# THE MONGE-AMPÈRE SYSTEM: CONVEX INTEGRATION IN ARBITRARY DIMENSION AND CODIMENSION

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ABSTRACT. In this paper, we are concerned with the flexibility of weak solutions to the Monge-Ampére system via convex integration. This system is a natural extension of the Monge-Ampére equation in $d = 2$ dimensions, arising from the problems of isometric immersion and nonlinear elasticity. The main technical ingredient consists of the "stage" construction, in which we achieve the Hölder regularity $\mathcal{C}^{1,\alpha}$ of the approximating fields, for all $\alpha < \frac{1}{1+d(d+1)/k}$ where $d$ is an arbitrary dimension and $k \geq 1$ is an arbitrary codimension.

## CONTENTS



## 1. INTRODUCTION

This paper concerns the weak formulation of a multidimensional version of the Monge-Ampére equation, which arises in applications to the problem of isometric immersions and elasticity of prestrained materials. Namely, given a symmetric matrix field $A : \omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ on a domain $\omega \subset \mathbb{R}^d$, we look for vector fields $w$ and $v$ on $\omega$, such that:

$$\begin{aligned} & v : \omega \to \mathbb{R}^k, \quad w : \omega \to \mathbb{R}^d, \\ & \frac{1}{2}(\nabla v)^T \nabla v + \mathrm{sym}\nabla w = A \quad \text{ in } \omega. \end{aligned} \tag{VK}$$

Our main result states that any $\mathcal{C}^1$-regular pair $(v, w)$ which is a subsolution of (VK), can be uniformly approximated by a sequence of solutions $\{(v_n, w_n)\}_{n=1}^\infty$ of regularity $\mathcal{C}^{1,\alpha}$, for any Hölder exponent $\alpha < \frac{1}{1+2d_*/k}$ where by $d_* = d(d + 1)/2$ we denote the dimension of the space $\mathbb{R}^{d\times d}_{\mathrm{sym}}$. We refer to this type of result as *flexibility* up to $\mathcal{C}^{1,\frac{1}{1+2d_*/k}}$. More precisely, there holds:

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**Theorem 1.1.** *Let $\omega \subset \mathbb{R}^d$ be an open, bounded domain. Given two vector fields $v \in \mathcal{C}^1(\bar\omega, \mathbb{R}^k)$, $w \in \mathcal{C}^1(\bar\omega, \mathbb{R}^d)$ and a matrix field $A \in \mathcal{C}^{0,\beta}(\bar\omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, assume that:*

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T \nabla v + \mathrm{sym}\nabla w\big) \quad \textit{satisfies} \quad \mathcal{D} > c\,\mathrm{Id}_d \ \ \textit{on} \ \ \bar\omega,$$

*for some $c > 0$, in the sense of matrix inequalities. Fix $\epsilon > 0$ and let:*

$$0 < \alpha < \min\Big\{\frac{\beta}{2}, \frac{1}{1 + d(d+1)/k}\Big\}.$$

*Then, there exists $\tilde v \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^k)$ and $\tilde w \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^d)$ such that the following holds:*

$$\|\tilde v - v\|_0 \leq \epsilon, \quad \|\tilde w - w\|_0 \leq \epsilon, \tag{1.1$_1$}$$

$$A - \big(\frac{1}{2}(\nabla \tilde v)^T \nabla \tilde v + \mathrm{sym}\nabla \tilde w\big) = 0 \quad \textit{in} \ \ \bar\omega. \tag{1.1$_2$}$$

Our result generalizes [7, Theorem 1.1], where we proved flexibility for (VK) up to $\mathcal{C}^{1,\frac{1}{7}}$ in dimensions $d = 2$, $k = 1$. In that special case, motivated by theory of elasticity, the left hand side of (VK) represents the Von Kármán stretching content $\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\nabla w$ written in terms of the scalar out-of-plane displacement $v$ and the in-plane displacement $w$ of a middle plate $\omega$ of a thin film. The case $d = 2$ is special and flexibility (in codimension 1) of (VK) actually holds up to $\mathcal{C}^{1,\frac{1}{5}}$, as shown in [1, Theorem 1.1] using the conformal structure of $\mathbb{R}^2$.

The closely related problem is the isometric immersion problem $(\nabla u)^T \nabla u = g$, which reduces to (VK) by taking the family of Riemannian metrics $\{g = \mathrm{Id}_d + \epsilon A\}_{\epsilon \to 0}$ which are small perturbations of $\mathrm{Id}_d$, and gathering the lowest order terms in the $\epsilon$-expansion. For the isomeric immersion problem in arbitrary dimension $d$ but low codimension $k = 1$ corresponding to $u : \omega \to \mathbb{R}^{d+1}$, the parallel version of Theorem 1.1 has been shown in [2, Theorem 1.1] with flexibility up to $\mathcal{C}^{1,\frac{1}{1+2d_*}}$. We expect that the construction in the present paper can be extended also towards flexibility of isometric immersions, in case of arbitrary codimension of $u : \omega \to \mathbb{R}^{d+k}$. In this context, we observe that for $A \in \mathcal{C}^{1,1}$, Theorem 1.1 yields flexibility up to $\mathcal{C}^{1,\alpha}$ for $\alpha$ arbitrarily close to 1 as $k \to \infty$, which is consistent with the result in [6, Theorem 1.1], where the dependence of regularity on $k$ has not been, however, precisely quantified.

The main new technical ingredient allowing for the flexibility range stated in Theorem 1.1, is the following "stage"-type construction in the convex integration algorithm for (VK):

**Theorem 1.2.** *Let the vector fields $v \in \mathcal{C}^2(\bar\omega, \mathbb{R}^k)$, $w \in \mathcal{C}^2(\bar\omega, \mathbb{R}^d)$ and the matrix field $A \in \mathcal{C}^{0,\beta}(\bar\omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ be given on an open, bounded domain $\omega \subset \mathbb{R}^d$. Assume that:*

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T \nabla v + \mathrm{sym}\nabla w\big) \quad \textit{satisfies} \ \ 0 < \|\mathcal{D}\|_0 \leq 1.$$

*Fix two constants $M, \sigma$ such that:*

$$M \geq \max\{\|v\|_2, \|w\|_2, 1\} \quad \textit{and} \quad \sigma \geq 1.$$

*Then, there exist $\tilde v \in \mathcal{C}^2(\bar\omega, \mathbb{R}^k)$ and $\tilde w \in \mathcal{C}^2(\bar\omega, \mathbb{R}^d)$ such that, denoting:*

$$\tilde{\mathcal{D}} = A - \big(\frac{1}{2}(\nabla \tilde v)^T \nabla \tilde v + \mathrm{sym}\nabla \tilde w\big),$$

*the following holds:*

$$\|\tilde v - v\|_1 \leq C\|\mathcal{D}\|_0^{1/2}, \quad \|\tilde w - w\|_1 \leq C\|\mathcal{D}\|_0^{1/2}(1+\|\nabla v\|_0), \tag{1.2$_1$}$$

$$\|\nabla^2 \tilde v\|_0 \leq CM\sigma^{d_*/k}, \quad \|\nabla^2 \tilde w\|_0 \leq CM\sigma^{d_*/k}(1+\|\nabla v\|_0), \tag{1.2$_2$}$$

$$\|\tilde{\mathcal{D}}\|_0 \leq C\Big(\frac{\|A\|_{0,\beta}}{M^\beta}\|\mathcal{D}\|_0^{\beta/2} + \frac{\|\mathcal{D}\|_0}{\sigma}\Big), \tag{1.2$_3$}$$

*where $d_* = d(d+1)/2$ and where the constants $C$ depend only on $d, k$ and $\omega$.*

We briefly outline how our construction differs from [7] and [2]. There, a stage consisted of precisely $d_*$ "steps", each cancelling one of the one-dimensional "primitive" deficits in the decomposition of $\mathcal{D}$. The initially chosen frequency of perturbation was multiplied by a factor $\sigma$ at each step, leading to the increase of the second derivative by $\sigma^{d_*}$ and thus to the exponent $d_*$ replacing $d_*/k$ in $(1.2)_2$, while the remaining error in $\mathcal{D}$ was of order $1/\sigma$, leading to $(1.2)_3$.

Presently, we first observe that $k$ such deficits may be cancelled at once, by using $k$ linearly independent codimensions. Further, when all the first order primitive deficits are cancelled, one may proceed to cancelling the second order deficits obtained as the one-dimensional decompositions of the error between the original and the decreased $\mathcal{D}$; the corresponding frequencies must be then increased by the factor $\sigma^{1/2}$, precisely due to the decrease of $\mathcal{D}$ by the factor $1/\sigma$. One may inductively proceed in this fashion, cancelling even higher order deficits, and adding $k$-tuples of single codimension perturbations, for a total of $N = lcm(k, d_*)$ steps. The frequencies get increased by the factor of $\sigma$ over each multiple of $k$, leading to the increase of the second derivatives by $\sigma$, and by the factor of $\sigma^{1/2}$ over each multiple of $d_*$, where the deficit decreases by the factor of $1/\sigma$. In the final count, the total increase of the second derivatives has the factor $\sigma^{N/k}$, while the decrease of the deficit has the factor $1/\sigma^{N/d_*}$. The relative change of order is precisely $(N/k)/(N/d_*) = d_*/k$, as presented in Theorem 1.2.

We point out that for this scheme to work, it is essential to use the optimal "step"-type construction in which the chosen one-dimensional primitive deficit is cancelled at the expense of introducing least error terms possible. Our previous definition from [7] would not work for this purpose, and we need to superpose three oscillatory perturbations rather than two.

We now proceed to interpreting the flexibility result in Theorem 1.1 in the context of the "Monge-Ampére system" which comes up as a canonically equivalent version of (VK), as follows. Recall that for a matrix field $A = [A_{ij}]_{i,j=1\ldots 2} : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$, the scalar field $\mathrm{curl}\,\mathrm{curl} A$ is defined by taking the curl operator on each row of $A$, and then applying another curl on thus formed two-dimensional vector field:

$$\begin{aligned}\mathrm{curl}\,\mathrm{curl} A &= \mathrm{curl}\big[\partial_1 A_{12} - \partial_2 A_{11}, \partial_1 A_{22} - \partial_2 A_{21}\big]\\ &= \partial_1\partial_1 A_{22} - \partial_1\partial_2 A_{21} - \partial_1\partial_2 A_{12} + \partial_2\partial_2 A_{11}.\end{aligned}$$

It is a well known fact that the kernel of $\mathrm{curl}\,\mathrm{curl}$ when restricted to symmetric matrix fields, consists precisely of symmetric gradients. We will be concerned with the following generalization of $\mathrm{curl}\,\mathrm{curl}$, serving the same characterisation in higher dimensions:

**Definition 1.3.** *Given a $d$-dimensional square matrix field $A = [A_{ij}]_{i,j=1\ldots d} : \omega \to \mathbb{R}^{d\times d}$ on a domain $\omega \subset \mathbb{R}^d$, we define $\mathfrak{C}^2(A) : \omega \to \mathbb{R}^{d^4}$ by:*

$$\mathfrak{C}^2(A)_{ij,st} = \partial_i\partial_s A_{jt} + \partial_j\partial_t A_{is} - \partial_i\partial_t A_{js} - \partial_j\partial_s A_{it} \quad \textit{for all } i,j,s,t = 1\ldots d. \tag{1.3}$$

It can be checked that the components of the Riemann curvature tensor of a metric $\mathrm{Id}_d + \epsilon A$ on $\omega$, are given, to the leading order, by the components of $\mathfrak{C}^2(A)$:

$$Riem(\mathrm{Id}_d + \epsilon A)_{ij,st} = -\frac{\epsilon}{2}\mathfrak{C}^2(A)_{ij,st} + O(\epsilon^2) \quad \text{for all } \ i,j,s,t = 1\dots d. \tag{1.4}$$

We have the following:

**Lemma 1.4.** *Let $\omega \subset \mathbb{R}^d$ be an open, bounded, contractible domain with Lipschitz boundary. Given a symmetric matrix field $A \in L^2(\omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, the following conditions are equivalent:*

(i) $A = \mathrm{sym}\nabla w$ *for some* $w \in W^{1,2}(\omega, \mathbb{R}^d)$,
(ii) $\mathfrak{C}^2(A) = 0$ *in the sense of distributions on* $\omega$.

Observe that for $A = (\nabla v)^T\nabla v$ given through a vector field $v : \omega \to \mathbb{R}^k$, there holds:

$$\mathfrak{C}^2\big((\nabla v)^T\nabla v\big)_{ij,st} = 2\langle \partial_i\partial_t v, \partial_j\partial_s v\rangle - 2\langle \partial_i\partial_s v, \partial_j\partial_t v\rangle.$$

When $d = 2$ and $k = 1$, the above reduces to the familiar formula: $\mathrm{curl}\,\mathrm{curl}(\nabla v \otimes \nabla v) = -2\det\nabla^2 v$. Following this motivation, we introduce:

**Definition 1.5.** *For $v : \omega \to \mathbb{R}^k$ defined on a domain $\omega \subset \mathbb{R}^d$, we set $\mathfrak{Det}\,\nabla^2 v : \omega \to \mathbb{R}^{d^4}$ in:*

$$\big(\mathfrak{Det}\,\nabla^2 v\big)_{ij,st} = \langle \partial_i\partial_s v, \partial_j\partial_t v\rangle - \langle \partial_i\partial_t v, \partial_j\partial_s v\rangle \quad \text{for all } \ i,j,s,t = 1\dots d. \tag{1.5}$$

*Given $F : \omega \to \mathbb{R}^{d^4}$, we call the following system of Pdes, the Mongé-Amperé system:*

$$\mathfrak{Det}\,\nabla^2 v = F \quad \text{on } \ \omega.$$

Lemma 1.4 can be restated in this context as follows. Given a matrix field $A : \omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ on a domain $\omega \subset \mathbb{R}^d$, problem (VK) is equivalent to (disregarding the regularity questions):

$$\begin{aligned} &v : \omega \to \mathbb{R}^k, \\ &\mathfrak{Det}\,\nabla^2 v = -\mathfrak{C}^2(A), \end{aligned} \tag{MA}$$

which, for $d = 2$ and $k = 1$, is precisely the Monge-Ampére constraint $\det\nabla^2 v = -\mathrm{curl}^T\mathrm{curl}A$ appearing in the linearized Kirchhoff's theory of thin plates [4]. For the family of immersions: $\bar\phi_\epsilon = id_d + \epsilon[0,v] + \epsilon^2[w,0] : \omega \to \mathbb{R}^{d+k}$, one further notes that:

$$(\nabla\bar\phi_\epsilon)^T\nabla\bar\phi_\epsilon = \mathrm{Id}_d + \epsilon^2\big((\nabla v)^T\nabla v + 2\,\mathrm{sym}\nabla w\big) + O(\epsilon^4).$$

From (1.4), we thus see that the problem of finding a vector field $v$ for which the Riemann curvatures of the metrics $\mathrm{Id}_d + \epsilon^2 A$ and the Riemann curvatures of the pull-back of $\mathrm{Id}_{d+k}$ via the reduced maps $\phi_\epsilon$ below, coincide at their lowest order terms in $\epsilon$ on $\omega$:

$$\begin{aligned} &\phi_\epsilon = id_d + \epsilon[0,v] : \omega \to \mathbb{R}^{d+k}, \\ &Riem\big(\mathrm{Id}_d + \epsilon^2 A\big) = Riem\big((\nabla\phi_\epsilon)^T\nabla\phi_\epsilon\big) + o(\epsilon^2) \end{aligned} \tag{1.6}$$

is equivalent to the problem of finding $v$ that can be matched by an auxiliary vector field $w$ so that the two Riemannian metrics families: $\mathrm{Id}_d + \epsilon^2 A$, and the pull-back of $\mathrm{Id}_{d+k}$ via the maps $\bar\phi_\epsilon$, coincide at their lowest order terms in $\epsilon$ on $\omega$:

$$\begin{aligned} &\bar\phi_\epsilon = id_d + \epsilon[0,v] + \epsilon^2[w,0] : \omega \to \mathbb{R}^{d+k}, \\ &\mathrm{Id}_d + \epsilon^2 A = (\nabla\phi_\epsilon)^T\nabla\phi_\epsilon\big) + o(\epsilon^2). \end{aligned} \tag{1.7}$$

Thus, the four problems (VK), (MA), (1.6) and (1.7) are equivalent.

The above discussion motivates the following:

**Definition 1.6.** *Assume that $F \in L^2(\omega, \mathbb{R}^{d^4})$ given on an open, bounded, contractible domain $\omega \subset \mathbb{R}^d$ with Lipschitz boundary, satisfies conditions (6.3). We say that $v \in W^{1,2}_{loc}(\omega, \mathbb{R}^k)$ is a very weak solution to the Monge-Amperé system:*

$$\mathfrak{Det}\,\nabla^2 v = F \quad \text{on } \omega, \tag{1.8}$$

*provided that there exists $w \in W^{1,1}_{loc}(\omega, \mathbb{R}^d)$ such that (VK) holds with $\mathfrak{C}^2(A) = -F$, namely:*

$$\frac{1}{2}(\nabla v)^T \nabla v + \mathrm{sym}\nabla w = -\big(\mathfrak{C}^2\big)^{-1}(F) \quad \text{on } \omega.$$

For $d = 2$, $k = 1$, any $F \in L^{1+}(\omega, \mathbb{R})$ can be expressed as the right hand side of (MA), because writing $A = \gamma \mathrm{Id}_2$ where $\Delta\gamma = -F$ in $\omega$, there holds: $F = -\mathrm{curl}\,\mathrm{curl}\, A$. In higher dimensions, the solvability conditions are given by the specific derived symmetry and Bianchi identities (6.3) in Theorem 6.1. In view of Theorem 1.1, we thus obtain the following extension of [7, Theorem 1.1] proved in dimension $d = 2$ and codimension $k = 1$, now to arbitrary $d, k \geq 1$:

**Theorem 1.7.** *Let $F \in L^\infty(\omega, \mathbb{R}^{d^4})$ on an open, bounded, contractible domain $\omega \subset \mathbb{R}^d$ with Lipschitz boundary, satisfy the compatibility conditions (6.3). Fix $k \geq 1$ and fix an exponent:*

$$0 < \alpha < \frac{1}{1 + d(d+1)/k}.$$

*Then the set of $\mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^k)$ very weak solutions to (1.8) is dense in $\mathcal{C}^0(\bar\omega, \mathbb{R}^k)$. Namely, every $v \in \mathcal{C}^0(\bar\omega, \mathbb{R}^k)$ is the uniform limit of some sequence $\{v_n \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^k)\}_{n=1}^\infty$, such that:*

$$\mathfrak{Det}\,\nabla^2 v_n = F \quad \text{on } \omega, \quad \text{for all } n = 1 \ldots \infty.$$

1.1. **Overview of the paper.** In section 2 we construct the single "step" of our convex integration algorithm, and recall a few auxiliary results. The proof of Theorem 1.2 and the "stage" construction is carried out in section 3. The Nash-Kuiper scheme involving induction on stages is presented in section 4, and Theorem 1.1 is proved in section 5. In section 6 we discuss the Monge-Ampére system and prove Lemma 1.4 and Theorem 1.7. Finally, in section 7 we present another application of Theorem 1.1 in the context of the scaling of elastic-type energies and the quantitative isometric immersion problem.

1.2. **Notation.** By $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ we denote the space of symmetric $d \times d$ matrices, and by $\mathbb{R}^{d\times d}_{\mathrm{sym},>}$ we denote the cone of symmetric, positive definite $d \times d$ matrices. The space of Hölder continuous vector fields $\mathcal{C}^{m,\alpha}(\bar\omega, \mathbb{R}^k)$ consists of restrictions of all $f \in \mathcal{C}^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^k)$ to the closure of an open domain $\omega \subset \mathbb{R}^d$. Then, the $\mathcal{C}^m(\bar\omega, \mathbb{R}^k)$ norm of such restriction is denoted by $\|f\|_m$, while its Hölder norm $\mathcal{C}^{m,\alpha}(\bar\omega, \mathbb{R}^k)$ is $\|f\|_{m,\alpha}$. By $C > 0$ we denote a universal constant which may change from line to line, but which is independent of all parameters, unless indicated otherwise.

## 2. Convex integration: the basic "step" and preparatory statements

The following single "step" construction is an important building block of our convex integration algorithm. A similar calculation in [7] essentially had $\bar\Gamma = 0$, resulting in the presence of the extra term $-\frac{2}{\lambda} a\bar{\bar\Gamma}(\lambda t_\eta)\mathrm{sym}(\nabla a \otimes \eta)$ in the right hand side of (2.2). With that term, achieving the construction of a stage in section 3 and obtaining error bounds leading to the Hölder exponent threshold in Theorems 1.1 and 1.7 would not be possible.

**Lemma 2.1.** *Let $v \in \mathcal{C}^2(\omega, \mathbb{R}^k)$ and $w \in \mathcal{C}^1(\omega, \mathbb{R}^d)$ be two vector fields on an open domain $\omega \subset \mathbb{R}^d$. Let $\eta \in \mathbb{R}^d$ and $E \in \mathbb{R}^k$ be two unit vectors and let $\lambda > 0$, $a \in \mathcal{C}^2(\omega, \mathbb{R})$. We denote:*

$$\Gamma(t) = 2\sin t, \quad \bar\Gamma(t) = -\frac{1}{2}\cos(2t), \quad \bar{\bar\Gamma}(t) = -\frac{1}{2}\sin(2t).$$

*Denoting further* $t_\eta = \langle x, \eta\rangle$, *we define:*

$$
\begin{aligned}
&\tilde v(x) = v(x) + \frac{1}{\lambda}a(x)\Gamma(\lambda t_\eta)E\\
&\tilde w(x) = w(x) - \frac{1}{\lambda}a(x)\Gamma(\lambda t_\eta)\nabla\langle v(x), E\rangle - \frac{1}{\lambda^2}a(x)\bar\Gamma(\lambda t_\eta)\nabla a(x) + \frac{1}{\lambda}a(x)^2\bar{\bar\Gamma}(\lambda t_\eta)\eta.
\end{aligned}
\tag{2.1}
$$

*Then we have:*

$$
\begin{aligned}
&\Big(\frac12(\nabla\tilde v)^T\nabla\tilde v + \mathrm{sym}\nabla\tilde w\Big) - \Big(\frac12(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\Big) - a^2\eta\otimes\eta\\
&= -\frac{1}{\lambda}a\Gamma(\lambda t_\eta)\nabla^2\langle v, E\rangle + \frac{1}{\lambda^2}\Big(\frac12\Gamma(\lambda t_\eta)^2 - \bar\Gamma(\lambda t_\eta)\Big)\nabla a\otimes\nabla a - \frac{1}{\lambda^2}a\bar\Gamma(\lambda t_\eta)\nabla^2 a,
\end{aligned}
\tag{2.2}
$$

*where* $\frac12\Gamma(t)^2 - \bar\Gamma(t) = 1 - \frac12\cos(2t)$.

*Proof.* By a direct calculation we obtain:

$$\nabla\tilde v = \nabla v + \frac{1}{\lambda}\Gamma(\lambda t_\eta)E\otimes\nabla a + a\Gamma'(\lambda t_\eta)E\otimes\eta,$$

which implies:

$$
\begin{aligned}
\frac12(\nabla\tilde v)^T\nabla\tilde v - \frac12(\nabla v)^T\nabla v = {}& \frac12 a^2\Gamma'(\lambda t_\eta)^2\eta\otimes\eta + \frac{1}{\lambda}a\Gamma'(\lambda t_\eta)\Gamma(\lambda t_\eta)\mathrm{sym}\big(\nabla a\otimes\eta\big)\\
&+ \frac{1}{2\lambda^2}\Gamma(\lambda t_\eta)^2\nabla a\otimes\nabla a.\\
&+ \Big(a\Gamma'(\lambda t_\eta)\mathrm{sym}\big((\eta\otimes E)\nabla v\big) + \frac{1}{\lambda}\Gamma(\lambda t_\eta)\mathrm{sym}\big((\nabla a\otimes E)\nabla v\big)\Big).
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
\mathrm{sym}\nabla\tilde w - \mathrm{sym}\nabla w = {}& a^2\bar{\bar\Gamma}'(\lambda t_\eta)\eta\otimes\eta + \frac{1}{\lambda}a\big(-\bar\Gamma'(\lambda t_\eta) + 2\bar{\bar\Gamma}(\lambda t_\eta)\big)\mathrm{sym}\big(\nabla a\otimes\eta\big)\\
&- \frac{1}{\lambda}a\Gamma(\lambda t_\eta)\nabla^2\langle v, E\rangle - \frac{1}{\lambda^2}\bar\Gamma(\lambda t_\eta)\nabla a\otimes\nabla a - \frac{1}{\lambda^2}a\bar\Gamma(\lambda t_\eta)\nabla^2 a\\
&- \Big(a\Gamma'(\lambda t_\eta)\mathrm{sym}\big(\eta\otimes\nabla\langle v, E\rangle\big) + \frac{1}{\lambda}\Gamma(\lambda t_\eta)\mathrm{sym}\big(\nabla a\otimes\nabla\langle v, E\rangle\big)\Big).
\end{aligned}
$$

Summing the above two identities and noting that:

$$\frac12(\Gamma')^2 + \bar{\bar\Gamma}' = 1 \quad\text{and}\quad \Gamma'\Gamma - \bar\Gamma' + 2\bar{\bar\Gamma} = 0,$$

we arrive at the claimed identity (2.2). The proof is done. ■

We next observe that taking several perturbations in $v$ of the form $\frac{1}{\lambda}a\Gamma(\lambda t_\eta)E$, and matching them with the perturbations of $w$ as in (2.1), accumulates the error in (2.2) in a linear fashion as long as the directions $E$ are linearly independent. The same calculations as above, lead to:

**Corollary 2.2.** *Let* $v \in \mathcal{C}^2(\omega, \mathbb{R}^k)$ *and* $w \in \mathcal{C}^1(\omega, \mathbb{R}^d)$ *be two vector fields on an open domain* $\omega \subset \mathbb{R}^d$. *Let* $\{\eta_i \in \mathbb{R}^d\}_{i=1}^k$ *be given unit vectors, and let* $\{E_i \in \mathbb{R}^k\}_{i=1}^k$ *also be unit vectors which are moreover linearly independent in* $\mathbb{R}^k$. *Given* $\{\lambda_i > 0\}_{i=1}^k$ *and* $\{a_i \in \mathcal{C}^1(\omega, \mathbb{R})\}_{i=1}^k$, *we set:*

$$
\begin{aligned}
&\tilde v = v + \sum_{i=1}^k \frac{1}{\lambda_i}a_i\Gamma(\lambda_i t_{\eta_i})E_i\\
&\tilde w = w - \sum_{i=1}^k \frac{1}{\lambda_i}a_i\Gamma(\lambda_i t_{\eta_i})\nabla\langle v, E_i\rangle - \sum_{i=1}^k \frac{1}{\lambda_i^2}a_i\bar\Gamma(\lambda_i t_{\eta_i})\nabla a_i + \sum_{i=1}^k \frac{1}{\lambda_i}a_i^2\bar{\bar\Gamma}(\lambda_i t_{\eta_i})\eta_i,
\end{aligned}
$$

*where the functions* $\Gamma$, $\bar\Gamma$, $\bar{\bar\Gamma}$ *and* $t_\eta$ *are defined as in Lemma 2.1. Then we have:*

$$\begin{aligned}&\big(\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v+\mathrm{sym}\nabla\tilde w\big)-\big(\frac{1}{2}(\nabla v)^T\nabla v+\mathrm{sym}\nabla w\big)-\sum_{i=1}^k a_i^2\eta_i\otimes\eta_i\\ &\qquad=-\sum_{i=1}^k\frac{1}{\lambda_i}a_i\Gamma(\lambda_i t_{\eta_i})\nabla^2\langle v,E_i\rangle+\sum_{i=1}^k\frac{1}{\lambda_i^2}\big(\frac{1}{2}\Gamma(\lambda_i t_{\eta_i})^2-\bar\Gamma(\lambda_i t_{\eta_i})\big)\nabla a_i\otimes\nabla a_i\\ &\qquad\quad-\sum_{i=1}^k\frac{1}{\lambda_i^2}a_i\bar\Gamma(\lambda_i t_{\eta_i})\nabla^2 a_i.\end{aligned}$$

We now recall two auxiliary results from [2]. The first one gathers the convolution and commutator estimates [2, Lemma 2.1]:

**Lemma 2.3.** *Let* $\phi\in\mathcal{C}_c^\infty(\mathbb{R}^d,\mathbb{R})$ *be a standard mollifier that is nonnegative, radially symmetric, supported on the unit ball* $B(0,1)\subset\mathbb{R}^d$ *and such that* $\int_{\mathbb{R}^d}\phi\,\mathrm{d}x=1$. *Denote:*

$$\phi_l(x)=\frac{1}{l^d}\phi(\frac{x}{l})\qquad\textit{for all }\ l\in(0,1],\ x\in\mathbb{R}^d.$$

*Then, for every* $f,g\in\mathcal{C}^0(\mathbb{R}^d,\mathbb{R})$ *and every* $m,n\geq 0$ *and* $\beta\in(0,1]$ *there holds:*

$$\|\nabla^{(m)}(f*\phi_l)\|_0\leq\frac{C}{l^m}\|f\|_0, \tag{2.3$_1$}$$

$$\|f-f*\phi_l\|_0\leq C\min\big\{l^2\|\nabla^2 f\|_0,l\|\nabla f\|_0,l^\beta\|f\|_{0,\beta}\big\}, \tag{2.3$_2$}$$

$$\|\nabla^{(m)}\big((fg)*\phi_l-(f*\phi_l)(g*\phi_l)\big)\|_0\leq Cl^{2-m}\|\nabla f\|_0\|\nabla g\|_0, \tag{2.3$_4$}$$

*with a constant* $C>0$ *depending only on the differentiability exponent* $m$.

The next result states the decomposition of symmetric positive definite matrices which are close to $\mathrm{Id}_d$, into "primitive matrices", as proved in [2, Lemma 5.2]:

**Lemma 2.4.** *Given the dimension* $d\geq 1$, *let* $d_*$ *be the dimension of the space* $\mathbb{R}^{d\times d}_{\mathrm{sym}}$, *namely:*

$$d_*=\frac{d(d+1)}{2}.$$

*There exist: a constant* $r_0>0$, *the linear maps* $\{\bar a_i:\mathbb{R}^{d\times d}_{\mathrm{sym}}\to\mathbb{R}\}_{i=1}^{d_*}$, *and the unit vectors* $\{\eta_i\in\mathbb{R}^d\}_{i=1}^{d_*}$, *such that for all* $A\in B(\mathrm{Id}_d,r_0)\subset\mathbb{R}^{d\times d}_{\mathrm{sym}}$, *there holds:*

$$A=\sum_{i=1}^{d_*}\bar a_i(A)\eta_i\otimes\eta_i\qquad\textit{and}\qquad\bar a_i(A)\geq r_0\ \ \textit{for all}\ \ i=1\dots d_*.$$

## 3. The "stage" for the $\mathcal{C}^{1,\alpha}$ approximations: a proof of Theorem 1.2

The following result is the main technical contribution of this paper:

**Proof of Theorem 1.2**

The proof consists of several steps in an inductive construction below.

**1. (Preparing the data)** Recall that $v,w,A$ are restrictions to $\bar\omega$ of some $v,w,A$ defined on and, without loss of generality, compactly supported in $\mathbb{R}^d$. We set the mollification scale:

$$l=\frac{\|\mathcal{D}\|_0^{1/2}}{M}\in(0,1], \tag{3.1}$$

and taking $\phi_l(x) = \frac{1}{l^d}\phi(x/l)$ as in Lemma 2.3, we define:

$$v_0 = v * \phi_l, \quad w_0 = w * \phi_l, \quad A_0 = A * \phi_l, \quad \mathcal{D}_0 = \big(\frac{1}{2}(\nabla v_0)^T\nabla v_0 + \mathrm{sym}\nabla w_0\big) - A_0.$$

From the estimates in Lemma 2.3, one deduces the initial bounds:

$$\|v_0 - v\|_1 + \|w_0 - w\|_1 \leq C\|\mathcal{D}\|_0^{1/2}, \tag{3.2$_1$}$$

$$\|A_0 - A\|_0 \leq Cl^\beta \|A\|_{0,\beta}, \tag{3.2$_2$}$$

$$\|\nabla^{(m+1)}v_0\|_0 + \|\nabla^{(m+1)}w_0\|_0 \leq \frac{C}{l^m}\|\mathcal{D}\|_0^{1/2} \quad \text{for all } \ m \geq 1, \tag{3.2$_3$}$$

$$\|\nabla^{(m)}\mathcal{D}_0\|_0 \leq \frac{C}{l^m}\|\mathcal{D}\|_0 \quad \text{for all } \ m \geq 0. \tag{3.2$_4$}$$

Indeed, $(3.2)_2$ follows directly from $(2.3)_2$, and $(3.2)_1$ similarly follows by applying $(2.3)_2$ to $v$, $\nabla v$, $w$, $\nabla w$ and noting that, in view of (3.1) we have:

$$l\|v\|_2 + l\|w\|_2 \leq 2\|\mathcal{D}\|_0^{1/2}. \tag{3.3}$$

Further, $(3.2)_3$ follows by applying $(2.3)_1$ to $\nabla^2 v$ and $\nabla^2 w$ with the differentiability exponent $m-1$ and again taking into account (3.3). To check $(3.2)_4$, we write:

$$\mathcal{D}_0 = \frac{1}{2}\big((\nabla v_0)^T\nabla v_0 - ((\nabla v)^T\nabla v) * \phi_l\big) - \mathcal{D} * \phi_l,$$

and apply $(2.3)_1$ to $\mathcal{D}$, and $(2.3)_4$ to $(\nabla v)^T$ and $\nabla v$, where the final bound is due to (3.3).

**2. (Induction definition: frequencies)** We now inductively define the main constants, frequencies and corrections in the construction of $(\tilde{v}, \tilde{w})$ from $(v, w)$. First, we write the least common multiple of the auxiliary dimension $d_*$ and the codimension $k$, as follows:

$$N = lcm(d_*, k) = Sd_* = Jk, \qquad S, J \geq 1. \tag{3.4}$$

Then, we set the initial perturbation frequencies as:

$$\lambda_0 = \frac{1}{l}, \qquad \lambda_1 = \lambda = \frac{\sigma^{1/S}}{l}.$$

For every $i = 2\ldots N$ we define $\lambda_i \geq 1$ according to the mutually exclusive cases in:

$$\lambda_i = \lambda_{i-1}\cdot\begin{cases} (\lambda l) & \text{if } \ k \mid (i-1), \\ (\lambda l)^{1/2} & \text{if } \ d_* \mid (i-1), \\ 1 & \text{otherwise.} \end{cases}$$

It follows that for all $j = 0\ldots J-1$ and $s = 0\ldots S-1$ there holds:

$$\lambda_i l = (\lambda l)^{1+j+s/2} \quad \text{for all } \ i \in (jk, (j+1)k] \cap (sd_*, (s+1)d_*]. \tag{3.5}$$

**3. (Induction definition: decomposition of deficits)** First, let $\{\eta_\delta \in \mathbb{R}^d\}_{\delta=1}^{d_*}$ be the unit vectors as in Lemma 2.4. For all $s = 0\ldots S-1$ we define constants $\tilde{C}_s$ and perturbation amplitudes vector $a^s = [a^s_\delta]_{\delta=1}^{d_*} \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{d_*})$ by:

$$\tilde{C}_s = \frac{2}{r_0}\Big(\frac{1}{(\lambda l)^s}\|\mathcal{D}\|_0 + \|\mathcal{D}_s\|_0\Big),$$

$$a^s_\delta(x) = \Big(\tilde{C}_s\bar{a}_\delta\big(\mathrm{Id}_d - \frac{1}{\tilde{C}_s}\mathcal{D}_s(x)\big)\Big)^{1/2} \quad \text{for all } \ \delta = 1\ldots d_*, \ \ x \in \bar{\omega}.$$

Above, $r_0 > 0$ and the maps $\bar{a}_\delta$ are as in Lemma 2.4, so our definition is correctly posed because $\mathrm{Id}_d - \frac{1}{\tilde{C}_s}\mathcal{D}_s(x) \in B(\mathrm{Id}_d, r_0) \subset \mathbb{R}^{d\times d}_{\mathrm{sym}}$ for all $x \in \bar{\omega}$. As $\tilde{C}_s\mathrm{Id}_d - \mathcal{D}_s = \tilde{C}_s\big(\mathrm{Id}_d - \frac{1}{\tilde{C}_s}\mathcal{D}_s\big)$, we get:

$$\tilde{C}_s\mathrm{Id}_d - \mathcal{D}_s = \sum_{\delta=1}^{d_*}(a^s_\delta)^2\eta_\delta\otimes\eta_\delta \quad \text{ and } \quad (a^s_\delta)^2 \geq r_0\tilde{C}_s \text{ in } \bar{\omega}, \text{ for all } \delta = 1\dots d_*. \tag{3.6}$$

Since $\{\eta_\delta\otimes\eta_\delta\}_{\delta=1}^{d_*}$ is a basis of the linear space $\mathbb{R}^{d\times d}_{\mathrm{sym}}$, we obtain:

$$\|a^s\|_0 \leq C\|\tilde{C}_s\mathrm{Id}_d - \mathcal{D}_s\|_0^{1/2} \leq C\tilde{C}_s^{1/2}. \tag{3.7}$$

We also right away observe that, by the Faá di Bruno formula, there holds, for $m \geq 1$:

$$\|\nabla^{(m)}a^s_\delta\|_0 \leq C\Big\|\sum_{p_1+2p_2+\dots mp_m=m}|a^s_\delta|^{2(1/2-p_1-\dots-p_m)}\prod_{t=1}^m\big|\nabla^{(t)}|a^s_\delta|^2\big|^{p_t}\Big\|_0 \quad \text{for all } \delta = 1\dots d_*.$$

Using the lower bound in (3.6) and the linearity of $\bar{a}_\delta$ in Lemma 2.4, we further get:

$$\begin{aligned}\|\nabla^{(m)}a^s\|_0 &\leq C\sum_{p_1+2p_2+\dots mp_m=m}\frac{1}{\tilde{C}_s^{(p_1+\dots+p_m)-1/2}}\prod_{t=1}^m\|\nabla^{(t)}\mathcal{D}_s\|_0^{p_t} \\ &\leq C\tilde{C}_s^{1/2}\sum_{p_1+2p_2+\dots mp_m=m}\prod_{t=1}^m\Big(\frac{\|\nabla^{(t)}\mathcal{D}_s\|_0}{\tilde{C}_s}\Big)^{p_t}.\end{aligned} \tag{3.8}$$

In particular, for $s = 0$ and any $\delta = 1\dots d_*$, the bounds (3.7), (3.8) and $(3.2)_4$ yield:

$$\tilde{C}_0 \leq C\|\mathcal{D}\|_0 \quad \text{ and } \quad \|\nabla^{(m)}a^0\|_0 \leq \frac{C}{l^m}\|\mathcal{D}\|_0^{1/2} \quad \text{for all } m \geq 0. \tag{3.9}$$

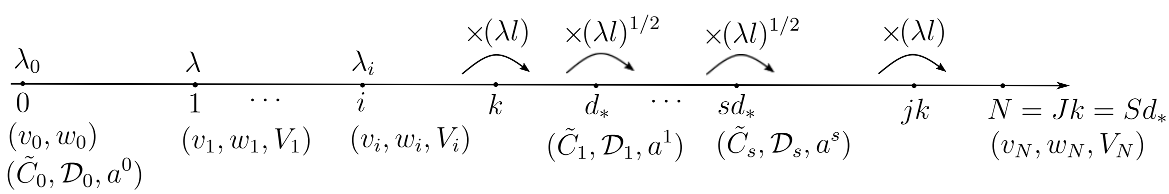


FIGURE 1. Progression of frequencies $\lambda_i$ and other intermediary quantities defined at integers $i = 1\dots N = lcm(k, d_*)$.

**4. (Induction definition: perturbations)** For each $i = 1\dots N$ we may uniquely write:

$$\begin{aligned} i = jk+\gamma = sd_*+\delta \quad \text{with} \quad & j = 0\dots J-1, \quad \gamma = 1\dots k, \\ & s = 0\dots S-1, \quad \delta = 1\dots d_*.\end{aligned} \tag{3.10}$$

Define $v_i \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^k)$ and $w_i \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^d)$ according to the "step" construction in Lemma 2.1, involving the periodic profile functions $\Gamma, \bar{\Gamma}, \bar{\bar{\Gamma}}$ and the notation $t_\eta = \langle x, \eta\rangle$:

$$\begin{aligned} v_i(x) &= v_{i-1}(x) + \frac{1}{\lambda_i}a^s_\delta(x)\Gamma(\lambda_i t_{\eta_\delta})e_\gamma, \\ w_i(x) &= w_{i-1}(x) - \frac{1}{\lambda_i}a^s_\delta(x)\Gamma(\lambda_i t_{\eta_\delta})\nabla v^\gamma_{i-1} - \frac{1}{\lambda_i^2}a^s_\delta(x)\bar{\Gamma}(\lambda_i t_{\eta_\delta})\nabla a^s_\delta + \frac{1}{\lambda_i}a^s_\delta(x)^2\bar{\bar{\Gamma}}(\lambda_i t_{\eta_\delta})\eta_\delta.\end{aligned}$$

We observe that by construction of $v_i$, the second term in $w_i$ can be rewritten as follows:

$$\frac{1}{\lambda_i}a^s_\delta(x)\Gamma(\lambda_i t_{\eta_\delta})\nabla v^\gamma_{i-1} = \frac{1}{\lambda_i}a^s_\delta(x)\Gamma(\lambda_i t_{\eta_\delta})\nabla v^\gamma_{jk}. \tag{3.11}$$

We eventually define:

$$\tilde v = v_N, \qquad \tilde w = w_N - \sum_{s=0}^{S-1}\tilde C_s x. \tag{3.12}$$

**5. (Induction definition: deficits)** For each $i = 1\dots N$, we define the partial deficit:

$$V_i = \big(\frac{1}{2}(\nabla v_i)^T\nabla v_i + \mathrm{sym}\nabla w_i\big) - \big(\frac{1}{2}(\nabla v_{i-1})^T\nabla v_{i-1} + \mathrm{sym}\nabla w_{i-1}\big),$$

and for each $s = 1\dots S$ we define the combined deficit: $\mathcal{D}_s \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ in:

$$\begin{aligned}\mathcal{D}_s = & \big(\frac{1}{2}(\nabla v_{sd_*})^T\nabla v_{sd_*} + \mathrm{sym}\nabla w_{sd_*}\big) - \big(\frac{1}{2}(\nabla v_{(s-1)d_*})^T\nabla v_{(s-1)d_*} + \mathrm{sym}\nabla w_{(s-1)d_*}\big) \\ & - \sum_{\delta=1}^{d_*}(a_\delta^{s-1})^2\eta_\delta\otimes\eta_\delta \\ = & \sum_{i=(s-1)d_*+1}^{sd_*}\Big(V_i - (a_\delta^{s-1})^2\eta_\delta\otimes\eta_\delta\Big),\end{aligned}$$

where in components of the last sum we used the convention (3.10), setting $\delta = \delta(i) = 1\dots d_*$. By Lemma 2.1 and noting (3.11), for each $i = (s-1)d_*\dots sd_*$ as above, we get:

$$\begin{aligned}V_i - (a_\delta^{s-1})^2\eta_\delta\otimes\eta_\delta = & -\frac{1}{\lambda_i}a_\delta^{s-1}\Gamma(\lambda_i t_{\eta_\delta})\nabla^2 v_{jk}^\gamma - \frac{1}{\lambda_i^2}a_\delta^{s-1}\bar\Gamma(\lambda_i t_{\eta_\delta})\nabla^2 a_\delta^{s-1} \\ & + \frac{1}{\lambda_i^2}\big(\frac{1}{2}\Gamma(\lambda_i t_{\eta_\delta})^2 - \bar\Gamma(\lambda_i t_{\eta_\delta})\big)\nabla a_\delta^{s-1}\otimes\nabla a_\delta^{s-1},\end{aligned} \tag{3.13}$$

where $j = 0\dots J-1$ is again set according to (3.10).

**6. (Inductive estimates)** In steps 7-10 below we will prove the following estimates:

$$\left.\begin{aligned}&\|v_i - v_{i-1}\|_1 \leq C\|\mathcal{D}\|_0^{1/2} \\ &\|w_i - w_{i-1}\|_1 \leq C\|\mathcal{D}\|_0^{1/2}(1+\|\nabla v\|_0)\end{aligned}\right\}\quad \text{for all } \ i = 1\dots N, \tag{3.14$_1$}$$

$$\left.\begin{aligned}&\|\nabla^{(m+1)}v_{kj}\|_0 \leq C\frac{\lambda_{kj}^{m-1}}{l}(\lambda l)^j\|\mathcal{D}\|_0^{1/2} \\ &\|\nabla^{(m+1)}w_{kj}\|_0 \leq C\frac{\lambda_{kj}^{m-1}}{l}(\lambda l)^j\|\mathcal{D}\|_0^{1/2}(1+\|\nabla v\|_0)\end{aligned}\right\}\quad \text{for all } \ j = 0\dots J, \ \ m\geq 1, \tag{3.14$_2$}$$

$$\tilde C_s \leq \frac{C}{(\lambda l)^s}\|\mathcal{D}\|_0, \qquad \|\nabla^{(m)}\mathcal{D}_s\|_0 \leq C\frac{\lambda_{sd_*}^m}{(\lambda l)^s}\|\mathcal{D}\|_0 \quad \text{for all } \ s = 0\dots S, \ \ m\geq 0, \tag{3.14$_3$}$$

$$\|\nabla^{(m)}a^s\|_0 \leq C\frac{\lambda_{sd_*}^m}{(\lambda l)^{s/2}}\|\mathcal{D}\|_0^{1/2} \quad \text{for all } \ s = 0\dots S-1, \ \ m\geq 0. \tag{3.14$_4$}$$

We observe that all the bounds are already valid at their lowest counter values: by $(3.2)_3$ there holds $(3.14)_2$ for $j=0$, the first bound in $(3.14)_3$ and the bound in $(3.14)_4$ at $s=0$ have been established in (3.9), while the second bound in $(3.14)_3$ at $s=0$ is exactly $(3.2)_4$. To show

$(3.14)_1$ at $i=1$, we use (3.9) and $(3.2)_3$ in:

$$\begin{aligned}\|v_1-v_0\|_1 &\leq C\Big(\|a^0\|_0+\frac{\|\nabla a^0\|_0}{\lambda}\Big)\leq C\|\mathcal{D}\|_0^{1/2}\big(1+\frac{1}{\lambda l}\big)\leq C\|\mathcal{D}\|_0^{1/2},\\ \|w_1-w_0\|_1 &\leq C\Big(\|a^0\|_0\|\nabla v_0\|_0+\|a^0\|_0^2+\frac{\|\nabla a^0\|_0^2+\|a^0\|_0\|\nabla^2a^0\|_0}{\lambda^2}\\ &\qquad +\frac{\|a^0\|_0\|\nabla a^0\|_0+\|\nabla a^0\|_0\|\nabla v_0\|_0+\|a^0\|_0\|\nabla^2 v_0\|_0}{\lambda}\Big)\\ &\leq C\|\mathcal{D}\|_0^{1/2}(1+\|\nabla v\|_0),\end{aligned}$$

because $\lambda l\geq 1$ and $\|\nabla v_0\|_0\leq\|\nabla v\|_0+C\|\mathcal{D}\|^{1/2}\leq C(1+\|\nabla v\|_0)$ from $(3.2)_1$.

**7. (Proof of estimate $(3.14)_1$)** For $i\in(1,N)$, we write:

$$i\in(jk,(j+1)k]\cap(sd_*,(s+1)d_*]$$

with $j,s$ as in (3.10). By $(3.14)_4$, we get:

$$\|v_i-v_{i-1}\|_1\leq C\Big(\|a^s\|_0+\frac{\|\nabla a^s\|_0}{\lambda_i}\Big)\leq\frac{C}{(\lambda l)^{s/2}}\|\mathcal{D}\|_0^{1/2}\big(1+\frac{\lambda_{sd_*}}{\lambda_i}\big)\leq C\|\mathcal{D}\|_0^{1/2},$$

where we used that $\lambda l\geq 1$ and $\lambda_{sd_*}l\leq\lambda_i l$, due to $i>sd_*$. The bound for the $w$-increment follows by $(3.14)_2$ at $m=1$, $(3.14)_4$, $(3.14)_1$ and $(3.2)_1$:

$$\begin{aligned}\|w_i-w_{i-1}\|_1 &\leq C\Big(\|a^s\|_0\|\nabla v_{jk}\|_0+\|a^s\|_0^2+\frac{\|\nabla a^s\|_0^2+\|a^s\|_0\|\nabla^2a^s\|_0}{\lambda_j^2}\\ &\qquad+\frac{\|\nabla a^s\|_0\|\nabla v_{jk}\|_0+\|a^s\|_0\|\nabla^2v_{jk}\|_0+\|a^s\|_0\|\nabla a^s\|_0}{\lambda_i}\Big)\\ &\leq\frac{C}{(\lambda l)^{s/2}}\|\mathcal{D}\|_0^{1/2}\big(1+\frac{\lambda_{sd_*}}{\lambda_i}+\frac{\lambda_{sd_*}^2}{\lambda_i^2}+\frac{(\lambda l)^j}{\lambda_i l}\big)\big(1+\|\nabla v\|_0\big)\leq C\|\mathcal{D}\|_0^{1/2}\big(1+\|\nabla v\|_0\big),\end{aligned}$$

where again we used $\lambda_{sd_*}l\leq\lambda_i l$ due to $i>sd_*$, and $(\lambda l)^j\leq\lambda_i l$ due to $i>jk$.

**8. (Proof of estimate $(3.14)_2$)** Let $i=1\dots N$ and $m\geq 1$. Write:

$$i\in((j-1)k,jk]\cap(sd_*,(s+1)d_*]$$

with $j=1\dots J$, $s=0\dots S-1$. Then:

$$\begin{aligned}\|\nabla^{(m+1)}(v_i-v_{i-1})\|_0 &\leq\sum_{p+q=m+1}\lambda_i^{p-1}\|\nabla^{(q)}a^s\|_0\leq C\lambda_i^{m-1}\sum_{q=0}^{m+1}\frac{\lambda_{sd_*}^q\lambda_i\|\mathcal{D}\|_0^{1/2}}{\lambda_i^q(\lambda l)^{s/2}}\\ &\leq C\lambda_i^{m-1}\frac{\lambda_i}{(\lambda l)^{s/2}}\|\mathcal{D}\|_0^{1/2}=C\lambda_i^{m-1}\frac{(\lambda l)^j}{l}\|\mathcal{D}\|_0^{1/2}\end{aligned}$$

because $\lambda_{sd_*}\leq\lambda_i$ due to $i>sd_*$, and in fact from (3.5):

$$\lambda_i=\frac{(\lambda l)^{j+s/2}}{l}.$$

The above justifies:

$$\|\nabla^{(m+1)}(v_{kj}-v_{(k-1)j})\|_0\leq\sum_{i=(k-1)j+1}^{kj}\|\nabla^{(m+1)}(v_i-v_{i-1})\|_0\leq C\lambda_{jk}^{m-1}\frac{(\lambda l)^j}{l}\|\mathcal{D}\|_0^{1/2},$$

since $i \mapsto \lambda_i$ is a nondecreasing function. Further, by $(3.2)_3$ we get:

$$\|\nabla^{(m+1)} v_0\|_0 \leq \frac{C}{l^m}\|\mathcal{D}\|_0^{1/2} = C\frac{l_0^{m-1}}{l}\|\mathcal{D}\|_0^{1/2} \leq C\lambda_{jk}^{m-1}\frac{(\lambda l)^j}{l}\|\mathcal{D}\|_0^{1/2}.$$

The above two bounds prove the first statement in $(3.14)_2$.

Towards proving the second bound, we note that the increment in $w$ is estimated:

$$\begin{split}\|\nabla^{(m+1)}(w_i - w_{i-1})\|_0 \leq C \sum_{p+q+t=m+1} \lambda_i^{p-1}\|\nabla^{(q)} a^s\|_0 \|\nabla^{(t+1)} v_{(j-1)k}\|_0 \\ + C \sum_{p+q+t=m+1}\Big(\lambda_i^{p-2}\|\nabla^{(q)} a^s\|_0\|\nabla^{(t+1)} a^s\|_0 + \lambda_i^{p-1}\|\nabla^{(q)} a^s\|_0\|\nabla^{(t)} a^s\|_0\Big).\end{split} \tag{3.15}$$

We split the first sum in the right hand side above into cases $t = 0$ and $t \geq 1$, so that by $(3.14)_4$ and $(3.14)_2$, together with $(3.14)_1$ and $(3.2)_1$:

$$\begin{split}&\sum_{p+q+t=m+1} \lambda_i^{p-1}\|\nabla^{(q)} a^s\|_0\|\nabla^{(t+1)} v_{(j-1)k}\|_0 \\ &\quad \leq \|\nabla v_{(j-1)k}\|_0 \sum_{p+q=m+1}\lambda_i^{p-1}\|\nabla^{(q)} a^s\|_0 + \sum_{p+q+t=m}\lambda_i^{p-1}\|\nabla^{(q)} a^s\|_0\|\nabla^{(t+2)} v_{(j-1)k}\|_0 \\ &\quad \leq C\lambda_i^m\big(1+\|\nabla v\|_0\big)\sum_{q=0}^{m+1}\frac{\lambda_{sd_*}^q\|\mathcal{D}\|_0^{1/2}}{\lambda_i^q(\lambda l)^{s/2}} + C\lambda_i^m \sum_{p+q+t=m}\frac{\lambda_{sd_*}^q}{\lambda_i^q}\frac{\lambda_{(j-1)k}^t}{\lambda_i^t}\frac{(\lambda l)^{j-1}\|\mathcal{D}\|_0}{(\lambda_i l)(\lambda l)^{s/2}} \\ &\quad \leq C\lambda_i^m\frac{\|\mathcal{D}\|_0^{1/2}}{(\lambda l)^{s/2}}\big(1+\|\nabla v\|_0\big) + C\lambda_i^{m-1}\|\mathcal{D}\|_0\frac{(\lambda l)^{j-1-s/2}}{l}\end{split}$$

where in the last bound we used the fact that $\lambda_{sd_*} \leq \lambda_i$ due to $i > sd_*$, and $\lambda_{(j-1)k} \leq \lambda_i$ due to $i > (j-1)k$. The second term in (3.15) is similarly estimated:

$$\begin{split}&\sum_{p+q+t=m+1}\Big(\lambda_i^{p-2}\|\nabla^{(q)} a^s\|_0\|\nabla^{(t+1)} a^s\|_0 + \lambda_i^{p-1}\|\nabla^{(q)} a^s\|_0\|\nabla^{(t)} a^s\|_0\Big) \\ &\quad \leq C\lambda_i^m\sum_{p+q+t=m+1}\Big(\frac{\lambda_{sd_*}^{q+t+1}}{\lambda_i^{q+t+1}}\frac{\|\mathcal{D}\|_0}{(\lambda l)^s} + \frac{\lambda_{sd_*}^{q+t}}{\lambda_i^{q+t}}\frac{\|\mathcal{D}\|_0}{(\lambda l)^s}\Big) \leq C\lambda_i^m\frac{\|\mathcal{D}\|_0}{(\lambda l)^s}.\end{split}$$

Summing the last two displayed formulas, gives in view of (3.15):

$$\begin{split}\|\nabla^{(m+1)}(w_i - w_{i-1})\|_0 &\leq C\lambda_i^{m-1}\|\mathcal{D}\|_0^{1/2}\Big(\frac{\lambda_i}{(\lambda l)^{s/2}} + \frac{(\lambda l)^{j-1-s/2}}{l}\Big)\big(1+\|\nabla v\|_0\big) \\ &\leq C\lambda_i^{m-1}\frac{(\lambda l)^j}{l}\|\mathcal{D}\|_0^{1/2}\big(1+\|\nabla v\|_0\big).\end{split}$$

The above implies the second statement in $(3.14)_2$, in view of $(3.2)_3$ resulting in:

$$\|\nabla^{(m+1)} w_0\|_0 \leq \frac{C}{l^m}\|\mathcal{D}\|_0^{1/2} \leq C\lambda_{jk}^{m-1}\frac{(\lambda l)^j}{l}\|\mathcal{D}\|_0^{1/2},$$

and since $i \mapsto \lambda_i$ is a nondecreasing function, which yields:

$$\begin{split}\|\nabla^{(m+1)}(w_{kj} - w_{(k-1)j})\|_0 &\leq \sum_{i=(k-1)j+1}^{kj}\|\nabla^{(m+1)}(w_i - w_{i-1})\|_0 \\ &\leq C\lambda_{jk}^{m-1}\frac{(\lambda l)^j}{l}\|\mathcal{D}\|_0^{1/2}\big(1+\|\nabla v\|_0\big).\end{split}$$

**9. (Proof of estimate (3.14)$_3$)** Let $i = 1 \dots N$ and $m \geq 0$. Write:

$$i \in (jk, (j+1)k] \cap ((s-1)d_*, sd_*]$$

with $j = 0 \dots J-1$, $s = 1 \dots S$. Denoting $\delta = i - (s-1)d_*$, we use (3.14)$_2$, (3.14)$_4$ in (3.13):

$$\begin{aligned}
\big\|\nabla^{(m)}\big(V_i - (a_\delta^{s-1})^2 \eta_\delta \otimes \eta_\delta\big)\big\|_0 &\leq C \sum_{p+q+t=m} \lambda_i^{p-1} \|\nabla^{(q)} a^{s-1}\|_0 \|\nabla^{(t+2)} v_{jk}\|_0 \\
&\quad + C \sum_{p+q+t=m} \lambda_i^{p-2} \Big( \|\nabla^{(q+1)} a^{s-1}\|_0 \|\nabla^{(t+1)} a^{s-1}\|_0 + \|\nabla^{(q)} a^{s-1}\|_0 \|\nabla^{(t+2)} a^{s-1}\|_0 \Big) \\
&\leq C\lambda_i^m \|\mathcal{D}\|_0 \Big( \sum_{p+q+t=m} \frac{\lambda_{(s-1)d_*}^q \lambda_{jk}^t}{\lambda_i^{q+t}} \frac{(\lambda l)^{j-(s-1)/2}}{\lambda_i l} + \sum_{p+q+t=m} \frac{\lambda_{(s-1)d_*}^{q+t+2}}{\lambda_i^{q+t+2}} \frac{1}{(\lambda l)^{s-1}} \Big).
\end{aligned}$$

Since $\lambda_{(s-1)d_*} \leq \lambda_i$ by $i > (s-1)d_*$, and $\lambda_{jk} \leq \lambda_i$ by $i > jk$, we simplify the above estimate:

$$\begin{aligned}
\big\|\nabla^{(m)}\big(V_i - (a_\delta^{s-1})^2 \eta_\delta \otimes \eta_\delta\big)\big\|_0 &\leq C\lambda_i^m \|\mathcal{D}\|_0 \Big( \frac{(\lambda l)^{j-(s-1)/2}}{\lambda_i l} + \frac{\lambda_{(s-1)d_*}^2}{\lambda_i^2} \frac{1}{(\lambda l)^{s-1}} \Big) \\
&\leq C \frac{\lambda_i^m}{(\lambda l)^s} \|\mathcal{D}\|_0,
\end{aligned}$$

where the last bound follows since $\lambda_i \geq \lambda_{(s-1)d_*} (\lambda l)^{1/2}$ by $i > (s-1)d_*$, and since (3.5) yields:

$$\lambda_i = \lambda (\lambda l)^{j+(s-1)/2}.$$

Note that having the second power of the quotient $\lambda_{(s-1)d_*}/\lambda_i$ was essential to provide the missing multiplier $\frac{1}{\lambda l}$ in order for both considered error terms to have the right order $C\frac{\lambda_i^m}{(\lambda l)^s}\|\mathcal{D}\|_0$. It is precisely at this point where we are using the optimal step construction in Lemma 2.1, and where the previous sub-par step construction in [7, Lemma 2.2] would not be sufficient.

Consequently, summing the partial deficits in $\mathcal{D}_s$, we obtain the second bound in (3.14)$_3$:

$$\|\nabla^{(m)} \mathcal{D}_s\|_0 \leq C \sum_{i=(s-1)d_*+1}^{sd_*} \frac{\lambda_i^m}{(\lambda l)^s} \|\mathcal{D}\|_0 \leq C \frac{\lambda_{sd_*}^m}{(\lambda l)^s} \|\mathcal{D}\|_0,$$

as $\lambda_i \leq \lambda_{sd_*}$ for all $i \leq sd_*$. The first bound in (3.14)$_3$ is now also immediate:

$$\tilde{C}_s \leq C\Big( \frac{\|\mathcal{D}\|_0}{(\lambda l)^s} + \|\mathcal{D}_s\|_0 \Big) \leq \frac{C}{(\lambda l)^s} \|\mathcal{D}\|_0.$$

**10. (Proof of estimate (3.14)$_4$)** Let $s = 1 \dots S-1$. From (3.7) and the first bound in (3.14)$_3$, we readily deduce (3.14)$_4$ at $m = 0$:

$$\|a^s\|_0 \leq \frac{C}{(\lambda l)^{s/2}} \|\mathcal{D}\|_0^{1/2}.$$

For $m \geq 1$, we use the preparatory bound (3.8) in which we take account of (3.14)$_3$ and (3.14)$_3$:

$$\|\nabla^{(m)} a^s\|_0 \leq \frac{C}{(\lambda l)^{s/2}} \|\mathcal{D}\|_0^{1/2} \sum_{p_1+2p_2+\ldots+mp_m=m} \prod_{t=1}^m \Big( \lambda_{sd_*}^{tp_t} \Big( \frac{\|\mathcal{D}\|_0}{(\lambda l)^s \tilde{C}_s} \Big)^{p_t} \Big) \leq C \frac{\lambda_{sd_*}^m}{(\lambda l)^{s/2}} \|\mathcal{D}\|_0^{1/2},$$

in virtue of having $\frac{\|\mathcal{D}\|_0}{(\lambda l)^s \tilde{C}_s} \leq C$. This completes the proof of all the inductive estimates.

**11. (End of proof)** We now show that $(3.14)_1$ - $(3.14)_4$ imply the bounds claimed in the Theorem. Recall (3.12), and use $(3.14)_1$, $(3.14)_3$ and $(3.2)_1$ to conclude $(1.2)_1$:

$$\|\tilde v - v\|_1 \leq \|v_0 - v\|_1 + \sum_{i=1}^N \|v_i - v_{i-1}\|_1 \leq C\|\mathcal{D}\|_0^{1/2},$$

$$\|\tilde w - w\|_1 \leq \|w_0 - w\|_1 + \sum_{i=1}^N \|w_i - w_{i-1}\|_1 + C\sum_{s=0}^{S-1}\tilde C_s \leq C\|\mathcal{D}\|_0^{1/2}(1+\|\nabla v\|_0).$$

By $(3.14)_2$ with $m=2$, there follows $(1.2)_2$:

$$\|\nabla^2\tilde v\|_0 = \|\nabla^2 v_N\|_0 \leq C\frac{(\lambda l)^{kJ}}{l}\|\mathcal{D}\|_0^{1/2} = CM(\lambda l)^J = CM\sigma^{J/S} = CM\sigma^{d_*/k},$$

$$\|\nabla^2\tilde w\|_0 = \|\nabla^2 w_N\|_0 \leq C\frac{(\lambda l)^{kJ}}{l}\|\mathcal{D}\|_0^{1/2}(1+\|\nabla v\|_0) = CM\sigma^{d_*/k}(1+\|\nabla v\|_0),$$

where we used the definition $\sigma = (\lambda l)^S$ and the fact that:

$$\frac{J}{S} = \frac{J}{N}\cdot\frac{N}{S} = \frac{d_*}{k}.$$

Finally, $(3.2)_2$, and $(3.14)_3$ applied with $m=0$ yield $(1.2)_3$:

$$\|\tilde{\mathcal{D}}\|_0 = \|(A-A_0) - \mathcal{D}_S\|_0 \leq \|A-A_0\|_0 + \|\mathcal{D}_S\|_0 \leq$$
$$C\Big(l^\beta\|A\|_{0,\beta} + \frac{\|\mathcal{D}\|_0}{(\lambda l)^S}\Big) = C\Big(\frac{\|A\|_{0,\beta}}{M^\beta}\|\mathcal{D}\|_0^{\beta/2} + \frac{\|\mathcal{D}\|_0}{\sigma}\Big),$$

in view of the following direct decomposition:

$$\begin{aligned}\tilde{\mathcal{D}} &= (A-A_0) - \mathcal{D}_0 - \Big(\big(\frac12(\nabla\tilde v)^T\nabla\tilde v + \mathrm{sym}\nabla\tilde w\big) - \big(\frac12(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big)\Big)\\ &= (A-A_0) - \mathcal{D}_0 + \sum_{s=0}^{S-1}\tilde C_s\mathrm{Id}_d - \sum_{s=1}^{S}\sum_{i=(s-1)d_*+1}^{sd_*} V_i\\ &= (A-A_0) + \sum_{s=0}^{S-1}\tilde C_s\mathrm{Id}_d - \sum_{s=0}^{S}\mathcal{D}_s - \sum_{s=1}^{S}\sum_{\delta=1}^{d_*}(a_\delta^{s-1})^2\eta_\delta\otimes\eta_\delta\\ &= (A-A_0) - \mathcal{D}_S\end{aligned}$$

The proof is done. ■

## 4. The Nash-Kuiper scheme in $\mathcal{C}^{1,\alpha}$

To perform induction on stages we need the following argument, similar to [2, Theorem 1.1]:

**Theorem 4.1.** *Let $\omega\subset\mathbb{R}^d$ be an open, bounded domain and let $k\geq 1$ and $\gamma>0$ be such that the statement of Theorem 1.2 holds true with $\gamma$ replacing the exponent $d_*/k$ in $(1.2)_2$. Then we have the following. For every $v\in\mathcal{C}^2(\bar\omega,\mathbb{R}^k)$, $\tilde w\in\mathcal{C}^2(\bar\omega,\mathbb{R}^d)$, $A\in\mathcal{C}^{0,\beta}(\bar\omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$, such that:*

$$\mathcal{D} = A - \big(\frac12(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big) \quad \textit{satisfies} \quad 0 < \|\mathcal{D}\|_0 \leq 1,$$

*and for every $\alpha$ in the range:*

$$0 < \alpha < \min\Big\{\frac{\beta}{2}, \frac{1}{1+2\gamma}\Big\}, \tag{4.1}$$

*there exist* $\tilde v\in\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^k)$ *and* $w\in\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^d)$ *with the following properties:*

$$\|\tilde v - v\|_1\leq C\|\mathcal{D}\|_0^{1/2},\quad \|\tilde w - w\|_1\leq C\|\mathcal{D}\|_0^{1/2}(1+\|\nabla v\|_0), \tag{4.2$_1$}$$

$$A-\big(\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v + \mathrm{sym}\nabla\tilde w\big) = 0\quad \text{ in }\ \bar\omega, \tag{4.2$_2$}$$

*where the constants* $C$ *depend only on* $d,k$ *and* $\omega$.

*Proof.* **1.** Because of the assumption (4.1), there exists an exponent:

$$\frac{2\gamma\alpha}{(1-\alpha)}<\delta<\min\Big\{1,\frac{2\gamma\beta}{(2-\beta)}\Big\}. \tag{4.3}$$

We let $\sigma>1$ be a sufficiently large constant, in function of $\delta,\alpha,\gamma$, $\|\nabla v\|_0$ and all constants $C$ in the assumed assertions of Theorem 1.2 (these constants depend only in $d,k,\omega$).

We further set $v_0=v$, $w_0=w$, $\mathcal{D}_0=\mathcal{D}$, and take $M_0\geq\max\{\|v\|_0,\|w\|_0,1\}$ that is again sufficiently large, now in function of $\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2-1}\sigma^\delta$ and constants $C$ indicated before. By successive applications of Theorem 1.2 with the chosen $\sigma$ and constants $\{M_i\geq 1\}_{i=1}^\infty$ in:

$$M_i = \Big(\tilde C(1+\|\nabla v\|_0)\sigma^\gamma\Big)^i M_0$$

where $\tilde C>1$ is again some large constant (in function of the aforementioned $C$), we obtain sequences $\{v_i\in\mathcal{C}^2(\bar\omega,\mathbb{R}^k)\}_{i=1}^\infty$, $\{w_i\in\mathcal{C}^2(\bar\omega,\mathbb{R}^d)\}_{i=1}^\infty$ and the related deficits $\{\mathcal{D}_i\in\mathcal{C}^0(\bar\omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})\}_{i=1}^\infty$:

$$\mathcal{D}_i = A-\big(\frac{1}{2}(\nabla v_i)^T\nabla v_i + \mathrm{sym}\nabla w_i\big).$$

We see that, as long as there holds:

$$0<\|\mathcal{D}_i\|_0\leq 1\qquad\text{and}\qquad M_i\geq\max\{\|v_i\|_2,\|w_i\|_2,1\}, \tag{4.4}$$

we have, with the constants $C$ depending only on $d,k$ and $\omega$:

$$\|v_{i+1}-v_i\|_1\leq C\|\mathcal{D}_i\|_0^{1/2},\quad \|w_{i+1}-w\|_1\leq C\|\mathcal{D}_i\|_0^{1/2}(1+\|\nabla v_i\|_0), \tag{4.5$_1$}$$

$$\|v_{i+1}\|_2\leq CM_i\sigma^\gamma,\quad \|w_{i+1}\|_2\leq CM_i\sigma^\gamma(1+\|\nabla v_i\|_0), \tag{4.5$_2$}$$

$$\|\mathcal{D}_{i+1}\|_0\leq C\Big(\frac{\|A\|_{0,\beta}}{M_i^\beta}\|\mathcal{D}_i\|_0^{\beta/2}+\frac{\|\mathcal{D}_i\|_0}{\sigma}\Big). \tag{4.5$_3$}$$

Below, we inductively validate (4.4) for all $i\geq 0$, and in fact we show that:

$$\|\mathcal{D}_i\|_0\leq\frac{1}{\sigma^{\delta i}}\|\mathcal{D}\|_0\qquad\text{for all }\ i=0\ldots\infty. \tag{4.6}$$

Before doing so, note that (4.6) actually implies both statements in (4.4). Indeed, by $(4.5)_2$:

$$\begin{aligned}&\|v_{i+1}\|_2\leq CM_i\sigma^\gamma\leq M_{i+1},\\ &\|w_{i+1}\|_2\leq CM_i\sigma^\gamma(1+\|\nabla v_i\|_0)\leq CM_i\sigma^\gamma(1+2C+\|\nabla v\|_0)\leq M_{i+1},\end{aligned}$$

since by the first bound in $(4.5)_1$ and (4.6) there follows, provided that $\sigma^{\delta/2}\geq 2$:

$$\begin{aligned}\|\nabla v_i\|_0 &\leq\|\nabla v\|_0+\sum_{j=0}^{i-1}\|\nabla v_{j+1}-\nabla v_j\|_0\leq\|\nabla v\|_0+C\sum_{j=0}^{i-1}\|\mathcal{D}_j\|_0^{1/2}\\ &\leq\|\nabla v\|_0+C\sum_{j=0}^{\infty}\frac{\|\mathcal{D}\|_0^{1/2}}{\sigma^{\delta j/2}}=\|\nabla v\|_0+\frac{C}{1-\sigma^{-\delta/2}}\|\mathcal{D}\|_0^{1/2}\\ &\leq\|\nabla v\|_0+2C\|\mathcal{D}\|_0^{1/2}\leq 2C+\|\nabla v\|_0.\end{aligned}\tag{4.7}$$

**2.** Clearly (4.6) holds at $i=0$. To prove it at $(i+1)$, use $(4.5)_3$ and the induction assumption:

$$\|\mathcal{D}_{i+1}\|_0 \leq C\Big(\frac{\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2}}{M_i^\beta \sigma^{\delta i\beta/2}} + \frac{\|\mathcal{D}\|_0}{\sigma^{\delta i+1}}\Big) = \frac{\|\mathcal{D}\|_0}{\sigma^{\delta(i+1)}}\Big(\frac{C\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2-1}}{M_i^\beta \sigma^{\delta i\beta/2-\delta(i+1)}} + \frac{C}{\sigma^{1-\delta}}\Big), \tag{4.8}$$

and check that both terms in parentheses in the right hand side above are not greater than 1/2. For the second term, this is readily implied by taking $\sigma$ large enough that $\sigma^{1-\delta}\geq 2C$, in view of $1-\delta>0$ in (4.3). For the first term, we note that

$$\frac{C\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2-1}}{M_i^\beta \sigma^{\delta i\beta/2-\delta(i+1)}} \leq \frac{C\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2-1}\sigma^\delta}{M_0^\beta}\cdot \sigma^{\delta i-\gamma\beta i-\delta\beta i/2} \leq \frac{C\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2-1}\sigma^\delta}{M_0^\beta},$$

since the exponent $\delta i-\gamma\beta i-\delta\beta i/2$ is non-positive, due to $\delta<\frac{2\gamma\beta}{2-\beta}$ in (4.3):

$$\delta i-\gamma\beta i-\delta\beta i/2 = \frac{i}{2}\big(\delta(2-\beta)-2\gamma\beta\big)\leq 0 \quad \text{ for all } \ i\geq 0.$$

In conclusion, the expression in parentheses in (4.8) is bounded by 1, provided $M_0$ has been chosen sufficiently large. This ends the proof of (4.6).

**3.** From $(4.5)_1$, (4.7) and (4.6), it follows that for all $i=0\ldots\infty$:

$$\|v_{i+1}-v_i\|_1\leq \frac{C}{\sigma^{\delta i/2}}\|\mathcal{D}\|_0^{1/2}, \qquad \|w_{i+1}-w_i\|_1\leq \frac{C}{\sigma^{\delta i/2}}\|\mathcal{D}\|_0^{1/2}\big(1+\|\nabla v\|_0\big),$$

Hence, both sequences $\{v_i\}_{i=1}^\infty$, $\{w_i\}_{i=1}^\infty$ are Cauchy in $\mathcal{C}^1(\bar\omega)$ and as such they converge to the limit fields, respectively:

$$\tilde v\in\mathcal{C}^1(\omega,\mathbb{R}^k), \qquad \tilde w\in\mathcal{C}^1(\omega,\mathbb{R}^d)$$

that satisfy $(4.2)_1$ and $(4.2)_2$, in virtue of $\|\mathcal{D}_i\|_0\to 0$ as $i\to\infty$.

It remains to show that $\tilde v$ and $\tilde w$ are $\mathcal{C}^{1,\alpha}$-regular. To this end, we use the estimate:

$$\begin{aligned}\|v_{i+1}-v_i\|_2+\|w_{i+1}-w_i\|_2 &\leq CM_i\sigma^\gamma\big(1+\|\nabla v\|_0\big)\\ &\leq C\Big(\tilde C(1+\|\nabla v\|_0)\sigma^\gamma\Big)^{i+1}M_0,\end{aligned}$$

resulting from $(4.5)_2$ and (4.7), in the interpolation inequality $\|\cdot\|_{1,\alpha}\leq\|\cdot\|_1^{1-\alpha}\|\cdot\|_2^\alpha$:

$$\begin{aligned}\|v_{i+1}-&v_i\|_{1,\alpha}+\|w_{i+1}-w_i\|_{1,\alpha}\\ &\leq C\|\mathcal{D}\|_0^{(1-\alpha)/2}\Big(\tilde C(1+\|\nabla v\|_0)\Big)^{(i+1)\alpha+(1-\alpha)}M_0^\alpha\sigma^{\alpha\gamma(i+1)-\delta i(1-\alpha)/2}\\ &= C\tilde C\cdot M_0^\alpha\|\mathcal{D}\|_0^{(1-\alpha)/2}\big(1+\|\nabla v\|_0\big)\sigma^{\alpha\gamma}\cdot\Big(\frac{\tilde C^\alpha(1+\|\nabla v\|_0)^\alpha}{\sigma^{\delta(1-\alpha)/2-\alpha\gamma}}\Big)^i.\end{aligned}$$

Since the exponent $\delta(1-\alpha)/2-\alpha\gamma$ is positive, in view of $\delta>\frac{2\gamma\alpha}{1-\alpha}$ in (4.3), we see that both sequences $\{v_i\}_{i=1}^\infty$, $\{w_i\}_{i=1}^\infty$ are Cauchy in $\mathcal{C}^{1,\alpha}(\bar\omega)$, provided that $\sigma$ is sufficiently large to have:

$$\frac{\tilde C^\alpha(1+\|\nabla v\|_0)^\alpha}{\sigma^{\delta(1-\alpha)/2-\alpha\gamma}}<1.$$

In conclusion, $\tilde v\in\mathcal{C}^{1,\alpha}(\omega,\mathbb{R}^k)$ and $\tilde w\in\mathcal{C}^{1,\alpha}(\omega,\mathbb{R}^d)$ as claimed. The proof is done. ■

Taking now $\gamma=d_*/k$ as guaranteed by Theorem 1.2, Theorem 4.1 implies:

**Corollary 4.2.** *For every $v \in \mathcal{C}^2(\bar\omega, \mathbb{R}^k)$, $w \in \mathcal{C}^2(\bar\omega, \mathbb{R}^d)$ and $A \in \mathcal{C}^{0,\beta}(\bar\omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ defined on an open, bounded domain $\omega \subset \mathbb{R}^d$, and such that:*

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big) \quad \textit{satisfies} \quad 0 < \|\mathcal{D}\|_0 \le 1,$$

*and for every exponent $\alpha$ in the range:*

$$0 < \alpha < \min\Big\{\frac{\beta}{2}, \frac{1}{1+2d_*/k}\Big\},$$

*there exist $\tilde v \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^k)$ and $\tilde w \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^d)$ with the following properties:*

$$\|\tilde v - v\|_1 \le C\|\mathcal{D}\|_0^{1/2}, \quad \|\tilde w - w\|_1 \le C\|\mathcal{D}\|_0^{1/2}(1 + \|\nabla v\|_0), \tag{4.9$_1$}$$

$$A - \big(\frac{1}{2}(\nabla \tilde v)^T\nabla \tilde v + \mathrm{sym}\nabla \tilde w\big) = 0 \quad \textit{in } \bar\omega, \tag{4.9$_2$}$$

*where the constants $C$ depend only on $d, k$ and $\omega$.*

## 5. A proof of Theorem 1.1

The final auxiliary result that we need, is a combination of the local decomposition into "primitive metrics" with a partition of unity - type statement from [10, Lemma 3.3]:

**Lemma 5.1.** *Given the dimension $d \ge 1$, there exists a constant $N_0$ and sequences of unit vectors $\{\eta_i \in \mathbb{R}^d\}_{i=1}^\infty$ and nonnegative functions $\{\varphi_i \in \mathcal{C}^\infty_c(\mathbb{R}^{d\times d}_{\mathrm{sym},>}, \mathbb{R})\}_{i=1}^\infty$, such that:*

$$A = \sum_{i=1}^\infty \varphi_i(A)^2 \eta_i \otimes \eta_i \quad \textit{for all } A \in \mathbb{R}^{d\times d}_{\mathrm{sym},>},$$

*and such that:*

(i) *at most $N_0$ terms in the above sum are nonzero,*
(ii) *every compact set of matrices $K \subset \mathbb{R}^{d\times d}_{\mathrm{sym},>}$ induces a finite set of indices $J(K) \subset \mathbb{N}$, such that $\varphi_i(A) = 0$ for all $A \in K$ and all $i \notin J(K)$.*

Equipped with Lemma 5.1 and the "step" construction in Lemma 2.1, one easily deduces the deficit decrease - approximation result in $\mathcal{C}^1$, which is the multidimensional version of the basic "stage" construction in [7, Proposition 3.2]:

**Theorem 5.2.** *Let $\omega \subset \mathbb{R}^d$ be an open, bounded domain. Given two vector fields $v \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^k)$, $w \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^d)$ and a matrix field $A \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, assume that:*

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big) \quad \textit{satisfies} \quad \mathcal{D} > c\,\mathrm{Id}_d \ \textit{ on } \bar\omega$$

*for some $c > 0$, in the sense of matrix inequalities. Fix $\epsilon > 0$. Then, there exists $\tilde v \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^k)$ and $\tilde w \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^d)$ such that, denoting:*

$$\tilde{\mathcal{D}} = A - \big(\frac{1}{2}(\nabla \tilde v)^T\nabla \tilde v + \mathrm{sym}\nabla \tilde w\big),$$

*the following holds with constants $C$ depending only on $d, k$ and $\omega$:*

$$\|\tilde v - v\|_0 \le \epsilon, \quad \|\tilde w - w\|_0 \le \epsilon, \tag{5.1$_1$}$$

$$\|\nabla(\tilde v - v)\|_0 \le C\|\mathcal{D}\|_0^{1/2}, \quad \|\nabla(\tilde w - w)\|_0 \le C\|\mathcal{D}\|_0^{1/2}\big(\|\mathcal{D}\|_0^{1/2} + \|\nabla v\|_0\big), \tag{5.1$_2$}$$

$$\|\tilde{\mathcal{D}}\|_0 \le \epsilon \quad \textit{and} \quad \tilde{\mathcal{D}} > \tilde c\,\mathrm{Id}_d \ \textit{ for some } \tilde c > 0. \tag{5.1$_3$}$$

*Proof.* **1.** Since $\mathcal{D}(\bar\omega)$ is a compact subset of $\mathbb{R}^{d\times d}_{\mathrm{sym},>}$, Lemma 5.1 yields a finite set of indices for which the decomposition into "primitive matrices" is active. Without loss of generality these indices are $\{1\dots N\}$. Then, with the unit vectors $\{\eta_i\in\mathbb{R}^d\}_{i=1}^N$ and the nonnegative functions $\{b_i\in\mathcal{C}^\infty(\bar\omega,\mathbb{R})\}_{i=1}^N$ defined by $b_i(x)=\varphi_i(\mathcal{D}(x))$, there holds:

$$\mathcal{D}(x)=\sum_{i=1}^N b_i(x)^2\eta_i\otimes\eta_i \qquad \text{for all } \ x\in\bar\omega.$$

We now define the modified nonnegative amplitude functions $\{a_i\in\mathcal{C}^\infty(\bar\omega,\mathbb{R})\}_{i=1}^N$ by:

$$a_i=(1-\delta)^{1/2}b_i \quad \text{on } \ \omega, \qquad \text{where } \ \delta=\min\Big\{\frac{1}{2},\frac{\epsilon}{2\|\mathcal{D}\|_0}\Big\}.$$

Observe that:

$$\mathcal{D}-\sum_{i=1}^N a_i^2\eta_i\otimes\eta_i=\delta\mathcal{D}>\delta c\,\mathrm{Id}_d.$$

**2.** We set $v_1=v$ and $w_1=w$. We then inductively define the vector fields $\{v_i\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^k)\}_{i=1}^{N+1}$ and $\{w_i\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^d)\}_{i=1}^{N+1}$ by applying Lemma 2.1 to each consecutive pair $(v_i,w_i)$ with the given unit vector $\eta_i$, an arbitrary unit vector $E\in\mathbb{R}^k$, the given amplitude $a_i$ and a frequency $\lambda_i>0$ that is sufficiently large as indicated below. We finally set:

$$\tilde v=v_{N+1},\quad \tilde w=w_{N+1}.$$

It is clear that by taking $\{\lambda_i\}_{i=1}^N$ large, one can ensure the validity of $(5.1)_1$. Further, by (2.2):

$$\begin{aligned}\tilde{\mathcal{D}}&=\mathcal{D}-\Big(\big(\frac{1}{2}(\nabla v_{N+1})^T\nabla v_{N+1}+\mathrm{sym}\nabla w_{N+1}\big)-\big(\frac{1}{2}(\nabla v_1)^T\nabla v_1+\mathrm{sym}\nabla w_1\big)\Big)\\&=\Big(\mathcal{D}-\sum_{i=1}^N a_i^2\eta_i\otimes\eta_i\Big)\\&\qquad-\sum_{i=1}^N\Big(\big(\frac{1}{2}(\nabla v_{i+1})^T\nabla v_{i+1}+\mathrm{sym}\nabla w_{i+1}\big)-\big(\frac{1}{2}(\nabla v_i)^T\nabla v_i+\mathrm{sym}\nabla w_i\big)-a_i^2\eta_i\otimes\eta_i\Big)\\&=\delta\mathcal{D}+\sum_{i=1}^N O\Big(\frac{\|a_i\|_0\|\nabla^2 v_i\|_0}{\lambda_i}+\frac{\|\nabla a_i\|^2+\|a_i\|_0\|\nabla^2 a\|_0}{\lambda_i^2}\Big),\end{aligned}$$

which implies $(5.1)_3$ with $\tilde c=\delta c/2$, provided that $\{\lambda_i\}_{i=1}^N$ are sufficiently large.

**3.** It remains to check $(5.1)_2$. By Lemma 5.1 (i), we get for each $x\in\bar\omega$:

$$0\le\sum_{i=1}^N a_i(x)\le\sum_{i=1}^N b_i(x)\le N_0^{1/2}\big(\sum_{i=1}^N b_i(x)^2\big)^{1/2}=N_0^{1/2}\big(\mathrm{Trace}\,\mathcal{D}(x)\big)^{1/2}\le CN_0^{1/2}\|\mathcal{D}\|_0^{1/2}.$$

Consequently, the definition (2.1) yields, with sufficiently large $\{\lambda_i\}_{i=1}^N$:

$$\|\nabla(\tilde v-v)\|_0\le\sum_{i=1}^N\|\nabla v_{i+1}-\nabla v_i\|_0\le 2\|\sum_{i=1}^N a_i\|_0+C\sum_{i=1}^N\frac{\|\nabla a_i\|_0}{\lambda_i}\le C\|\mathcal{D}\|_0^{1/2}.$$

In a similar fashion, and using the above bound, we obtain:

$$\begin{aligned}\|\nabla(\tilde{w}-w)\|_0 \leq \sum_{i=1}^{N}\|\nabla w_{i+1}-\nabla w_i\|_0 &\leq C\Big\|\sum_{i=1}^{N}\big(\|\nabla v_i\|_0+\|a_i\|_0\big)a_i\Big\|_0\\ &+C\sum_{i=1}^{N}\frac{\|\nabla a_i\|_0\|\nabla v_i\|_0+\|a_i\|_0\|\nabla^2 v_i\|_0+\|a_i\|_0^2+\|a_i\|_0\|\nabla a_i\|_0}{\lambda_i}\\ &+C\sum_{i=1}^{N}\frac{\|\nabla a_i\|_0\|\nabla^2 a_i\|_0+\|a_i\|_0\|\nabla^3 a_i\|_0}{\lambda_i^2}\\ &\leq C\|\mathcal{D}\|_0^{1/2}\cdot\sup_{i=1\dots N}\big(\|\nabla v_i\|_0+\|a_i\|_0\big)\leq C\|\mathcal{D}\|_0^{1/2}\big(\|\nabla v_0\|_0+\|\mathcal{D}\|_0^{1/2}\big).\end{aligned}$$

This ends the proof of $(5.1)_2$ and of the theorem. ∎

We remark that having the upgraded version of the "step" in Lemma 2.1 was irrelevant to the proof above, and that the sub-optimal construction in [7, Lemma 2.2] would still suffice.

We are now ready to give:

**Proof of Theorem 1.1**

In order to apply Corollary 4.2, we need to increase the regularity of $v$, $w$ and decrease the deficit $\mathcal{D}$. Take $\epsilon < 1$ that is sufficiently small, as indicated below. First, we let $v_1 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^k)$, $w_1 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^d)$ and $A_1 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ be such that:

$$\begin{aligned}&\|v_1-v\|_1\leq\epsilon^3,\quad \|w_1-w\|_1\leq\epsilon^3,\quad \|A_1-A\|_0\leq\epsilon^3,\\ &\mathcal{D}_1=A_1-\big(\frac{1}{2}(\nabla v_1)^T\nabla v_1+\mathrm{sym}\nabla w_1\big)>c_1\mathrm{Id}_d\quad\text{for some }\ c_1>0.\end{aligned}$$

The last property follows from the fact that:

$$\|\mathcal{D}_1-\mathcal{D}\|_0\leq 3\epsilon^3+\epsilon^3\|\nabla v\|_0 \tag{5.2}$$

Second, use Theorem 5.2 to get $v_2 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^k)$, $w_2 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^d)$ such that:

$$\begin{aligned}&\|v_2-v_1\|_0\leq\epsilon^3,\quad \|w_2-w_1\|_0\leq\epsilon^3,\\ &\|\nabla(v_2-v_1)\|_0\leq C\|\mathcal{D}_1\|_0^{1/2}\leq C\big(\|\mathcal{D}\|_0^{1/2}+\epsilon^{3/2}+\|\nabla v\|_0^{1/2}\big),\\ &\mathcal{D}_2=A_1-\big(\frac{1}{2}(\nabla v_2)^T\nabla v_2+\mathrm{sym}\nabla w_2\big)\quad\text{satisfies}\ \ \|\mathcal{D}_2\|_0\leq\epsilon^3,\end{aligned}$$

where we applied (5.2) in the gradient increment bound of $v$.

Clearly, if the deficit $\mathcal{D}_3$, defined below:

$$\mathcal{D}_3=A-\big(\frac{1}{2}(\nabla v_2)^T\nabla v_2+\mathrm{sym}\nabla w_2\big)$$

is equivalently zero on $\bar{\omega}$, then we may simply take $\tilde{v} = v_2$ and $\tilde{w} = w_2$ to satisfy the claim of the theorem. Otherwise, we use Corollary 4.2 to $v_2$, $w_2$ and $A$, since:

$$0<\|\mathcal{D}_3\|_0\leq\|A-A_1\|_0+\|\mathcal{D}_2\|\leq 2\epsilon^3\leq 1,$$

and consequently obtain $\tilde v \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^k)$, $\tilde w \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^d)$ such that:

$$\begin{aligned}
&\|\tilde v - v_2\|_0 \leq C\epsilon^{3/2},\\
&\|\tilde w - w_2\|_0 \leq C\epsilon^{3/2}(1+\|\nabla v_2\|_0) \leq C\epsilon^{3/2}(1+\|\mathcal{D}\|_0^{1/2} + \|\nabla v\|_0),\\
&A - \big(\frac12 (\nabla \tilde v)^T\nabla\tilde v + \mathrm{sym}\nabla\tilde w\big) = 0 \quad \text{ in } \bar\omega.
\end{aligned}$$

It now suffices to observe that, taking $\epsilon$ sufficiently small (in function of $\|\mathcal{D}\|_0^{1/2}$, $\|\nabla v\|_0$ and of constants $C$ that depend only on $k$, $d$ and $\omega$), we get:

$$\begin{aligned}
&\|\tilde v - v\|_0 \leq \|\tilde v - v_2\|_0 + \|v_2 - v_1\|_0 + \|v_1 - v\|_0 \leq C\epsilon^{3/2} \leq \epsilon,\\
&\|\tilde w - w\|_0 \leq \|\tilde w - w_2\|_0 + \|w_2 - w_1\|_0 + \|w_1 - w\|_0 \leq C\epsilon^{3/2}(1+\|\mathcal{D}\|_0^{1/2} + \|\nabla v\|_0) \leq \epsilon.
\end{aligned}$$

The proof is done. ■

## 6. The Monge-Ampére system: proofs of Lemma 1.4 and Theorem 1.7

We remark that when $d=2$ then the formula (1.4) rests in agreement with the expansion of the Gaussian curvature: $\kappa(\mathrm{Id}_2 + \epsilon A) = -\frac{\epsilon}{2}\mathrm{curl}\,\mathrm{curl} A + O(\epsilon^2)$, where we have:

$$\mathfrak{C}^2(A)_{ij,st} = \begin{cases} \mathrm{curl}\,\mathrm{curl} A & \text{if } (ij,st)\in\{(12,12),(21,21)\},\\ -\mathrm{curl}\,\mathrm{curl} A & \text{if } (ij,st)\in\{(12,21),(21,12)\},\\ 0 & \text{otherwise.}\end{cases}$$

We now give:

**Proof of Lemma 1.4**

The implication (i)⇒(ii) follows by a direct inspection:

$$\begin{aligned}
2\mathfrak{C}^2(\mathrm{sym}\nabla w)_{ij,st} = {} & \partial_i\partial_s(\partial_j w^t + \partial_t w^j) + \partial_j\partial_t(\partial_i w^s + \partial_c w^i)\\
& - \partial_i\partial_t(\partial_j w^s + \partial_s w^j) - \partial_j\partial_s(\partial_i w^t + \partial_t w^i) = 0.
\end{aligned}$$

To prove that (ii)⇒(i), note that condition:

$$\mathfrak{C}^2(A)_{ij,st} = \partial_i\big(\partial_s A_{jt} - \partial_t A_{js}\big) - \partial_j\big(\partial_s A_{it} - \partial_t A_{is}\big) = 0$$

implies, for each fixed $s,t: 1\ldots d$, that the vector field $[\partial_s A_{jt} - \partial_t A_{js}]_{j=1\ldots d}$ must be a gradient of some scalar field $-\phi_{st}$, where we used the Poincaré Lemma on the contractible domain $\omega$. We thus write:

$$\partial_s A_{jt} - \partial_t A_{js} = -\partial_j\phi_{st} \quad \text{ for all } j,s,t = 1\ldots d. \tag{6.1}$$

Observe that $\nabla(\phi_{st} + \phi_{ts}) = 0$, so without loss of generality, $\phi_{st} = -\phi_{ts}$ and $\phi_{ss} = 0$. The following matrix field is thus skew-symmetric:

$$\phi = [\phi_{st}]_{s,t=1\ldots d} : \omega \to \mathrm{so}(d). \tag{6.2}$$

Consequently, permuting the indices in (6.1), leads to:

$$\partial_s A_{jt} - \partial_t A_{js} + \partial_j\phi_{st} = 0 \quad \text{ and } \quad \partial_t A_{js} - \partial_j A_{st} + \partial_s\phi_{tj} = 0.$$

Summing the above two expressions, we get:

$$\partial_s A_{tj} - \partial_j A_{ts} + \partial_s\phi_{tj} - \partial_j\phi_{ts} = 0 \quad \text{ for all } j,s,t = 1\ldots d.$$

Hence for any $i = 1\ldots d$, the $i$-th row $[A_{ij} + \phi_{ji}]_{j=1\ldots d}$ of the matrix field $A+\phi$ is a gradient of some scalar field $w^i$ on $\omega$, where we again used Poincaré's Lemma. Writing $w = [w^i]_{i=1\ldots d} : \omega \to \mathbb{R}^d$, we obtain the claim by taking the symmetric parts of the resulting identity:

$$A + \phi = \nabla w,$$

and invoking the skew-symmetry in (6.2). The proof is done. ■

Note that the operator $\mathfrak{C}^2$ can be interpreted as curl curl, in any dimension $d$. Recall that when $d = 2, 3$ then the coefficients of curl $w$ for a vector field $w = [w^i]_{i=1\dots d}$, coincide with the coefficients of the exterior derivative $\mathrm{d}\alpha = \sum_{i<j} \big(\partial_i w^j - \partial_j w^i\big)\mathrm{d}x_i \wedge \mathrm{d}x_j$ of the 1-form $\alpha = \sum_{i=1}^d w^i \mathrm{d}x_i$. Given a matrix field $A$ in any dimension $d$, we may still apply d row-wise:

$$\mathrm{d}A = \Big[\sum_{s<t} \big(\partial_s A_{it} - \partial_t A_{is}\big)\mathrm{d}x_s \wedge \mathrm{d}x_t\Big]_{i=1\dots d},$$

returning a vector or 2-forms, and then apply d to each vector of coefficients in $\mathrm{d}A$:

$$\begin{aligned}\mathrm{d}^2 A &= \Big[\sum_{i<j} \big(\partial_i(\partial_s A_{jt} - \partial_t A_{js}) - \partial_j(\partial_s A_{it} - \partial_t A_{is})\big)\mathrm{d}x_i \wedge \mathrm{d}x_j\Big]_{s<t:1\dots d} \\ &= \Big[\sum_{i<j} \mathfrak{C}^2(A)_{ij,st}\mathrm{d}x_i \wedge \mathrm{d}x_j\Big]_{s<t:1\dots d}.\end{aligned}$$

Our next result formulates the solvability conditions determining the range of $\mathfrak{C}^2$:

**Lemma 6.1.** *Let $\omega \subset \mathbb{R}^d$ be an open, bounded, contractible domain with Lipschitz boundary. Given $F = [F_{ij,st}]_{i,j,s,t=1\dots d} \in L^2(\omega, \mathbb{R}^{d^4})$, the following conditions are equivalent:*

(i) *$F = \mathfrak{C}^2(A)$ for some $A \in W^{2,2}(\omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$,*

(ii) *$F$ satisfies the conditions below, for all $i, j, s, t, q = 1 \dots d$:*

$$\begin{aligned}&F_{ij,st} = -F_{ji,st} = -F_{ij,ts}, \qquad F_{ij,st} = F_{st,ij}, \\ &F_{ij,st} + F_{is,tj} + F_{it,js} = 0, \\ &\partial_q F_{ij,st} + \partial_s F_{ij,tq} + \partial_t F_{ij,qs} = 0 \quad \textit{in the sense of distributions on } \omega.\end{aligned} \tag{6.3}$$

*Proof.* The implication (i)⇒(ii) follows by a direct inspection. To prove (ii)⇒(i), observe first that for a skew-symmetric matrix field $B : \omega \to \mathbb{R}^{d\times d}_{\mathrm{skew}}$ to be of the form: $B = (\nabla w)^T - \nabla w$ for some $w = [w^i]_{i=1\dots d} : \omega \to \mathbb{R}^d$, the sufficient and necessary condition is:

$$\partial_i B_{jq} + \partial_j B_{qi} + \partial_q B_{ij} = 0 \quad \text{for all } i, j, q = 1 \dots d. \tag{6.4}$$

This claim follows by taking the exterior derivative of the 2-form in:

$$\begin{aligned}\mathrm{d}\Big(\sum_{j,q=1\dots d} B_{jq}\mathrm{d}x_j \wedge \mathrm{d}x_q\Big) &= \sum_{i,j,q=1\dots d} \partial_i B_{jq}\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_q \\ &= 2\sum_{i<j<q:1\dots d} (\partial_i B_{jq} + \partial_j B_{qi} + \partial_q B_{ij})\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_q,\end{aligned}$$

where we used the skew-symmetry assumption, and invoking Poincaré's Lemma on the contractible domain $\omega$.

For every $s, t = 1 \dots d$ we apply the above criterion to $B = [F_{ij,st}]_{i,j=1\dots d}$. Since the first and third conditions in (6.3) validate the skew-symmetry of $B$ and (6.4), we get existence of $d^2$ vector fields $\phi_{st} = [\phi^j_{st}]_{j=1\dots d}$ on $\omega$, satisfying:

$$F_{ij,st} = \partial_i \phi^j_{st} - \partial_j \phi^i_{st} \quad \text{for all } i, j, s, t = 1 \dots d. \tag{6.5}$$

By the first condition in (6.3), we note that $\partial_i(\phi^j_{st} + \phi^j_{st}) - \partial_j(\phi^i_{st} + \phi^i_{st}) = 0$ for all $i, j, s, t$, which implies that each $\phi_{st} + \phi_{ts}$ is a gradient. Thus, without loss of generality we may take:

$$\phi_{st} = -\phi_{ts} \quad \text{for all } s, t = 1 \dots d.$$

For every $t = 1 \dots d$ consider now the skew-symmetric matrix field $B = [\phi^j_{st} - \phi^s_{jt}]_{j,s=1\dots d}$. Condition (6.4) holds, in virtue of (6.5) and the second condition in (6.3):

$$\partial_i(\phi^j_{st} - \phi^s_{jt}) + \partial_j(\phi^s_{it} - \phi^i_{st}) + \partial_s(\phi^i_{jt} - \phi^j_{it}) = F_{ij,st} + F_{si,jt} + F_{js,it} = 0,$$

and so there follows existence of vector fields $\eta_t = [\eta^s_t]_{s=1\dots d}$ on $\omega$, such that:

$$\phi^j_{st} - \phi^s_{jt} = \partial_j \eta^s_t - \partial_s \eta^t_j \quad \text{for all } \ j, s, t = 1 \dots d. \tag{6.6}$$

We now finally define:

$$A_{ij} = -\frac{1}{2}(\eta^i_j + \eta^j_i) \quad \text{for all } \ i, j = 1 \dots d.$$

The matrix field $A = [A_{ij}]_{i,j=1\dots d}$ is obviously symmetric, and from (6.6) and (6.5) we get:

$$\begin{aligned} 2\mathfrak{C}^2(A)_{ij,st} &= -\partial_i\partial_s(\eta^j_t + \eta^t_j) - \partial_j\partial_t(\eta^i_s + \eta^s_i) + \partial_i\partial_t(\eta^j_s + \eta^s_j) + \partial_j\partial_s(\eta^i_t + \eta^t_i) \\ &= \partial_t(\phi^i_{js} - \phi^j_{is}) + \partial_j(\phi^s_{ti} - \phi^t_{si}) + \partial_s(\phi^j_{it} - \phi^i_{jt}) + \partial_i(\phi^t_{sj} - \phi^s_{tj}) \\ &= F_{ti,js} + F_{js,ti} + F_{tj,si} + F_{si,tj} = 2(F_{ti,js} + F_{tj,si}) = 2F_{st,ij} = 2F_{ij,st} \end{aligned}$$

for all $i, j, s, t = 1 \dots d$, where in the last three equalities above we used the first and second conditions in (6.3). The proof is done. ■

Note that for $d = 3$ and $k = 1$, any choice of 6 functions $F_{12,12}, F_{12,13}, F_{12,23}, F_{13,13}, F_{23,23} \in L^2(\omega, \mathbb{R})$ gives raise to $F \in L^2(\omega, \mathbb{R}^{81})$ satisfying (6.3). Indeed, the first condition holds by defining the remaining components of $F$ appropriately, while the second and the third conditions are implied automatically by these symmetries. In this case, (MA) consists of 6 equations in a single unknown $v \in \mathbb{R}$, while (VK) consists of 6 equations but now in 4 unknowns $(v, w) \in \mathbb{R}^4$. Even though both formulations seem to be largely overdetermined, this paper actually shows that the set of their solutions is dense in the space of continuous functions on $\bar{\omega}$.

We are now ready to give:

**Proof of Theorem 1.7**

By the construction in Theorem 6.1, there exists a matrix field $A \in \mathcal{C}^{1,1}(\bar{\omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ such that $\mathfrak{C}^2(A) = -F$. Given $v \in \mathcal{C}^1(\bar{\omega}, \mathbb{R}^k)$, we apply Theorem 1.1 to $\epsilon = 1/n$ and $v$, $w = 0$, $A + C\mathrm{Id}_d$ where $C > 0$ is taken large enough to have, in the sense of matrix inequalities:

$$A + C\mathrm{Id}_d > \frac{1}{2}(\nabla v_0)^T \nabla v_0 \quad \text{on } \ \bar{\omega}.$$

The resulting $v_n = \tilde{v}$ provides the $n$-th member of the claimed approximating sequence for $v$. When $v \in \mathcal{C}^0(\bar{\omega}, \mathbb{R}^k)$, the sequence is obtained using a density argument. ■

## 7. Application: energy scaling bound for thin multidimensional films

In this section, we present another application of Theorem 1.1, in which we estimate the infimum of an energy functional that is the generalisation to arbitrary dimension and codimension, of the so-called non-Euclidean elasticity. For $d = 2$, $k = 1$, this functional models the elastic energy of deformations of prestrained films, and various techniques have been applied to its study in the last decade, notably the dimension reduction via the $\Gamma$-convergence [4]. From another point of view, given the Riemannian metric $g$ on a reference configuration $\Omega$, the energy $\mathcal{E}$ below measures the averaged pointwise deficit of an immersion from being an orientation preserving isometric immersion of $g$, for all weakly regular immersions.

More precisely, given $\omega\subset\mathbb{R}^d$ we define the family of "thin films", parametrised by $h\ll 1$:

$$\Omega^h=\big\{(x,z);x\in\omega,\ z\in B(0,h)\subset\mathbb{R}^k\big\}\subset\mathbb{R}^{d+k}.$$

Consider the Riemannian metrics on $\Omega^h$ of the form:

$$g^h=\mathrm{Id}_{d+k}+2h^{\gamma/2}S,\qquad \text{where } \gamma>0 \text{ and }\qquad S\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^{(d+k)\times(d+k)}_{\mathrm{sym}}).$$

We then pose the problem of minimizing the following energy functionals, as $h\to 0$:

$$\mathcal{E}^h(u)=\fint_{\Omega^h}W\big((\nabla u)(g^h)^{-1/2}\big)\ \mathrm{d}(x,z)\qquad \text{for all }\ u\in W^{1,2}(\Omega^h,\mathbb{R}^{d+k})$$

Extending the analysis for $d=2$, $k=1$ in [5, Theorem 1.4], we get:

**Theorem 7.1.** *Assume that $\omega\subset\mathbb{R}^d$ is an open, bounded domain and let $k\geq 1$. Denote $\beta=d(d+1)/k$. Then, there holds:*

(i) *if $\gamma\geq 4$, then $\inf\mathcal{E}^h\leq Ch^s$, for every $s<2+\frac{\gamma}{2}$,*
(ii) *if $\gamma\in\big[\frac{4}{3+\beta},4\big)$, then $\inf\mathcal{E}^h\leq Ch^s$ for every $s<\frac{4+\gamma(1+\beta)}{2+\beta}$,*
(iii) *if $\gamma\in\big(0,\frac{4}{3+\beta}\big)$, then $\inf\mathcal{E}^h\leq Ch^{2\gamma}$.*

*Proof.* **1.** Fix $\alpha\in\big(0,\frac{1}{1+\beta}\big)$. By Theorem 1.1, there exists $v\in\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^k)$ and $w\in\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^d)$, solving (VK) with the right hand side given by the $d\times d$ principal minor of $S$:

$$\frac{1}{2}(\nabla v)^T\nabla v+\mathrm{sym}\nabla w=S_{d\times d}. \tag{7.1}$$

We regularize $v,w$ to $v_\epsilon\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^k)$, $w_\epsilon\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^d)$ convolving with the kernels $\{\phi_\epsilon(x)\}_{\epsilon\to 0}$ as in Lemma 2.3, where $\epsilon$ is a positive power $t$ of $h$, to be chosen later:

$$v_\epsilon=v*\phi_\epsilon,\quad w_\epsilon=w*\phi_\epsilon,\quad \epsilon=h^t.$$

By $(2.3)_2$ and a version of $(2.3)_4$ in: $\|(fg)*\phi_\epsilon-(f*\phi_\epsilon)(g*\phi_\epsilon)\|_0\leq C\epsilon^{2\alpha}\|f\|_{0,\alpha}\|g\|_{0,\alpha}$, we get:

$$\begin{aligned}\Big\|\frac{1}{2}(\nabla v_\epsilon)^T\nabla v_\epsilon&+\mathrm{sym}\nabla w_\epsilon-S_{d\times d}\Big\|_0\\ &\leq\Big\|\frac{1}{2}(\nabla v_\epsilon)^T\nabla v_\epsilon+\mathrm{sym}\,\nabla w_\epsilon-S_{d\times d}*\phi_\epsilon\Big\|_0+\|S_{d\times d}*\phi_\epsilon-S_{d\times d}\|_0\\ &=\frac{1}{2}\Big\|(\nabla v_\epsilon)^T\nabla v_\epsilon-\big((\nabla v)^T\nabla v\big)*\phi_\epsilon\Big\|_0+\|S_{d\times d}*\phi_\epsilon-S_{d\times d}\|_0\\ &\leq C\epsilon^{2\alpha}\|\nabla v\|^2_{0,\alpha}+C\epsilon^2\|\nabla^2S_{d\times d}\|_0\leq C\epsilon^{2\alpha}.\end{aligned}\tag{7.2}$$

Further, by $(2.3)_1$ and a version of $(2.3)_2$ in: $\|\nabla(f-f*\phi_\epsilon)\|_0\leq C\epsilon^{\alpha-1}\|f\|_{0,\alpha}$, we obtain:

$$\|\nabla v_\epsilon\|_0+\|\nabla w_\epsilon\|_0\leq C,\qquad \|\nabla^2v_\epsilon\|_0+\|\nabla^2w_\epsilon\|_0\leq C\epsilon^{\alpha-1}. \tag{7.3}$$

**2.** Denote $\delta=\gamma/2$ and define $u^h\in\mathcal{C}^\infty(\bar\Omega^h,\mathbb{R}^{d+k})$ as follows:

$$\begin{aligned}u^h(x,z)=id_{d+k}&+h^{\delta/2}\begin{bmatrix}0\\ v_\epsilon\end{bmatrix}+h^\delta\begin{bmatrix}w_\epsilon\\ 0\end{bmatrix}\\ &+\Big(h^{\delta/2}\begin{bmatrix}-(\nabla v_\epsilon)^T\\ 0\end{bmatrix}+h^\delta\begin{bmatrix}2S_{d\times k}\\ S_{k\times k}-\frac{1}{2}(\nabla v_\epsilon)(\nabla v_\epsilon)^T\end{bmatrix}+h^{3\delta/2}B(x)\Big)z\end{aligned}$$

where we denote: $S=\left[\begin{array}{c|c}S_{d\times d}&S_{d\times k}\\ \hline S_{k\times d}&S_{k\times k}\end{array}\right]$, and where the higher order correction field $B \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^{(d+k)\times k})$ is given by:

$$B(x) = \left[\begin{array}{c} -(\nabla v_\epsilon)^T S_{k\times k} + \frac{1}{2}(\nabla v_\epsilon)^T(\nabla v_\epsilon)(\nabla v_\epsilon)^T + (\nabla w_\epsilon)^T(\nabla v_\epsilon)^T \\ \hline 2\mathrm{sym}\big((\nabla v_\epsilon)S_{d\times k}\big) \end{array}\right].$$

It follows that for all $x \in \bar\omega$ and $z \in B(0,1) \subset \mathbb{R}^k$ there holds:

$$\begin{aligned} \nabla u^h(x,hz) = \mathrm{Id}_{d+k} &+ h^{\delta/2}\left[\begin{array}{c|c} 0 & -(\nabla v_\epsilon)^T \\ \hline \nabla v_\epsilon & 0 \end{array}\right] + h^\delta \left[\begin{array}{c|c} \nabla w_\epsilon & 2S_{d\times k} \\ \hline 0 & S_{k\times k} - \frac{1}{2}(\nabla v_\epsilon)(\nabla v_\epsilon)^T \end{array}\right] \\ &+ h^{3\delta/2}\left[\begin{array}{c|c} 0 & B \end{array}\right] - h^{1+\delta/2}\left[\begin{array}{c|c} \big[\langle \partial_i\partial_j v_\epsilon, z\rangle\big]_{i,j=1\ldots d} & 0 \\ \hline 0 & 0 \end{array}\right] \\ &+ O\big(h^{1+\delta}\big)(1 + \|\nabla^2 v_\epsilon\|_0) + O(h^{1+3\delta/2})\|\nabla^2 w_\epsilon\|_0. \end{aligned}$$

We now observa that: $(g^h)^{-1/2} = \mathrm{Id}_{d+k} - h^\delta S + O(h^{2\delta})$, and proceed with computing:

$$\begin{aligned} \big(\nabla u^h (g^h)^{-1/2}\big)(x,hz) = \mathrm{Id}_{d+k} + P^h &+ h^\delta\left[\begin{array}{c|c} \nabla w_\epsilon - S_{d\times d} & 0 \\ \hline 0 & -\frac{1}{2}(\nabla v_\epsilon)(\nabla v_\epsilon)^T \end{array}\right] \\ &+ h^{3\delta/2}\left[\begin{array}{c|c} (\nabla v_\epsilon)^T S_{k\times d} & \frac{1}{2}(\nabla v_\epsilon)(\nabla v_\epsilon)^T\nabla v_\epsilon + (\nabla w_\epsilon)^T(\nabla v_\epsilon)^T \\ \hline -(\nabla v_\epsilon)S_{d\times d} & S_{k\times d}(\nabla v_\epsilon)^T \end{array}\right] \\ &- h^{1+\delta/2}\left[\begin{array}{c|c} \big[\langle \partial_i\partial_j v_\epsilon, z\rangle\big]_{i,j=1\ldots d} & 0 \\ \hline 0 & 0 \end{array}\right] \\ &+ O(h^{2\delta}) + O(h^{1+\delta})(1 + \|\nabla^2 v_\epsilon\|_0 + \|\nabla^2 w_\epsilon\|_0). \end{aligned}$$

Above, we used the following skew-symmetric matrix field:

$$P^h = \left[\begin{array}{c|c} 0 & p^h \\ \hline -(p^h)^T & 0 \end{array}\right], \qquad p^h = -h^{\delta/2}(\nabla v_\epsilon)^T + h^\delta S_{d\times k}.$$

For future purpose, it is convenient to compute:

$$(P^h)^2 = -h^\delta\left[\begin{array}{c|c} (\nabla v_\epsilon)^T\nabla v_\epsilon & 0 \\ \hline 0 & (\nabla v_\epsilon)(\nabla v_\epsilon)^T \end{array}\right] + 2h^{3\delta/2}\mathrm{sym}\left[\begin{array}{c|c} (\nabla v_\epsilon)^T S_{k\times d} & 0 \\ \hline 0 & (\nabla v_\epsilon)S_{d\times k} \end{array}\right] + O(h^{2\delta}).$$

**3.** Consider the rotation fields $Q^h \in \mathcal{C}^\infty(\bar\omega, \mathrm{SO}(d+k))$, defined by:

$$Q^h = \exp(-P^h) = \mathrm{Id}_{d+k} - P^h + \frac{1}{2}(P^h)^2 - \frac{1}{6}(P^h)^3 + O(h^{2\delta}).$$

Then we get:

$$\begin{aligned}\big(Q^h\nabla u^h(g^h)^{-1/2}\big)(x,hz) = \mathrm{Id}_{d+k} + h^\delta & \left[\begin{array}{c|c}\frac{1}{2}(\nabla v_\epsilon)^T\nabla v_\epsilon + \nabla w_\epsilon - S_{d\times d} & 0 \\ \hline 0 & 0\end{array}\right] \\ & + h^{3\delta/2}\left[\begin{array}{c|c}\mathrm{skew}\big((\nabla v_\epsilon)^T S_{k\times d}\big) & (\nabla w_\epsilon)^T(\nabla v_\epsilon)^T \\ \hline -(\nabla v_\epsilon)\nabla w_\epsilon & \mathrm{skew}\big(S_{k\times d}(\nabla v_\epsilon)^T\big)\end{array}\right] + \frac{1}{3}(P^h)^3 \\ & - h^{1+\delta/2}\left[\begin{array}{c|c}\big[\langle\partial_i\partial_j v_\epsilon, z\rangle\big]_{i,j=1\ldots d} & 0 \\ \hline 0 & 0\end{array}\right] \\ & + O(h^{2\delta}) + O(h^{1+\delta})(1+\|\nabla^2 v_\epsilon\|_0 + \|\nabla^2 w_\epsilon\|_0).\end{aligned}$$

Finally, we apply another rotation field $\bar{Q}^h \in \mathcal{C}^\infty(\bar\omega, \mathrm{SO}(d+k))$:

$$\bar{Q}^h = \exp(-\bar{P}^h) = \mathrm{Id}_{d+k} - \bar{P}^h + O(h^{2\delta}),$$

$$\text{where}\quad \bar{P}^h = \left[\begin{array}{c|c}\mathrm{skew}\big(h^\delta\nabla w_\epsilon + h^{3\delta/2}(\nabla v_\epsilon)^T S_{k\times d}\big) & h^{3\delta/2}(\nabla w_\epsilon)^T(\nabla v_\epsilon)^T \\ \hline -h^{3\delta/2}(\nabla v_\epsilon)\nabla w_\epsilon & h^{3\delta/2}\mathrm{skew}\big(S_{k\times d}(\nabla v_\epsilon)^T\big)\end{array}\right] + \frac{1}{3}(P^h)^3,$$

to get:

$$\begin{aligned}\big(\bar{Q}^h Q^h\nabla u^h(g^h)^{-1/2}\big)(x,hz) = \mathrm{Id}_{d+k} + h^\delta & \left[\begin{array}{c|c}\frac{1}{2}(\nabla v_\epsilon)^T\nabla v_\epsilon + \mathrm{sym}\nabla w_\epsilon - S_{d\times d} & 0 \\ \hline 0 & 0\end{array}\right] \\ & - h^{1+\delta/2}\left[\begin{array}{c|c}\big[\langle\partial_i\partial_j v_\epsilon, z\rangle\big]_{i,j=1\ldots d} & 0 \\ \hline 0 & 0\end{array}\right] \\ & + O(h^{2\delta}) + O(h^{1+\delta})(1+\|\nabla^2 v_\epsilon\|_0 + \|\nabla^2 w_\epsilon\|_0).\end{aligned}$$

**4.** In conclusion, we obtain the following energy bound below, valid provided that we may use Taylor's expansion of $W$ up to second order in perturbation of $\mathrm{Id}_{d+k}$, which here holds when $h^{1+\delta/2}(\|\nabla^2 v_\epsilon\|_0 + \|\nabla^2 w_\epsilon\|_0) \to 0$ as $h\to 0$, implied by $\lim_{h\to 0}\big(h^{1+\delta/2)\epsilon^{\alpha-1}}\big) = 0$:

$$\begin{aligned}\inf\mathcal{E}^h &\le \mathcal{E}^h(u^h) = \fint_{\Omega^1} W\Big(\bar{Q}^h Q^h\nabla u^h(g^h)^{-1/2}(x,hz)\Big)\,\mathrm{d}(x,z) \\ &\le C\fint_{\Omega^1}\Big(h^{2\delta}\big|\frac{1}{2}(\nabla v_\epsilon)^T\nabla v_\epsilon + \mathrm{sym}\nabla w_\epsilon - S_{d\times d}\big|^2 + h^{2+\delta}(\|\nabla^2 v_\epsilon\|_0^2 + \|\nabla^2 w_\epsilon\|_0^2) + h^{4\delta}\Big)\,\mathrm{d}(x,z).\end{aligned}$$

Recalling (7.2) and (7.3), the obtained bound further leads to:

$$\inf\mathcal{E}^h \le C\big(h^{2\delta}\epsilon^{4\alpha} + h^{2+\delta}\epsilon^{2\alpha-2} + h^{4\delta}\big) = C\big(h^{2\delta+4\alpha t} + h^{2+\delta+(2\alpha-2)t} + h^{4\delta}\big).$$

Minimizing the right hand side above is equivalent to maximizing the minimal of the three exponents. For $\delta < 2$, we hence choose the exponent $t$ in $\epsilon = h^t$ so that $2\delta + 4\alpha t = 2+\delta+(2\alpha-2)t$, namely $t = \frac{2-\delta}{2\alpha+2}$. Consequently, we get:

$$\inf\mathcal{E}^h \le C\big(h^{2\frac{\delta+2\alpha}{\alpha+1}} + h^{4\delta}\big) \le C\big(h^{\left(4+2\frac{(1+\beta)(\delta-2)}{2+\beta}\right)-} + h^{4\delta}\big)$$

upon recalling the range of admissible exponents $\alpha$. On the other hand, when $\delta \ge 2$, then we choose $t$ close to 0. The conclusion of Theorem 7.1 follows by a direct inspection. ■

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