that (v, λ) is a critical point for Λ.*

Proof. Recall that since f is smooth and positive, then by [18, Theorem 6.3] any $W^{2,2}$ solution of the Monge-Ampère equation $\det \nabla^2 v = f$ in $B(0,1)$ satisfies $v \in \mathcal{C}^\infty(B(0,1))$. On the other hand, by radial symmetry, $v = v_f$ given in (6.3), so we conclude that in fact: $v \in \mathcal{C}^\infty(\bar{B}(0,1))$. In particular $v \in \mathcal{C}^\infty([0,1])$ and $v'(0) = (\Delta v)'(0) = 0$.

Let R_θ denote the planar rotation by angle θ. In polar coordinates, we have:

$$\nabla v(r,\theta) = v'(r) R_\theta e_1 = v'(r)\vec{n}, \qquad \nabla^2 v(r,\theta) = R_\theta \begin{bmatrix} v'' & 0 \\ 0 & \frac{v'}{r} \end{bmatrix} R_\theta^T,$$

and also note that: $\mathrm{cof}(R_\theta A R_\theta^T) = R_\theta (\mathrm{cof} A) R_\theta^T$. We now rewrite (7.3) (7.4) using the ansatz $\lambda = \lambda(r)$ and assuming sufficient regularity. First, (7.3) becomes: $\frac{1}{r}(v''\lambda' + v'\lambda'') = -2\big((\Delta v)'' + \frac{(\Delta v)'}{r}\big)$, where we used that $\Delta v = v'' + \frac{v'}{r}$. Equivalently: $(\lambda' v')' = -2\big(r(\Delta v)'\big)'$, which becomes:

$$\lambda'(r) = -2\frac{r}{v'(r)}(\Delta v)' \quad \text{in } (0,1). \tag{7.5}$$

Note that this is consistent with $\lambda'(0) = 0$, because:

$$\lim_{r\to 0} \frac{v'(r)}{r} = \big(\lim_{r\to 0} \frac{(v'(r))^2}{r^2}\big)^{1/2} = \big(\lim_{r\to 0} \frac{2\int_0^r s f(s)\,\mathrm{d}s}{r^2}\big)^{1/2} = \big(\lim_{r\to 0}\frac{2rf(r)}{2r}\big)^{1/2} = \sqrt{f(0)} \neq 0. \tag{7.6}$$

We now examine the boundary equations (7.4). We have:

$$\Big(2\nabla^2 v + \lambda \mathrm{cof}\nabla^2 v\Big) : (\tau \otimes \vec{n}) = R_\theta A(r) R_\theta^T : (\tau\otimes\vec{n}) = A(r) : (R_\theta^T \tau \otimes R_\theta^T \vec{n}) = A(r) : (e_2 \otimes e_1)$$

for a matrix field A depending only on r, and hence:

$$\partial_\tau \Big[\Big(2\nabla^2 v + \lambda \mathrm{cof}\nabla^2 v\Big) : (\tau\otimes\vec{n})\Big] = 0.$$

Also, in view of (7.5):

$$\Big(2\nabla\Delta v + (\mathrm{cof}\nabla^2 v)\nabla\lambda\Big)\vec{n} = 2(\Delta v)' + \lambda' \Big\langle \begin{bmatrix} \frac{v'}{r} & 0 \\ 0 & v'' \end{bmatrix} R_\theta^T\vec{n}, R_\theta^T\vec{n}\Big\rangle = 2(\Delta v)' + \frac{v'}{r}\lambda' = 0,$$

so that the first equation in (7.4) is automatically satisfied. Similarly:

$$\Big(2\nabla^2 v + \lambda\mathrm{cof}\nabla^2 v\Big) : (\vec{n}\otimes\vec{n}) = \Big(2\begin{bmatrix} v'' & 0 \\ 0 & \frac{v'}{r}\end{bmatrix} + \lambda \begin{bmatrix} \frac{v'}{r} & 0 \\ 0 & v''\end{bmatrix}\Big) : (e_1\otimes e_1) = 2v'' + \lambda v',$$

so that the second equation in (7.4) is satisfied if and only if:

$$2v''(1) + \lambda(1) v'(1) = 0. \tag{7.7}$$

Let $\lambda \in \mathcal{C}^1([0,1])$ be the solution of the initial value problem (7.5) (7.7). As a side note, we remark that λ possesses the following limits: $\lim_{r\to 0}\lambda''(r) = \lim_{r\to 0}\frac{\lambda'(r)}{r} = -2\lim_{r\to 0}\frac{(\Delta v)'}{v'} = (\Delta v)''(0)$, so it follows directly that $\lambda = \lambda(r) \in W^{2,\infty}(B(0,1))$. In fact, λ is a distributional solution of (7.3) so in view of the elliptic regularity: $\lambda \in \mathcal{C}^\infty(B(0,1))$. Since (7.3) (7.4) hold, the proof of Proposition 7.3 is accomplished. ■

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