THE MONGE-AMPÈRE SYSTEM: CONVEX INTEGRATION IN ARBITRARY DIMENSION AND CODIMENSION

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Abstract. In this paper, we study flexibility of the weak solutions to the Monge-Ampère system (MA) via convex integration. This system of Pdes is an extension of the Monge-Ampère equation in $d = 2$ dimensions, naturally arising from the prescribed curvature problem, and closely related to the classical problem of isometric immersions (II).

Our main result achieves density, in the set of subsolutions, of the Hölder regular $\mathcal{C}^{1,\alpha}$ solutions to the weak formulation (VK) of (MA), for all $\alpha < \frac{1}{1+d(d+1)/k}$ where d is an arbitrary dimension and k is an arbitrary codimension of the problem. This result seems to be optimal, from the technical viewpoint, for the corrugation-based convex integration scheme. In particular, it covers the codimension interval $k \in \big(1, d(d+1)\big)$, so far uncharted even for the system (II), since the regularity $\mathcal{C}^{1,\alpha}$ with any $\alpha < 1$ proved by Källen in [7], strictly requires a large codimension $k \geq d(d+1)$. We also reproduce Källen's result in the context of (MA). At $k = 1$, our result agrees with the regularity $\mathcal{C}^{1,\alpha}$ for (II) with any $\alpha < \frac{1}{1+d(d+1)}$, proved by Conti, Delellis and Szekelyhidi in [2]. Finally, our results extend the initial findings by the author and Pakzad in [10] for (MA) and $d = 2, k = 1$.

As an application of our results for (VK), we derive an energy scaling bound in the quantitative immersability of Riemannian metrics, for nonlinear energy functionals modelled on the energies of deformations of thin prestrained films in the nonlinear elasticity [8].

1. Introduction

This paper concerns regularity of weak solutions to a multi-dimensional version of the Monge-Ampère equation, which arises from the prescribed curvature problem and is closely related to the problem of isometric immersions and the dimension reduction of thin films. Namely, given a matrix field $A : \omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ on a domain $\omega \subset \mathbb{R}^d$, we look for vector fields w and v such that:

$$\begin{aligned} & v : \omega \to \mathbb{R}^k, \quad w : \omega \to \mathbb{R}^d, \\ & \frac{1}{2}(\nabla v)^T \nabla v + \mathrm{sym}\nabla w = A \quad \text{ in } \omega. \end{aligned} \tag{VK}$$

When $d = 2, k = 1$, the left hand side above is the von Kármán content whose energy measures the stretching of a thin film with midplate ω, subject to the out of plane displacement v and the in plane displacement w. Taking *curl curl* of both sides of (VK) yields then the Monge-Ampère equation: $\det^2 v = -\mathrm{curl}\,\mathrm{curl}\, A$ on ω, which reflects matching the leading order term in the expansion of the Gaussian curvature κ of surfaces $\{(x, \epsilon v(x));\ x \in \omega\} \subset \mathbb{R}^3$ with the given scalar function $f = -\mathrm{curl}\,\mathrm{curl}\, A$. We note that one can achieve any sufficiently regular f by imposing $A = -(\Delta^{-1} f)\mathrm{Id}_2$. For arbitrary $d, k \geq 1$, the situation is similar: applying a multi-dimensional operator $\mathfrak{C}^2$ (see the formula in (1.4)) to both sides of (VK), leads to the system which equates the leading order term in the expansion of the full Riemann curvature

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tensor of the manifold $\{(x, \epsilon v(x));\ x\in\omega\}\subset\mathbb{R}^{d+k}$ with the table $F=-\mathfrak{C}^2(A):\omega\to\mathbb{R}^{d^4}$:

$$\begin{aligned} & v:\omega\to\mathbb{R}^k,\\ & \mathfrak{Det}\,\nabla^2 v \doteq \big[\langle\partial_i\partial_j v, \partial_j\partial_t v\rangle - \langle\partial_i\partial_t v - \partial_j\partial_s v\rangle\big]_{ij,st:1\dots d} = F \quad \text{in } \omega. \end{aligned} \tag{MA}$$

Again, any F satisfying the symmetries and Bianchi's identities can be achieved this way.

The closely related problem to (VK) and (MA) is the problem of finding an isometric immersion u of the given Riemannian metric $g:\omega\to\mathbb{R}^{d\times d}_{\mathrm{sym},>}$, into a higher dimensional space $\mathbb{R}^{d+k}$:

$$\begin{aligned} & u:\omega\to\mathbb{R}^{d+k},\\ & (\nabla u)^T\nabla u = g \quad \text{in } \omega. \end{aligned} \tag{II}$$

This problem reduces to (VK) when gathering the leading (lowest order) terms in the ϵ-expansions, for the family of Riemannian metrics $\{g^\epsilon=\mathrm{Id}_d+2\epsilon^2A\}_{\epsilon\to 0}$ and the family of metrics generated by the immersions $\{u^\epsilon=(id_d+\epsilon^2 w, \epsilon v)\}_{\epsilon\to 0}$. When posed in arbitrary dimension d but low codimension $k=1$, it has been shown in [3, Theorem 1.1] that any local subsolution to (II) can be uniformly approximated by a sequence of solutions $\{u_n\}_{n=1}^\infty$ of regularity $\mathcal{C}^{1,\alpha}$, for any Hölder exponent $\alpha<\frac{1}{1+2d_*}$ where $d_*=d(d+1)/2$ is the dimension of $\mathbb{R}^{d\times d}_{\mathrm{sym}}$. On the other hand, as showed in [7], regularity $\mathcal{C}^{1,\alpha}$ with any $\alpha<1$ can be achieved in sufficiently high codimension k; however even for the local result this argument strictly requires $k\geq 2d_*$, whereas it yields no outcome for $k<2d_*$. These two results, albait both relying on convex integration, use different constructions of the cascade of perturbations: Kuiper's corrugations in [3] and Nash's spirals in [7], and one cannot be deduced from the other.

To our knowledge, there has been no result for the codimension interval $k\in(1,2d_*)$ interpolating the regularity in [3] and [7], or even improving the exponent $\frac{1}{1+2d_*}$ without the requirement $k\geq 2d_*$. The purpose of our paper is to achieve precisely this goal, for the system (VK). Our main result states that any $\mathcal{C}^1$-regular pair (v,w) which is a subsolution of (VK), can be uniformly approximated by a sequence of solutions $\{(v_n,w_n)\}_{n=1}^\infty$ of regularity $\mathcal{C}^{1,\alpha}$ for any Hölder exponent $\alpha<\frac{1}{1+2d_*/k}$, in case of *arbitrary d and k*. Our proof only uses corrugations, extending the construction in [3] in an optimal manner. Clearly, the obtained critical regularity exponent $1/2$ at $k=2d_*$ is inferior to the exponent 1 from a version of the same construction as in [7], that we also demonstrate in our paper. We expect that the superposition of both techniques should yield a tighter interpolation, which is the subject of the ongoing research.

We state our results and offer further discussion in the subsections below.

1.1. **Convex integration by corrugations, arbitrary d and k.**

The following theorem is our main result. We refer to it as *flexibility* of (VK) up to $\mathcal{C}^{1,\frac{1}{1+2d_*/k}}$:

Theorem 1.1. *Let $\omega\subset\mathbb{R}^d$ be an open, bounded domain. Given two vector fields $v\in\mathcal{C}^1(\bar\omega,\mathbb{R}^k)$, $w\in\mathcal{C}^1(\bar\omega,\mathbb{R}^d)$ and a matrix field $A\in\mathcal{C}^{0,\beta}(\bar\omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$, assume that:*

$$\mathcal{D}=A-\big(\frac{1}{2}(\nabla v)^T\nabla v+\mathrm{sym}\nabla w\big) \quad \textit{satisfies} \quad \mathcal{D}>c\,\mathrm{Id}_d \ \textit{ on } \bar\omega,$$

for some $c>0$, in the sense of matrix inequalities. Fix $\epsilon>0$ and let:

$$0<\alpha<\min\Big\{\frac{\beta}{2},\frac{1}{1+d(d+1)/k}\Big\}.$$

Then, there exists $\tilde v \in \mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^k)$ *and* $\tilde w \in \mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^d)$ *such that the following holds:*

$$\|\tilde v - v\|_0 \leq \epsilon, \quad \|\tilde w - w\|_0 \leq \epsilon, \tag{1.1$_1$}$$

$$A - \big(\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v + \mathrm{sym}\nabla\tilde w\big) = 0 \quad \textit{in } \bar\omega. \tag{1.1$_2$}$$

This result generalizes [10, Theorem 1.1], where we proved flexibility for (VK) up to $\mathcal{C}^{1,\frac{1}{7}}$ in dimensions $d = 2$, $k = 1$. In that special case, motivated by theory of elasticity, the left hand side of (VK) represents the von Kármán stretching content $\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{sym}\nabla w$ written in terms of the scalar out-of-plane displacement v and the in-plane displacement w of the middle plate ω of a thin film. The case $d = 2$ is special and flexibility (in codimension 1) of (VK) holds up to $\mathcal{C}^{1,\frac{1}{5}}$ as shown in [2, Theorem 1.1] using the conformal equivalence of 2-dimensional metrics to the Euclidean metric. In our follow-up extension [9] of the present work, we likewise show that any $k \geq 1$ allow for flexibility up to $\mathcal{C}^{1,\frac{1}{1+4/k}}$ when $d = 2$. After the completion of our both works, we learned of the very recent preprint [1] in which flexibility of (VK) for $d = 2, k = 1$ has been futher improved to hold up to $\mathcal{C}^{1,\frac{1}{3}}$.

The main new technical ingredient allowing for the flexibility range stated in Theorem 1.1, is the following "stage"-type construction in the convex integration algorithm for (VK):

Theorem 1.2. *Let the vector fields* $v \in \mathcal{C}^2(\bar\omega,\mathbb{R}^k)$, $w \in \mathcal{C}^2(\bar\omega,\mathbb{R}^d)$ *and the matrix field* $A \in \mathcal{C}^{0,\beta}(\bar\omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$ *be given on an open, bounded domain* $\omega \subset \mathbb{R}^d$. *Assume that:*

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big) \quad \textit{satisfies } 0 < \|\mathcal{D}\|_0 \leq 1.$$

Fix two constants M, σ *such that:*

$$M \geq \max\{\|v\|_2, \|w\|_2, 1\} \quad \textit{and} \quad \sigma \geq 1.$$

Then, there exist $\tilde v \in \mathcal{C}^2(\bar\omega,\mathbb{R}^k)$ *and* $\tilde w \in \mathcal{C}^2(\bar\omega,\mathbb{R}^d)$ *such that, denoting:*

$$\tilde{\mathcal{D}} = A - \big(\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v + \mathrm{sym}\nabla\tilde w\big),$$

the following holds:

$$\|\tilde v - v\|_1 \leq C\|\mathcal{D}\|_0^{1/2}, \quad \|\tilde w - w\|_1 \leq C\|\mathcal{D}\|_0^{1/2}(1 + \|\nabla v\|_0), \tag{1.2$_1$}$$

$$\|\nabla^2\tilde v\|_0 \leq CM\sigma^{d_*/k}, \quad \|\nabla^2\tilde w\|_0 \leq CM\sigma^{d_*/k}(1 + \|\nabla v\|_0), \tag{1.2$_2$}$$

$$\|\tilde{\mathcal{D}}\|_0 \leq C\Big(\frac{\|A\|_{0,\beta}}{M^\beta}\|\mathcal{D}\|_0^{\beta/2} + \frac{\|\mathcal{D}\|_0}{\sigma}\Big), \tag{1.2$_3$}$$

where $d_* = d(d+1)/2$ *and where the constants* C *depend only on* d, k *and* ω.

We briefly outline how our construction differs from [10] and [3]. There, a stage consisted of precisely d_* "steps", each cancelling one of the rank-one "primitive" deficits in the decomposition of $\mathcal{D}$. The initially chosen frequency of perturbation was multiplied by a factor σ at each step, leading to the increase of the second derivative by σ^{d_*} and thus to the exponent d_* replacing d_*/k in $(1.2)_2$, while the remaining error in $\mathcal{D}$ was of order $1/\sigma$, leading to $(1.2)_3$.

Presently, we first observe that k such deficits may be cancelled at once, by using k linearly independent codimensions. Further, when all the first order primitive deficits are cancelled, one may proceed to cancelling the second order deficits obtained as the one-dimensional decompositions of the error between the original and the decreased $\mathcal{D}$; the corresponding frequencies must be then increased by the factor $\sigma^{1/2}$, precisely due to the decrease of $\mathcal{D}$ by the factor

$1/\sigma$. One may inductively proceed in this fashion, cancelling even higher order deficits, and adding k-tuples of single codimension perturbations, for a total of $N = lcm(k, d_*)$ steps. The frequencies get increased by the factor of σ over each multiple of k, leading to the increase of the second derivatives by σ, and by the factor of $\sigma^{1/2}$ over each multiple of d_*, where the deficit decreases by the factor of $1/\sigma$. In the final count, the total increase of the second derivatives has the factor $\sigma^{N/k}$, while the decrease of the deficit has the factor $1/\sigma^{N/d_*}$. The relative change of order is thus $(N/k)/(N/d_*) = d_*/k$, as stated in Theorem 1.2.

We point out that for this scheme to work, it is essential to use the optimal "step"-type construction in which the chosen one-dimensional primitive deficit is cancelled at the expense of introducing least error possible. Our previous definition from [10] would not work for this purpose, and we need to superpose three corrugations rather than two.

1.2. **Convex integration by spirals, $\mathbf{k \geq 2d_*}$.** For a large codimension, one can reach flexibility of (VK) up to $\mathcal{C}^{1,1}$, motivated by a similar result for (II) in [7]:

Theorem 1.3. *In the context of Theorem 1.1, assume that the codimension k satisfies $k \geq 2d_* = d(d+1)$. Then, the same result is valid for any exponent in the range:*

$$0 < \alpha < \min\Big\{\frac{\beta}{2}, 1\Big\}.$$

The "stage" construction allowing for flexibility as above, is the counterpart of Theorem 1.4:

Theorem 1.4. *Let $\omega \subset \mathbb{R}^d$ and k be as in Theorem 1.3. Fix an exponent $\delta > 0$. Then, there exists $\sigma_0 > 1$ depending only on ω and δ, such that we have the following. Given $v \in \mathcal{C}^2(\bar\omega, \mathbb{R}^k)$, $w \in \mathcal{C}^2(\bar\omega, \mathbb{R}^d)$, $A \in \mathcal{C}^{0,\beta}(\bar\omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and given two constants M, σ wih the properties:*

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big) \qquad \textit{satisfies} \;\; 0 < \|\mathcal{D}\|_0 \leq 1,$$

$$M \geq \max\{\|v\|_2, \|w\|_2, 1\}, \qquad \sigma \geq \sigma_0,$$

there exist $\tilde v \in \mathcal{C}^2(\bar\omega, \mathbb{R}^k)$, $\tilde w \in \mathcal{C}^2(\bar\omega, \mathbb{R}^d)$ such that, denoting:

$$\tilde{\mathcal{D}} = A - \big(\frac{1}{2}(\nabla \tilde v)^T\nabla \tilde v + \mathrm{sym}\nabla \tilde w\big),$$

the following bounds are valid, with constants C depending only on d, k, ω and δ:

$$\|\tilde v - v\|_1 \leq C\|\mathcal{D}\|_0^{1/2}, \quad \|\tilde w - w\|_1 \leq C\|\mathcal{D}\|_0^{1/2}(1 + \|\nabla v\|_0), \tag{1.3$_1$}$$

$$\|\nabla^2 \tilde v\|_0 \leq CM\sigma^\delta, \quad \|\nabla^2 \tilde w\|_0 \leq CM\sigma^\delta(1 + \|\nabla v\|_0), \tag{1.3$_2$}$$

$$\|\tilde{\mathcal{D}}\|_0 \leq C\Big(\frac{\|A\|_{0,\beta}}{M^\beta}\|\mathcal{D}\|_0^{\beta/2} + \frac{\|\mathcal{D}\|_0}{\sigma}\Big). \tag{1.3$_3$}$$

An outline of this construction, based on the approach in [7], is as follows. Firstly, each rank-one "primitive" deficit is cancelled using two codimensions, via spiral-like perturbations of the fields v, w, rather than via one-dimensional corrugations. This allows for a better order in the second order deficit. Since we now have $2d_*$ codimensions available, we may rank-one decompose the new deficit as well and cancel it right away by adjusting the original perturbations. Proceeding this way, it is possible to cancel arbitrarily high order of deficits, keeping the frequency at a chosen value σ while assuring that $\mathcal{D}$ is decreased by the factor $1/\sigma^N$, for arbitrarily large N.

1.3. **The Monge-Ampére system.** We now proceed to interpreting Theorem 1.1 in the context of the Monge-Ampére system. We introduce this new system of partial differential equations as the strong formulation of (VK). Recall that for a matrix field $A = [A_{ij}]_{i,j=1\dots 2} : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$, the scalar field *curl curlA* is defined by taking the *curl* operator on each row of A, and then applying another *curl* on thus formed two-dimensional vector field:

$$\begin{aligned}\operatorname{curl}\operatorname{curl}A &= \operatorname{curl}\big[\partial_1 A_{12} - \partial_2 A_{11}, \partial_1 A_{22} - \partial_2 A_{21}\big]\\ &= \partial_1\partial_1 A_{22} - \partial_1\partial_2 A_{21} - \partial_1\partial_2 A_{12} + \partial_2\partial_2 A_{11}.\end{aligned}$$

It is well known that the kernel of *curl curl* when restricted to $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ matrix fields, consists precisely of symmetric gradients. We will be concerned with the following generalization of *curl curl*, serving the same characterisation in higher dimensions:

Definition 1.5. *Given a d-dimensional square matrix field $A = [A_{ij}]_{i,j=1\dots d} : \omega \to \mathbb{R}^{d\times d}$ on a domain $\omega \subset \mathbb{R}^d$, we define $\mathfrak{C}^2(A) : \omega \to \mathbb{R}^{d^4}$ by:*

$$\mathfrak{C}^2(A)_{ij,st} = \partial_i\partial_s A_{jt} + \partial_j\partial_t A_{is} - \partial_i\partial_t A_{js} - \partial_j\partial_s A_{it} \quad \textit{for all } \ i,j,s,t = 1\dots d. \tag{1.4}$$

It can be checked that the components of the Riemann curvature tensor of a family of metrics $\mathrm{Id}_d + \epsilon A$ on ω, are given, to the leading order, by the components of $\mathfrak{C}^2(A)$:

$$Riem(\mathrm{Id}_d + \epsilon A)_{ij,st} = -\frac{\epsilon}{2}\mathfrak{C}^2(A)_{ij,st} + O(\epsilon^2) \quad \text{for all } \ i,j,s,t = 1\dots d. \tag{1.5}$$

For dimension $d = 2$, the above formula yields the linearization of the Gaussian curvature: $\kappa(\mathrm{Id}_2 + \epsilon A) = -\frac{\epsilon}{2}\operatorname{curl}\operatorname{curl}A + O(\epsilon^2)$. We have the following:

Lemma 1.6. *Let $\omega \subset \mathbb{R}^d$ be an open, bounded, contractible domain with Lipschitz boundary. Given a symmetric matrix field $A \in L^2(\omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, the following conditions are equivalent:*

(i) $A = \mathrm{sym}\nabla w$ *for some* $w \in H^1(\omega, \mathbb{R}^d)$,
(ii) $\mathfrak{C}^2(A) = 0$ *in the sense of distributions on* ω.

Observe that for $A = (\nabla v)^T\nabla v$ given through a vector field $v : \omega \to \mathbb{R}^k$, there holds:

$$\mathfrak{C}^2\big((\nabla v)^T\nabla v\big)_{ij,st} = 2\langle\partial_i\partial_t v, \partial_j\partial_s v\rangle - 2\langle\partial_i\partial_s v, \partial_j\partial_t v\rangle.$$

When $d = 2$ and $k = 1$, the above reduces to the familiar formula: $\operatorname{curl}\operatorname{curl}(\nabla v\otimes\nabla v) = -2\det\nabla^2 v$. Following this motivation, we introduce:

Definition 1.7. *For $v : \omega \to \mathbb{R}^k$ defined on a domain $\omega \subset \mathbb{R}^d$, we set $\mathfrak{Det}\,\nabla^2 v : \omega \to \mathbb{R}^{d^4}$ in:*

$$\big(\mathfrak{Det}\,\nabla^2 v\big)_{ij,st} = \langle\partial_i\partial_s v, \partial_j\partial_t v\rangle - \langle\partial_i\partial_t v, \partial_j\partial_s v\rangle \quad \textit{for all } \ i,j,s,t = 1\dots d. \tag{1.6}$$

Given $F : \omega \to \mathbb{R}^{d^4}$, we call the following system of Pdes, the Mongé-Amperé system:

$$\mathfrak{Det}\,\nabla^2 v = F \quad \textit{on } \ \omega.$$

Lemma 1.6 can be restated in this context as follows. Given a matrix field $A : \omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ on a domain $\omega \subset \mathbb{R}^d$, the problem (VK) is equivalent to (disregarding the regularity questions):

$$\begin{aligned}&v : \omega \to \mathbb{R}^k,\\ &\mathfrak{Det}\,\nabla^2 v = -\mathfrak{C}^2(A),\end{aligned} \tag{MA}$$

which, for $d=2$ and $k=1$, is precisely the Monge-Ampére constraint $\det\nabla^2 v = -\mathrm{curl}\,\mathrm{curl}A$ appearing in the dimensionally reduced, linearized Kirchhoff's theory of thin plates [5]. For the family of immersions: $\bar\phi_\epsilon = id_d + \epsilon[0,v] + \epsilon^2[w,0] : \omega \to \mathbb{R}^{d+k}$, one further notes that:

$$(\nabla\bar\phi_\epsilon)^T\nabla\bar\phi_\epsilon = \mathrm{Id}_d + 2\epsilon^2\big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big) + O(\epsilon^4).$$

From (1.5), we thus see that the problem of finding a vector field v for which the Riemann curvatures of the metrics $\mathrm{Id}_d+\epsilon^2 A$ and the Riemann curvatures of the pull-back of Id_{d+k} via the reduced maps ϕ_ϵ below, coincide at their lowest order terms in ϵ on ω:

$$\begin{aligned}&\phi_\epsilon = id_d + \epsilon[0,v] : \omega\to\mathbb{R}^{d+k},\\ &Riem\big(\mathrm{Id}_d+\epsilon^2 A\big) = Riem\big((\nabla\phi_\epsilon)^T\nabla\phi_\epsilon\big) + o(\epsilon^2)\end{aligned}\tag{1.7}$$

is equivalent to the problem of finding v that can be matched by an auxiliary vector field w so that the two Riemannian metrics families: $\mathrm{Id}_d+\epsilon^2 A$, and the pull-back of Id_{d+k} via the maps $\bar\phi_\epsilon$, coincide at their lowest order terms in ϵ on ω:

$$\begin{aligned}&\bar\phi_\epsilon = id_d + \epsilon[0,v] + \epsilon^2[w,0] : \omega\to\mathbb{R}^{d+k},\\ &\mathrm{Id}_d+\epsilon^2 A = (\nabla\phi_\epsilon)^T\nabla\phi_\epsilon + o(\epsilon^2).\end{aligned}\tag{1.8}$$

Thus, the four problems (VK), (MA), (1.7) and (1.8) are equivalent.

We further identify the range of $\mathfrak{C}^2$, in terms of the derived symmetry and Bianchi identities:

Lemma 1.8. *Let $\omega\subset\mathbb{R}^d$ be an open, bounded, contractible domain with Lipschitz boundary. Given $F=[F_{ij,st}]_{i,j,s,t=1\dots d}\in L^2(\omega,\mathbb{R}^{d^4})$, the following are equivalent:*

(i) *$F=\mathfrak{C}^2(A)$ for some $A\in H^2(\omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$,*

(ii) *F satisfies the compatibility conditions, for all $i,j,s,t,q=1\dots d$:*

$$\begin{aligned}&F_{ij,st} = -F_{ji,st} = -F_{ij,ts}, \qquad F_{ij,st}=F_{st,ij},\\ &F_{ij,st}+F_{is,tj}+F_{it,js}=0,\\ &\partial_q F_{ij,st} + \partial_s F_{ij,tq} + \partial_t F_{ij,qs} = 0 \quad \textit{in the sense of distributions on } \omega.\end{aligned}\tag{1.9}$$

The above discussion motivates then the following:

Definition 1.9. *Assume that $F\in L^2(\omega,\mathbb{R}^{d^4})$ given on an open, bounded, contractible domain $\omega\subset\mathbb{R}^d$ with Lipschitz boundary, satisfies conditions (1.9). We say that $v\in H^1_{loc}(\omega,\mathbb{R}^k)$ is a weak solution to the Monge-Amperé system:*

$$\mathfrak{Det}\,\nabla^2 v = F \quad \textit{on } \omega,\tag{1.10}$$

provided that there exists $w\in W^{1,1}_{loc}(\omega,\mathbb{R}^d)$ such that (VK) holds with $\mathfrak{C}^2(A)=-F$, namely:

$$\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w = -\big(\mathfrak{C}^2\big)^{-1}(F) \quad \textit{on } \omega.$$

For $d=2$, $k=1$, any $F\in L^{1+}(\omega,\mathbb{R})$ can be expressed as the right hand side of (MA), because writing $A=\gamma\mathrm{Id}_2$ where $\Delta\gamma=-F$ in ω, there holds: $F=-\mathrm{curl}\,\mathrm{curl}\,A$. In higher dimensions, the solvability conditions are nontrivial and precisely given by (1.9) in Theorem 1.8. In view of Theorems 1.1 and 1.3, we thus obtain the following extension of [10, Theorem 1.1] proved there in dimension $d=2$ and codimension $k=1$, now to arbitrary d,k:

Theorem 1.10. *Let $F \in L^\infty(\omega, \mathbb{R}^{d^4})$ on an open, bounded, contractible domain $\omega \subset \mathbb{R}^d$ with Lipschitz boundary, satisfy (1.9). Fix $k \geq 1$ and fix an exponent α in:*

$$0 < \alpha < \frac{1}{1 + d(d+1)/k}, \qquad \text{or } \ 0 < \alpha < 1 \ \text{ in case of } \ k \geq d(d+1).$$

Then the set of $\mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^k)$ weak solutions to (1.10) is dense in $\mathcal{C}^0(\bar{\omega}, \mathbb{R}^k)$. Namely, every $v \in \mathcal{C}^0(\bar{\omega}, \mathbb{R}^k)$ is the uniform limit of some sequence $\{v_n \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^k)\}_{n=1}^\infty$, such that:

$$\mathfrak{Det}\, \nabla^2 v_n = F \quad \text{ on } \ \omega, \quad \text{ for all } \ n = 1 \ldots \infty.$$

1.4. **Energy scaling bound for thin multidimensional films.** As an application, we now present an estimate on the energy functional that is the generalisation to arbitrary dimension and codimension, of the non-Euclidean elasticity. For $d = 2$, $k = 1$, this functional models the elastic energy of deformations of prestrained films, and various techniques have been applied to its study [8]. From another point of view, given the Riemannian metric g on a reference configuration Ω, the energy $\mathcal{E}$ below measures the averaged pointwise deficit of an immersion from being an orientation preserving isometric immersion of g, for all weakly regular immersions.

More precisely, given $\omega \subset \mathbb{R}^d$ we define the family of "thin films", parametrised by $h \ll 1$:

$$\Omega^h = \big\{(x, z); \ x \in \omega, \ z \in B(0, h) \subset \mathbb{R}^k\big\} \subset \mathbb{R}^{d+k}.$$

Consider the Riemannian metrics on Ω^h of the form:

$$g^h = \mathrm{Id}_{d+k} + 2h^{\gamma/2} S, \qquad \text{where } \gamma > 0 \text{ and } S \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{(d+k)\times(d+k)}_{\mathrm{sym}}).$$

We then pose the problem of minimizing the following energy functionals, as $h \to 0$:

$$\mathcal{E}^h(u) = \fint_{\Omega^h} W\big((\nabla u)(g^h)^{-1/2}\big)\, \mathrm{d}(x, z) \qquad \text{for all } \ u \in H^1(\Omega^h, \mathbb{R}^{d+k}). \tag{1.11}$$

The function $W : \mathbb{R}^{(d+k)\times(d+k)} \to [0, \infty]$ is assumed to be $\mathcal{C}^2$-regular in the vicinity of $\mathrm{SO}(d+k)$, equal to 0 at Id_{d+k}, and frame-invariant in the sense that $W(RF) = W(F)$ for all $R \in \mathrm{SO}(d+k)$. Questions on asymptotics of minimizing configurations to $\mathcal{E}^h$ as $h \to 0$, in function of the scaling exponent β in: $\inf \mathcal{E}^h \sim Ch^\beta$, received a lot of attention in the last decade, via the techniques of dimension reduction and Γ-convergence, starting with the seminal paper [5] (see also [8] and references therein). Extending the analysis in [6, Theorem 1.4], we get:

Theorem 1.11. *Assume that $\omega \subset \mathbb{R}^d$ is an open, bounded domain and let $k \geq 1$. Denote $s = d(d+1)/k$, or $s = 1$ when $k \geq d(d+1)$. Then, there holds:*

(i) *if $\gamma \geq 4$, then $\inf \mathcal{E}^h \leq Ch^\beta$, for every $\beta < 2 + \frac{\gamma}{2}$,*
(ii) *if $\gamma \in \big[\frac{4}{3+s}, 4\big)$, then $\inf \mathcal{E}^h \leq Ch^\beta$ for every $\beta < \frac{4+\gamma(1+s)}{2+s}$,*
(iii) *if $\gamma \in \big(0, \frac{4}{3+s}\big)$, then $\inf \mathcal{E}^h \leq Ch^\beta$, with $\beta = 2\gamma$.*

We recall that for $d = 2, k = 1$, the asymptotic behaviour of the minimizing sequences to (1.11) as $h \to 0$, is fully understood in the scaling regime corresponding to $\beta \geq 2$ (see [8]).

1.5. **Organization of the paper and notation.** In section 2 we give two different constructions of the single "step" of the convex integration algorithm, and recall a few auxiliary results. The proof of Theorem 1.2 and the "stage" construction is carried out in section 3, based on the corrugation "step" in Lemma 2.1. The proof of Theorem 1.4 and the corresponding "stage" construction based on the spirals "step" in Lemma 2.3 is given in section 4. The Nash-Kuiper scheme involving induction on stages is presented in section 5, and Theorems 1.1 and 1.3 are then deduced in section 6. In section 7 we discuss the Monge-Ampére system and prove Lemmas 1.6, 1.8, and Theorem 1.10. Finally, in section 8 we prove Theorem 1.11.

By $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ we denote the space of symmetric $d\times d$ matrices, and by $\mathbb{R}^{d\times d}_{\mathrm{sym},>}$ we denote the cone of symmetric, positive definite $d\times d$ matrices. The space of Hölder continuous vector fields $\mathcal{C}^{m,\alpha}(\bar\omega,\mathbb{R}^k)$ consists of restrictions of all $f\in\mathcal{C}^{m,\alpha}(\mathbb{R}^d,\mathbb{R}^k)$ to the closure of an open domain $\omega\subset\mathbb{R}^d$. Then, the $\mathcal{C}^m(\bar\omega,\mathbb{R}^k)$ norm of such restriction is denoted by $\|f\|_m$, while its Hölder norm $\mathcal{C}^{m,\alpha}(\bar\omega,\mathbb{R}^k)$ is $\|f\|_{m,\alpha}$. By $C>0$ we denote a universal constant which may change from line to line, but which is independent of all parameters, unless indicated otherwise.

2. Convex integration: the basic "step" and preparatory statements

In this section, we give two different constructions of the basic building block in the convex integration algorithm towards the proof of Theorems 1.1 and 1.3. The first construction below is based on Kuiper's corrugations. A similar calculation in [10] essentially had $\bar\Gamma=0$, resulting in the presence of the extra term $-\frac{2}{\lambda}a\bar{\bar\Gamma}(\lambda t_\eta)\mathrm{sym}(\nabla a\otimes\eta)$ in the right hand side of (2.3). With that term, achieving the construction of a stage in section 3 and obtaining error bounds leading to the Hölder exponent threshold in Theorems 1.1 and 1.10 would not be possible.

Lemma 2.1. *Let $v\in\mathcal{C}^2(\omega,\mathbb{R}^k)$ and $w\in\mathcal{C}^1(\omega,\mathbb{R}^d)$ be two vector fields on an open domain $\omega\subset\mathbb{R}^d$. Let $\eta\in\mathbb{R}^d$ and $E\in\mathbb{R}^k$ be two unit vectors and let $\lambda>0$, $a\in\mathcal{C}^2(\omega,\mathbb{R})$. We denote:*

$$\Gamma(t)=2\sin t,\quad \bar\Gamma(t)=-\frac{1}{2}\cos(2t),\quad \bar{\bar\Gamma}(t)=-\frac{1}{2}\sin(2t). \tag{2.1}$$

Denoting further $t_\eta=\langle x,\eta\rangle$, we define:

$$\begin{aligned}\tilde v(x)&=v(x)+\frac{1}{\lambda}a(x)\Gamma(\lambda t_\eta)E\\ \tilde w(x)&=w(x)-\frac{1}{\lambda}a(x)\Gamma(\lambda t_\eta)\nabla\langle v(x),E\rangle-\frac{1}{\lambda^2}a(x)\bar\Gamma(\lambda t_\eta)\nabla a(x)+\frac{1}{\lambda}a(x)^2\bar{\bar\Gamma}(\lambda t_\eta)\eta.\end{aligned} \tag{2.2}$$

Then we have:

$$\begin{aligned}&\big(\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v+\mathrm{sym}\nabla\tilde w\big)-\big(\frac{1}{2}(\nabla v)^T\nabla v+\mathrm{sym}\nabla w\big)-a^2\eta\otimes\eta\\ &=-\frac{1}{\lambda}a\Gamma(\lambda t_\eta)\nabla^2\langle v,E\rangle+\frac{1}{\lambda^2}\big(\frac{1}{2}\Gamma(\lambda t_\eta)^2-\bar\Gamma(\lambda t_\eta)\big)\nabla a\otimes\nabla a-\frac{1}{\lambda^2}a\bar\Gamma(\lambda t_\eta)\nabla^2 a,\end{aligned} \tag{2.3}$$

where $\frac{1}{2}\Gamma(t)^2-\bar\Gamma(t)=1-\frac{1}{2}\cos(2t)$.

Proof. By a direct calculation, it follows that:

$$\nabla\tilde v=\nabla v+\frac{1}{\lambda}\Gamma(\lambda t_\eta)E\otimes\nabla a+a\Gamma'(\lambda t_\eta)E\otimes\eta,$$

which implies:

$$\begin{aligned}\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v-\frac{1}{2}(\nabla v)^T\nabla v=&\ \frac{1}{2}a^2\Gamma'(\lambda t_\eta)^2\eta\otimes\eta+\frac{1}{\lambda}a\Gamma'(\lambda t_\eta)\Gamma(\lambda t_\eta)\mathrm{sym}\big(\nabla a\otimes\eta\big)\\ &+\frac{1}{2\lambda^2}\Gamma(\lambda t_\eta)^2\nabla a\otimes\nabla a.\\ &+\Big(a\Gamma'(\lambda t_\eta)\mathrm{sym}\big((\eta\otimes E)\nabla v\big)+\frac{1}{\lambda}\Gamma(\lambda t_\eta)\mathrm{sym}\big((\nabla a\otimes E)\nabla v\big)\Big).\end{aligned}$$

Similarly:

$$\begin{aligned}\mathrm{sym}\nabla\tilde{w} - \mathrm{sym}\nabla w = {} & a^2\bar{\bar{\Gamma}}'(\lambda t_\eta)\eta\otimes\eta + \frac{1}{\lambda}a\big(-\bar{\Gamma}'(\lambda t_\eta) + 2\bar{\bar{\Gamma}}(\lambda t_\eta)\big)\mathrm{sym}\big(\nabla a\otimes\eta\big) \\ & - \frac{1}{\lambda}a\Gamma(\lambda t_\eta)\nabla^2\langle v, E\rangle - \frac{1}{\lambda^2}\bar{\Gamma}(\lambda t_\eta)\nabla a\otimes\nabla a - \frac{1}{\lambda^2}a\bar{\Gamma}(\lambda t_\eta)\nabla^2 a \\ & - \Big(a\Gamma'(\lambda t_\eta)\mathrm{sym}\big(\eta\otimes\nabla\langle v, E\rangle\big) + \frac{1}{\lambda}\Gamma(\lambda t_\eta)\mathrm{sym}\big(\nabla a\otimes\nabla\langle v, E\rangle\big)\Big).\end{aligned}$$

Summing the above two identities and noting that:

$$\frac{1}{2}(\Gamma')^2 + \bar{\bar{\Gamma}}' = 1 \quad \text{ and } \quad \Gamma'\Gamma - \bar{\Gamma}' + 2\bar{\bar{\Gamma}} = 0, \tag{2.4}$$

we arrive at the claimed identity (2.3). The proof is done. ■

We next observe that taking several perturbations in v of the form $\frac{1}{\lambda}a\Gamma(\lambda t_\eta)E$, and matching them with the perturbations of w as in (2.2), accumulates the error in (2.3) in a linear fashion as long as the directions E are mutually orthogonal. The same calculations as above, lead to:

Corollary 2.2. *Let $v\in\mathcal{C}^2(\omega,\mathbb{R}^k)$ and $w\in\mathcal{C}^1(\omega,\mathbb{R}^d)$ be two vector fields on an open domain $\omega\subset\mathbb{R}^d$. Let $\{\eta_i\in\mathbb{R}^d\}_{i=1}^k$ be given unit vectors, and let $\{E_i\in\mathbb{R}^k\}_{i=1}^k$ be some orthonormal basis of $\mathbb{R}^k$. Given $\{\lambda_i>0\}_{i=1}^k$ and $\{a_i\in\mathcal{C}^1(\omega,\mathbb{R})\}_{i=1}^k$, we set:*

$$\begin{aligned}\tilde{v} &= v + \sum_{i=1}^k \frac{1}{\lambda_i}a_i\Gamma(\lambda_i t_{\eta_i})E_i \\ \tilde{w} &= w - \sum_{i=1}^k \frac{1}{\lambda_i}a_i\Gamma(\lambda_i t_{\eta_i})\nabla\langle v, E_i\rangle - \sum_{i=1}^k\frac{1}{\lambda_i^2}a_i\bar{\Gamma}(\lambda_i t_{\eta_i})\nabla a_i + \sum_{i=1}^k\frac{1}{\lambda_i}a_i^2\bar{\bar{\Gamma}}(\lambda_i t_{\eta_i})\eta_i,\end{aligned}$$

where the functions Γ, $\bar{\Gamma}$, $\bar{\bar{\Gamma}}$ and t_η are defined as in Lemma 2.1. Then we have:

$$\begin{aligned}&\Big(\frac{1}{2}(\nabla\tilde{v})^T\nabla\tilde{v} + \mathrm{sym}\nabla\tilde{w}\Big) - \Big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\Big) - \sum_{i=1}^k a_i^2\eta_i\otimes\eta_i \\ &\quad = -\sum_{i=1}^k\frac{1}{\lambda_i}a_i\Gamma(\lambda_i t_{\eta_i})\nabla^2\langle v, E_i\rangle + \sum_{i=1}^k\frac{1}{\lambda_i^2}\Big(\frac{1}{2}\Gamma(\lambda_i t_{\eta_i})^2 - \bar{\Gamma}(\lambda_i t_{\eta_i})\Big)\nabla a_i\otimes\nabla a_i \\ &\qquad - \sum_{i=1}^k\frac{1}{\lambda_i^2}a_i\bar{\Gamma}(\lambda_i t_{\eta_i})\nabla^2 a_i.\end{aligned}$$

The second basic "step" construction, exhibited below and parallel to that in Lemma 2.1, utilizes Nash's spirals. Observe that another choice of periodic functions satisfying (2.4), is:

$$\Gamma_2(t) = 2\sin t, \quad \bar{\Gamma}_2(t) = -\frac{1}{2}\cos(2t), \quad \bar{\bar{\Gamma}}_2(t) = -\frac{1}{2}\sin(2t).$$

The above triple is conjugate to the triple in (2.1), in the sense that superposing the two perturbations they respectively induce, in some two orthonormal directions E_1 and E_2, in: $\tilde{v}(x) = v(x) + \frac{1}{\lambda}a(x)\Gamma(\lambda t_x)E_1 + \frac{1}{\lambda}a(x)\Gamma_2(\lambda t_x)E_2$, together with the matching adjustments in $\tilde{w}$, results in the cancellation of the error term $\frac{1}{\lambda^2}a\bar{\Gamma}(\lambda t_\eta)\nabla^2 a$ in the right hand side of (2.3). This is precisly the reason why the Newton iteration scheme in the proof of Theorem 1.4 via Källen's approach (more precisely: the validity of the inductive estimate $(4.7)_3$ in section 4), can be carried out. On the other hand, we emphasize that this contruction requires a pair of codimensions to cancel each single rank-one defect of the form $a\eta\otimes\eta$.

Lemma 2.3. *Let $\omega\subset\mathbb{R}^d$ be an open domain and let $k\geq 2$. Given $v\in\mathcal{C}^2(\omega,\mathbb{R}^k)$, $w\in\mathcal{C}^1(\omega,\mathbb{R}^d)$, $a\in\mathcal{C}^2(\omega,\mathbb{R})$, $\eta\in\mathbb{S}^{d-1}$, $\lambda>0$ and two orthogonal unit vectors $E_1,E_2\in\mathbb{R}^k$, denote:*

$$G(t)=\sin t,\qquad \bar G(t)=\cos t,$$

and denoting further $t_\eta=\langle x,\eta\rangle$, define:

$$\begin{aligned}\tilde v(x) &= v(x)+\frac{1}{\lambda}a(x)\Big(G(\lambda t_\eta)E_1+\bar G(\lambda t_\eta)E_2\Big)\\ \tilde w(x) &= w(x)-\frac{1}{\lambda}a(x)\Big(G(\lambda t_\eta)\nabla\langle v(x),E_1\rangle+\bar G(\lambda t_\eta)\nabla\langle v(x),E_2\rangle\Big).\end{aligned}\tag{2.5}$$

Then, there holds:

$$\begin{aligned}&\big(\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v+\mathrm{sym}\nabla\tilde w\big)-\big(\frac{1}{2}(\nabla v)^T\nabla v+\mathrm{sym}\nabla w\big)-\frac{1}{2}a^2\eta\otimes\eta\\ &=-\frac{a}{\lambda}\Big(G(\lambda t_\eta)\nabla^2\langle v,E_1\rangle+\bar G(\lambda t_\eta)\nabla^2\langle v,E_2\rangle\Big)+\frac{1}{2\lambda^2}\nabla a\otimes\nabla a.\end{aligned}\tag{2.6}$$

Proof. By a direct calculation, it follows that:

$$\nabla\tilde v=\nabla v+\frac{1}{\lambda}\Big(G(\lambda t_\eta)E_1+\bar G(\lambda t_\eta)E_2\Big)\otimes\nabla a+a\Big(G'(\lambda t_\eta)E_1+\bar G'(\lambda t_\eta)E_2\Big)\otimes\eta,$$

which implies:

$$\begin{aligned}&\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v-\frac{1}{2}(\nabla v)^T\nabla v-\frac{1}{2}a^2\eta\otimes\eta\\ &=a\Big(G'(\lambda t_\eta)\mathrm{sym}(\nabla\langle v,E_1\rangle\otimes\eta)+\bar G'(\lambda t_\eta)\mathrm{sym}(\nabla\langle v,E_2\rangle\otimes\eta)\Big)\\ &\quad+\frac{1}{\lambda}\Big(G(\lambda t_\eta)\mathrm{sym}(\nabla\langle v,E_1\rangle\otimes\nabla a)+\bar G(\lambda t_\eta)\mathrm{sym}(\nabla\langle v,E_2\rangle\otimes\nabla a)\Big)+\frac{1}{2\lambda^2}\nabla a\otimes\nabla a.\end{aligned}$$

Similarly:

$$\begin{aligned}\mathrm{sym}\nabla\tilde w-\mathrm{sym}\nabla w= &-a\Big(G'(\lambda t_\eta)\mathrm{sym}(\nabla\langle v,E_1\rangle\otimes\eta)+\bar G'(\lambda t_\eta)\mathrm{sym}(\nabla\langle v,E_2\rangle\otimes\eta)\Big)\\ &-\frac{1}{\lambda}\Big(G(\lambda t_\eta)\mathrm{sym}(\nabla\langle v,E_1\rangle\otimes\nabla a)+\bar G(\lambda t_\eta)\mathrm{sym}(\nabla\langle v,E_2\rangle\otimes\nabla a)\Big)\\ &-\frac{1}{\lambda}a\Big(G(\lambda t_\eta)\nabla^2\langle v,E_1\rangle+\bar\Gamma(\lambda t_\eta)\nabla^2\langle v,E_2\rangle\Big).\end{aligned}$$

Summing the above two identities we arrive at (2.6). The proof is done. ∎

Similarly as in Corollary 2.2, we note that:

Corollary 2.4. *Let $\omega\subset\mathbb{R}^d$ be an open domain and let $k\geq 2d_*$. Given $v\in\mathcal{C}^2(\omega,\mathbb{R}^k)$, $w\in\mathcal{C}^1(\omega,\mathbb{R}^d)$, $\{a_i\in\mathcal{C}^2(\omega,\mathbb{R})\}_{i=1}^{d_*}$, the unit vectors $\{\eta_i\in\mathbb{R}^d\}_{i=1}^{d_*}$ and the frequency $\lambda>0$, set:*

$$\begin{aligned}\tilde v(x) &= v(x)+\frac{1}{\lambda}\sum_{i=1}^{d_*}a_i(x)\Big(G(\lambda t_{\eta_i})e_i+\bar G(\lambda t_{\eta_i})e_{d_*+i}\Big)\\ \tilde w(x) &= w(x)-\frac{1}{\lambda}\sum_{i=1}^{d_*}a_i(x)\Big(G(\lambda t_{\eta_i})\nabla v^i(x)+\bar G(\lambda t_{\eta_i})\nabla v^{d_*+i}(x)\Big),\end{aligned}$$

where the functions $G, \bar G$ and t_η are defined in Lemma 2.3. Then, we have:

$$\begin{aligned}&\Big(\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v + \mathrm{sym}\nabla\tilde w\Big) - \Big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\Big) - \frac{1}{2}\sum_{i=1}^{d_*} a_i^2\eta_i\otimes\eta_i\\ &= -\frac{1}{\lambda}\sum_{i=1}^{d_*} a_i\Big(G(\lambda t_{\eta_i})\nabla^2 v^i + \bar G(\lambda t_{\eta_i})\nabla^2 v^{d_*+i}\Big) + \frac{1}{2\lambda^2}\sum_{i=1}^{d_*}\nabla a_i\otimes\nabla a_i.\end{aligned}$$

We now recall two auxiliary results from [3]. The first one gathers the convolution and commutator estimates [3, Lemma 2.1]:

Lemma 2.5. *Let $\phi\in\mathcal{C}_c^\infty(\mathbb{R}^d,\mathbb{R})$ be a standard mollifier that is nonnegative, radially symmetric, supported on the unit ball $B(0,1)\subset\mathbb{R}^d$ and such that $\int_{\mathbb{R}^d}\phi\ \mathrm{d}x = 1$. Denote:*

$$\phi_l(x) = \frac{1}{l^d}\phi(\frac{x}{l}) \quad \text{for all } \ l\in(0,1],\ x\in\mathbb{R}^d.$$

Then, for every $f,g\in\mathcal{C}^0(\mathbb{R}^d,\mathbb{R})$ and every $m,n\geq 0$ and $\beta\in(0,1]$ there holds:

$$\|\nabla^{(m)}(f*\phi_l)\|_0 \leq \frac{C}{l^m}\|f\|_0, \tag{2.7$_1$}$$

$$\|f - f*\phi_l\|_0 \leq C\min\big\{l^2\|\nabla^2 f\|_0, l\|\nabla f\|_0, l^\beta\|f\|_{0,\beta}\big\}, \tag{2.7$_2$}$$

$$\|\nabla^{(m)}\big((fg)*\phi_l - (f*\phi_l)(g*\phi_l)\big)\|_0 \leq Cl^{2-m}\|\nabla f\|_0\|\nabla g\|_0, \tag{2.7$_3$}$$

with a constant $C>0$ depending only on the differentiability exponent m.

The next result states the decomposition of symmetric positive definite matrices which are close to Id_d, into "primitive matrices", as proved in [3, Lemma 5.2]:

Lemma 2.6. *Given the dimension $d\geq 1$, let d_* be the dimension of the space $\mathbb{R}^{d\times d}_{\mathrm{sym}}$, namely:*

$$d_* = \frac{d(d+1)}{2}.$$

There exist: a constant $r_0>0$, the linear maps $\{\bar a_i:\mathbb{R}^{d\times d}_{\mathrm{sym}}\to\mathbb{R}\}_{i=1}^{d_*}$, and the unit vectors $\{\eta_i\in\mathbb{R}^d\}_{i=1}^{d_*}$, such that for all $A\in B(\mathrm{Id}_d, r_0)\subset\mathbb{R}^{d\times d}_{\mathrm{sym}}$, there holds:

$$A = \sum_{i=1}^{d_*}\bar a_i(A)\eta_i\otimes\eta_i \quad \text{and} \quad \bar a_i(A)\geq r_0 \ \text{ for all } \ i=1\dots d_*.$$

3. The "stage" for the $\mathcal{C}^{1,\alpha}$ approximations: a proof of Theorem 1.2

The following construction is the main technical contribution of this paper:

Proof of Theorem 1.2

The proof consists of several steps in an inductive construction below.

1. (Preparing the data) Recall that v, w, A are restrictions to $\bar\omega$ of some v, w, A defined on and, without loss of generality, compactly supported in $\mathbb{R}^d$. We set the mollification scale:

$$l = \frac{\|\mathcal{D}\|_0^{1/2}}{M}\in(0,1], \tag{3.1}$$

and taking $\phi_l(x) = \frac{1}{l^d}\phi(x/l)$ as in Lemma 2.5, we define:

$$v_0 = v * \phi_l, \quad w_0 = w * \phi_l, \quad A_0 = A * \phi_l, \quad \mathcal{D}_0 = A_0 - \big(\frac{1}{2}(\nabla v_0)^T\nabla v_0 + \mathrm{sym}\nabla w_0\big).$$

From the estimates in Lemma 2.5, one deduces the initial bounds:

$$\|v_0 - v\|_1 + \|w_0 - w\|_1 \leq C\|\mathcal{D}\|_0^{1/2}, \qquad (3.2)_1$$

$$\|A_0 - A\|_0 \leq Cl^\beta \|A\|_{0,\beta}, \qquad (3.2)_2$$

$$\|\nabla^{(m+1)}v_0\|_0 + \|\nabla^{(m+1)}w_0\|_0 \leq \frac{C}{l^m}\|\mathcal{D}\|_0^{1/2} \quad \text{for all } m \geq 1, \qquad (3.2)_3$$

$$\|\nabla^{(m)}\mathcal{D}_0\|_0 \leq \frac{C}{l^m}\|\mathcal{D}\|_0 \quad \text{for all } m \geq 0. \qquad (3.2)_4$$

Indeed, $(3.2)_2$ follows directly from $(2.7)_2$, and $(3.2)_1$ similarly follows by applying $(2.7)_2$ to v, ∇v, w, ∇w and noting that, in view of (3.1) we have:

$$l\|v\|_2 + l\|w\|_2 \leq 2\|\mathcal{D}\|_0^{1/2}. \qquad (3.3)$$

Further, $(3.2)_3$ follows by applying $(2.7)_1$ to $\nabla^2 v$ and $\nabla^2 w$ with the differentiability exponent $m-1$ and again taking into account (3.3). To check $(3.2)_4$, we write:

$$\mathcal{D}_0 = \mathcal{D} * \phi_l - \frac{1}{2}\big((\nabla v_0)^T\nabla v_0 - ((\nabla v)^T\nabla v) * \phi_l\big),$$

and apply $(2.7)_1$ to $\mathcal{D}$, and $(2.7)_3$ to $(\nabla v)^T$ and ∇v, where the final bound is due to (3.3).

2. (Induction definition: frequencies) We now inductively define the main constants, frequencies and corrections in the construction of $(\tilde{v}, \tilde{w})$ from (v, w). First, we write the least common multiple of the auxiliary dimension d_* and the codimension k, as follows:

$$N = lcm(d_*, k) = Sd_* = Jk, \qquad S, J \geq 1. \qquad (3.4)$$

Then, we set the initial perturbation frequencies as:

$$\lambda_0 = \frac{1}{l}, \qquad \lambda_1 = \lambda = \frac{\sigma^{1/S}}{l}.$$

For every $i = 2 \dots N$ we define $\lambda_i \geq 1$ according to the mutually exclusive cases in:

$$\lambda_i = \lambda_{i-1} \cdot \begin{cases} (\lambda l) & \text{if } k \mid (i-1), \\ (\lambda l)^{1/2} & \text{if } d_* \mid (i-1), \\ 1 & \text{otherwise.} \end{cases}$$

It follows that for all $j = 0 \dots J-1$ and $s = 0 \dots S-1$ there holds:

$$\lambda_i l = (\lambda l)^{1+j+s/2} \quad \text{for all } i \in (jk, (j+1)k] \cap (sd_*, (s+1)d_*]. \qquad (3.5)$$

3. (Induction definition: decomposition of deficits) First, let $\{\eta_\delta \in \mathbb{R}^d\}_{\delta=1}^{d_*}$ be the unit vectors as in Lemma 2.6. For all $s = 0 \dots S-1$ we define constants $\tilde{C}_s$ and perturbation amplitudes vector $a^s = [a^s_\delta]_{\delta=1}^{d_*} \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{d_*})$ by:

$$\tilde{C}_s = \frac{2}{r_0}\Big(\frac{1}{(\lambda l)^s}\|\mathcal{D}\|_0 + \|\mathcal{D}_s\|_0\Big),$$

$$a^s_\delta(x) = \Big(\tilde{C}_s\bar{a}_\delta\big(\mathrm{Id}_d + \frac{1}{\tilde{C}_s}\mathcal{D}_s(x)\big)\Big)^{1/2} \quad \text{for all } \delta = 1 \dots d_*, \ x \in \bar{\omega}.$$

Above, $r_0 > 0$ and the maps $\bar{a}_\delta$ are as in Lemma 2.6, so our definition is correctly posed because $\mathrm{Id}_d + \frac{1}{\tilde{C}_s}\mathcal{D}_s(x) \in B(\mathrm{Id}_d, r_0) \subset \mathbb{R}^{d\times d}_{\mathrm{sym}}$ for all $x \in \bar{\omega}$. As $\tilde{C}_s\mathrm{Id}_d + \mathcal{D}_s = \tilde{C}_s\big(\mathrm{Id}_d + \frac{1}{\tilde{C}_s}\mathcal{D}_s\big)$, we get:

$$\tilde{C}_s\mathrm{Id}_d + \mathcal{D}_s = \sum_{\delta=1}^{d_*}(a^s_\delta)^2\eta_\delta\otimes\eta_\delta \quad \text{ and } \quad (a^s_\delta)^2 \geq r_0\tilde{C}_s \ \text{ in } \ \bar{\omega}, \ \text{ for all } \ \delta = 1\dots d_*. \tag{3.6}$$

Since $\{\eta_\delta\otimes\eta_\delta\}_{\delta=1}^{d_*}$ is a basis of the linear space $\mathbb{R}^{d\times d}_{\mathrm{sym}}$, we obtain:

$$\|a^s\|_0 \leq C\|\tilde{C}_s\mathrm{Id}_d + \mathcal{D}_s\|_0^{1/2} \leq C\tilde{C}_s^{1/2}. \tag{3.7}$$

We also right away observe that, by the Faá di Bruno formula, there holds, for $m \geq 1$:

$$\|\nabla^{(m)}a^s_\delta\|_0 \leq C\Big\|\sum_{p_1+2p_2+\dots mp_m=m}|a^s_\delta|^{2(1/2-p_1-\dots-p_m)}\prod_{t=1}^m\big|\nabla^{(t)}|a^s_\delta|^2\big|^{p_t}\Big\|_0 \quad \text{ for all } \ \delta = 1\dots d_*.$$

Using the lower bound in (3.6) and the linearity of $\bar{a}_\delta$ in Lemma 2.6, we further get:

$$\begin{aligned}\|\nabla^{(m)}a^s\|_0 &\leq C\sum_{p_1+2p_2+\dots mp_m=m}\frac{1}{\tilde{C}_s^{(p_1+\dots+p_m)-1/2}}\prod_{t=1}^m\|\nabla^{(t)}\mathcal{D}_s\|_0^{p_t}\\ &\leq C\tilde{C}_s^{1/2}\sum_{p_1+2p_2+\dots mp_m=m}\prod_{t=1}^m\Big(\frac{\|\nabla^{(t)}\mathcal{D}_s\|_0}{\tilde{C}_s}\Big)^{p_t}.\end{aligned} \tag{3.8}$$

In particular, for $s = 0$ and any $\delta = 1\dots d_*$, the bounds (3.7), (3.8) and $(3.2)_4$ yield:

$$\tilde{C}_0 \leq C\|\mathcal{D}\|_0 \quad \text{ and } \quad \|\nabla^{(m)}a^0\|_0 \leq \frac{C}{l^m}\|\mathcal{D}\|_0^{1/2} \quad \text{ for all } \ m \geq 0. \tag{3.9}$$

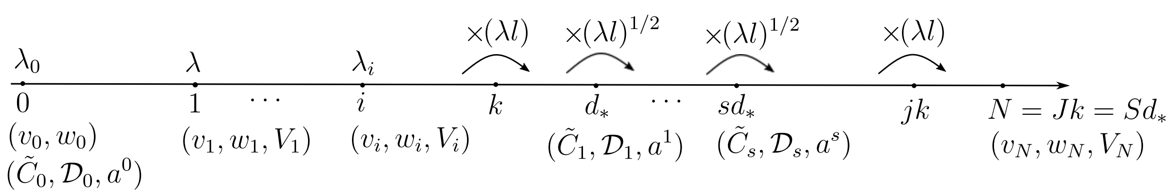

FIGURE 1. Progression of frequencies λ_i and other intermediary quantities defined at integers $i = 1\dots N$, where $N = lcm(k, d_*)$.

4. (Induction definition: perturbations) For each $i = 1\dots N$ we may uniquely write:

$$i = jk + \gamma = sd_* + \delta \quad \text{ with } \quad \begin{array}{l} j = 0\dots J-1, \quad \gamma = 1\dots k,\\ s = 0\dots S-1, \quad \delta = 1\dots d_*.\end{array} \tag{3.10}$$

Define $v_i \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^k)$ and $w_i \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^d)$ according to the "step" construction in Lemma 2.1, involving the periodic profile functions $\Gamma, \bar{\Gamma}, \bar{\bar{\Gamma}}$ and the notation $t_\eta = \langle x, \eta\rangle$:

$$\begin{aligned} v_i(x) &= v_{i-1}(x) + \frac{1}{\lambda_i}a^s_\delta(x)\Gamma(\lambda_i t_{\eta_\delta})e_\gamma,\\ w_i(x) &= w_{i-1}(x) - \frac{1}{\lambda_i}a^s_\delta(x)\Gamma(\lambda_i t_{\eta_\delta})\nabla v^\gamma_{i-1} - \frac{1}{\lambda_i^2}a^s_\delta(x)\bar{\Gamma}(\lambda_i t_{\eta_\delta})\nabla a^s_\delta + \frac{1}{\lambda_i}a^s_\delta(x)^2\bar{\bar{\Gamma}}(\lambda_i t_{\eta_\delta})\eta_\delta.\end{aligned}$$

We observe that by construction of v_i, the second term in w_i can be rewritten as follows:

$$\frac{1}{\lambda_i}a^s_\delta(x)\Gamma(\lambda_i t_{\eta_\delta})\nabla v^\gamma_{i-1} = \frac{1}{\lambda_i}a^s_\delta(x)\Gamma(\lambda_i t_{\eta_\delta})\nabla v^\gamma_{jk}. \tag{3.11}$$

We eventually define:

$$\tilde{v} = v_N, \qquad \tilde{w} = w_N - \sum_{s=0}^{S-1} \tilde{C}_s x. \tag{3.12}$$

5. (Induction definition: deficits) For each $i = 1 \dots N$, we define the partial deficit:

$$V_i = \big(\frac{1}{2}(\nabla v_i)^T \nabla v_i + \mathrm{sym}\nabla w_i\big) - \big(\frac{1}{2}(\nabla v_{i-1})^T \nabla v_{i-1} + \mathrm{sym}\nabla w_{i-1}\big),$$

and for each $s = 1 \dots S$ we define the combined deficit $\mathcal{D}_s \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ in:

$$\begin{aligned}\mathcal{D}_s = & -\big(\frac{1}{2}(\nabla v_{sd_*})^T \nabla v_{sd_*} + \mathrm{sym}\nabla w_{sd_*}\big) + \big(\frac{1}{2}(\nabla v_{(s-1)d_*})^T \nabla v_{(s-1)d_*} + \mathrm{sym}\nabla w_{(s-1)d_*}\big) \\ & + \sum_{\delta=1}^{d_*} (a_\delta^{s-1})^2 \eta_\delta \otimes \eta_\delta = - \sum_{i=(s-1)d_*+1}^{sd_*} \Big(V_i - (a_\delta^{s-1})^2 \eta_\delta \otimes \eta_\delta\Big),\end{aligned}$$

where in components of the last sum we used the convention (3.10), setting $\delta = \delta(i) = 1 \dots d_*$. By Lemma 2.1 and noting (3.11), for each $i = (s-1)d_* \dots sd_*$ as above, we get:

$$\begin{aligned}V_i - (a_\delta^{s-1})^2 \eta_\delta \otimes \eta_\delta = & -\frac{1}{\lambda_i} a_\delta^{s-1} \Gamma(\lambda_i t_{\eta_\delta}) \nabla^2 v^\gamma_{jk} - \frac{1}{\lambda_i^2} a_\delta^{s-1} \bar{\Gamma}(\lambda_i t_{\eta_\delta}) \nabla^2 a_\delta^{s-1} \\ & + \frac{1}{\lambda_i^2} \big(\frac{1}{2}\Gamma(\lambda_i t_{\eta_\delta})^2 - \bar{\Gamma}(\lambda_i t_{\eta_\delta})\big) \nabla a_\delta^{s-1} \otimes \nabla a_\delta^{s-1},\end{aligned} \tag{3.13}$$

where $j = 0 \dots J-1$ is again set according to (3.10).

6. (Inductive estimates) In steps 7-10 below we will prove the following estimates:

$$\left.\begin{aligned}&\|v_i - v_{i-1}\|_1 \leq C\|\mathcal{D}\|_0^{1/2} \\ &\|w_i - w_{i-1}\|_1 \leq C\|\mathcal{D}\|_0^{1/2}(1 + \|\nabla v\|_0)\end{aligned}\right\} \quad \text{for all } \ i = 1 \dots N, \tag{3.14$_1$}$$

$$\left.\begin{aligned}&\|\nabla^{(m+1)} v_{kj}\|_0 \leq C\frac{\lambda_{kj}^{m-1}}{l}(\lambda l)^j \|\mathcal{D}\|_0^{1/2} \\ &\|\nabla^{(m+1)} w_{kj}\|_0 \leq C\frac{\lambda_{kj}^{m-1}}{l}(\lambda l)^j \|\mathcal{D}\|_0^{1/2}(1 + \|\nabla v\|_0)\end{aligned}\right\} \quad \text{for all } \ j = 0 \dots J, \ \ m \geq 1, \tag{3.14$_2$}$$

$$\tilde{C}_s \leq \frac{C}{(\lambda l)^s}\|\mathcal{D}\|_0, \qquad \|\nabla^{(m)} \mathcal{D}_s\|_0 \leq C\frac{\lambda_{sd_*}^m}{(\lambda l)^s}\|\mathcal{D}\|_0 \quad \text{for all } \ s = 0 \dots S, \ \ m \geq 0, \tag{3.14$_3$}$$

$$\|\nabla^{(m)} a^s\|_0 \leq C\frac{\lambda_{sd_*}^m}{(\lambda l)^{s/2}}\|\mathcal{D}\|_0^{1/2} \quad \text{for all } \ s = 0 \dots S-1, \ \ m \geq 0. \tag{3.14$_4$}$$

We observe that all the bounds are already valid at their lowest counter values: by $(3.2)_3$ there holds $(3.14)_2$ for $j = 0$, the first bound in $(3.14)_3$ and the bound in $(3.14)_4$ at $s = 0$ have been established in (3.9), while the second bound in $(3.14)_3$ at $s = 0$ is exactly $(3.2)_4$. To show

$(3.14)_1$ at $i=1$, we use (3.9) and $(3.2)_3$ in:

$$\|v_1-v_0\|_1 \leq C\Big(\|a^0\|_0 + \frac{\|\nabla a^0\|_0}{\lambda}\Big) \leq C\|\mathcal{D}\|_0^{1/2}\big(1+\frac{1}{\lambda l}\big) \leq C\|\mathcal{D}\|_0^{1/2},$$

$$\begin{aligned}\|w_1-w_0\|_1 \leq C\Big(&\|a^0\|_0\|\nabla v_0\|_0 + \|a^0\|_0^2 + \frac{\|\nabla a^0\|_0^2+\|a^0\|_0\|\nabla^2 a^0\|_0}{\lambda^2} \\ &+ \frac{\|a^0\|_0\|\nabla a^0\|_0+\|\nabla a^0\|_0\|\nabla v_0\|_0+\|a^0\|_0\|\nabla^2 v_0\|_0}{\lambda}\Big) \\ \leq C&\|\mathcal{D}\|_0^{1/2}(1+\|\nabla v\|_0),\end{aligned}$$

because $\lambda l \geq 1$ and $\|\nabla v_0\|_0 \leq \|\nabla v\|_0 + C\|\mathcal{D}\|^{1/2} \leq C(1+\|\nabla v\|_0)$ from $(3.2)_1$.

7. (Proof of estimate $(3.14)_1$) For $i\in(1,N]$, we write:

$$i\in(jk,(j+1)k]\cap(sd_*,(s+1)d_*]$$

with j,s as in (3.10). By $(3.14)_4$, we get:

$$\|v_i-v_{i-1}\|_1 \leq C\Big(\|a^s\|_0 + \frac{\|\nabla a^s\|_0}{\lambda_i}\Big) \leq \frac{C}{(\lambda l)^{s/2}}\|\mathcal{D}\|_0^{1/2}\big(1+\frac{\lambda_{sd_*}}{\lambda_i}\big) \leq C\|\mathcal{D}\|_0^{1/2},$$

where we used that $\lambda l\geq 1$ and $\lambda_{sd_*}l \leq \lambda_i l$, due to $i>sd_*$. The bound for the w-increment follows by $(3.14)_2$ at $m=1$, $(3.14)_4$, $(3.14)_1$ and $(3.2)_1$:

$$\begin{aligned}\|w_i-w_{i-1}\|_1 \leq C\Big(&\|a^s\|_0\|\nabla v_{jk}\|_0 + \|a^s\|_0^2 + \frac{\|\nabla a^s\|_0^2+\|a^s\|_0\|\nabla^2 a^s\|_0}{\lambda_i^2} \\ &+ \frac{\|\nabla a^s\|_0\|\nabla v_{jk}\|_0+\|a^s\|_0\|\nabla^2 v_{jk}\|_0+\|a^s\|_0\|\nabla a^s\|_0}{\lambda_i}\Big) \\ \leq \frac{C}{(\lambda l)^{s/2}}&\|\mathcal{D}\|_0^{1/2}\big(1+\frac{\lambda_{sd_*}}{\lambda_i}+\frac{\lambda_{sd_*}^2}{\lambda_i^2}+\frac{(\lambda l)^j}{\lambda_i l}\big)\big(1+\|\nabla v\|_0\big) \leq C\|\mathcal{D}\|_0^{1/2}\big(1+\|\nabla v\|_0\big),\end{aligned}$$

where again we used $\lambda_{sd_*}l\leq\lambda_i l$ due to $i>sd_*$, and $(\lambda l)^j\leq\lambda_i l$ due to $i>jk$.

8. (Proof of estimate $(3.14)_2$) Let $i=1\dots N$ and $m\geq 1$. Write:

$$i\in((j-1)k,jk]\cap(sd_*,(s+1)d_*]$$

with $j=1\dots J$, $s=0\dots S-1$. Then:

$$\begin{aligned}\|\nabla^{(m+1)}(v_i-v_{i-1})\|_0 &\leq C\sum_{p+q=m+1}\lambda_i^{p-1}\|\nabla^{(q)}a^s\|_0 \leq C\lambda_i^{m-1}\sum_{q=0}^{m+1}\frac{\lambda_{sd_*}^q\lambda_i\|\mathcal{D}\|_0^{1/2}}{\lambda_i^q(\lambda l)^{s/2}} \\ &\leq C\lambda_i^{m-1}\frac{\lambda_i}{(\lambda l)^{s/2}}\|\mathcal{D}\|_0^{1/2} = C\lambda_i^{m-1}\frac{(\lambda l)^j}{l}\|\mathcal{D}\|_0^{1/2}\end{aligned}$$

because $\lambda_{sd_*}\leq\lambda_i$ due to $i>sd_*$, and in fact from (3.5):

$$\lambda_i = \frac{(\lambda l)^{j+s/2}}{l}.$$

The above justifies:

$$\|\nabla^{(m+1)}(v_{kj}-v_{(j-1)k})\|_0 \leq \sum_{i=(j-1)k+1}^{jk}\|\nabla^{(m+1)}(v_i-v_{i-1})\|_0 \leq C\lambda_{jk}^{m-1}\frac{(\lambda l)^j}{l}\|\mathcal{D}\|_0^{1/2},$$

since $i\mapsto \lambda_i$ is a nondecreasing function. Further, by $(3.2)_3$ we get:

$$\|\nabla^{(m+1)}v_0\|_0\leq \frac{C}{l^m}\|\mathcal{D}\|_0^{1/2} = C\frac{\lambda_0^{m-1}}{l}\|\mathcal{D}\|_0^{1/2}\leq C\lambda_{jk}^{m-1}\frac{(\lambda l)^j}{l}\|\mathcal{D}\|_0^{1/2}.$$

The above two bounds prove the first statement in $(3.14)_2$.

Towards proving the second bound, we note that the increment in w is estimated:

$$\begin{split}\|\nabla^{(m+1)}(w_i-w_{i-1})\|_0\leq C\sum_{p+q+t=m+1}\lambda_i^{p-1}\|\nabla^{(q)}a^s\|_0\|\nabla^{(t+1)}v_{(j-1)k}\|_0\\ + C\sum_{p+q+t=m+1}\Big(\lambda_i^{p-2}\|\nabla^{(q)}a^s\|_0\|\nabla^{(t+1)}a^s\|_0+\lambda_i^{p-1}\|\nabla^{(q)}a^s\|_0\|\nabla^{(t)}a^s\|_0\Big).\end{split}\tag{3.15}$$

We split the first sum in the right hand side above into cases $t=0$ and $t\geq 1$, so that by $(3.14)_4$ and $(3.14)_2$, together with $(3.14)_1$ and $(3.2)_1$:

$$\begin{split}&\sum_{p+q+t=m+1}\lambda_i^{p-1}\|\nabla^{(q)}a^s\|_0\|\nabla^{(t+1)}v_{(j-1)k}\|_0\\ &\quad\leq \|\nabla v_{(j-1)k}\|_0\sum_{p+q=m+1}\lambda_i^{p-1}\|\nabla^{(q)}a^s\|_0+\sum_{p+q+t=m}\lambda_i^{p-1}\|\nabla^{(q)}a^s\|_0\|\nabla^{(t+2)}v_{(j-1)k}\|_0\\ &\quad\leq C\lambda_i^m\big(1+\|\nabla v\|_0\big)\sum_{q=0}^{m+1}\frac{\lambda_{sd_*}^q\|\mathcal{D}\|_0^{1/2}}{\lambda_i^q(\lambda l)^{s/2}}+C\lambda_i^m\sum_{p+q+t=m}\frac{\lambda_{sd_*}^q}{\lambda_i^q}\frac{\lambda_{(j-1)k}^t}{\lambda_i^t}\frac{(\lambda l)^{j-1}\|\mathcal{D}\|_0}{(\lambda_i l)(\lambda l)^{s/2}}\\ &\quad\leq C\lambda_i^m\frac{\|\mathcal{D}\|_0^{1/2}}{(\lambda l)^{s/2}}\big(1+\|\nabla v\|_0\big)+C\lambda_i^{m-1}\|\mathcal{D}\|_0\frac{(\lambda l)^{j-1-s/2}}{l}\end{split}$$

where in the last bound we used the fact that $\lambda_{sd_*}\leq\lambda_i$ due to $i>sd_*$, and $\lambda_{(j-1)k}\leq\lambda_i$ due to $i>(j-1)k$. The second term in (3.15) is similarly estimated:

$$\begin{split}&\sum_{p+q+t=m+1}\Big(\lambda_i^{p-2}\|\nabla^{(q)}a^s\|_0\|\nabla^{(t+1)}a^s\|_0+\lambda_i^{p-1}\|\nabla^{(q)}a^s\|_0\|\nabla^{(t)}a^s\|_0\Big)\\ &\qquad\leq C\lambda_i^m\sum_{p+q+t=m+1}\Big(\frac{\lambda_{sd_*}^{q+t+1}}{\lambda_i^{q+t+1}}\frac{\|\mathcal{D}\|_0}{(\lambda l)^s}+\frac{\lambda_{sd_*}^{q+t}}{\lambda_i^{q+t}}\frac{\|\mathcal{D}\|_0}{(\lambda l)^s}\Big)\leq C\lambda_i^m\frac{\|\mathcal{D}\|_0}{(\lambda l)^s}.\end{split}$$

Summing the last two displayed formulas, gives in view of (3.15):

$$\begin{split}\|\nabla^{(m+1)}(w_i-w_{i-1})\|_0&\leq C\lambda_i^{m-1}\|\mathcal{D}\|_0^{1/2}\Big(\frac{\lambda_i}{(\lambda l)^{s/2}}+\frac{(\lambda l)^{j-1-s/2}}{l}\Big)\big(1+\|\nabla v\|_0\big)\\ &\leq C\lambda_i^{m-1}\frac{(\lambda l)^j}{l}\|\mathcal{D}\|_0^{1/2}\big(1+\|\nabla v\|_0\big).\end{split}$$

The above implies the second statement in $(3.14)_2$, in view of $(3.2)_3$ resulting in:

$$\|\nabla^{(m+1)}w_0\|_0\leq\frac{C}{l^m}\|\mathcal{D}\|_0^{1/2}\leq C\lambda_{jk}^{m-1}\frac{(\lambda l)^j}{l}\|\mathcal{D}\|_0^{1/2},$$

and since $i\mapsto\lambda_i$ is a nondecreasing function, which yields:

$$\begin{split}\|\nabla^{(m+1)}(w_{kj}-w_{(k-1)j})\|_0&\leq\sum_{i=(k-1)j+1}^{kj}\|\nabla^{(m+1)}(w_i-w_{i-1})\|_0\\ &\leq C\lambda_{jk}^{m-1}\frac{(\lambda l)^j}{l}\|\mathcal{D}\|_0^{1/2}\big(1+\|\nabla v\|_0\big).\end{split}$$

9. (Proof of estimate (3.14)$_3$) Let $i = 1 \dots N$ and $m \geq 0$. Write:

$$i \in (jk, (j+1)k] \cap ((s-1)d_*, sd_*]$$

with $j = 0 \dots J-1$, $s = 1 \dots S$. Denoting $\delta = i - (s-1)d_*$, we use (3.14)$_2$, (3.14)$_4$ in (3.13):

$$\begin{aligned} \big\|\nabla^{(m)}\big(V_i - (a_\delta^{s-1})^2\eta_\delta\otimes\eta_\delta\big)\big\|_0 &\leq C\sum_{p+q+t=m}\lambda_i^{p-1}\|\nabla^{(q)}a^{s-1}\|_0\|\nabla^{(t+2)}v_{jk}\|_0 \\ &\quad + C\sum_{p+q+t=m}\lambda_i^{p-2}\Big(\|\nabla^{(q+1)}a^{s-1}\|_0\|\nabla^{(t+1)}a^{s-1}\|_0 + \|\nabla^{(q)}a^{s-1}\|_0\|\nabla^{(t+2)}a^{s-1}\|_0\Big) \\ &\leq C\lambda_i^m\|\mathcal{D}\|_0\Big(\sum_{p+q+t=m}\frac{\lambda_{(s-1)d_*}^q\lambda_{jk}^t}{\lambda_i^{q+t}}\frac{(\lambda l)^{j-(s-1)/2}}{\lambda_i l} + \sum_{p+q+t=m}\frac{\lambda_{(s-1)d_*}^{q+t+2}}{\lambda_i^{q+t+2}}\frac{1}{(\lambda l)^{s-1}}\Big). \end{aligned}$$

Since $\lambda_{(s-1)d_*} \leq \lambda_i$ by $i > (s-1)d_*$, and $\lambda_{jk} \leq \lambda_i$ by $i > jk$, we simplify the above estimate:

$$\begin{aligned} \big\|\nabla^{(m)}\big(V_i - (a_\delta^{s-1})^2\eta_\delta\otimes\eta_\delta\big)\big\|_0 &\leq C\lambda_i^m\|\mathcal{D}\|_0\Big(\frac{(\lambda l)^{j-(s-1)/2}}{\lambda_i l} + \frac{\lambda_{(s-1)d_*}^2}{\lambda_i^2}\frac{1}{(\lambda l)^{s-1}}\Big) \\ &\leq C\frac{\lambda_i^m}{(\lambda l)^s}\|\mathcal{D}\|_0, \end{aligned}$$

where the last bound follows since $\lambda_i \geq \lambda_{(s-1)d_*}(\lambda l)^{1/2}$ by $i > (s-1)d_*$, and since (3.5) yields:

$$\lambda_i = \lambda(\lambda l)^{j+(s-1)/2}.$$

Note that having the second power of the quotient $\lambda_{(s-1)d_*}/\lambda_i$ was essential to provide the missing multiplier $\frac{1}{\lambda l}$ in order for both considered error terms to have the right order $C\frac{\lambda_i^m}{(\lambda l)^s}\|\mathcal{D}\|_0$. It is precisely at this point where we are using the optimal step construction in Lemma 2.1, and where the previous sub-par step construction in [10, Lemma 2.2] would not be sufficient.

Consequently, summing the partial deficits in $\mathcal{D}_s$, we obtain the second bound in (3.14)$_3$:

$$\|\nabla^{(m)}\mathcal{D}_s\|_0 \leq C\sum_{i=(s-1)d_*+1}^{sd_*}\frac{\lambda_i^m}{(\lambda l)^s}\|\mathcal{D}\|_0 \leq C\frac{\lambda_{sd_*}^m}{(\lambda l)^s}\|\mathcal{D}\|_0,$$

as $\lambda_i \leq \lambda_{sd_*}$ for all $i \leq sd_*$. The first bound in (3.14)$_3$ is now also immediate:

$$\tilde{C}_s \leq C\Big(\frac{\|\mathcal{D}\|_0}{(\lambda l)^s} + \|\mathcal{D}_s\|_0\Big) \leq \frac{C}{(\lambda l)^s}\|\mathcal{D}\|_0.$$

10. (Proof of estimate (3.14)$_4$) Let $s = 1 \dots S-1$. From (3.7) and the first bound in (3.14)$_3$, we readily deduce (3.14)$_4$ at $m = 0$:

$$\|a^s\|_0 \leq \frac{C}{(\lambda l)^{s/2}}\|\mathcal{D}\|_0^{1/2}.$$

For $m \geq 1$, we use the preparatory bound (3.8) in which we take account of (3.14)$_3$ and (3.14)$_3$:

$$\|\nabla^{(m)}a^s\|_0 \leq \frac{C}{(\lambda l)^{s/2}}\|\mathcal{D}\|_0^{1/2}\sum_{p_1+2p_2+\ldots+mp_m=m}\prod_{t=1}^{m}\Big(\lambda_{sd_*}^{tp_t}\Big(\frac{\|\mathcal{D}\|_0}{(\lambda l)^s\tilde{C}_s}\Big)^{p_t}\Big) \leq C\frac{\lambda_{sd_*}^m}{(\lambda l)^{s/2}}\|\mathcal{D}\|_0^{1/2},$$

in virtue of having $\frac{\|\mathcal{D}\|_0}{(\lambda l)^s\tilde{C}_s} \leq C$. This completes the proof of all the inductive estimates.

11. (End of proof) We now show that $(3.14)_1$ - $(3.14)_4$ imply the bounds claimed in the Theorem. Recall (3.12), and use $(3.14)_1$, $(3.14)_3$ and $(3.2)_1$ to conclude $(1.2)_1$:

$$\|\tilde{v}-v\|_1 \leq \|v_0-v\|_1 + \sum_{i=1}^N \|v_i - v_{i-1}\|_1 \leq C\|\mathcal{D}\|_0^{1/2},$$
$$\|\tilde{w}-w\|_1 \leq \|w_0-w\|_1 + \sum_{i=1}^N \|w_i - w_{i-1}\|_1 + C\sum_{s=0}^{S-1}\tilde{C}_s \leq C\|\mathcal{D}\|_0^{1/2}(1+\|\nabla v\|_0).$$

By $(3.14)_2$ with $m=1$, there follows $(1.2)_2$:

$$\|\nabla^2\tilde{v}\|_0 = \|\nabla^2 v_N\|_0 \leq C\frac{(\lambda l)^J}{l}\|\mathcal{D}\|_0^{1/2} = CM(\lambda l)^J = CM\sigma^{J/S} = CM\sigma^{d_*/k},$$
$$\|\nabla^2\tilde{w}\|_0 = \|\nabla^2 w_N\|_0 \leq C\frac{(\lambda l)^J}{l}\|\mathcal{D}\|_0^{1/2}(1+\|\nabla v\|_0) = CM\sigma^{d_*/k}(1+\|\nabla v\|_0),$$

where we used the definition $\sigma = (\lambda l)^S$ and the fact that:

$$\frac{J}{S} = \frac{J}{N}\cdot\frac{N}{S} = \frac{d_*}{k}.$$

Finally, $(3.2)_2$, and $(3.14)_3$ applied with $m=0$ yield $(1.2)_3$:

$$\begin{aligned}\|\tilde{\mathcal{D}}\|_0 &= \|(A-A_0)-\mathcal{D}_S\|_0 \leq \|A-A_0\|_0 + \|\mathcal{D}_S\|_0 \\ &\leq C\Big(l^\beta\|A\|_{0,\beta} + \frac{\|\mathcal{D}\|_0}{(\lambda l)^S}\Big) \\ &= C\Big(\frac{\|A\|_{0,\beta}}{M^\beta}\|\mathcal{D}\|_0^{\beta/2} + \frac{\|\mathcal{D}\|_0}{\sigma}\Big),\end{aligned}$$

in view of the following direct decomposition:

$$\begin{aligned}\tilde{\mathcal{D}} &= (A-A_0) + \mathcal{D}_0 - \Big(\big(\frac{1}{2}(\nabla\tilde{v})^T\nabla\tilde{v} + \mathrm{sym}\nabla\tilde{w}\big) - \big(\frac{1}{2}(\nabla v_0)^T\nabla v_0 + \mathrm{sym}\nabla w_0\big)\Big) \\ &= (A-A_0) + \mathcal{D}_0 + \sum_{s=0}^{S-1}\tilde{C}_s\mathrm{Id}_d - \sum_{s=1}^{S}\sum_{i=(s-1)d_*+1}^{sd_*} V_i \\ &= (A-A_0) + \sum_{s=0}^{S-1}\tilde{C}_s\mathrm{Id}_d + \sum_{s=0}^{S}\mathcal{D}_s - \sum_{s=1}^{S}\sum_{\delta=1}^{d_*}(a_\delta^{s-1})^2\eta_\delta\otimes\eta_\delta \\ &= (A-A_0) + \mathcal{D}_S\end{aligned}$$

The proof is done. ∎

4. The "stage" using Källen's approach: a proof of Theorem 1.4

In this section, we prove flexibility of (VK) up to regularity $\mathcal{C}^{1,1}$, provided that $k \geq 2d_*$.

Proof of Theorem 1.4

1. (Preparing the data) We set the mollification scale l and the frequency λ:

$$l = \frac{\|\mathcal{D}\|_0^{1/2}}{M} \in (0,1], \qquad \lambda = \frac{\sigma^{1/N}}{l} > 1 \quad \text{where } \frac{1}{\delta} \leq N \in \mathbb{N}. \tag{4.1}$$

Taking $\phi_l(x)=\frac{1}{l^d}\phi(x/l)$ as in Lemma 2.5, we define:

$$v_0=v*\phi_l,\quad w_0=w*\phi_l,\quad A_0=A*\phi_l,\quad \mathcal{D}_0=A_0-\big(\frac{1}{2}(\nabla v_0)^T\nabla v_0+\mathrm{sym}\nabla w_0\big).$$

and observe the initial bounds $(3.2)_1$-$(3.2)_4$ exactly as in the proof of Theorem 1.2 in section 3.

2. (Induction definition: improving the deficit decomposition) Let $\{\eta_i\in\mathbb{S}^{d-1}\}_{i=1}^{d_*}$, $r_0>0$ and the linear maps $\bar a_i$ be as in Lemma 2.6. For $r=0\dots N$ we iteratively define the perturbation amplitude vectors $a^r=[a_i^r]_{i=1}^{d_*}\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^{d_*})$, by setting:

$$\begin{aligned}&a_i^0(x)=0\quad \text{ for all }\ i=1\dots d_*,\ \ x\in\bar\omega,\\ &a_i^r(x)=\Big(2\bar a_i\big(\tilde C\mathrm{Id}_d+\mathcal{D}_0(x)-\mathcal{E}_{r-1}(x)\big)\Big)^{1/2}\quad \text{ for all }\ i=1\dots d_*,\ \ r=1\dots N,\ \ x\in\bar\omega,\\ &\text{with }\ \tilde C=\frac{2}{r_0}\Big(\|\mathcal{D}\|_0+\|\mathcal{D}_0\|_0\Big),\end{aligned}$$

where the error fields $\mathcal{E}_r\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$ are given by the right hand side of (2.6):

$$\mathcal{E}_r=-\frac{1}{\lambda}\sum_{i=1}^{d_*}a_i^r\Big(G(\lambda t_{\eta_i})\nabla^2v_0^i+\bar G(\lambda t_{\eta_i})\nabla^2v_0^{d_*+i}\Big)+\frac{1}{2\lambda^2}\sum_{i=1}^{d_*}\nabla a_i^r\otimes\nabla a_i^r\quad \text{ for all }\ r=0\dots N.$$

Our definition of a^r is correctly posed if only $\mathrm{Id}_d+\frac{1}{\tilde C}\big(\mathcal{D}_0(x)-\mathcal{E}_{r-1}(x)\big)\in B(\mathrm{Id}_d,r_0)\subset\mathbb{R}^{d\times d}_{\mathrm{sym}}$ for all $x\in\bar\omega$. To this end, we will prove that λl large enough (in function of ω and N) implies:

$$\|\mathcal{E}_r\|_0\leq\frac{r_0\tilde C}{2}\quad \text{ for all }\ r=0\dots N-1. \tag{4.2}$$

Note that then automatically there holds for all $r=1\dots N$:

$$\begin{aligned}&\tilde C\mathrm{Id}_d+\mathcal{D}_0-\mathcal{E}_{r-1}=\frac{1}{2}\sum_{i=1}^{d_*}(a_i^r)^2\eta_i\otimes\eta_i\\ &\text{and}\qquad (a_i^r)^2\geq 2r_0\tilde C\ \text{ in }\ \bar\omega,\ \text{ for all }\ i=1\dots d_*.\end{aligned}\tag{4.3}$$

We now develop some preliminary estimates. Firstly, by the linearity of $\bar a_i$ in Lemma 2.6:

$$\|a^r\|_0\leq C\|\tilde C\mathrm{Id}_d+\mathcal{D}_0-\mathcal{E}_{r-1}\|_0^{1/2}\leq C\big(\|\mathcal{D}\|_0+\|\mathcal{E}_{r-1}\|_0\big)^{1/2}. \tag{4.4}$$

Secondly, by the Faá di Bruno formula, exactly as proved in (3.8), there holds for $m\geq1$:

$$\|\nabla^{(m)}a^r\|_0\leq C\|a^r\|_0\sum_{p_1+2p_2+\dots mp_m=m}\ \prod_{t=1}^m\Big(\frac{\|\nabla^{(t)}(\mathcal{D}_0-\mathcal{E}_{r-1})\|_0}{\tilde C}\Big)^{p_t}. \tag{4.5}$$

Thirdly, applying Faá di Bruno's formula to the inverse rather than the square root, we get:

$$\begin{aligned}&\|\nabla^{(m)}\Big(\frac{1}{a_i^r+a_i^{r-1}}\Big)\|_0\\ &\leq\frac{C}{\tilde C^{1/2}}\sum_{p_1+2p_2+\dots mp_m=m}\ \prod_{t=1}^m\Big(\frac{\|\nabla^{(t)}(a^r+a^{r-1})\|_0}{\tilde C^{1/2}}\Big)^{p_t}\quad \text{ for all }\ i=1\dots d_*.\end{aligned}\tag{4.6}$$

The formulas (4.4), (4.5), (4.6) hold for all $r=1\dots N$ with constants C depending on ω, m.

3. (Inductive estimates) In the next step we will prove the following estimates:

$$\|a^r\|_0 \leq C\|\mathcal{D}\|_0^{1/2} \quad \text{for all } \ r = 1\dots N, \tag{4.7$_1$}$$

$$\|\nabla^{(m)}a^r\|_0 \leq C\frac{\lambda^m}{\lambda l}\|\mathcal{D}\|_0^{1/2} \quad \text{for all } \ r = 1\dots N, \ \ m \geq 1, \tag{4.7$_2$}$$

$$\|\nabla^{(m)}\big(\mathcal{E}_r - \mathcal{E}_{r-1}\big)\|_0 \leq C\frac{\lambda^m}{(\lambda l)^r}\|\mathcal{D}\|_0 \quad \text{for all } \ r = 1\dots N, \ \ m \geq 0, \tag{4.7$_3$}$$

with constants C that depend only on ω, r and m. In general, $C \to \infty$ as $m \to \infty$ or $r \to \infty$, so it is crucial that eventually only finitely many of bounds above are used. In particular, we note that $(4.7)_3$ implies (4.2) provided that λl surpasses the sum of constants C corresponding to $m = 0$ and $r = 1\dots N$. This is achieved by taking $\sigma = (\lambda l)^N \geq \sigma_0$ where the constant $\sigma_0 \sim C^{1/\delta} \gg 1$ depends only on ω and δ.

We now check that $(4.7)_1$-$(4.7)_3$ are already valid at their lowest counter value $r = 1$. Indeed, $(4.7)_1$ is a consequence of (4.4), while $(4.7)_2$ further follows from (4.5) in view of $(3.2)_4$:

$$\begin{aligned}\|\nabla^{(m)}a^1\|_0 &\leq C\|a^1\|_0 \sum_{p_1+2p_2+\dots mp_m=m}\prod_{t=1}^m\Big(\frac{\|\nabla^{(t)}\mathcal{D}_0\|_0}{\tilde{C}}\Big)^{p_t} \\ &\leq C\|\mathcal{D}\|_0^{1/2}\sum_{p_1+2p_2+\dots mp_m=m}\prod_{t=1}^m\Big(\frac{\|\mathcal{D}\|_0}{\tilde{C}l^t}\Big)^{p_t} \leq \frac{C}{l^m}\|\mathcal{D}\|_0^{1/2} = C\frac{\lambda^m}{(\lambda l)^m}\|\mathcal{D}\|_0^{1/2} \leq C\frac{\lambda^m}{\lambda l}\|\mathcal{D}\|_0^{1/2}.\end{aligned}$$

For the estimate $(4.7)_3$, with the help of the above we compute at $m = 0$:

$$\|\mathcal{E}_1\|_0 \leq C\Big(\frac{\|a^1\|_0\|\nabla^2 v_0\|_0}{\lambda} + \frac{\|\nabla a^1\|_0^2}{\lambda^2}\Big) \leq C\Big(\frac{\|\mathcal{D}\|_0}{\lambda l} + \frac{\|\mathcal{D}\|_0}{(\lambda l)^2}\Big) \leq C\frac{\|\mathcal{D}\|_0}{\lambda l},$$

and further, for all $m \geq 1$ in view of $(3.2)_3$ and with C depending on ω and m:

$$\begin{aligned}\|\nabla^{(m)}\mathcal{E}_1\|_0 &\leq C\sum_{p+q+t=m}\lambda^{p-1}\|\nabla^{(q)}a^1\|_0\|\nabla^{(t+2)}v_0\|_0 + C\sum_{q+t=m}\lambda^{-2}\|\nabla^{(q+1)}a^1\|_0\|\nabla^{(t+1)}a^1\|_0 \\ &\leq C\Big(\sum_{p+q=m}\frac{\lambda^{p-1}}{l^{t+1}}\|\mathcal{D}\|_0 + \sum_{p+q+t=m, q\neq 0}\frac{\lambda^{p+q-1}}{(\lambda l)l^{t+1}}\|\mathcal{D}\|_0 + \sum_{q+t=m}\frac{\lambda^{q+t+2}}{\lambda^2(\lambda l)^2}\|\mathcal{D}\|_0\Big) \leq C\frac{\lambda^m}{\lambda l}\|\mathcal{D}\|_0.\end{aligned}$$

4. (Proof of the inductive estimates) Assume that $(4.7)_1$–$(4.7)_3$ hold, up to some counter value $1 \leq r \leq N-1$ and all $m \geq 0$. We will prove their validity at $r+1$. By (4.4) and $(4.7)_3$ we directly get $(4.7)_1$:

$$\|a^{r+1}\|_0 \leq C\Big(\|\mathcal{D}\|_0 + \sum_{j=1}^r\|\mathcal{E}_j - \mathcal{E}_{j-1}\|_0\Big)^{1/2} \leq C\big(1 + \frac{1}{\lambda l}\big)^{1/2}\|\mathcal{D}\|_0^{1/2} \leq C\|\mathcal{D}\|_0^{1/2}.$$

Similarly, from (4.5) and $(3.2)_4$ we conclude $(4.7)_2$ with C depending on ω and m:

$$\begin{aligned}\|\nabla^{(m)}a^{r+1}\|_0 &\leq C\|a^{r+1}\|_0\sum_{p_1+2p_2+\dots mp_m=m}\prod_{t=1}^m\Big(\frac{\|\nabla^{(t)}\mathcal{D}_0\|_0 + \sum_{j=1}^r\|\nabla^{(t)}(\mathcal{E}_j - \mathcal{E}_{j-1})\|_0}{\tilde{C}}\Big)^{p_t} \\ &\leq C\|\mathcal{D}\|_0^{1/2}\sum_{p_1+2p_2+\dots mp_m=m}\prod_{t=1}^m\Big(\frac{1}{l^t} + \frac{\lambda^t}{\lambda l}\Big)^{p_t} \\ &\leq C\|\mathcal{D}\|_0^{1/2}\sum_{p_1+2p_2+\dots mp_m=m}\frac{\lambda^m}{(\lambda l)^{p_1+p_2+\dots p_m}} \leq C\frac{\lambda^m}{\lambda l}\|\mathcal{D}\|_0^{1/2}.\end{aligned}$$

Towards showing $(4.7)_3$, we first deduce from the identity in (4.3) that:

$$\mathcal{E}_r - \mathcal{E}_{r-1} = -\frac{1}{2}\sum_{i=1}^{d_*}\big((a_i^{r+1})^2 - (a_i^r)^2\big)\eta_i\otimes\eta_i,$$

and hence for all $m\geq 0$ we get:

$$\|\nabla^{(m)}\big((a_i^{r+1})^2 - (a_i^r)^2\big)\|_0 \leq C\|\nabla^{(m)}\big(\mathcal{E}_r - \mathcal{E}_{r-1}\big)\|_0 \leq C\frac{\lambda^m}{(\lambda l)^r}\|\mathcal{D}\|_0 \quad \text{for all } \ i = 1\dots d_*. \tag{4.8}$$

This in particular implies that by recalling the lower bound in (4.3):

$$\|a_i^{r+1} - a_i^r\|_0 \leq \frac{C}{\tilde{C}^{1/2}}\|(a_i^{r+1})^2 - (a_i^r)^2\|_0 \leq \frac{C}{(\lambda l)^r}\|\mathcal{D}\|_0^{1/2} \quad \text{for all } \ i = 1\dots d_*. \tag{4.9}$$

To estimate derivatives of $(a^{r+1} - a^r)$, we recall (4.6) and observe that for every $m\geq 1$:

$$\|\nabla^{(m)}\Big(\frac{1}{a_i^{r+1} + a_i^r}\Big)\|_0 \leq \frac{C}{\tilde{C}^{1/2}}\sum_{p_1+2p_2+\dots mp_m = m}\prod_{t=1}^{m}\Big(\frac{\lambda^t}{\lambda l}\Big)^{p_t} \leq \frac{C}{\tilde{C}^{1/2}}\frac{\lambda^m}{\lambda l} \qquad \text{for all } \ i = 1\dots d_*,$$

which in combination with (4.8) yields for all $m\geq 1$:

$$\begin{aligned}
\|\nabla^{(m)}\big(a_i^{r+1} - a_i^r\big)\|_0 &\leq C\sum_{q+t=m}\|\nabla^{(q)}\big((a_i^{r+1})^2 - (a_i^r)^2\big)\|_0\|\nabla^{(t)}\Big(\frac{1}{a_i^{r+1} + a_i^r}\Big)\|_0 \\
&\leq C\frac{\lambda^m}{(\lambda l)^r}\frac{\|\mathcal{D}\|_0}{\tilde{C}^{1/2}} + \sum_{q+t=m, t\neq 0} C\frac{\lambda^q}{(\lambda l)^r}\frac{\|\mathcal{D}\|_0}{\tilde{C}^{1/2}}\frac{\lambda^t}{\lambda l} \\
&\leq C\frac{\lambda^m}{(\lambda l)^r}\|\mathcal{D}\|_0^{1/2} \qquad \text{for all } \ i = 1\dots d_*,
\end{aligned} \tag{4.10}$$

We are now ready to estimate the derivatives of:

$$\begin{aligned}
\mathcal{E}_{r+1} - \mathcal{E}_r = &-\frac{1}{\lambda}\sum_{i=1}^{d_*}\big(a_i^{r+1} - a_i^r\big)\Big(\Gamma(\lambda t_{\eta_i})\nabla^2 v_0^i + \bar{\Gamma}(\lambda t_{\eta_i})\nabla^2 v_0^{d_*+i}\Big) \\
&+ \frac{1}{2\lambda^2}\sum_{i=1}^{d_*}\Big(\big(\nabla a_i^{r+1} - \nabla a_i^r\big)\otimes\nabla a_i^{r+1} + \nabla a_i^r\otimes\big(\nabla a_i^{r+1} - \nabla a_i^r\big)\Big).
\end{aligned}$$

Namely, we get for all $m\geq 0$:

$$\begin{aligned}
\|\nabla^{(m)}(\mathcal{E}_{r+1} - \mathcal{E}_r)\|_0 &\leq C\sum_{p+q+t=m}\lambda^{p-1}\|\nabla^{(q)}\big(a^{r+1} - a^r\big)\|_0\|\nabla^{(t+2)}v_0\|_0 \\
&\qquad + C\sum_{q+t=m}\lambda^{-2}\|\nabla^{(q+1)}(a^{r+1} - a^r)\|_0\big(\|\nabla^{(t+1)}a^{r+1}\|_0 + \|\nabla^{(t+1)}a^r\|_0\big) \\
&\leq \sum_{p+q+t=m}\lambda^{p-1}\frac{\lambda^q}{(\lambda l)^r l^{t+1}}\|\mathcal{D}\|_0 + C\sum_{q+t=m}\lambda^{-2}\frac{\lambda^{q+1}}{(\lambda l)^r}\frac{\lambda^{t+1}}{\lambda l}\|\mathcal{D}\|_0 \leq C\frac{\lambda^m}{(\lambda l)^{r+1}}\|\mathcal{D}\|_0.
\end{aligned}$$

This ends the proof of the last inductive estimate $(4.7)_3$. Observe that the closing of the bounds as above, was possible due to the absence of the error term $\frac{1}{\lambda^2}a\bar{\Gamma}(\lambda t_\eta)\nabla^2 a$ in the right hand side of (2.6), in the step construction generated by Nash's spirals. This term, appearing in (2.3) as the second order deficit generated by Kuiper's corrugations, would result in $\mathcal{E}_{r+1} - \mathcal{E}_r$ containing expressions of the form $\frac{1}{\lambda^2}a_i^r\bar{\Gamma}(\lambda t_{\eta_i})(\nabla^2 a^{r+1} - \nabla^2 a^r)$ which do not allow for the gain in the power of (λl), because each $\|a^r\|_0$ is only of order 1 in $\tilde{C}$, see the bound in $(4.7)_1$.

5. (End of proof) Define $\tilde v \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^k)$ and $\tilde w \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^d)$ according to the "step" construction in Lemma 2.3, involving the periodic functions $G, \bar G$ and the notation $t_\eta = \langle x, \eta\rangle$:

$$\begin{aligned} \tilde v &= v_1, & v_1(x) &= v_0(x) + \frac{1}{\lambda}\sum_{i=1}^{d_*} a_i^N(x)\big(G(\lambda t_{\eta_i})e_i + \bar G(\lambda t_{\eta_i})e_{d_*+i}\big),\\ \tilde w &= w_1 - \tilde C id_d, & w_1(x) &= w_0(x) - \frac{1}{\lambda}\sum_{i=1}^{d_*} a_i^N(x)\big(G(\lambda t_{\eta_i})\nabla v_0^i + \bar G(\lambda t_{\eta_i})\nabla v_0^{d_*+i}\big). \end{aligned} \tag{4.11}$$

We now show that $(4.7)_1$ - $(4.7)_3$ imply the bounds claimed in the Theorem. To prove $(1.3)_1$, we use $(4.7)_1$, $(4.7)_2$ and $(3.2)_1$, $(3.2)_3$:

$$\begin{aligned} \|\tilde v - v\|_1 &\le \|v_0 - v\|_1 + C\Big(\|a^N\|_0 + \frac{\|\nabla a^N\|_0}{\lambda}\Big) \le C\|\mathcal{D}\|_0^{1/2}\big(1 + \frac{1}{\lambda l}\big) \le C\|\mathcal{D}\|_0^{1/2},\\ \|\tilde w - w\|_1 &\le \|w_0 - w\|_1 + C\Big(\tilde C + \|a^N\|_0\|\nabla v_0\|_0 + \frac{\|\nabla a^N\|_0\|\nabla v_0\|_0 + \|a^N\|_0\|\nabla^2 v_0\|_0}{\lambda}\Big)\\ &\le C\|\mathcal{D}\|_0^{1/2} + C\|\mathcal{D}\|_0^{1/2}\Big(\|\mathcal{D}\|_0^{1/2} + (\|\mathcal{D}\|_0^{1/2} + \|\nabla v\|_0) + \frac{(\|\mathcal{D}\|_0^{1/2} + \|\nabla v\|_0) + \|\mathcal{D}\|_0^{1/2}}{\lambda l}\Big)\\ &\le C\|\mathcal{D}\|_0^{1/2}\big(1 + \|\nabla v\|_0\big). \end{aligned}$$

Similarly, there follows $(1.3)_2$ when we recall that $\lambda l = \sigma^{1/N} \le \sigma^\delta$ from (4.1):

$$\begin{aligned} \|\nabla^2\tilde v\|_0 &\le \|\nabla^2 v_0\|_0 + C\Big(\lambda\|a^N\|_0 + \|\nabla a^N\|_0 + \frac{\|\nabla^2 a^N\|_0}{\lambda}\Big)\\ &\le C\|\mathcal{D}\|_0^{1/2}\Big(\frac{1}{l} + \lambda\Big) = CM(1 + \lambda l) \le CM\sigma^{1/N},\\ \|\nabla^2\tilde w\|_0 &\le \|\nabla^2 w_0\|_0 + C\Big(\lambda\|a^N\|_0\|\nabla v_0\|_0 + \big(\|\nabla a^N\|_0\|\nabla v_0\|_0 + \|a^N\|_0\|\nabla^2 v_0\|_0\big)\\ &\qquad + \frac{\|\nabla^2 a^N\|_0\|\nabla v_0\|_0 + \|\nabla a^N\|_0\|\nabla^2 v_0\|_0 + \|a^N\|_0\|\nabla^3 v_0\|_0}{\lambda}\Big)\\ &\le C\|\mathcal{D}\|_0^{1/2}\Big(\frac{1}{l} + \big(\lambda + \frac{1}{l}\big)(\|\mathcal{D}\|_0^{1/2} + \|\nabla v\|_0) + \frac{\|\mathcal{D}\|_0^{1/2}}{\lambda l^2}\Big)\\ &\le CM(1 + \lambda l)(1 + \|\nabla v\|_0) \le CM\sigma^{1/N}(1 + \|\nabla v\|_0). \end{aligned}$$

Finally, (2.6) and (4.3) yield $(1.3)_3$, because we decompose:

$$\tilde{\mathcal{D}} = (A - A_0) + \mathcal{D}_0 - \Big(\frac{1}{2}\sum_{i=1}^{d_*}(a_i^N)^2\eta_i\otimes\eta_i + \mathcal{E}_N - \tilde C\mathrm{Id}_d\Big) = (A - A_0) - (\mathcal{E}_N - \mathcal{E}_{N-1}),$$

and further, in view of $(3.2)_2$, $(4.7)_3$:

$$\begin{aligned} \|\tilde{\mathcal{D}}\|_0 &\le \|A - A_0\|_0 + \|\mathcal{E}_N - \mathcal{E}_{N-1}\|_0\\ &\le C\Big(l^\beta\|A\|_{0,\beta} + \frac{\|\mathcal{D}\|_0}{(\lambda l)^N}\Big) = C\Big(\frac{\|A\|_{0,\beta}}{M^\beta}\|\mathcal{D}\|_0^{\beta/2} + \frac{\|\mathcal{D}\|_0}{\sigma}\Big). \end{aligned}$$

The proof is done. ∎

5. The Nash-Kuiper scheme in $\mathcal{C}^{1,\alpha}$

To perform induction on stages we need the following argument, similar to [3, Theorem 1.1]:

Theorem 5.1. *Let $\omega \subset \mathbb{R}^d$ be an open, bounded domain and let $k \geq 1$ and $\gamma > 0$ be such that the statement of Theorem 1.2 holds true with γ replacing the exponent d_*/k in $(1.2)_2$, provided that $\sigma > \sigma_0$ where $\sigma_0 > 1$ depends only on ω and γ. Then we have the following. For every $v \in \mathcal{C}^2(\bar\omega, \mathbb{R}^k)$, $w \in \mathcal{C}^2(\bar\omega, \mathbb{R}^d)$, $A \in \mathcal{C}^{0,\beta}(\bar\omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, such that:*

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big) \quad \textit{satisfies} \quad 0 < \|\mathcal{D}\|_0 \leq 1,$$

and for every α in the range:

$$0 < \alpha < \min\Big\{\frac{\beta}{2}, \frac{1}{1+2\gamma}\Big\}, \tag{5.1}$$

there exist $\tilde v \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^k)$ and $\tilde w \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^d)$ with the following properties:

$$\|\tilde v - v\|_1 \leq C\|\mathcal{D}\|_0^{1/2}, \quad \|\tilde w - w\|_1 \leq C\|\mathcal{D}\|_0^{1/2}(1 + \|\nabla v\|_0), \tag*{$(5.2)_1$}$$

$$A - \big(\frac{1}{2}(\nabla \tilde v)^T\nabla \tilde v + \mathrm{sym}\nabla \tilde w\big) = 0 \quad \textit{in } \ \bar\omega, \tag*{$(5.2)_2$}$$

where the constants C depend only on d, k and ω.

Proof. **1.** Because of the assumption (5.1), there exists an exponent:

$$\frac{2\gamma\alpha}{(1-\alpha)} < \delta < \min\Big\{1, \frac{2\gamma\beta}{(2-\beta)}\Big\}. \tag{5.3}$$

We let $\sigma > \sigma_0$ be a sufficiently large constant, in function of δ, α, γ, $\|\nabla v\|_0$ and all constants C in the assumed assertions of Theorem 1.2 (these constants depend only in d, k, ω).

We further set $v_0 = v$, $w_0 = w$, $\mathcal{D}_0 = \mathcal{D}$, and take $M_0 \geq \max\{\|v_0\|_2, \|w_0\|_2, 1\}$ that is again sufficiently large, now in function of $\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2-1}\sigma^\delta$ and constants C indicated before. By successive applications of Theorem 1.2 with the chosen σ and constants $\{M_i \geq 1\}_{i=1}^\infty$ in:

$$M_i = \Big(\tilde C(1 + \|\nabla v\|_0)\sigma^\gamma\Big)^i M_0$$

where $\tilde C > 1$ is again some large constant (in function of the aforementioned C), we obtain sequences $\{v_i \in \mathcal{C}^2(\bar\omega, \mathbb{R}^k)\}_{i=1}^\infty$, $\{w_i \in \mathcal{C}^2(\bar\omega, \mathbb{R}^d)\}_{i=1}^\infty$ and the related deficits $\{\mathcal{D}_i \in \mathcal{C}^0(\bar\omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})\}_{i=1}^\infty$:

$$\mathcal{D}_i = A - \big(\frac{1}{2}(\nabla v_i)^T\nabla v_i + \mathrm{sym}\nabla w_i\big).$$

We see that, as long as there holds:

$$0 < \|\mathcal{D}_i\|_0 \leq 1 \quad \text{ and } \quad M_i \geq \max\{\|v_i\|_2, \|w_i\|_2, 1\}, \tag{5.4}$$

we have, with the constants C depending only on d, k and ω:

$$\|v_{i+1} - v_i\|_1 \leq C\|\mathcal{D}_i\|_0^{1/2}, \quad \|w_{i+1} - w\|_1 \leq C\|\mathcal{D}_i\|_0^{1/2}(1 + \|\nabla v_i\|_0), \tag*{$(5.5)_1$}$$

$$\|v_{i+1}\|_2 \leq CM_i\sigma^\gamma, \quad \|w_{i+1}\|_2 \leq CM_i\sigma^\gamma(1 + \|\nabla v_i\|_0), \tag*{$(5.5)_2$}$$

$$\|\mathcal{D}_{i+1}\|_0 \leq C\Big(\frac{\|A\|_{0,\beta}}{M_i^\beta}\|\mathcal{D}_i\|_0^{\beta/2} + \frac{\|\mathcal{D}_i\|_0}{\sigma}\Big). \tag*{$(5.5)_3$}$$

Below, we inductively validate (5.4) for all $i \geq 0$, and in fact we show that:

$$\|\mathcal{D}_i\|_0 \leq \frac{1}{\sigma^{\delta i}}\|\mathcal{D}\|_0 \quad \text{ for all } \ i = 0\ldots\infty. \tag{5.6}$$

Before doing so, note that (5.6) actually implies both statements in (5.4). Indeed, by $(5.5)_2$:

$$\|v_{i+1}\|_2 \leq CM_i\sigma^\gamma \leq M_{i+1},$$
$$\|w_{i+1}\|_2 \leq CM_i\sigma^\gamma(1+\|\nabla v_i\|_0) \leq CM_i\sigma^\gamma(1+2C+\|\nabla v\|_0) \leq M_{i+1},$$

since by the first bound in $(5.5)_1$ and (5.6) there follows, provided that $\sigma^{\delta/2}\geq 2$:

$$\begin{aligned}\|\nabla v_i\|_0 &\leq \|\nabla v\|_0 + \sum_{j=0}^{i-1}\|\nabla v_{j+1}-\nabla v_j\|_0 \leq \|\nabla v\|_0 + C\sum_{j=0}^{i-1}\|\mathcal{D}_j\|_0^{1/2} \\ &\leq \|\nabla v\|_0 + C\sum_{j=0}^{\infty}\frac{\|\mathcal{D}\|_0^{1/2}}{\sigma^{\delta j/2}} = \|\nabla v\|_0 + \frac{C}{1-\sigma^{-\delta/2}}\|\mathcal{D}\|_0^{1/2} \\ &\leq \|\nabla v\|_0 + 2C\|\mathcal{D}\|_0^{1/2} \leq 2C+\|\nabla v\|_0.\end{aligned} \tag{5.7}$$

2. Clearly (5.6) holds at $i=0$. To prove it at $(i+1)$, use $(5.5)_3$ and the induction assumption:

$$\|\mathcal{D}_{i+1}\|_0 \leq C\Big(\frac{\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2}}{M_i^\beta\sigma^{\delta i\beta/2}} + \frac{\|\mathcal{D}\|_0}{\sigma^{\delta i+1}}\Big) = \frac{\|\mathcal{D}\|_0}{\sigma^{\delta(i+1)}}\Big(\frac{C\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2-1}}{M_i^\beta\sigma^{\delta i\beta/2-\delta(i+1)}} + \frac{C}{\sigma^{1-\delta}}\Big), \tag{5.8}$$

and check that both terms in parentheses in the right hand side above are not greater than 1/2. For the second term, this is readily implied by taking σ large enough that $\sigma^{1-\delta}\geq 2C$, in view of $1-\delta>0$ in (5.3). For the first term, we note that

$$\frac{C\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2-1}}{M_i^\beta\sigma^{\delta i\beta/2-\delta(i+1)}} \leq \frac{C\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2-1}\sigma^\delta}{M_0^\beta}\cdot\sigma^{\delta i-\gamma\beta i-\delta\beta i/2} \leq \frac{C\|A\|_{0,\beta}\|\mathcal{D}\|_0^{\beta/2-1}\sigma^\delta}{M_0^\beta},$$

since the exponent $\delta i-\gamma\beta i-\delta\beta i/2$ is non-positive, due to $\delta<\frac{2\gamma\beta}{2-\beta}$ in (5.3):

$$\delta i-\gamma\beta i-\delta\beta i/2 = \frac{i}{2}\big(\delta(2-\beta)-2\gamma\beta\big)\leq 0 \quad \text{for all } i\geq 0.$$

In conclusion, the expression in parentheses in (5.8) is bounded by 1, provided M_0 has been chosen sufficiently large. This ends the proof of (5.6).

3. From $(5.5)_1$, (5.7) and (5.6), it follows that for all $i=0\ldots\infty$:

$$\|v_{i+1}-v_i\|_1 \leq \frac{C}{\sigma^{\delta i/2}}\|\mathcal{D}\|_0^{1/2}, \qquad \|w_{i+1}-w_i\|_1 \leq \frac{C}{\sigma^{\delta i/2}}\|\mathcal{D}\|_0^{1/2}\big(1+\|\nabla v\|_0\big),$$

Hence, both sequences $\{v_i\}_{i=1}^\infty$, $\{w_i\}_{i=1}^\infty$ are Cauchy in $\mathcal{C}^1(\bar\omega)$ and as such they converge to the limit fields, respectively:

$$\tilde v\in\mathcal{C}^1(\omega,\mathbb{R}^k), \qquad \tilde w\in\mathcal{C}^1(\omega,\mathbb{R}^d)$$

that satisfy $(5.2)_1$ and $(5.2)_2$, in virtue of $\|\mathcal{D}_i\|_0\to 0$ as $i\to\infty$.

It remains to show that $\tilde v$ and $\tilde w$ are $\mathcal{C}^{1,\alpha}$-regular. To this end, we use the estimate:

$$\begin{aligned}\|v_{i+1}-v_i\|_2 + \|w_{i+1}-w_i\|_2 &\leq CM_i\sigma^\gamma\big(1+\|\nabla v\|_0\big) \\ &\leq C\Big(\tilde C(1+\|\nabla v\|_0)\sigma^\gamma\Big)^{i+1}M_0,\end{aligned}$$

resulting from $(5.5)_2$ and (5.7), in the interpolation inequality $\|\cdot\|_{1,\alpha} \leq C\|\cdot\|_1^{1-\alpha}\|\cdot\|_2^{\alpha}$:

$$\begin{aligned}\|v_{i+1}-v_i\|_{1,\alpha} &+ \|w_{i+1}-w_i\|_{1,\alpha} \\ &\leq C\|\mathcal{D}\|_0^{(1-\alpha)/2}\Big(\tilde{C}(1+\|\nabla v\|_0)\Big)^{(i+1)\alpha+(1-\alpha)} M_0^{\alpha}\sigma^{\alpha\gamma(i+1)-\delta i(1-\alpha)/2} \\ &= C\tilde{C}\cdot M_0^{\alpha}\|\mathcal{D}\|_0^{(1-\alpha)/2}(1+\|\nabla v\|_0)\sigma^{\alpha\gamma}\cdot\Big(\frac{\tilde{C}^{\alpha}(1+\|\nabla v\|_0)^{\alpha}}{\sigma^{\delta(1-\alpha)/2-\alpha\gamma}}\Big)^{i}.\end{aligned}$$

Since the exponent $\delta(1-\alpha)/2-\alpha\gamma$ is positive, in view of $\delta > \frac{2\gamma\alpha}{1-\alpha}$ in (5.3), we see that both sequences $\{v_i\}_{i=1}^{\infty}$, $\{w_i\}_{i=1}^{\infty}$ are Cauchy in $\mathcal{C}^{1,\alpha}(\bar{\omega})$, provided that σ is sufficiently large to have:

$$\frac{\tilde{C}^{\alpha}(1+\|\nabla v\|_0)^{\alpha}}{\sigma^{\delta(1-\alpha)/2-\alpha\gamma}} < 1.$$

In conclusion, $\tilde{v} \in \mathcal{C}^{1,\alpha}(\omega, \mathbb{R}^k)$ and $\tilde{w} \in \mathcal{C}^{1,\alpha}(\omega, \mathbb{R}^d)$ as claimed. The proof is done. ■

Taking now $\gamma = d_*/k$ or taking an arbitrary $\gamma < 1$ as guaranteed by Theorem 1.2 and Theorem 1.4, respectively, Theorem 5.1 implies:

Corollary 5.2. *For every $v \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^k)$, $w \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^d)$ and $A \in \mathcal{C}^{0,\beta}(\bar{\omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ defined on an open, bounded domain $\omega \subset \mathbb{R}^d$, and such that:*

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big) \quad \textit{satisfies} \quad 0 < \|\mathcal{D}\|_0 \leq 1,$$

and for every exponent α in the range:

$$0 < \alpha < \min\Big\{\frac{\beta}{2}, \frac{1}{1+2d_*/k}\Big\}, \qquad \textit{or} \;\; 0 < \alpha < \min\Big\{\frac{\beta}{2}, 1\Big\} \;\; \textit{when} \;\; k \geq d(d+1),$$

there exist $\tilde{v} \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^k)$ and $\tilde{w} \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^d)$ with the following properties:

$$\|\tilde{v}-v\|_1 \leq C\|\mathcal{D}\|_0^{1/2}, \quad \|\tilde{w}-w\|_1 \leq C\|\mathcal{D}\|_0^{1/2}(1+\|\nabla v\|_0), \tag{5.9$_1$}$$

$$A - \big(\frac{1}{2}(\nabla\tilde{v})^T\nabla\tilde{v} + \mathrm{sym}\nabla\tilde{w}\big) = 0 \quad \textit{in} \;\; \bar{\omega}, \tag{5.9$_2$}$$

where the constants C depend only on d, k and ω.

6. Proofs of Theorem 1.1 and Theorem 1.3

The final auxiliary result that we need, is a combination of the local decomposition into "primitive metrics" with a partition of unity - type statement from [13, Lemma 3.3]:

Lemma 6.1. *Given the dimension $d \geq 1$, there exists a constant N_0 and sequences of unit vectors $\{\eta_i \in \mathbb{R}^d\}_{i=1}^{\infty}$ and nonnegative functions $\{\varphi_i \in \mathcal{C}_c^{\infty}(\mathbb{R}^{d\times d}_{\mathrm{sym},>}, \mathbb{R})\}_{i=1}^{\infty}$, such that:*

$$A = \sum_{i=1}^{\infty}\varphi_i(A)^2\eta_i\otimes\eta_i \quad \textit{for all} \;\; A \in \mathbb{R}^{d\times d}_{\mathrm{sym},>},$$

and such that:

(i) *at most N_0 terms in the above sum are nonzero,*
(ii) *every compact set of matrices $K \subset \mathbb{R}^{d\times d}_{\mathrm{sym},>}$ induces a finite set of indices $J(K) \subset \mathbb{N}$, such that $\varphi_i(A) = 0$ for all $A \in K$ and all $i \notin J(K)$.*

Equipped with Lemma 6.1 and the "step" construction in Lemma 2.1, one easily deduces the deficit decrease - approximation result in $\mathcal{C}^1$, which is the multidimensional version of the basic "stage" construction in [10, Proposition 3.2]:

Theorem 6.2. *Let $\omega\subset\mathbb{R}^d$ be an open, bounded domain. Given two vector fields $v\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^k)$, $w\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^d)$ and a matrix field $A\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$, assume that:*

$$\mathcal{D} = A - \big(\frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\nabla w\big) \quad \textit{satisfies} \quad \mathcal{D} > c\,\mathrm{Id}_d \;\; \textit{on} \;\; \bar\omega$$

for some $c>0$, in the sense of matrix inequalities. Fix $\epsilon>0$. Then, there exists $\tilde v\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^k)$ and $\tilde w\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^d)$ such that, denoting:

$$\tilde{\mathcal{D}} = A - \big(\frac{1}{2}(\nabla \tilde v)^T\nabla \tilde v + \mathrm{sym}\nabla \tilde w\big),$$

the following holds with constants C depending only on d,k and ω:

$$\|\tilde v - v\|_0 \leq \epsilon, \quad \|\tilde w - w\|_0\leq\epsilon, \tag{6.1$_1$}$$

$$\|\nabla(\tilde v - v)\|_0 \leq C\|\mathcal{D}\|_0^{1/2}, \quad \|\nabla(\tilde w - w)\|_0 \leq C\|\mathcal{D}\|_0^{1/2}\big(\|\mathcal{D}\|_0^{1/2} + \|\nabla v\|_0\big), \tag{6.1$_2$}$$

$$\|\tilde{\mathcal{D}}\|_0\leq \epsilon \quad \textit{and} \quad \tilde{\mathcal{D}} > \tilde c\,\mathrm{Id}_d \;\; \textit{for some } \tilde c>0. \tag{6.1$_3$}$$

Proof. **1.** Since $\mathcal{D}(\bar\omega)$ is a compact subset of $\mathbb{R}^{d\times d}_{\mathrm{sym},>}$, Lemma 6.1 yields a finite set of indices for which the decomposition into "primitive matrices" is active. Without loss of generality these indices are $\{1\dots N\}$. Then, with the unit vectors $\{\eta_i\in\mathbb{R}^d\}_{i=1}^N$ and the nonnegative functions $\{b_i\in\mathcal{C}^\infty(\bar\omega,\mathbb{R})\}_{i=1}^N$ defined by $b_i(x)=\varphi_i(\mathcal{D}(x))$, there holds:

$$\mathcal{D}(x) = \sum_{i=1}^N b_i(x)^2\eta_i\otimes\eta_i \qquad \text{for all} \;\; x\in\bar\omega.$$

We now define the modified nonnegative amplitude functions $\{a_i\in\mathcal{C}^\infty(\bar\omega,\mathbb{R})\}_{i=1}^N$ by:

$$a_i = (1-\delta)^{1/2}b_i \quad \text{on} \;\; \omega, \qquad \text{where} \;\; \delta = \min\big\{\frac{1}{2}, \frac{\epsilon}{2\|\mathcal{D}\|_0}\big\}.$$

Observe that:

$$\mathcal{D} - \sum_{i=1}^N a_i^2\eta_i\otimes\eta_i = \delta\mathcal{D} > \delta c\,\mathrm{Id}_d.$$

2. We set $v_1=v$ and $w_1=w$. We then inductively define the vector fields $\{v_i\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^k)\}_{i=1}^{N+1}$ and $\{w_i\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^d)\}_{i=1}^{N+1}$ by applying Lemma 2.1 to each consecutive pair (v_i,w_i) with the given unit vector η_i, an arbitrary unit vector $E\in\mathbb{R}^k$, the given amplitude a_i and a frequency $\lambda_i>0$ that is sufficiently large as indicated below. We finally set:

$$\tilde v = v_{N+1}, \quad \tilde w = w_{N+1}.$$

It is clear that by taking $\{\lambda_i\}_{i=1}^N$ large, one can ensure the validity of $(6.1)_1$. Further, by (2.3):

$$\begin{aligned}\tilde{\mathcal{D}} &= \mathcal{D} - \Big(\big(\frac{1}{2}(\nabla v_{N+1})^T\nabla v_{N+1} + \mathrm{sym}\nabla w_{N+1}\big) - \big(\frac{1}{2}(\nabla v_1)^T\nabla v_1 + \mathrm{sym}\nabla w_1\big)\Big)\\ &= \Big(\mathcal{D} - \sum_{i=1}^N a_i^2\eta_i\otimes\eta_i\Big)\\ &\qquad - \sum_{i=1}^N \Big(\big(\frac{1}{2}(\nabla v_{i+1})^T\nabla v_{i+1} + \mathrm{sym}\nabla w_{i+1}\big) - \big(\frac{1}{2}(\nabla v_i)^T\nabla v_i + \mathrm{sym}\nabla w_i\big) - a_i^2\eta_i\otimes\eta_i\Big)\\ &= \delta\mathcal{D} + \sum_{i=1}^N O\Big(\frac{\|a_i\|_0\|\nabla^2 v_i\|_0}{\lambda_i} + \frac{\|\nabla a_i\|^2 + \|a_i\|_0\|\nabla^2 a\|_0}{\lambda_i^2}\Big),\end{aligned}$$

which implies $(6.1)_3$ with $\tilde{c} = \delta c/2$, provided that $\{\lambda_i\}_{i=1}^N$ are sufficiently large.

3. It remains to check $(6.1)_2$. By Lemma 6.1 (i), we get for each $x \in \bar{\omega}$:

$$0 \leq \sum_{i=1}^N a_i(x) \leq \sum_{i=1}^N b_i(x) \leq N_0^{1/2}\big(\sum_{i=1}^N b_i(x)^2\big)^{1/2} = N_0^{1/2}\big(\mathrm{Trace}\,\mathcal{D}(x)\big)^{1/2} \leq CN_0^{1/2}\|\mathcal{D}\|_0^{1/2}.$$

Consequently, the definition (2.2) yields, with sufficiently large $\{\lambda_i\}_{i=1}^N$:

$$\|\nabla(\tilde{v} - v)\|_0 \leq \sum_{i=1}^N \|\nabla v_{i+1} - \nabla v_i\|_0 \leq 2\|\sum_{i=1}^N a_i\|_0 + C\sum_{i=1}^N \frac{\|\nabla a_i\|_0}{\lambda_i} \leq C\|\mathcal{D}\|_0^{1/2}.$$

In a similar fashion, and using the above bound, we obtain:

$$\begin{aligned}\|\nabla(\tilde{w} - w)\|_0 &\leq \sum_{i=1}^N \|\nabla w_{i+1} - \nabla w_i\|_0 \leq C\Big\|\sum_{i=1}^N \big(\|\nabla v_i\|_0 + \|a_i\|_0\big)a_i\Big\|_0\\ &\qquad + C\sum_{i=1}^N \frac{\|\nabla a_i\|_0\|\nabla v_i\|_0 + \|a_i\|_0\|\nabla^2 v_i\|_0 + \|a_i\|_0^2 + \|a_i\|_0\|\nabla a_i\|_0}{\lambda_i}\\ &\qquad + C\sum_{i=1}^N \frac{\|\nabla a_i\|_0\|\nabla^2 a_i\|_0 + \|a_i\|_0\|\nabla^3 a_i\|_0}{\lambda_i^2}\\ &\leq C\|\mathcal{D}\|_0^{1/2}\cdot\sup_{i=1\ldots N}\big(\|\nabla v_i\|_0 + \|a_i\|_0\big) \leq C\|\mathcal{D}\|_0^{1/2}\big(\|\nabla v_0\|_0 + \|\mathcal{D}\|_0^{1/2}\big).\end{aligned}$$

This ends the proof of $(6.1)_2$ and of the theorem. ■

We remark that having the upgraded version of the "step" in Lemma 2.1 was irrelevant to the proof above, and that the sub-optimal construction in [10, Lemma 2.2] would still suffice.

We are now ready to give:

Proofs of Theorem 1.1 and Theorem 1.3

In order to apply Corollary 5.2, we need to increase the regularity of v, w and decrease the deficit $\mathcal{D}$. Take $\epsilon < 1$ that is sufficiently small, as indicated below. First, we let $v_1 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^k)$, $w_1 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^d)$ and $A_1 \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ be such that:

$$\begin{aligned}&\|v_1 - v\|_1 \leq \epsilon^3, \quad \|w_1 - w\|_1 \leq \epsilon^3, \quad \|A_1 - A\|_0 \leq \epsilon^3,\\ &\mathcal{D}_1 = A_1 - \big(\frac{1}{2}(\nabla v_1)^T\nabla v_1 + \mathrm{sym}\nabla w_1\big) > c_1\mathrm{Id}_d \quad \text{for some } c_1 > 0.\end{aligned}$$

The last property follows from the fact that:

$$\|\mathcal{D}_1 - \mathcal{D}\|_0 \leq 3\epsilon^3 + \epsilon^3\|\nabla v\|_0 \tag{6.2}$$

Second, use Theorem 6.2 to get $v_2 \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^k)$, $w_2 \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^d)$ such that:

$$\begin{aligned}
&\|v_2 - v_1\|_0 \leq \epsilon^3, \quad \|w_2 - w_1\|_0 \leq \epsilon^3,\\
&\|\nabla(v_2 - v_1)\|_0 \leq C\|\mathcal{D}_1\|_0^{1/2} \leq C\big(\|\mathcal{D}\|_0^{1/2} + \epsilon^{3/2} + \|\nabla v\|_0^{1/2}\big),\\
&\mathcal{D}_2 = A_1 - \big(\frac{1}{2}(\nabla v_2)^T\nabla v_2 + \mathrm{sym}\nabla w_2\big) \quad \text{ satisfies } \ \|\mathcal{D}_2\|_0 \leq \epsilon^3,
\end{aligned}$$

where we applied (6.2) in the gradient increment bound of v.

Clearly, if the deficit $\mathcal{D}_3$, defined below:

$$\mathcal{D}_3 = A - \big(\frac{1}{2}(\nabla v_2)^T\nabla v_2 + \mathrm{sym}\nabla w_2\big)$$

is equivalently zero on $\bar\omega$, then we may simply take $\tilde v = v_2$ and $\tilde w = w_2$ to satisfy the claim of the theorem. Otherwise, we use Corollary 5.2 to v_2, w_2 and A, since:

$$0 < \|\mathcal{D}_3\|_0 \leq \|A - A_1\|_0 + \|\mathcal{D}_2\| \leq 2\epsilon^3 \leq 1,$$

and consequently obtain $\tilde v \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^k)$, $\tilde w \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^d)$ such that:

$$\begin{aligned}
&\|\tilde v - v_2\|_0 \leq C\epsilon^{3/2},\\
&\|\tilde w - w_2\|_0 \leq C\epsilon^{3/2}(1 + \|\nabla v_2\|_0) \leq C\epsilon^{3/2}(1 + \|\mathcal{D}\|_0^{1/2} + \|\nabla v\|_0),\\
&A - \big(\frac{1}{2}(\nabla\tilde v)^T\nabla\tilde v + \mathrm{sym}\nabla\tilde w\big) = 0 \quad \text{ in } \ \bar\omega.
\end{aligned}$$

It now suffices to observe that, taking ϵ sufficiently small (in function of $\|\mathcal{D}\|_0^{1/2}$, $\|\nabla v\|_0$ and of constants C that depend only on k, d and ω), we get:

$$\begin{aligned}
&\|\tilde v - v\|_0 \leq \|\tilde v - v_2\|_0 + \|v_2 - v_1\|_0 + \|v_1 - v\|_0 \leq C\epsilon^{3/2} \leq \epsilon,\\
&\|\tilde w - w\|_0 \leq \|\tilde w - w_2\|_0 + \|w_2 - w_1\|_0 + \|w_1 - w\|_0 \leq C\epsilon^{3/2}(1 + \|\mathcal{D}\|_0^{1/2} + \|\nabla v\|_0) \leq \epsilon.
\end{aligned}$$

The proof is done. ■

7. The Monge-Ampère system: proofs of Lemma 1.6, Lemma 1.8 and Theorem 1.10

We remark that when $d = 2$ then the formula (1.5) rests in agreement with the expansion of the Gaussian curvature: $\kappa(\mathrm{Id}_2 + \epsilon A) = -\frac{\epsilon}{2}\mathrm{curl}\,\mathrm{curl}A + O(\epsilon^2)$, because we have:

$$\mathfrak{C}^2(A)_{ij,st} = \begin{cases} \mathrm{curl}\,\mathrm{curl}A & \text{if } (ij,st) \in \{(12,12),(21,21)\},\\ -\mathrm{curl}\,\mathrm{curl}A & \text{if } (ij,st) \in \{(12,21),(21,12)\},\\ 0 & \text{otherwise.}\end{cases}$$

We now give:

Proof of Lemma 1.6

The implication (i)⇒(ii) follows by a direct inspection:

$$\begin{aligned}
2\mathfrak{C}^2(\mathrm{sym}\nabla w)_{ij,st} = {} & \partial_i\partial_s(\partial_j w^t + \partial_t w^j) + \partial_j\partial_t(\partial_i w^s + \partial_s w^i)\\
& - \partial_i\partial_t(\partial_j w^s + \partial_s w^j) - \partial_j\partial_s(\partial_i w^t + \partial_t w^i) = 0.
\end{aligned}$$

To prove that (ii)⇒(i), note that condition:

$$\mathfrak{C}^2(A)_{ij,st} = \partial_i\big(\partial_s A_{jt} - \partial_t A_{js}\big) - \partial_j\big(\partial_s A_{it} - \partial_t A_{is}\big) = 0$$

implies, for each fixed $s,t : 1\dots d$, that the vector field $[\partial_s A_{jt} - \partial_t A_{js}]_{j=1\dots d}$ must be a gradient of some scalar field $-\phi_{st}$, where we used the Poincaré Lemma on the contractible domain ω. We thus write:

$$\partial_s A_{jt} - \partial_t A_{js} = -\partial_j \phi_{st} \quad \text{ for all } \; j,s,t = 1\dots d. \tag{7.1}$$

Observe that $\nabla(\phi_{st} + \phi_{ts}) = 0$, so without loss of generality, $\phi_{st} = -\phi_{ts}$ and $\phi_{ss} = 0$. The following matrix field is thus skew-symmetric:

$$\phi = [\phi_{st}]_{s,t=1\dots d} : \omega \to \mathrm{so}(d). \tag{7.2}$$

Consequently, permuting the indices in (7.1), leads to:

$$\partial_s A_{jt} - \partial_t A_{js} + \partial_j \phi_{st} = 0 \quad \text{ and } \quad \partial_t A_{js} - \partial_j A_{st} + \partial_s \phi_{tj} = 0.$$

Summing the above two expressions, we get:

$$\partial_s A_{tj} - \partial_j A_{ts} + \partial_s \phi_{tj} - \partial_j \phi_{ts} = 0 \quad \text{ for all } \; j,s,t = 1\dots d.$$

Hence for any $i = 1\dots d$, the i-th row $[A_{ij} + \phi_{ji}]_{j=1\dots d}$ of the matrix field $A+\phi$ is a gradient of some scalar field w^i on ω, where we again used Poincaré's Lemma. Writing $w = [w^i]_{i=1\dots d} : \omega \to \mathbb{R}^d$, we obtain the claim by taking the symmetric parts of the resulting identity:

$$A + \phi = \nabla w,$$

and invoking the skew-symmetry in (7.2). The proof is done. ■

Note that the operator $\mathfrak{C}^2$ can be interpreted as curl curl, in any dimension d. To see this, recall that when $d = 2,3$ then the coefficients of $\operatorname{curl} w$ for a vector field $w = [w^i]_{i=1\dots d}$, coincide with the coefficients of the exterior derivative $\mathrm{d}\alpha = \sum_{i<j}\big(\partial_i w^j - \partial_j w^i\big)\mathrm{d}x_i \wedge \mathrm{d}x_j$ of the 1-form $\alpha = \sum_{i=1}^d w^i \mathrm{d}x_i$. Given a matrix field A in any dimension d, we may still apply d row-wise:

$$\mathrm{d}A = \Big[\sum_{s<t}\big(\partial_s A_{it} - \partial_t A_{is}\big)\mathrm{d}x_s \wedge \mathrm{d}x_t\Big]_{i=1\dots d},$$

returning a vector of 2-forms, and then apply d to each vector of coefficients in $\mathrm{d}A$:

$$\begin{aligned}\mathrm{d}^2 A &= \Big[\sum_{i<j}\big(\partial_i(\partial_s A_{jt} - \partial_t A_{js}) - \partial_j(\partial_s A_{it} - \partial_t A_{is})\big)\mathrm{d}x_i \wedge \mathrm{d}x_j\Big]_{s<t:1\dots d} \\ &= \Big[\sum_{i<j}\mathfrak{C}^2(A)_{ij,st}\mathrm{d}x_i \wedge \mathrm{d}x_j\Big]_{s<t:1\dots d}.\end{aligned}$$

Next, we show the equivalent solvability conditions determining the range of $\mathfrak{C}^2$:

Proof of Lemma 1.8

The implication (i)⇒(ii) follows by a direct inspection. To prove (ii)⇒(i), observe first that for a skew-symmetric matrix field $B : \omega \to \mathbb{R}^{d\times d}_{\text{skew}}$ to be of the form: $B = (\nabla w)^T - \nabla w$ for some $w = [w^i]_{i=1\dots d} : \omega \to \mathbb{R}^d$, the sufficient and necessary condition is:

$$\partial_i B_{jq} + \partial_j B_{qi} + \partial_q B_{ij} = 0 \quad \text{ for all } \; i,j,q = 1\dots d. \tag{7.3}$$

This claim follows by taking the exterior derivative of the 2-form in:

$$\begin{aligned} \mathrm{d}\Big(\sum_{j,q=1\dots d} B_{jq}\mathrm{d}x_j\wedge \mathrm{d}x_q\Big) &= \sum_{i,j,q=1\dots d}\partial_i B_{jq}\mathrm{d}x_i\wedge\mathrm{d}x_j\wedge\mathrm{d}x_q \\ &= 2\sum_{i<j<q:1\dots d}(\partial_i B_{jq}+\partial_j B_{qi}+\partial_q B_{ij})\mathrm{d}x_i\wedge\mathrm{d}x_j\wedge\mathrm{d}x_q, \end{aligned}$$

where we used the skew-symmetry assumption, and invoking Poincaré's Lemma on the contractible domain ω.

For every $s,t=1\dots d$ we apply the above criterion to $B=[F_{ij,st}]_{i,j=1\dots d}$. Since the first and third conditions in (1.9) validate the skew-symmetry of B and (7.3), we get existence of d^2 vector fields $\phi_{st}=[\phi^j_{st}]_{j=1\dots d}$ on ω, satisfying:

$$F_{ij,st}=\partial_i\phi^j_{st}-\partial_j\phi^i_{st}\quad \text{for all } \ i,j,s,t=1\dots d. \tag{7.4}$$

By the first condition in (1.9), we note that $\partial_i(\phi^j_{st}+\phi^j_{ts})-\partial_j(\phi^i_{st}+\phi^i_{ts})=0$ for all i,j,s,t, which implies that each $\phi_{st}+\phi_{ts}$ is a gradient. Thus, without loss of generality we may take:

$$\phi_{st}=-\phi_{ts}\quad \text{for all } \ s,t=1\dots d.$$

For every $t=1\dots d$ consider now the skew-symmetric matrix field $B=[\phi^j_{st}-\phi^s_{jt}]_{j,s=1\dots d}$. Condition (7.3) holds, in virtue of (7.4) and the second condition in (1.9):

$$\partial_i(\phi^j_{st}-\phi^s_{jt})+\partial_j(\phi^s_{it}-\phi^i_{st})+\partial_s(\phi^i_{jt}-\phi^j_{it})=F_{ij,st}+F_{si,jt}+F_{js,it}=0,$$

and so there follows existence of vector fields $\eta_t=[\eta^s_t]_{s=1\dots d}$ on ω, such that:

$$\phi^j_{st}-\phi^s_{jt}=\partial_j\eta^s_t-\partial_s\eta^t_j\quad \text{for all } \ j,s,t=1\dots d. \tag{7.5}$$

We now finally define:

$$A_{ij}=-\frac{1}{2}(\eta^i_j+\eta^j_i)\quad \text{for all } \ i,j=1\dots d.$$

The matrix field $A=[A_{ij}]_{i,j=1\dots d}$ is obviously symmetric, and from (7.5) and (7.4) we get:

$$\begin{aligned} 2\mathfrak{C}^2(A)_{ij,st} &= -\partial_i\partial_s(\eta^j_t+\eta^t_j)-\partial_j\partial_t(\eta^i_s+\eta^s_i)+\partial_i\partial_t(\eta^j_s+\eta^s_j)+\partial_j\partial_s(\eta^i_t+\eta^t_i) \\ &= \partial_t(\phi^i_{js}-\phi^j_{is})+\partial_j(\phi^s_{ti}-\phi^t_{si})+\partial_s(\phi^j_{it}-\phi^i_{jt})+\partial_i(\phi^t_{sj}-\phi^s_{tj}) \\ &= F_{ti,js}+F_{js,ti}+F_{tj,si}+F_{si,tj}=2(F_{ti,js}+F_{tj,si})=2F_{st,ij}=2F_{ij,st} \end{aligned}$$

for all $i,j,s,t=1\dots d$, where in the last three equalities above we used the first and second conditions in (1.9). The proof is done. ∎

Observe that for $d=3$ and $k=1$, any choice of 6 functions $F_{12,12},F_{12,13},F_{12,23},F_{13,13}$, $F_{13,23}$, $F_{23,23}\in L^2(\omega,\mathbb{R})$ gives raise to $F\in L^2(\omega,\mathbb{R}^{81})$ satisfying (1.9). Indeed, the first condition holds by defining the remaining components of F appropriately, while the second and the third conditions are implied automatically by these symmetries. In this case, (MA) consists of 6 equations in a single unknown $v\in\mathbb{R}$, while (VK) consists of 6 equations in 4 unknowns $(v,w)\in\mathbb{R}^4$. Although both formulations seem to be largely overdetermined, this paper actually shows that the set of their solutions is dense in the space of continuous functions on $\bar\omega$.

We are now ready to give:

Proof of Theorem 1.10

By the construction in Theorem 1.8, there exists a matrix field $A\in\mathcal{C}^{1,1}(\bar\omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$ such that $\mathfrak{C}^2(A)=-F$. Given $v\in\mathcal{C}^1(\bar\omega,\mathbb{R}^k)$, we apply Theorem 1.1, or Theorem 1.3 in case of codimension $k \geq 2d_*$, to $\epsilon = 1/n$ and $v,\ w = 0,\ A + C\mathrm{Id}_d$. The constant $C > 0$ is taken large enough to have, in the sense of matrix inequalities:

$$A + C\mathrm{Id}_d > \frac{1}{2}(\nabla v)^T \nabla v \quad \text{ on } \bar{\omega}.$$

The resulting $v_n = \tilde{v}$ provides the n-th member of the claimed approximating sequence for v. When $v \in \mathcal{C}^0(\bar{\omega}, \mathbb{R}^k)$, the sequence is obtained using a density argument. ■

8. Energy scaling bound for thin multidimensional films: proof of Theorem 1.11

In this section, we estimate the infimum of the energy $\mathcal{E}^h(u)$ defined in (1.11), interpreted as the averaged pointwise deficit of a weakly regular immersion u from being the orientation preserving isometric immersion of the metric g^h on Ω^h. When $d = 2$, $k = 1$ then $\mathcal{E}^h(u)$ is the elastic energy (per unit thickness) of the deformation u of a thin film with midplate ω, thickness $2h$, elastic energy density W, and the prestrain tensor g^h.

Proof of Theorem 1.11

1. Fix $\alpha \in \big(0, \frac{1}{1+s}\big)$. By Theorem 1.1, there exists $v \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^k)$ and $w \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^d)$, solving (VK) with the right hand side given by the $d \times d$ principal minor of S:

$$\frac{1}{2}(\nabla v)^T \nabla v + \mathrm{sym}\nabla w = S_{d\times d}. \tag{8.1}$$

We regularize v, w to $v_\epsilon \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^k)$, $w_\epsilon \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^d)$ convolving with the kernels $\{\phi_\epsilon(x)\}_{\epsilon\to 0}$ as in Lemma 2.5, where ϵ is a positive power t of h, to be chosen later:

$$v_\epsilon = v * \phi_\epsilon, \quad w_\epsilon = w * \phi_\epsilon, \quad \epsilon = h^t.$$

By $(2.7)_2$ and a version of $(2.7)_3$ in: $\|(fg) * \phi_\epsilon - (f * \phi_\epsilon)(g * \phi_\epsilon)\|_0 \leq C\epsilon^{2\alpha}\|f\|_{0,\alpha}\|g\|_{0,\alpha}$, we get:

$$\begin{aligned}&\Big\|\frac{1}{2}(\nabla v_\epsilon)^T \nabla v_\epsilon + \mathrm{sym}\nabla w_\epsilon - S_{d\times d}\Big\|_0\\ &\qquad \leq \Big\|\frac{1}{2}(\nabla v_\epsilon)^T \nabla v_\epsilon + \mathrm{sym}\,\nabla w_\epsilon - S_{d\times d} * \phi_\epsilon\Big\|_0 + \|S_{d\times d} * \phi_\epsilon - S_{d\times d}\|_0.\end{aligned}$$

Since $\mathrm{sym}\,\nabla w_\epsilon - S_{d\times d} * \phi_\epsilon = -\frac{1}{2}\big((\nabla v)^T \nabla v\big) * \phi_\epsilon$, this leads to:

$$\Big\|\frac{1}{2}(\nabla v_\epsilon)^T \nabla v_\epsilon + \mathrm{sym}\nabla w_\epsilon - S_{d\times d}\Big\|_0 \leq C\epsilon^{2\alpha}\|\nabla v\|_{0,\alpha}^2 + C\epsilon^2\|\nabla^2 S_{d\times d}\|_0 \leq C\epsilon^{2\alpha}. \tag{8.2}$$

Further, by $(2.7)_1$ and a version of $(2.7)_2$ in: $\|\nabla(f - f * \phi_\epsilon)\|_0 \leq C\epsilon^{\alpha-1}\|f\|_{0,\alpha}$, we obtain:

$$\|\nabla v_\epsilon\|_0 + \|\nabla w_\epsilon\|_0 \leq C, \qquad \|\nabla^2 v_\epsilon\|_0 + \|\nabla^2 w_\epsilon\|_0 \leq C\epsilon^{\alpha-1}. \tag{8.3}$$

2. Denote $\delta = \gamma/2$ and define $u^h \in \mathcal{C}^\infty(\bar{\Omega}^h, \mathbb{R}^{d+k})$ as follows:

$$\begin{aligned}u^h(x,z) = id_{d+k} &+ h^{\delta/2}\begin{bmatrix} 0 \\ v_\epsilon \end{bmatrix} + h^\delta \begin{bmatrix} w_\epsilon \\ 0 \end{bmatrix}\\ &+ \Big(h^{\delta/2}\begin{bmatrix} -(\nabla v_\epsilon)^T \\ 0 \end{bmatrix} + h^\delta \begin{bmatrix} 2S_{d\times k} \\ S_{k\times k} - \frac{1}{2}(\nabla v_\epsilon)(\nabla v_\epsilon)^T \end{bmatrix} + h^{3\delta/2}B(x)\Big)z\end{aligned}$$

where we denote: $S = \left[\begin{array}{c|c} S_{d\times d} & S_{d\times k} \\ \hline S_{k\times d} & S_{k\times k} \end{array}\right]$, and where the higher order correction field $B \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^{(d+k)\times k})$ is given by:

$$B(x) = \left[\begin{array}{c} -(\nabla v_\epsilon)^T S_{k\times k} + \frac{1}{2}(\nabla v_\epsilon)^T(\nabla v_\epsilon)(\nabla v_\epsilon)^T + (\nabla w_\epsilon)^T(\nabla v_\epsilon)^T \\ \hline 2\mathrm{sym}\big((\nabla v_\epsilon)S_{d\times k}\big) \end{array}\right].$$

It follows that for all $x \in \bar\omega$ and $z \in B(0,1) \subset \mathbb{R}^k$ there holds:

$$\begin{aligned} \nabla u^h(x,hz) = {} & \mathrm{Id}_{d+k} + h^{\delta/2}\left[\begin{array}{c|c} 0 & -(\nabla v_\epsilon)^T \\ \hline \nabla v_\epsilon & 0 \end{array}\right] + h^\delta \left[\begin{array}{c|c} \nabla w_\epsilon & 2S_{d\times k} \\ \hline 0 & S_{k\times k} - \frac{1}{2}(\nabla v_\epsilon)(\nabla v_\epsilon)^T \end{array}\right] \\ & + h^{3\delta/2}\left[\begin{array}{c|c} 0 & B \end{array}\right] - h^{1+\delta/2}\left[\begin{array}{c|c} \big[\langle \partial_i\partial_j v_\epsilon, z\rangle\big]_{i,j=1\ldots d} & 0 \\ \hline 0 & 0 \end{array}\right] \\ & + O\big(h^{1+\delta}\big)(1 + \|\nabla^2 v_\epsilon\|_0) + O(h^{1+3\delta/2})\|\nabla^2 w_\epsilon\|_0. \end{aligned}$$

We now observe that: $(g^h)^{-1/2} = \mathrm{Id}_{d+k} - h^\delta S + O(h^{2\delta})$, and proceed with computing:

$$\begin{aligned} \big(\nabla u^h (g^h)^{-1/2}\big)(x,hz) = {} & \mathrm{Id}_{d+k} + P^h + h^\delta \left[\begin{array}{c|c} \nabla w_\epsilon - S_{d\times d} & 0 \\ \hline 0 & -\frac{1}{2}(\nabla v_\epsilon)(\nabla v_\epsilon)^T \end{array}\right] \\ & + h^{3\delta/2}\left[\begin{array}{c|c} (\nabla v_\epsilon)^T S_{k\times d} & \frac{1}{2}(\nabla v_\epsilon)(\nabla v_\epsilon)^T \nabla v_\epsilon + (\nabla w_\epsilon)^T(\nabla v_\epsilon)^T \\ \hline -(\nabla v_\epsilon)S_{d\times d} & S_{k\times d}(\nabla v_\epsilon)^T \end{array}\right] \\ & - h^{1+\delta/2}\left[\begin{array}{c|c} \big[\langle \partial_i\partial_j v_\epsilon, z\rangle\big]_{i,j=1\ldots d} & 0 \\ \hline 0 & 0 \end{array}\right] \\ & + O(h^{2\delta}) + O(h^{1+\delta})(1 + \|\nabla^2 v_\epsilon\|_0 + \|\nabla^2 w_\epsilon\|_0). \end{aligned}$$

Above, we used the following skew-symmetric matrix field:

$$P^h = \left[\begin{array}{c|c} 0 & p^h \\ \hline -(p^h)^T & 0 \end{array}\right], \qquad p^h = -h^{\delta/2}(\nabla v_\epsilon)^T + h^\delta S_{d\times k}.$$

For future purpose, it is convenient to compute:

$$(P^h)^2 = -h^\delta \left[\begin{array}{c|c} (\nabla v_\epsilon)^T \nabla v_\epsilon & 0 \\ \hline 0 & (\nabla v_\epsilon)(\nabla v_\epsilon)^T \end{array}\right] + 2h^{3\delta/2}\mathrm{sym}\left[\begin{array}{c|c} (\nabla v_\epsilon)^T S_{k\times d} & 0 \\ \hline 0 & (\nabla v_\epsilon)S_{d\times k} \end{array}\right] + O(h^{2\delta}).$$

3. Consider the rotation fields $Q^h \in \mathcal{C}^\infty(\bar\omega, \mathrm{SO}(d+k))$, defined by:

$$Q^h = \exp(-P^h) = \mathrm{Id}_{d+k} - P^h + \frac{1}{2}(P^h)^2 - \frac{1}{6}(P^h)^3 + O(h^{2\delta}).$$

Then we get:

$$\begin{aligned}\big(Q^h\nabla u^h(g^h)^{-1/2}\big)(x,hz) = {} & \mathrm{Id}_{d+k} + h^\delta \left[\begin{array}{c|c}\frac{1}{2}(\nabla v_\epsilon)^T\nabla v_\epsilon + \nabla w_\epsilon - S_{d\times d} & 0\\ \hline 0 & 0\end{array}\right] \\ & + h^{3\delta/2}\left[\begin{array}{c|c}\mathrm{skew}\big((\nabla v_\epsilon)^T S_{k\times d}\big) & (\nabla w_\epsilon)^T(\nabla v_\epsilon)^T \\ \hline -(\nabla v_\epsilon)\nabla w_\epsilon & \mathrm{skew}\big(S_{k\times d}(\nabla v_\epsilon)^T\big)\end{array}\right] + \frac{1}{3}(P^h)^3 \\ & - h^{1+\delta/2}\left[\begin{array}{c|c}\big[\langle\partial_i\partial_j v_\epsilon, z\rangle\big]_{i,j=1\ldots d} & 0\\ \hline 0 & 0\end{array}\right] \\ & + O(h^{2\delta}) + O(h^{1+\delta})(1+\|\nabla^2 v_\epsilon\|_0 + \|\nabla^2 w_\epsilon\|_0).\end{aligned}$$

Finally, we apply another rotation field $\bar Q^h \in \mathcal{C}^\infty(\bar\omega, \mathrm{SO}(d+k))$:

$$\bar Q^h = \exp(-\bar P^h) = \mathrm{Id}_{d+k} - \bar P^h + O(h^{2\delta}),$$

$$\text{where}\ \ \bar P^h = \left[\begin{array}{c|c}\mathrm{skew}\big(h^\delta\nabla w_\epsilon + h^{3\delta/2}(\nabla v_\epsilon)^T S_{k\times d}\big) & h^{3\delta/2}(\nabla w_\epsilon)^T(\nabla v_\epsilon)^T \\ \hline -h^{3\delta/2}(\nabla v_\epsilon)\nabla w_\epsilon & h^{3\delta/2}\mathrm{skew}\big(S_{k\times d}(\nabla v_\epsilon)^T\big)\end{array}\right] + \frac{1}{3}(P^h)^3,$$

to get:

$$\begin{aligned}\big(\bar Q^h Q^h\nabla u^h(g^h)^{-1/2}\big)(x,hz) = {} & \mathrm{Id}_{d+k} + h^\delta \left[\begin{array}{c|c}\frac{1}{2}(\nabla v_\epsilon)^T\nabla v_\epsilon + \mathrm{sym}\nabla w_\epsilon - S_{d\times d} & 0\\ \hline 0 & 0\end{array}\right] \\ & - h^{1+\delta/2}\left[\begin{array}{c|c}\big[\langle\partial_i\partial_j v_\epsilon, z\rangle\big]_{i,j=1\ldots d} & 0\\ \hline 0 & 0\end{array}\right] \\ & + O(h^{2\delta}) + O(h^{1+\delta})(1+\|\nabla^2 v_\epsilon\|_0 + \|\nabla^2 w_\epsilon\|_0).\end{aligned}$$

4. In conclusion, we obtain the following energy bound below, valid provided that we may use Taylor's expansion of W up to second order in perturbation of Id_{d+k}, which here holds when $h^{1+\delta/2}\big(\|\nabla^2 v_\epsilon\|_0 + \|\nabla^2 w_\epsilon\|_0\big) \to 0$ as $h\to 0$, implied by $\lim_{h\to 0}\big(h^{1+\delta/2}\epsilon^{\alpha-1}\big) = 0$:

$$\begin{aligned}\inf \mathcal{E}^h & \leq \mathcal{E}^h(u^h) = \fint_{\Omega^1} W\Big(\bar Q^h Q^h\nabla u^h(g^h)^{-1/2}(x,hz)\Big)\,\mathrm{d}(x,z) \\ & \leq C\fint_{\Omega^1}\Big(h^{2\delta}\big|\frac{1}{2}(\nabla v_\epsilon)^T\nabla v_\epsilon + \mathrm{sym}\nabla w_\epsilon - S_{d\times d}\big|^2 + h^{2+\delta}(\|\nabla^2 v_\epsilon\|_0^2 + \|\nabla^2 w_\epsilon\|_0^2) + h^{4\delta}\Big)\,\mathrm{d}(x,z).\end{aligned}$$

Recalling (8.2) and (8.3), the obtained bound further leads to:

$$\inf\mathcal{E}^h \leq C\big(h^{2\delta}\epsilon^{4\alpha} + h^{2+\delta}\epsilon^{2\alpha-2} + h^{4\delta}\big) = C\big(h^{2\delta+4\alpha t} + h^{2+\delta+(2\alpha-2)t} + h^{4\delta}\big).$$

Minimizing the right hand side above is equivalent to maximizing the minimal of the three exponents. For $\delta < 2$, we hence choose the exponent t in $\epsilon = h^t$ so that $2\delta + 4\alpha t = 2+\delta+(2\alpha-2)t$, namely $t = \frac{2-\delta}{2\alpha+2}$. Consequently, we get:

$$\inf\mathcal{E}^h \leq C\big(h^{2\frac{\delta+2\alpha}{\alpha+1}} + h^{4\delta}\big) \leq C\big(h^{\left(4+2\frac{(1+s)(\delta-2)}{2+s}\right)-} + h^{4\delta}\big)$$

upon recalling the range of admissible exponents α. On the other hand, when $\delta\geq 2$, then we choose t close to 0. The conclusion of Theorem 1.11 follows by a direct inspection. ∎

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