NON-LOCAL TUG-OF-WAR WITH NOISE FOR THE GEOMETRIC FRACTIONAL p-LAPLACIAN

MARTA LEWICKA

Abstract. This paper concerns the fractional $\mathbf{p}$-Laplace operator $\Delta_{\mathbf{p}}^s$ in non-divergence form, which has been introduced in [2]. For any $\mathbf{p}\in[2,\infty)$ and $s\in(\frac{1}{2},1)$ we first define two families of non-local, non-linear averaging operators, parametrised by ε and defined for all bounded, Borel functions $u:\mathbb{R}^N\to\mathbb{R}$. We prove that $\Delta_{\mathbf{p}}^s u(x)$ emerges as the ε^{2s}-order coefficient in the expansion of the deviation of each ε-average from the value $u(x)$, in the limit of the domain of averaging exhausting an appropriate cone in $\mathbb{R}^N$ at the rate $\varepsilon\to 0$.

Second, we consider the ε-dynamic programming principles modeled on the first average, and show that their solutions converge uniformly as $\varepsilon\to 0$, to viscosity solutions of the homogeneous non-local Dirichlet problem for $\Delta_{\mathbf{p}}^s$, when posed in a domain $\mathcal{D}$ that satisfies the external cone condition and subject to bounded, uniformly continuous data on $\mathbb{R}^N\setminus\mathcal{D}$.

Finally, we interpret such ε-approximating solutions as values to the non-local Tug-of-War game with noise. In this game, players choose directions while the game position is updated randomly within the infinite cone that aligns with the specified direction, whose aperture angle depends on $\mathbf{p}$ and N, and whose ε-tip has been removed.

1. Introduction

This paper concerns a version of the fractional $\mathbf{p}$-Laplace operator, which has been introduced in [2]. More precisely, for $\mathbf{p}\geq 2$, $s\in(\frac{1}{2},1)$, and for a given bounded function $u:\mathbb{R}^N\to\mathbb{R}$ that is of regularity $C^{1,1}(x)$ with $\nabla u(x)\neq 0$, one defines:

$$\Delta_{\mathbf{p}}^s u(x) \doteq C_{N,\mathbf{p},s}\int_{T_{\mathbf{p}}^{0,\infty}(\frac{\nabla u(x)}{|\nabla u(x)|})} \frac{u(x+z)+u(x-z)-2u(x)}{|z|^{N+2s}}\,\mathrm{d}z. \tag{1.1}$$

Above, $C_{N,\mathbf{p},s}$ is a specific constant depending on $N,\mathbf{p},s$, whereas the integration occurs on the infinite cone $T_{\mathbf{p}}^{0,\infty}(\frac{\nabla u(x)}{|\nabla u(x)|})\subset\mathbb{R}^N$ whose centerline is aligned with the vector $\frac{\nabla u(x)}{|\nabla u(x)|}$ and whose aperture angle α depends on $N,\mathbf{p}$. In particular, for $\mathbf{p}=2$ we have $\alpha=\frac{\pi}{2}$ so that the said cone becomes the half-space and (1.1) is consistent with the familiar formula: $-(-\Delta)^s u(x) = C_{N,s}\int_{\mathbb{R}^N}\frac{u(z)-u(x)}{|x-z|^{N+2s}}\,\mathrm{d}z$. On the other hand, when $\mathbf{p}\to\infty$ then $\alpha\to 0$ and the cone reduces to a line, consistently with the parallel definition for fractional infinity Laplacian $\Delta_\infty^s u(x)$ in [1].

As pointed out in [2], definition (1.1) arises naturally when extending the game-theoretical interpretation to the non-local, non-divergence version of the classical $\mathbf{p}$-Laplace operator $\Delta_{\mathbf{p}}$. The interpretation for $\Delta_{\mathbf{p}}$ has been originally put forward in [12] and it is based on the Tug-of-War game with random noise, which in its turn can be seen as the interpolation between the pure Tug-of-War developed for the ∞-Laplacian Δ_∞ in [11], and the random walk description of the linear harmonic operator Δ, which is classical. In order to emphasise the importance of the choice of the integration cone $T_{\mathbf{p}}^{0,\infty}$ and to distinguish the formula (1.1) from the divergence form

Marta Lewicka was supported by NSF grants DMS-1613153 and DMS-2006439.

of the fractional **p**-Laplacian arising through the Euler-Lagrange equations of an appropriate non-local energy [5], we call the operator $\Delta_{\mathbf{p}}^s$ above the "geometric" **p**-s-Laplacian.

The purpose of this paper is to rigorously define the non-local version of the noisy Tug-of-War game and prove that its values converge to viscosity solutions of the Dirichlet problem for $\Delta_{\mathbf{p}}^s$, posed on a sufficiently regular domain $\mathcal{D}\subset\mathbb{R}^N$:

$$\Delta_{\mathbf{p}}^s u = 0 \ \text{ in } \ \mathcal{D}, \qquad u = F \ \text{ in } \ \mathbb{R}^N\setminus\mathcal{D}. \tag{1.2}$$

We remark that condition $\mathbf{p}\geq 2$ which we assume throughout, can be relaxed to cover the full range $\mathbf{p}\in(1,\infty)$, by replacing the cone $T_{\mathbf{p}}$ with the complement of its doubled version for $\mathbf{p}\in(1,2)$. This construction has been proposed in [2, Remark 4.5]. We now describe our main results. The said game will be modeled on the dynamic programming principle that involves an appropriate average, in whose asymptotic expansion the operator $\Delta_{\mathbf{p}}^s$ arises as the highest order term, in the vanishing limit of the expansion parameter ε. Hence, our first set of results develops such asymptotic expansions, reminiscent of the well known local and linear formula:

$$\fint_{B_\varepsilon(x)} u(y)\,\mathrm{d}y = u(x) + \frac{\varepsilon^2}{2(N+2)}\Delta u(x) + o(\varepsilon^2) \qquad \text{as } \ \varepsilon\to 0+\,. \tag{1.3}$$

1.1. **Asymptotic expansions.** More precisely, we define the following non-local and non-linear averaging operator:

$$\mathcal{A}_\varepsilon u(x) \doteq \frac{1}{2}\Big(\sup_{|y|=1}\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}\frac{u(x+z)}{|z|^{N+2s}}\,\mathrm{d}z + \inf_{|y|=1}\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}\frac{u(x+z)}{|z|^{N+2s}}\,\mathrm{d}z\Big),$$

where the integration takes place on the truncated infinite cones $T_{\mathbf{p}}^{\varepsilon,\infty}(y) = T_{\mathbf{p}}^{0,\infty}(y)\setminus B_\varepsilon(0)$, each oriented along its indicated unit direction vector y and having the aperture angle α as in (1.1). The integral averages $\fint$ are taken with respect to the singular measure $|z|^{-N-2s}\,\mathrm{d}z$. Note that $\mathcal{A}_\varepsilon u$ is well defined for any bounded, Borel function u, and in particular it does not necessitate the existence or the knowledge of $\nabla u(x)$, which was essential in (1.1). The form of $\mathcal{A}_\varepsilon$ is justified by the following expansion, which we prove to be valid for functions u that are C^2 in the vicinity of a given $x\in\mathbb{R}^N$ with $\nabla u(x)\neq 0$, and uniformly continuous away from x:

$$\mathcal{A}_\varepsilon u(x) = u(x) + \frac{s}{(2-2s)(N+\mathbf{p}-2)}\varepsilon^{2s}\cdot\Delta_{\mathbf{p}}^s u(x) + o(\varepsilon^{2s}) \qquad \text{as } \ \varepsilon\to 0+\,. \tag{1.4}$$

We also propose another nonlinear average of a combined local - non-local nature:

$$\begin{aligned}\bar{\mathcal{A}}_\varepsilon u(x) \doteq\ & \frac{(1-s)(N+\mathbf{p}-2)}{N+\mathbf{p}-2+2s}\cdot\mathcal{A}_\varepsilon u(x)\\ & + \frac{s(N+2)}{N+\mathbf{p}-2+2s}\cdot\fint_{B_\varepsilon(x)}u(y)\,\mathrm{d}y + \frac{s(\mathbf{p}-2)}{N+\mathbf{p}-2+2s}\cdot\frac{1}{2}\Big(\sup_{B_\varepsilon(x)}u + \inf_{B_\varepsilon(x)}u\Big).\end{aligned}$$

Note that the three positive multiplication factors above add up to 1. We prove that the result as in (1.4) similarly holds for $\bar{\mathcal{A}}_\varepsilon$:

$$\bar{\mathcal{A}}_\varepsilon u(x) = u(x) + \frac{s}{2(N+\mathbf{p}-2+2s)}\varepsilon^{2s}\cdot\Delta_{\mathbf{p}}^s u(x) + o(\varepsilon^{2s}) \qquad \text{as } \ \varepsilon\to 0+\,. \tag{1.5}$$

Expansion (1.5) is superior to (1.4), because the error quantity $o(\varepsilon^{2s})$ in (1.4), which we make precise in the paper, blows up to ∞ as $s\to 1-$, whereas $o(\varepsilon^{2s})$ in (1.5) is uniform in the whole considered range $s \in (\frac{1}{2}, 1)$. When $s \to 1$, the expansion (1.5) becomes:

$$\begin{aligned} \frac{N+2}{N+\mathbf{p}} \cdot \fint_{B_\varepsilon(x)} u(y)\,\mathrm{d}y + \frac{\mathbf{p}-2}{N+\mathbf{p}} \cdot \frac{1}{2}\big(\sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u\big) \\ = u(x) + \frac{\varepsilon^2}{2(N+\mathbf{p})} |\nabla u(x)|^{2-\mathbf{p}} \Delta_{\mathbf{p}} u(x) + o(\varepsilon^2), \end{aligned} \tag{1.6}$$

which in turn yields (1.3) for $\mathbf{p} = 2$. We recall in passing that (1.6) is a convex combination of (1.3) and the asymptotic expansion for the infinity Laplacian in:

$$\frac{1}{2}\big(\sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u\big) = u(x) + \frac{\varepsilon^2}{2} \Delta_\infty u(x) + o(\varepsilon^2) \qquad \text{as } \varepsilon \to 0+,$$

with the weights corresponding to the following identity for the classical **p**-Laplacian in non-divergence form: $|\nabla u|^{2-\mathbf{p}} \Delta_{\mathbf{p}} u = \Delta u + (\mathbf{p}-2)\Delta_\infty u$.

Asymptotic expansions for gradient-dependent operators have been recently discussed in [3, 4]. However, the averages in there depended on $\nabla u(x)$, which is a drawback in the context of our further applications, based on solutions to the truncated expansions $(\mathrm{DPP})_\varepsilon$. We seek these solutions among the natural class of Borel functions. Indeed, they are at most continuous and become higher regular only generically and in the limit as $\varepsilon \to 0$, so no notion of pointwise gradient may be feasible in the definition of an average.

The idea of the local correction in the average $\bar{\mathcal{A}}_\varepsilon$, and of the expansion (1.4) with no reference to the gradient, first appeared in [7] in the context of the fractional ∞-Laplacian Δ^s_∞. We also observe that the case $\mathbf{p} = \infty$ where $\alpha = 0$ in $T^{0,\infty}_{\mathbf{p}}$, is independent and cannot be deduced from the present work. Expansion (1.6) when $\Delta_{\mathbf{p}} u = 0$, and the related characterisation of **p**-harmonic functions in the viscosity sense, have been studied in [10]. This expansion informs a game-theoretical interpretation of the **p**-Laplacian (alternative to the one originally carried out in [12]) only for $\mathbf{p} \geq 2$, when the weight coefficients are nonnegative. Another expansion, yielding a family of Tug-of-War games in the whole range $\mathbf{p} \in (1, \infty)$, was proposed in [8].

1.2. **Dynamic programming and Tug-of-War.** The second set of results in this paper concerns the operator $\mathcal{A}_\varepsilon$ and the truncated version of the expansion (1.4), aiming at an approximation scheme for solutions to (1.2). More precisely, given an open bounded domain $\mathcal{D} \subset \mathbb{R}^N$ and a bounded Borel data function $F : \mathbb{R}^N \setminus \mathcal{D} \to \mathbb{R}$, we consider the following family of non-local averaging problems:

$$u_\varepsilon(x) = \begin{cases} \mathcal{A}_\varepsilon u_\varepsilon(x) & \text{for } x \in \mathcal{D} \\ F(x) & \text{for } x \in \mathbb{R}^N \setminus \mathcal{D}. \end{cases} \tag{DPP$_\varepsilon$}$$

We prove that for every $\varepsilon > 0$ there exists exactly one u_ε satisfying the above, which is bounded Borel on $\mathbb{R}^N$ (and continuous in $\mathcal{D}$). We then show, for $\mathcal{D}$ satisfying the exterior cone condition and for uniformly continuous F, that any sequence $\{u_\varepsilon\}_{\varepsilon \to 0}$ has a further subsequence converging uniformly in $\mathbb{R}^N$ to a continuous limit u that is a viscosity solution to (1.2). To this end, each $u_\varepsilon(x)$ is shown to be the value of the following zero-sum two-players game, which is a non-local version of the Tug-of-War with noise introduced in [12].

In this game, each Player chooses a unit direction vector according to their own strategy, based on the knowledge of all prior moves and random outcomes. With equal probabilities, direction from Player 1 or Player 2 is picked; this resulting direction is called y. The current game position x_n is then updated to a next position x_{n+1} within the shifted and truncated cone $x_n + T^{\varepsilon,\infty}_{\mathbf{p}}(y)$, randomly according to the probability-normalisation of the measure $|z|^{-N-2s}\mathrm{d}z$

on $T_{\mathbf{p}}^{\varepsilon,\infty}(y)$. Such process, started at a point $x_0 \in \mathbb{R}^N$ is stopped the first time when $x_n \notin \mathcal{D}$, whereas Player 1 collects from their opponent the payoff given by the value $F(x_n)$. We show that the expected value of the payoff, under condition that both Players play optimally, has the min-max property, yielding the solution u_ε to the dynamic programming principle $(\mathrm{DPP})_\varepsilon$.

Convergence as $\varepsilon \to 0$ is obtained by showing the approximate equicontinuity of the family $\{u_\varepsilon\}_{\varepsilon\to 0}$, for which the sufficient condition is expressed via "game-regularity" of the boundary points. This general condition, implied in particular by the exterior cone condition on $\partial\mathcal{D}$, is similar in spirit to the celebrated Doob's boundary regularity criterion for Brownian motion. The order of arguments follows then a general program in the context of Tug-of-War games (see a recent textbook [9]), which has been put forward in [11] and which has so far yielded results for **p**-Laplacian, obstacle problems, subriemannian geometries and time-dependent problems. The fact that this program can be carried out in the present non-local setting, is not obvious, and it is another main result of this work.

1.3. **Outline of the paper.** We set the notation and introduce the main integral operators in section 2. The non-local asymptotic expansions (1.4) and (1.5), together with precise bounds on their error terms, are proved in sections 3 and 4, respectively. The dynamic programming principles $(\mathrm{DPP})_\varepsilon$ are discussed in section 5. The fact that the uniform limits of their solutions $\{u_\varepsilon\}_{\varepsilon\to 0}$ are automatically viscosity solutions to (1.2), is shown in section 6. The non-local Tug-of-War game is defined and proved to yield solutions u_ε in section 7. Proofs of the asymptotic equicontinuity and game-regularity are carried out in sections 8 and 9, where we rely on further analysis of a barrier function from [2].

2. The fractional quotients and the fractional **p**-Laplacian

We consider the following measure on the Borel subsets of $\mathbb{R}^N$:

$$\mathrm{d}\mu_s^N(z) \doteq \frac{C(N,s)}{|z|^{N+2s}}\,\mathrm{d}z \quad \text{ where } \ C(N,s) = \frac{4^s s\Gamma\big(\frac{N}{2}+s\big)}{\pi^{N/2}\Gamma\big(1-s\big)} = \Big(\int_{\mathbb{R}^N} \frac{1-\cos\langle z, e_1\rangle}{|z|^{N+2s}}\,\mathrm{d}z\Big)^{-1},$$

where the exponent s in this paper is assumed to belong to the range:

$$s \in \big(\frac{1}{2}, 1\big).$$

One can show [6] that $C(N,s) = s(1-s)c_{N,s}$ where $c_{N,s}$ is bounded and positive uniformly in s. The role of the normalizing constant $C(N,s)$ is to ensure that the operator $-(-\Delta)^s$, given by: $-(-\Delta)^s u(x) \doteq -\int_{\mathbb{R}^N} u(x+z) + u(x-z) - 2u(x)\,\mathrm{d}\mu_s^N(z)$, is a pseudo-differential operator with symbol $|\xi|^{2s}$.

Definition 2.1. Fix $\mathbf{p} \in [2,\infty)$. We define the infinite cone $T_{\mathbf{p}}$ and the spherical cup $A_{\mathbf{p}}$:

$$T_{\mathbf{p}} \doteq \big\{z \in \mathbb{R}^N;\ \angle(e_1, z) < \alpha_{\mathbf{p}}\big\}, \qquad A_{\mathbf{p}} \doteq T_{\mathbf{p}} \cap \{|z| = 1\},$$

where $\alpha_{\mathbf{p}} \in (0, \frac{\pi}{2}]$ is the angle such that:

$$\mathbf{p} - 1 = \frac{\int_{A_{\mathbf{p}}} \langle z, e_1\rangle^2\,\mathrm{d}\sigma(z)}{\int_{A_{\mathbf{p}}} \langle z, e_2\rangle^2\,\mathrm{d}\sigma(z)}. \tag{2.1}$$

For every $0 \le a \le b \le \infty$ and $|y| = 1$, we also have the truncated cones:

$$T_{\mathbf{p}}^{a,b}(y) \doteq \big\{z \in \mathbb{R}^N;\ \angle(y, z) < \alpha_{\mathbf{p}} \ \text{ and } \ a < |z| < b\big\}, \qquad T_{\mathbf{p}}^{a,b} \doteq T_{\mathbf{p}}^{a,b}(e_1).$$

Further, for two unit vectors $y \neq \tilde{y}$ we define the rotation $R_{\tilde{y},y} \in SO(N)$ as the unique orientation preserving rotation, in plane spanned by $y, \tilde{y}$, and such that $R_{\tilde{y},y} y = \tilde{y}$. When $y = \tilde{y}$, we set: $R_{y,y} = Id_N$. Note that: $T_{\mathbf{p}}^{a,b}(y) = R_{y,e_1} T_{\mathbf{p}}^{a,b}$ and $T_{\mathbf{p}} = T_{\mathbf{p}}^{0,\infty} = T_{\mathbf{p}}^{0,\infty}(e_1)$.

The well posedness of this definition and its rationale will be explained in Lemma 2.5.

Definition 2.2. Fix an exponent $\mathbf{p} \in [2,\infty)$. Given a bounded, Borel function $u : \mathbb{R}^N \to \mathbb{R}$, we define the following family of integral operators, parametrised by $\varepsilon > 0$:

$$\mathcal{L}_{s,\mathbf{p}}^{\varepsilon}[u](x) \doteq \sup_{|y|=1} \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)} u(x+z) - u(x) \, \mathrm{d}\mu_s^N(z) + \inf_{|y|=1} \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)} u(x+z) - u(x) \, \mathrm{d}\mu_s^N(z).$$

When additionally $u \in C^{1,1}(x)$ and the corresponding gradient-like vector $p_x \neq 0$, we have:

$$\mathcal{L}_{s,\mathbf{p}}[u](x) \doteq \int_{T_{\mathbf{p}}^{0,\infty}(\frac{p_x}{|p_x|})} L_u(x,z,z) \, \mathrm{d}\mu_s^N(z),$$

where for each $x, z, \tilde{z} \in \mathbb{R}^N$ we set:

$$L_u(x,z,\tilde{z}) \doteq u(x+z) + u(x-\tilde{z}) - 2u(x).$$

Remark 2.3. Recall that $u \in C^{1,1}(x)$ provided that there are $p_x \in \mathbb{R}^N$ and $C_x, r_x > 0$ with:

$$\big|u(x+z) - u(x) - \langle p_x, z\rangle\big| \leq C_x |z|^2 \qquad \text{for all } \ |z| < r_x. \tag{2.2}$$

One immediate consequence of (2.2) is that:

$$\big|L_u(x,z,\tilde{z}) - \langle p_x, z - \tilde{z}\rangle\big| \leq C_x\big(|z|^2 + |\tilde{z}|^2\big) \qquad \text{for all } \ |z|, |\tilde{z}| < r_x. \tag{2.3}$$

Also, when $u \in C^2(\bar{B}_{r_x})$, where B_{r_x} denotes the open ball centered at x and with radius r_x, then condition (2.2) holds automatically with $p_x = \nabla u(x)$ and $C_x = \frac{1}{2}\|\nabla^2 u\|_{L^\infty(B_{r_x})}$.

Since $\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty}) < \infty$, each integral $\int_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)} u(x+z) \, \mathrm{d}\mu_s^N(z)$ and consequently $\mathcal{L}_{s,\mathbf{p}}^{\varepsilon}[u](x)$ are well defined and finite for any bounded, Borel u. On the other hand, $\mu_s^N(T_{\mathbf{p}}) = \infty$, so a corresponding formulation $\mathcal{L}_{s,\mathbf{p}}^0$ is in general not valid. We now observe:

Proposition 2.4. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a bounded, Borel function. Then:*

$$|\mathcal{L}_{s,\mathbf{p}}^{\varepsilon}[u](x)| \leq 2C(N,s)|A_{\mathbf{p}}| \cdot \frac{\|u\|_{L^\infty}}{s\varepsilon^{2s}} \quad \text{for all } x \in \mathbb{R}^N.$$

If moreover $u \in C^{1,1}(x)$, then $\mathcal{L}_{s,\mathbf{p}}^{\varepsilon}[u](x)$ are uniformly bounded in ε and:

$$|\mathcal{L}_{s,\mathbf{p}}^{\varepsilon}[u](x)|, \ |\mathcal{L}_{s,\mathbf{p}}[u](x)| \leq C(N,s)|A_{\mathbf{p}}| \cdot \Big(\frac{C_x r_x^{2-2s}}{1-s} + \frac{2\|u\|_{L^\infty}}{s r_x^{2s}}\Big).$$

Proof. The first claim is self-evident, because: $\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty}) = C(N,s)\int_\varepsilon^\infty \frac{t^{N-1}|A_{\mathbf{p}}|}{t^{N+2s}} \, \mathrm{d}t = C(N,s)\frac{|A_{\mathbf{p}}|}{2s\varepsilon^{2s}}$. For the second claim, by changing variables we deduce that:

$$\mathcal{L}_{s,\mathbf{p}}^{\varepsilon}[u](x) = \sup_{|y|=1} \inf_{|\tilde{y}|=1} \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)} u(x+z) + u(x - R_{\tilde{y},y} z) - 2u(x) \, \mathrm{d}\mu_s^N(z).$$

Then, by (2.3) we get, for any $|y|=|\tilde y|=1$:

$$\begin{split} \Big|\int_{T_{\mathbf p}^{\varepsilon,\infty}(y)} u(x+z)+ u(x-R_{\tilde y,y}z) - 2u(x)\;\mathrm{d}\mu_s^N(z) - \int_{T_{\mathbf p}^{\varepsilon,r_x}(y)}\langle p_x, z-R_{\tilde y,y}z\rangle\;\mathrm{d}\mu_s^N(z)\Big| \\ \leq \int_{T_{\mathbf p}^{\varepsilon,r_x}(y)} 2C_x|z|^2\;\mathrm{d}\mu_s^N(z) + \int_{T_{\mathbf p}^{r_x,\infty}(y)} 4\|u\|_{L^\infty}\;\mathrm{d}\mu_s^N(z) \\ = C(N,s)|A_{\mathbf p}|\cdot\Big(\frac{C_x r_x^{2-2s}}{1-s}+\frac{2\|u\|_{L^\infty}}{s r_x^{2s}}\Big). \end{split}$$

On the other hand:

$$\begin{split} \sup_{|y|=1}\inf_{|\tilde y|=1}\int_{T_{\mathbf p}^{\varepsilon,r_x}(y)}\langle p_x, z-R_{\tilde y,y}z\rangle\;\mathrm{d}\mu_s^N(z) \\ = \sup_{|y|=1}\inf_{|\tilde y|=1}\Big\langle p_x, \int_{T_{\mathbf p}^{\varepsilon,r_x}(y)} z\;\mathrm{d}\mu_s^N(z) - \int_{T_{\mathbf p}^{\varepsilon,r_x}(\tilde y)} z\;\mathrm{d}\mu_s^N(z)\Big\rangle = 0. \end{split}$$

This results in:

$$\begin{split} |\mathcal{L}_{s,\mathbf p}^\varepsilon[u](x)| = \Big|\mathcal{L}_{s,\mathbf p}^\varepsilon[u](x) - \sup_{|y|=1}\inf_{|\tilde y|=1}\int_{T_{\mathbf p}^{\varepsilon,r_x}(y)}\langle p_x, z-R_{\tilde y,y}z\rangle\;\mathrm{d}\mu_s^N(z)\Big| \\ \leq \sup_{|y|=|\tilde y|=1}\Big|\int_{T_{\mathbf p}^{\varepsilon,\infty}(y)} u(x+z)+u(x-R_{\tilde y,y}z)-2u(x)\;\mathrm{d}\mu_s^N(z) - \int_{T_{\mathbf p}^{\varepsilon,r_x}(y)}\langle p_x, z-R_{\tilde y,y}z\rangle\;\mathrm{d}\mu_s^N(z)\Big|, \end{split}$$

ending the proof of the bound for $|\mathcal{L}_{s,\mathbf p}^\varepsilon[u](x)|$. The statement for $|\mathcal{L}_{s,\mathbf p}[u](x)|$ follows similarly. ■

We close this section by noting some useful identities:

Lemma 2.5. *For every $\mathbf p\in[2,\infty)$ there exists $\alpha_{\mathbf p}\in(0,\frac{\pi}{2}]$ such that (2.1) holds. Moreover:*

(i) $\displaystyle\int_{A_{\mathbf p}}\langle z,e_2\rangle^2\;\mathrm{d}\sigma(z) = \frac{|A_{\mathbf p}|}{N+\mathbf p-2}$, *and* $\displaystyle\int_{A_{\mathbf p}}\langle z,e_1\rangle^2\;\mathrm{d}\sigma(z) = \frac{\mathbf p-1}{N+\mathbf p-2}|A_{\mathbf p}|$. *We also have:* $\displaystyle\int_{T_{\mathbf p}^{0,\varepsilon}}\langle z,e_2\rangle^2\;\mathrm{d}\mu_s^N(z) = \frac{C(N,s)|A_{\mathbf p}|}{(N+\mathbf p-2)(2-2s)}\varepsilon^{2-2s}$.

(ii) When $\nabla^2 u(x)$ and $p_x\doteq\nabla u(x)\neq 0$ are well defined, then:

$$\int_{T_{\mathbf p}^{0,\varepsilon}(\frac{p_x}{|p_x|})}\big\langle\nabla^2 u(x): z\otimes z\big\rangle\;\mathrm{d}\mu_s^N(z) = \frac{C(N,s)|A_{\mathbf p}|}{(N+\mathbf p-2)(2-2s)}\varepsilon^{2-2s}\cdot|p_x|^{2-\mathbf p}\Delta_{\mathbf p}u(x).$$

Proof. We consider the following function, which is continuous on $(0,\pi)$:

$$\alpha\mapsto Q(\alpha)\doteq\frac{\int_{A(\alpha)}\langle z,e_1\rangle^2\;\mathrm{d}\sigma(z)}{\int_{A(\alpha)}\langle z,e_2\rangle^2\;\mathrm{d}\sigma(z)},\qquad\text{where}\;\; A(\alpha)=\{|z|=1;\;\angle(e_1,z)<\alpha\}.$$

Since $Q(\frac{\pi}{2})=\frac12\int_{\{|z|=1\}}\langle z,e_1\rangle^2\;\mathrm{d}\sigma(z)/\big(\frac12\int_{\{|z|=1\}}\langle z,e_2\rangle^2\;\mathrm{d}\sigma(z)\big)=1$, while $\lim_{\alpha\to0}Q(\alpha)=\infty$, it follows that for each $\mathbf p-1\in[1,\infty)$ there indeed exists $\alpha_{\mathbf p}\in(0,\frac{\pi}{2}]$ satisfying $Q(\alpha_{\mathbf p})=\mathbf p-1$.

To prove (i), we compute, putting $A_{\mathbf p}=A(\alpha_{\mathbf p})$:

$$Q(\alpha_{\mathbf p}) = \frac{\int_{A_{\mathbf p}}1\;\mathrm{d}\sigma(z)-(N-1)\int_{A_{\mathbf p}}\langle z,e_2\rangle^2\;\mathrm{d}\sigma(z)}{\int_{A_{\mathbf p}}\langle z,e_2\rangle^2\;\mathrm{d}\sigma(z)} = \frac{1}{⨍_{A_{\mathbf p}}\langle z,e_2\rangle^2\;\mathrm{d}\sigma(z)}-(N-1),$$

which implies that: $\fint_{A_\mathbf{p}} \langle z, e_2\rangle^2 \ \mathrm{d}\sigma(z) = \frac{1}{N+\mathbf{p}-2}$, and consequently: $\fint_{A_\mathbf{p}} \langle z, e_1\rangle^2 \ \mathrm{d}\sigma(z) = 1 - (N-1)\fint_{A_\mathbf{p}} \langle z, e_2\rangle^2 \ \mathrm{d}\sigma(z) = \frac{\mathbf{p}-1}{N+\mathbf{p}-2}$. On the other hand:

$$\begin{aligned}\int_{T_\mathbf{p}^{0,\varepsilon}} \langle z, e_2\rangle^2 \ \mathrm{d}\mu_s^N(z) &= C(N,s)\int_0^\varepsilon \frac{t^2 t^{N-1}}{t^{N+2s}} \int_{A_\mathbf{p}} \langle z, e_2\rangle^2 \ \mathrm{d}\sigma(z) \ \mathrm{d}t \\ &= C(N,s)\int_{A_\mathbf{p}} \langle z, e_2\rangle^2 \ \mathrm{d}\sigma(z) \cdot \frac{\varepsilon^{2-2s}}{2-2s}.\end{aligned}$$

To prove (ii), observe that:

$$\begin{aligned}&\int_{T_\mathbf{p}^{0,\varepsilon}} z\otimes z \ \mathrm{d}\mu_s^N(z) = diag\Big(\int_{T_\mathbf{p}^{0,\varepsilon}} \langle z, e_1\rangle^2 \ \mathrm{d}\mu_s^N(z), \int_{T_\mathbf{p}^{0,\varepsilon}} \langle z, e_2\rangle^2 \ \mathrm{d}\mu_s^N(z), \dots\Big) \\ &= \int_{T_\mathbf{p}^{0,\varepsilon}} \langle z, e_2\rangle^2 \ \mathrm{d}\mu_s^N(z)\cdot Id_N + \Big(\int_{T_\mathbf{p}^{0,\varepsilon}} \langle z, e_1\rangle^2 \ \mathrm{d}\mu_s^N(z) - \int_{T_\mathbf{p}^{0,\varepsilon}} \langle z, e_2\rangle^2 \ \mathrm{d}\mu_s^N(z)\Big) e_1\otimes e_1 \\ &= \int_{T_\mathbf{p}^{0,\varepsilon}} \langle z, e_2\rangle^2 \ \mathrm{d}\mu_s^N(z)\cdot \Big(Id_N + (\mathbf{p}-2)\, e_1\otimes e_1\Big) \\ &= \frac{C(N,s)|A_\mathbf{p}|}{(N+\mathbf{p}-2)(2-2s)}\varepsilon^{2-2s}\cdot\Big(Id_N + (\mathbf{p}-2)\, e_1\otimes e_1\Big),\end{aligned}$$

where we used:

$$\frac{\int_{T_\mathbf{p}^{0,\varepsilon}} \langle z, e_1\rangle^2 \ \mathrm{d}\mu_s^N(z)}{\int_{T_\mathbf{p}^{0,\varepsilon}} \langle z, e_2\rangle^2 \ \mathrm{d}\mu_s^N(z)} = Q(\alpha_\mathbf{p}) = \mathbf{p}-1.$$

It thus follows that:

$$\begin{aligned}&\Big\langle \nabla^2 u(x) : \int_{T_\mathbf{p}^{0,\varepsilon}(\frac{p_x}{|p_x|})} z\otimes z \ \mathrm{d}\mu_s^N(z)\Big\rangle \\ &= \frac{C(N,s)|A_\mathbf{p}|}{(N+\mathbf{p}-2)(2-2s)}\varepsilon^{2-2s}\Big\langle \nabla^2 u(x) : Id_N + (\mathbf{p}-2)\frac{p_x}{|p_x|}\otimes\frac{p_x}{|p_x|}\Big\rangle,\end{aligned}$$

which completes the argument, upon recalling the formula: $\Delta_\mathbf{p} u = |\nabla u|^{\mathbf{p}-2}\big(\Delta u + (\mathbf{p}-2)\Delta_\infty u\big)$ and the definition of the ∞-Laplacian: $\Delta_\infty u = \langle \nabla^2 u : \frac{\nabla u}{|\nabla u|}\otimes\frac{\nabla u}{|\nabla u|}\rangle$. ■

Remark 2.6. In [2][Section 4.1.2], the fractional **p**-Laplacian $\Delta_\mathbf{p}^s u$ has been introduced by means of a scaled version of the operator $\mathcal{L}_{s,\mathbf{p}}[u]$. In particular, when $p_x \neq 0$ it follows that:

$$\Delta_\mathbf{p}^s u(x) = \frac{2-2s}{C(N,s)}\cdot\frac{N+\mathbf{p}-2}{|A_\mathbf{p}|}\mathcal{L}_{s,\mathbf{p}}[u](x).$$

3. A non-local asymptotic expansion

In this section we prove the formula (1.4) with a precise form of the error term. In what follows, we denote $B_r = B_r(x)$ for a fixed referential point $x \in \mathbb{R}^N$. Given a function $u : \mathbb{R}^N \to \mathbb{R}$ that is uniformly continuous on $\mathbb{R}^N \setminus \bar{B}_r$, we denote its modulus of continuity by:

$$\omega_u(a) = \sup\big\{|u(z) - u(\bar{z})|;\ z, \bar{z} \in \mathbb{R}^N \setminus \bar{B}_{r_x},\ |z - \bar{z}| \le a\big\}.$$

Theorem 3.1. *Let $u \in C^2(\bar B_{r_x})$ satisfy $p_x \doteq \nabla u(x) \neq 0$, and denote $C_x \doteq \frac{1}{2}\|\nabla^2 u\|_{L^\infty(B_{r_x})}$. Assume that u is uniformly continuous on $\mathbb{R}^N \setminus \bar B_{r_x}$ with modulus of continuity ω_u. Recall that:*

$$\mathcal{A}_\varepsilon u(x) \doteq \frac{1}{2}\Big(\sup_{|y|=1} \fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)} u(x+z)\;\mathrm{d}\mu_s^N(z) + \inf_{|y|=1} \fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)} u(x+z)\;\mathrm{d}\mu_s^N(z)\Big).$$

Then there holds:

$$\begin{aligned}&\Big|\mathcal{A}_\varepsilon u(x) - u(x) - \frac{s}{C(N,s)|A_{\mathbf{p}}|}\varepsilon^{2s}\mathcal{L}_{s,\mathbf{p}}[u](x)\Big| \\ &\qquad \leq \frac{s}{1-s}\cdot C_x\varepsilon^2 + \varepsilon^{2s}\Big(4sC_x\frac{r_x^{2-2s}-\varepsilon^{2-2s}}{1-s}\cdot m_\varepsilon + \big(r_x^{-2s} + \frac{2s}{2s-1}r_x^{1-2s}\big)\cdot\omega_u(m_\varepsilon)\Big),\end{aligned} \tag{3.1}$$

where we define:

$$\begin{aligned}& m_\varepsilon = \max\Big\{\frac{16(N+\mathbf{p}-2)}{\mathbf{p}-1}\cdot\frac{C_x}{|p_x|}\cdot\frac{2s-1}{1-s}\cdot\frac{r_x^{2-2s}-\varepsilon^{2-2s}}{\varepsilon^{1-2s}-{r_x}^{1-2s}},\ \kappa_\varepsilon\Big\}, \\ & \kappa_\varepsilon = \sup\Big\{m;\ m\in[0,2]\ \text{ and }\ m^2 \leq \frac{N+\mathbf{p}-2}{\mathbf{p}-1}\cdot\frac{8\omega_u(m)}{|p_x|}\cdot\frac{\frac{2s-1}{2s}r_x^{-2s}+r_x^{1-2s}}{\varepsilon^{1-2s}-r_x^{1-2s}}\Big\}.\end{aligned} \tag{3.2}$$

Before giving the proof, a few observations are in order:

Remark 3.2. (i) Using the crude bound $\omega_u \leq 2\|u\|_{L^\infty}$, it follows that for all $\varepsilon < \frac{r_x}{2}$ we have:

$$\begin{aligned}\kappa_\varepsilon &\leq \Big(\frac{16\|u\|_{L^\infty}}{|p_x|}\cdot\frac{N+\mathbf{p}-2}{\mathbf{p}-1}\cdot\frac{\frac{2s-1}{2s}r_x^{-2s}+r_x^{1-2s}}{\varepsilon^{1-2s}-r_x^{1-2s}}\Big)^{1/2} \\ &\leq 8\Big(\frac{\|u\|_{L^\infty}}{|p_x|}\cdot\frac{N+\mathbf{p}-2}{\mathbf{p}-1}\cdot\frac{r_x^{-2s}+r_x^{1-2s}}{2s-1}\Big)^{1/2}\varepsilon^{s-1/2}.\end{aligned}$$

In the second inequality above we used that $\varepsilon < \frac{r_x}{2}$ implies: $\varepsilon^{1-2s}-r_x^{1-2s} > \varepsilon^{1-2s}(1-2^{1-2s})$, and that $1-2^{1-2s} \geq (2s-1)\ln\sqrt{2} > \frac{2s-1}{4}$ in the range $s \in (\frac{1}{2},1)$. Since the first quantity in m_ε is of order ε^{2s-1}, the bounding coefficient of order ε^{2s} in (3.1) becomes:

$$C_{N,\mathbf{p},s}C(r_x)\cdot C\big(\frac{\|u\|_{L^\infty}}{|p_x|}\big)\cdot\big(C_x\varepsilon^{s-1/2}+\omega_u(\varepsilon^{s-1/2})\big),$$

where $C(\alpha)$ depends only on the indicated quantity α.

(ii) When $u \in C^{0,\alpha}(\mathbb{R}^N\setminus\bar B_{r_x})$ with $\alpha\in(0,1)$, then $\omega_u(m) = [u]_\alpha m^\alpha$ and we similarly obtain:

$$\kappa_\varepsilon \leq \Big(\frac{32\,[u]_\alpha}{|p_x|}\cdot\frac{N+\mathbf{p}-2}{\mathbf{p}-1}\cdot\frac{r_x^{-2s}+r_x^{1-2s}}{2s-1}\Big)^{\frac{1}{2-\alpha}}\varepsilon^{\frac{2s-1}{2-\alpha}},$$

resulting in the following bounding constant:

$$C_{N,\mathbf{p},s}C(r_x)\cdot C\big(\frac{[u]_\alpha}{|p_x|}\big)\varepsilon^{\alpha\cdot\frac{2s-1}{2-\alpha}}.$$

(iii) When u is Lipschitz on $\mathbb{R}^N\setminus\bar B_{r_x}$ with the Lipschitz constant Lip_u, we get:

$$\kappa_\varepsilon \leq \frac{32\,\mathrm{Lip}_u}{|p_x|}\cdot\frac{N+\mathbf{p}-2}{\mathbf{p}-1}\cdot\frac{r_x^{-2s}+r_x^{1-2s}}{2s-1}\varepsilon^{2s-1},$$

so both quantities in m_ε are of the same order ε^{2s-1} and:

$$m_\varepsilon \leq \frac{32}{|p_x|}\cdot\frac{N+\mathbf{p}-2}{\mathbf{p}-1}\cdot\max\Big\{\frac{2C_x r_x^{2-2s}}{1-s},\frac{\mathrm{Lip}_u\cdot(r_x^{-2s}+r_x^{1-2s})}{2s-1}\Big\}\varepsilon^{2s-1}.$$

Consequently, the discussed bounding expression becomes:

$$C_{N,\mathbf{p},s}C(r_x)\cdot C\big(C_x,\frac{1}{|p_x|},\mathrm{Lip}_u\big)\varepsilon^{2s-1}.$$

In order to estimate the difference of $\mathcal{L}^\varepsilon_{s,\mathbf{p}}[u]$ and $\mathcal{L}_{s,\mathbf{p}}[u]$ when $p_x\neq 0$, we will analyze the behaviour of approximations to the extremizers $y,\tilde{y}$ in Definition 2.2. The proof below follows the outline of the proof of [7, Theorem 1] in the fractional ∞-Laplacian setting.

Lemma 3.3. *Under the same assumptions and notation as in Theorem 3.1, for every $\varepsilon<r_x$ there holds, with the quantity m_ε is as (3.2).*

$$\begin{aligned}\Big|\mathcal{L}^\varepsilon_{s,\mathbf{p}}[u](x)-&\int_{T^{\varepsilon,\infty}_{\mathbf{p}}(\frac{p_x}{|p_x|})}L_u(x,z,z)\;\mathrm{d}\mu^N_s(z)\Big|\\ &\leq C(N,s)|A_{\mathbf{p}}|\cdot\Big(4m_\varepsilon C_x\frac{r_x^{2-2s}-\varepsilon^{2-2s}}{1-s}+2\omega_u(m_\varepsilon)\Big(\frac{r_x^{-2s}}{2s}+\frac{r_x^{1-2s}}{2s-1}\Big)\Big),\end{aligned}\tag{3.3}$$

Proof. **1.** For every $\varepsilon<\eta_x$ and every $\delta>0$ satisfying:

$$\delta\leq\frac{\big(16C_x\int_{T^{\varepsilon,r_x}_{\mathbf{p}}}|z|^2\;\mathrm{d}\mu^N_s(z)\big)^2}{|p_x|\int_{T^{\varepsilon,r_x}_{\mathbf{p}}}\langle z,e_1\rangle\;\mathrm{d}\mu^N_s(z)},\tag{3.4}$$

let $|y^\varepsilon_\delta|=1$ be such that $\sup_{|y|=1}\int_{T^{\varepsilon,\infty}_{\mathbf{p}}(y)}u(x+z)\;\mathrm{d}\mu^N_s(z)\leq\int_{T^{\varepsilon,\infty}_{\mathbf{p}}(y^\varepsilon_\delta)}u(x+z)\;\mathrm{d}\mu^N_s(z)+\delta$. Then:

$$\begin{aligned}\delta&\geq\int_{T^{\varepsilon,\infty}_{\mathbf{p}}(\frac{p_x}{|p_x|})}u(x+z)\;\mathrm{d}\mu^N_s(z)-\int_{T^{\varepsilon,\infty}_{\mathbf{p}}(y^\varepsilon_\delta)}u(x+z)\;\mathrm{d}\mu^N_s(z)\\ &\geq\int_{T^{\varepsilon,r_x}_{\mathbf{p}}(\frac{p_x}{|p_x|})}u(x+z)-u(x+R_{y^\varepsilon_\delta,\frac{p_x}{|p_x|}}z)\;\mathrm{d}\mu^N_s(z)-\omega_u\big(|Id_N-R_{y^\varepsilon_\delta,\frac{p_x}{|p_x|}}|\big)\cdot\int_{T^{r_x,\infty}_{\mathbf{p}}}(1+|z|)\;\mathrm{d}\mu^N_s(z),\end{aligned}$$

because:

$$\begin{aligned}&\int_{T^{r_x,\infty}_{\mathbf{p}}(\frac{p_x}{|p_x|})}\big|u(x+z)-u(x+R_{y^\varepsilon_\delta,\frac{p_x}{|p_x|}}z)\big|\;\mathrm{d}\mu^N_s(z)\\ &\quad\leq\int_{T^{r_x,\infty}_{\mathbf{p}}(\frac{p_x}{|p_x|})}(1+|z|)\cdot\sup\Big\{|u(y_1)-u(y_2)|;\;y_1,y_2\notin\bar{B}_{r_x},\;|y_1-y_2|\leq|Id_N-R_{y^\varepsilon_\delta,\frac{p_x}{|p_x|}}|\Big\}\;\mathrm{d}\mu^N_s(z)\\ &\quad\leq\omega_u(|Id_N-R_{y^\varepsilon_\delta,\frac{p_x}{|p_x|}}|)\cdot\int_{T^{r_x,\infty}_{\mathbf{p}}}(1+|z|)\;\mathrm{d}\mu^N_s(z).\end{aligned}$$

Call $m\doteq\big|\frac{p_x}{|p_x|}-y^\varepsilon_\delta\big|$ and note that $|Id_N-R_{y^\varepsilon_\delta,\frac{p_x}{|p_x|}}|=m$. Recalling (2.2) further leads to:

$$\begin{aligned}\delta&\geq\int_{T^{\varepsilon,r_x}_{\mathbf{p}}(\frac{p_x}{|p_x|})}\Big\langle\nabla u(x+z),(Id_N-R_{y^\varepsilon_\delta,\frac{p_x}{|p_x|}})z\Big\rangle\;\mathrm{d}\mu^N_s(z)\\ &\quad-C_x\int_{T^{\varepsilon,r_x}_{\mathbf{p}}}m^2|z|^2\;\mathrm{d}\mu^N_s(z)-\omega_u(m)\int_{T^{r_x,\infty}_{\mathbf{p}}}(1+|z|)\;\mathrm{d}\mu^N_s(z)\\ &\geq\Big\langle p_x,(Id_N-R_{y^\varepsilon_\delta,\frac{p_x}{|p_x|}})\int_{T^{\varepsilon,r_x}_{\mathbf{p}}(\frac{p_x}{|p_x|})}z\;\mathrm{d}\mu^N_s(z)\Big\rangle\\ &\quad-C_xm(2+m)\cdot\int_{T^{\varepsilon,r_x}_{\mathbf{p}}}|z|^2\;\mathrm{d}\mu^N_s(z)-\omega_u(m)\cdot\int_{T^{r_x,\infty}_{\mathbf{p}}}(1+|z|)\;\mathrm{d}\mu^N_s(z),\end{aligned}\tag{3.5}$$

since $\big|\nabla u(x+z) - \nabla u(x)\big| \leq 2C_x|z|$ for all $z \in B_{r_x}$.

We now observe that: $\int_{T_{\mathbf{p}}^{\varepsilon,r_x}(\frac{p_x}{|p_x|})} z \,\mathrm{d}\mu_s^N(z) = \int_{T_{\mathbf{p}}^{\varepsilon,r_x}} \langle z, e_1\rangle \,\mathrm{d}\mu_s^N(z) \cdot \frac{p_x}{|p_x|}$, which yields that the first term in the right hand side of (3.5) equals:

$$\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} \langle z, e_1\rangle \,\mathrm{d}\mu_s^N(z) \cdot \Big\langle p_x, (Id_N - R_{y_\delta^\varepsilon, \frac{p_x}{|p_x|}})\frac{p_x}{|p_x|}\Big\rangle = \int_{T_{\mathbf{p}}^{\varepsilon,r_x}} \langle z, e_1\rangle \,\mathrm{d}\mu_s^N(z) \cdot \big\langle p_x, \frac{p_x}{|p_x|} - y_\delta^\varepsilon\big\rangle.$$

Finally, noting the identity:

$$m^2 = \Big|\frac{p_x}{|p_x|} - y_\delta^\varepsilon\Big|^2 = 2 - 2\Big\langle \frac{p_x}{|p_x|}, y_\delta^\varepsilon\Big\rangle = \frac{2}{|p_x|}\Big\langle p_x, \frac{p_x}{|p_x|} - y_\delta^\varepsilon\Big\rangle,$$

the discussed term becomes:

$$\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} \langle z, e_1\rangle \,\mathrm{d}\mu_s^N(z) \cdot \frac{m^2|p_x|}{2}.$$

Consequently, it follows by (3.5) that:

$$m^2 \leq 2 \cdot \frac{\delta + 4C_x m \int_{T_{\mathbf{p}}^{\varepsilon,r_x}} |z|^2 \,\mathrm{d}\mu_s^N(z) + \omega_u(m)\int_{T_{\mathbf{p}}^{r_x,\infty}} (1+|z|)\,\mathrm{d}\mu_s^N(z)}{|p_x|\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} \langle z, e_1\rangle \,\mathrm{d}\mu_s^N(z)}. \tag{3.6}$$

2. We now analyze the bound (3.6) in the following distinct cases. In the first case:

$$m^2 \leq \frac{4\delta}{|p_x|\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} \langle z, e_1\rangle \,\mathrm{d}\mu_s^N(z)} \leq \Big(\frac{32C_x\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} |z|^2 \,\mathrm{d}\mu_s^N(z)}{|p_x|\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} \langle z, e_1\rangle \,\mathrm{d}\mu_s^N(z)}\Big)^2, \tag{3.7}$$

where we used (3.4) in the second inequality. In the reverse case, we get:

$$m^2 \leq \frac{16C_x m\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} |z|^2 \,\mathrm{d}\mu_s^N(z) + 4\omega_u(m)\int_{T_{\mathbf{p}}^{r_x,\infty}} (1+|z|)\,\mathrm{d}\mu_s^N(z)}{|p_x|\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} \langle z, e_1\rangle \,\mathrm{d}\mu_s^N(z)} \doteq I_1 + I_2.$$

When $I_2 \leq I_1$, then the above yields the same bound as in (3.7), namely:

$$m \leq \frac{32C_x\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} |z|^2 \,\mathrm{d}\mu_s^N(z)}{|p_x|\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} \langle z, e_1\rangle \,\mathrm{d}\mu_s^N(z)} = \frac{16C_x}{|p_x|\fint_{A_{\mathbf{p}}}\langle z, e_1\rangle \,\mathrm{d}\sigma(z)} \cdot \frac{2s-1}{1-s} \cdot \frac{r_x^{2-2s} - \varepsilon^{2-2s}}{\varepsilon^{1-2s} - {r_x}^{1-2s}}.$$

When $I_1 < I_2$, then:

$$m^2 \leq \frac{8\omega_u(m)\int_{T_{\mathbf{p}}^{r_x,\infty}} (1+|z|)\,\mathrm{d}\mu_s^N(z)}{|p_x|\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} \langle z, e_1\rangle \,\mathrm{d}\mu_s^N(z)} = \frac{8\omega_u(m)}{|p_x|\fint_{A_{\mathbf{p}}}\langle z, e_1\rangle \,\mathrm{d}\sigma(z)} \cdot \frac{\frac{2s-1}{2s}r_x^{-2s} + r_x^{1-2s}}{\varepsilon^{1-2s} - r_x^{1-2s}},$$

so that $m \leq \kappa_\varepsilon$ as $\fint_{A_{\mathbf{p}}}\langle z, e_1\rangle \,\mathrm{d}\sigma(z) \geq \fint_{A_{\mathbf{p}}}\langle z, e_1\rangle^2 \,\mathrm{d}\sigma(z) = \frac{\mathbf{p}-1}{N+\mathbf{p}-2}$. Hence $m \leq m_\varepsilon$ in both cases.

3. By the same analysis as in step 1, we see that the unit vector $\tilde y_\delta^\varepsilon$ satisfying:

$$\inf_{|y|=1}\int_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)} u(x+z) - u(x)\,\mathrm{d}\mu_s^N(z) \geq \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(-\tilde y_\delta^\varepsilon)} u(x+z) - u(x)\,\mathrm{d}\mu_s^N(z) - \delta,$$

differs from the unit vector $\frac{p_x}{|p_x|}$ at most by m_ε. Note that:

$$\begin{aligned}
&\int_{T_{\mathbf{p}}^{\varepsilon,\infty}(\tilde y_\delta^\varepsilon)} L_u(x,z,z)\,\mathrm{d}\mu_s^N(z) - \delta \\
&= \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(\tilde y_\delta^\varepsilon)} u(x+z)-u(x)\,\mathrm{d}\mu_s^N(z) + \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(-\tilde y_\delta^\varepsilon)} u(x+z)-u(x)\,\mathrm{d}\mu_s^N(z) - \delta \leq \mathcal{L}_{s,\mathbf{p}}^\varepsilon[u](x) \\
&\leq \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(y_\delta^\varepsilon)} u(x+z)-u(x)\,\mathrm{d}\mu_s^N(z) + \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(-y_\delta^\varepsilon)} u(x+z)-u(x)\,\mathrm{d}\mu_s^N(z) + \delta \\
&= \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(y_\delta^\varepsilon)} L_u(x,z,z)\,\mathrm{d}\mu_s^N(z) + \delta,
\end{aligned}$$

which yields:

$$\begin{aligned}
&\Big|\mathcal{L}_{s,\mathbf{p}}^\varepsilon[u](x) - \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(\frac{p_x}{|p_x|})} L_u(x,z,z)\,\mathrm{d}\mu_s^N(z)\Big| \leq \delta + \max\big\{|J(y_\delta^\varepsilon)|, |J(\tilde y_\delta^\varepsilon|)\big\}, \\
&\text{where: } J(y) \doteq \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(\frac{p_x}{|p_x|})} u(x+z) - u(x+R_{y,\frac{p_x}{|p_x|}}z) + u(x-z) - u(x-R_{y,\frac{p_x}{|p_x|}}z)\,\mathrm{d}\mu_s^N(z).
\end{aligned} \tag{3.8}$$

We now estimate the two terms $J(y_\delta^\varepsilon)$, $J(\tilde y_\delta^\varepsilon)$ and show that they are bounded independently of δ. This will allow to pass $\delta\to 0$ in (3.8) and directly conclude the claimed estimate (3.3). We start by splitting the integral in $J(y_\delta^\varepsilon)$ in two terms: $|J(y_\delta^\varepsilon)| \leq J_1 + J_2$, where:

$$\begin{aligned}
J_1 &= \Big|\int_{T_{\mathbf{p}}^{\varepsilon,r_x}(\frac{p_x}{|p_x|})} u(x+z) - u(x+R_{y_\delta^\varepsilon,\frac{p_x}{|p_x|}}z) + u(x-z) - u(x-R_{y_\delta^\varepsilon,\frac{p_x}{|p_x|}}z)\,\mathrm{d}\mu_s^N(z)\Big| \\
&\leq \int_{T_{\mathbf{p}}^{\varepsilon,r_x}(\frac{p_x}{|p_x|})} \Big|\big\langle \nabla u(x+z), z - R_{y_\delta^\varepsilon,\frac{p_x}{|p_x|}}z\big\rangle - \big\langle \nabla u(x-z), z - R_{y_\delta^\varepsilon,\frac{p_x}{|p_x|}}z\big\rangle\Big|\,\mathrm{d}\mu_s^N(z) \\
&\quad + \int_{T_{\mathbf{p}}^{\varepsilon,r_x}} 2C_x m^2|z|^2\,\mathrm{d}\mu_s^N(z) \\
&\leq (4m+2m^2)\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} C_x|z|^2\,\mathrm{d}\mu_s^N(z) \leq 8mC_x\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} |z|^2\,\mathrm{d}\mu_s^N(z).
\end{aligned}$$

The remaining estimate is:

$$\begin{aligned}
J_2 &= \Big|\int_{T_{\mathbf{p}}^{r_x,\infty}(\frac{p_x}{|p_x|})} u(x+z) - u(x+R_{y_\delta^\varepsilon,\frac{p_x}{|p_x|}}z) + u(x-z) - u(x-R_{y_\delta^\varepsilon,\frac{p_x}{|p_x|}}z)\,\mathrm{d}\mu_s^N(z)\Big| \\
&\leq 2\omega_u(m)\int_{T_{\mathbf{p}}^{r_x,\infty}} (1+|z|)\,\mathrm{d}\mu_s^N(z)
\end{aligned}$$

In conclusion, we obtain:

$$|J(y_\delta^\varepsilon)| \leq 8m_\varepsilon C_x\int_{T_{\mathbf{p}}^{\varepsilon,r_x}} |z|^2\,\mathrm{d}\mu_s^N(z) + 2\omega_u(m_\varepsilon)\int_{T_{\mathbf{p}}^{r_x,\infty}} (1+|z|)\,\mathrm{d}\mu_s^N(z).$$

Clearly, $|J(\tilde y_\delta^\varepsilon)|$ enjoys the same bound. The result in Lemma now follows by (3.8). ∎

By Remark 3.2 (iii) we see that in case of u which is Lipschitz on $\mathbb{R}^N\setminus\bar B_{r_x}$, the order of the error bounding quantity in Theorem 3.1 is $C(s)\cdot(\varepsilon^{4s-1}+\varepsilon^2)$ as $\varepsilon\to 0+$, where $C(s)$ blows up as $s\to 1-$. This drawback will be remedied by means of another asymptotic expansion in Theorem 4.2, proved in section 4. We are now ready to give:

Proof of Theorem 3.1

In view of (2.3) we get:

$$\Big|\mathcal{L}_{s,\mathbf{p}}[u](x) - \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(\frac{p_x}{|p_x|})} L_u(x,z,z)\;\mathrm{d}\mu_s^N(z)\Big| \leq \int_{T_{\mathbf{p}}^{0,\varepsilon}(\frac{p_x}{|p_x|})} |L_u(x,z,z)|\;\mathrm{d}\mu_s^N(z) \\ \leq C(N,s)|A_{\mathbf{p}}|\cdot C_x\frac{\varepsilon^{2-2s}}{1-s}.$$

Consequently, Lemma 3.3 yields:

$$\Big|\mathcal{L}_{s,\mathbf{p}}^{\varepsilon}[u](x) - \mathcal{L}_{s,\mathbf{p}}[u](x)\Big| \leq \frac{4C(N,s)|A_{\mathbf{p}}|}{1-s}\cdot C_x\big(r_x^{2-2s}-\varepsilon^{2-2s}\big)\cdot m_\varepsilon \\ + \frac{C(N,s)|A_{\mathbf{p}}|}{s}\cdot\Big(r_x^{-2s}+\frac{2s}{2s-1}r_x^{1-2s}\Big)\cdot\omega_u(m_\varepsilon) + \frac{C(N,s)|A_{\mathbf{p}}|}{1-s}\cdot C_x\varepsilon^{2-2s}.$$

The result follows by collecting terms and scaling by the factor: $\frac{s}{C(N,s)|A_{\mathbf{p}}|}\varepsilon^{2s}$. ■

4. A local - non-local asymptotic expansion

In this section we present a refined version of the argument in Theorem 3.1. We need one more estimate before giving the proof of expansion (1.5) in Theorem 4.2.

Proposition 4.1. *Let $u\in C^2(\bar B_{r_x})$ satisfy: $p_x\doteq\nabla u(x)\neq 0$. For every $\varepsilon<r_x$ such that $\varepsilon|\nabla^2 u(x)|\leq|p_x|$, denote $B_\varepsilon = B_\varepsilon(x)$. Then there holds:*

$$\Big|\frac{C(N,s)|A_{\mathbf{p}}|}{(N+p-2)(1-s)}\cdot\varepsilon^{-2s}\Big(\frac{\mathbf{p}-2}{2}\big(\sup_{B_\varepsilon}u+\inf_{B_\varepsilon}u\big)+(N+2)\fint_{B_\varepsilon}u(y)\;\mathrm{d}y-(N+\mathbf{p})u(x)\Big) \\ - \int_{T_{\mathbf{p}}^{0,\varepsilon}(\frac{p_x}{|p_x|})} L_u(x,z,z)\;\mathrm{d}\mu_s^N(z)\Big| \leq \frac{C(N,s)|A_{\mathbf{p}}|}{1-s}\cdot\varepsilon^{2-2s}\sup_{y\in B_\varepsilon}|\nabla^2u(y)-\nabla^2u(x)| \\ + \frac{2C(N,s)|A_{\mathbf{p}}|}{1-s}\cdot\frac{\mathbf{p}-2}{N+\mathbf{p}-2}\cdot\varepsilon^{3-2s}\frac{|\nabla^2u(x)|^2}{|p_x|}.$$

Proof. An application of Taylor's expansion and the identity in Lemma 2.5 (iii) results in the following estimate:

$$\Big|\int_{T_{\mathbf{p}}^{0,\varepsilon}(\frac{p_x}{|p_x|})} L_u(x,z,z)\;\mathrm{d}\mu_s^N(z) - \frac{C(N,s)|A_{\mathbf{p}}|}{(N+\mathbf{p}-2)(2-2s)}\cdot\varepsilon^{2-2s}|p_x|^{2-\mathbf{p}}\Delta_{\mathbf{p}}u(x)\Big| \\ \leq \int_{T_{\mathbf{p}}^{0,\varepsilon}(\frac{p_x}{|p_x|})} |L_u(x,z,z)-\langle\nabla^2u(x):z\otimes z\rangle|\;\mathrm{d}\mu_s^N(z) \\ \leq \int_{T_{\mathbf{p}}^{0,\varepsilon}} |z|^2\;\mathrm{d}\mu_s^N(z)\cdot\sup_{y\in B_\varepsilon}|\nabla^2u(y)-\nabla^2u(x)| = \frac{C(N,s)|A_{\mathbf{p}}|}{2-2s}\cdot\varepsilon^{2-2s}\sup_{y\in B_\varepsilon}|\nabla^2u(y)-\nabla^2u(x)|.$$

We now invoke the following folklore claim (whose proof we recall below):

$$\Big|(\mathbf{p}-2)\big(\sup_{B_\varepsilon}u+\inf_{B_\varepsilon}u\big)+2(N+2)\fint_{B_\varepsilon}u(y)\;\mathrm{d}y-2(N+\mathbf{p})u(x)-\varepsilon^2|p_x|^{2-\mathbf{p}}\Delta_{\mathbf{p}}u(x)\Big| \\ \leq 4\varepsilon^3(\mathbf{p}-2)\frac{|\nabla^2u(x)|^2}{|p_x|}+\varepsilon^2(N+\mathbf{p}-2)\cdot\sup_{y\in B_\varepsilon}|\nabla^2u(y)-\nabla^2u(x)|. \tag{4.1}$$

Summing up the scaled versions of the last two displayed formulas completes the argument. ■

Proof of claim (4.1).

1. We first deduce the bound in the particular case when $\varepsilon = 1$, $x = 0$ and u is a quadratic polynomial with gradient given by a unit vector $p \in \mathbb{R}^N$ and Hessian given by a symmetric matrix $B \in \mathbb{R}^{N\times N}$ satisfying $|B| \leq 1$:

$$u(z) = \langle p, z\rangle + \frac{1}{2}\langle B : z \otimes z\rangle.$$

It is straightforward that:

$$\fint_{B_1} u(z)\ \mathrm{d}z = \frac{1}{2}\Big\langle B : \fint_{B_1} z\otimes z\ \mathrm{d}z\Big\rangle = \frac{1}{2(N+2)}\big\langle Id_N : B\big\rangle = \frac{1}{2(N+2)}\Delta u(0). \tag{4.2}$$

In order to address the nonlinear averaging quantity $\sup_{B_1} u + \inf_{B_1} u$, consider $z_{max} \in \bar{B}_1$ that is a maximizer of u. Note that $|z_{max}| = 1$, because $|\nabla u(z)| = |p + Bz| > 1 - |B| \geq 0$ for all $|z| \leq 1$. Further, since $u(z_{max}) \geq u(p)$, there holds:

$$\langle p, z_{max}\rangle \geq 1 + \frac{1}{2}\langle B : p\otimes p - z_{max}\otimes z_{max}\rangle \geq 1 - |z_{max} - p|\cdot|B|.$$

Consequently: $|z_{max} - p|^2 = 2 - 2\langle z_{max}, p\rangle \leq 2|z_{max} - p|\cdot|B|$, so that:

$$|z_{max} - p| \leq 2|B|.$$

Noting that $\langle p, z_{max} - p\rangle \leq 0$, we hence arrive at:

$$0 \leq u(z_{max}) - u(p) = \langle p, z_{max} - p\rangle + \frac{1}{2}\langle B : z_{max}\otimes z_{max} - p\otimes p\rangle \leq |z_{max} - p|\cdot|B| \leq 2|B|^2.$$

An entirely similar argument yields the bound for a minimizer z_{min} of u on $\bar{B}_1$:

$$0 \geq u(z_{min}) - u(-p) \geq -2|B|^2,$$

which results in the bound:

$$\begin{aligned}\Big|\sup_{B_1} u + \inf_{B_1} u - \Delta_\infty u(0)\Big| &= \Big|u(z_{max}) + u(z_{min}) - \langle B : p\otimes p\rangle\Big| \\ &= \Big|u(z_{max}) - u(p) + u(z_{min}) - u(-p)\Big| \leq 4|B|^2.\end{aligned} \tag{4.3}$$

The estimate (4.1) follows in the present case summing (4.2) scaled by $2(N+2)$ and (4.3) scaled by $\mathbf{p} - 2$.

2. For the general case, we may still assume $x = 0$ and $u(0) = 0$. Define $\bar{u}(y) = \langle p_x, y\rangle + \frac{1}{2}\langle\nabla^2 u(0) : y\otimes y\rangle$ and note that:

$$\begin{aligned}&\frac{2(N+2)}{N}\cdot\Big|\fint_{B_\varepsilon} u(y)\ \mathrm{d}y - \fint_{B_\varepsilon}\bar{u}(y)\ \mathrm{d}y\Big|, \quad \Big|\big(\sup_{B_\varepsilon} u + \inf_{B_\varepsilon} u\big) - \big(\sup_{B_\varepsilon}\bar{u} + \inf_{B_\varepsilon}\bar{u}\big)\Big| \\ &\qquad\qquad \leq \varepsilon^2 \sup_{y\in B_\varepsilon}|\nabla^2 u(y) - \nabla^2 u(0)|,\end{aligned} \tag{4.4}$$

by means of Taylor's expansion. We now apply the conclusion of step 1 to the rescaled function:

$$\frac{1}{\varepsilon|p_x|}\bar{u}(\varepsilon z) = \big\langle\frac{p_x}{|p_x|}, z\big\rangle + \frac{1}{2}\Big\langle\varepsilon\frac{\nabla^2 u(0)}{|p_x|} : z\otimes z\Big\rangle \quad \text{for } \ z\in\bar{B}_1.$$

It follows that for all ε satisfying $\varepsilon|\nabla^2 u(0)| \leq |p_x|$, we get:

$$\begin{aligned}&\Big|\frac{\mathbf{p}-2}{\varepsilon|p_x|}\big(\sup_{B_\varepsilon}\bar{u} + \inf_{B_\varepsilon}\bar{u}\big) + \frac{2(N+2)}{\varepsilon|p_x|}\fint_{B_\varepsilon}\bar{u}(y)\ \mathrm{d}y - \Big(\varepsilon\frac{\Delta u(0)}{|p_x|} + (\mathbf{p}-2)\Big\langle\varepsilon\frac{\nabla^2 u(0)}{|p_x|} : \frac{p_x}{|p_x|}\otimes\frac{p_x}{|p_x|}\Big\rangle\Big)\Big| \\ &\leq 4\varepsilon^2\cdot(\mathbf{p}-2)\frac{|\nabla^2 u(0)|^2}{|p_x|^2},\end{aligned}$$

which yields:

$$\Big|(\mathbf{p}-2)\big(\sup_{B_\varepsilon}\bar u+\inf_{B_\varepsilon}\bar u\big)+2(N+2)\fint_{B_\varepsilon}\bar u(y)\;\mathrm{d}y-\varepsilon^2|p_x|^{2-\mathbf{p}}\Delta_{\mathbf{p}}u(0)\Big|\leq 4\varepsilon^3(\mathbf{p}-2)\frac{|\nabla^2u(0)|^2}{|p_x|}.$$

Combined together with (4.4), the above estimate ends the proof of (4.1). ∎

Theorem 4.2. *Under the same assumptions and notation as in Theorem 3.1, for every $\varepsilon<r_x$ such that $\varepsilon|\nabla^2\phi(x)|\leq|p_x|$, denote $B_\varepsilon=B_\varepsilon(x)$ and consider the average:*

$$\begin{split}\bar{\mathcal{A}}_\varepsilon u(x)=\frac{(1-s)(N+\mathbf{p}-2)}{N+\mathbf{p}-2+2s}\cdot\frac{1}{2}\Big(\sup_{|y|=1}\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}u(x+z)\;\mathrm{d}\mu_s^N(z)+\inf_{|y|=1}\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}u(x+z)\;\mathrm{d}\mu_s^N(z)\Big)\\ +\frac{s(\mathbf{p}-2)}{N+\mathbf{p}-2+2s}\cdot\frac{1}{2}\big(\sup_{B_\varepsilon}u+\inf_{B_\varepsilon}u\big)+\frac{s(N+2)}{N+\mathbf{p}-2+2s}\fint_{B_\varepsilon}u(y)\;\mathrm{d}y.\end{split}$$

Then there holds:

$$\begin{split}&\Big|\bar{\mathcal{A}}_\varepsilon u(x)-u(x)-\frac{(1-s)s}{C(N,s)|A_{\mathbf{p}}|}\cdot\frac{N+\mathbf{p}-2}{N+\mathbf{p}-2+2s}\cdot\varepsilon^{2s}\mathcal{L}_{s,\mathbf{p}}[u](x)\Big|\\ &\leq\frac{4s(N+\mathbf{p}-2)}{N+\mathbf{p}-2+2s}C_x(r_x^{2-2s}-\varepsilon^{2-2s})\cdot\varepsilon^{2s}m_\varepsilon\\ &\quad+\frac{(1-s)(N+\mathbf{p}-2)}{N+\mathbf{p}-2+2s}\big(r_x^{-2s}+\frac{2s}{2s-1}r_x^{1-2s}\big)\cdot\varepsilon^{2s}\omega_u(m_\varepsilon)\\ &\quad+\frac{s(N+\mathbf{p}-2)}{N+\mathbf{p}-2+2s}\cdot\varepsilon^2\sup_{y\in B_\varepsilon}\big|\nabla^2u(y)-\nabla^2u(x)\big|+\frac{2s(\mathbf{p}-2)}{N+\mathbf{p}-2+2s}\cdot\varepsilon^3\frac{|\nabla^2u(x)|^2}{|p_x|},\end{split}$$

where the quantity m_ε is as in the statement of Lemma 3.3.

Proof. We sum up formulas in Lemma 3.3 and Proposition 4.1, and multiply the result by the factor: $\frac{(1-s)s}{C(N,s)|A_{\mathbf{p}}|}\cdot\frac{N+\mathbf{p}-2}{N+\mathbf{p}-2+2s}\cdot\varepsilon^{2s}$. ∎

Remark 4.3. (i) Analysis similar to Remark 3.2 allows for computing the order of the error bound in Theorem 4.2 when $u\in C^{0,\alpha}\big(\mathbb{R}^N\setminus\bar B_{r_x}\big)$. In particular, when $\alpha=1$ then the bounding quantity becomes:

$$C_{N,\mathbf{p},s}\cdot\Big(C\big(C_x,\frac{1}{|p_x|},\mathrm{Lip}_u\big)\cdot C(r_x)\varepsilon^{4s-1}+C\big(|\nabla^2u(x)|,\frac{1}{|p_x|}\big)\varepsilon^3+o(\varepsilon^2)\Big).$$

When additionally $u\in C^{2,1}(B_{r_x})$, the above quantity has order $\varepsilon^{4s-1}+\varepsilon^3$, which further reduces to ε^3 when $s=1$.

(ii) For a more precise analysis of the asymptotic expansion when $s\to1-$, note that:

$$\kappa_\varepsilon\leq\sup\Big\{m;\ m\in[0,2]\ \text{ and }\ m^2\leq\frac{32\,\omega_u(m)}{|p_x|}\cdot\frac{N+\mathbf{p}-2}{\mathbf{p}-1}\cdot\frac{r_x^{-2s}+r_x^{1-2s}}{2s-1}\varepsilon^{2s-1}\Big\},$$

$$\frac{8\|\nabla^2u\|_{L^\infty(B_{r_x})}}{|p_x|}\cdot\frac{2s-1}{1-s}\cdot\frac{r_x^{2-2s}-\varepsilon^{2-2s}}{\varepsilon^{1-2s}-{r_x}^{1-2s}}\leq\frac{8\|\nabla^2u\|_{L^\infty(B_{r_x})}}{|p_x|}\cdot16r_x^{2-2s}|\ln\varepsilon|\varepsilon^{2s-1}.$$

The first bound above is valid when $\varepsilon<\frac{r_x}{2}$ (see Remark 3.2 (i)), while for the second bound we used that: $r_x^{2-2s}-\varepsilon^{2-2s}\leq(2-2s)(\ln r_x-\ln\varepsilon)r_x^{2-2s}\leq4(1-s)|\ln\varepsilon|r_x^{2-2s}$, valid for all $\varepsilon<e^{-|\ln r_x|}$. It is thus clear that $m_\varepsilon\leq o(1)$ as $\varepsilon\to0+$, independently of $s\in(\frac{1}{2},1)$

bounded away from $\frac{1}{2}$. In particular, for each fixed ε, the bounding quantity in Theorem 4.2 converges to:

$$2\frac{\mathbf{p}-2}{N+\mathbf{p}}\cdot\varepsilon^3\frac{|\nabla^2u(x)|^2}{|p_x|}+\frac{N+\mathbf{p}-2}{N+\mathbf{p}}\cdot\varepsilon^2\sup_{y\in B_\varepsilon}\big|\nabla^2u(y)-\nabla^2u(x)\big|$$

as $s\to 1-$, which is consistent with (4.1).

5. The averaging operator $\mathcal{A}_\varepsilon$ and its dynamic programming principle

Let $\mathcal{D}\subset\mathbb{R}^N$ be open, bounded and let $F:\mathbb{R}^N\to\mathbb{R}$ be bounded, Borel. In this section we discuss the non-local Dirichlet-type problem in:

$$u(x)=\begin{cases}\mathcal{A}_\varepsilon u(x) & \text{for } x\in\mathcal{D}\\ F(x) & \text{for } x\in\mathbb{R}^N\setminus\mathcal{D}.\end{cases}\tag{DPP$_\varepsilon$}$$

Equivalently, the above equation can be written as $u=S_\varepsilon u$, where the operator S_ε applied on a bounded Borel function $v:\mathbb{R}^N\to\mathbb{R}$ returns the bounded Borel function:

$$S_\varepsilon v=\mathbb{1}_{\mathcal{D}}\cdot\mathcal{A}_\varepsilon v+\mathbb{1}_{\mathbb{R}^N\setminus\mathcal{D}}\cdot F\tag{5.1}$$

The main result of this section is the following observation:

Theorem 5.1. *For any bounded Borel data $F:\mathbb{R}^N\to\mathbb{R}$, the problem* $(\mathrm{DPP})_\varepsilon$ *has the unique bounded Borel solution $u_\varepsilon^F:\mathbb{R}^N\to\mathbb{R}$ and there holds: $\|u_\varepsilon^F\|_{L^\infty}\le\|F\|_{L^\infty}$. Moreover, the solution operator to* $(\mathrm{DPP})_\varepsilon$ *is monotone, that is $F\le\bar{F}$ implies $u_\varepsilon^F\le u_\varepsilon^{\bar{F}}$.*

Before the proof, we derive another useful property:

Lemma 5.2. *Let $u:\mathbb{R}^N\to\mathbb{R}$ be bounded Borel. Then the following functions of x:*

$$\inf_{|y|=1}\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}u(x+z)\,\mathrm{d}\mu_s^N(z),\qquad\sup_{|y|=1}\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}u(x+z)\,\mathrm{d}\mu_s^N(z),\qquad\mathcal{A}_\varepsilon u,$$

are uniformly continuous on $\mathbb{R}^N$.

Proof. Denote any of the three listed functions by f and observe that, for a fixed $x,\bar{x}\in\mathbb{R}^N$ satisfying $|x-\bar{x}|<1$ there holds:

$$|f(x)-f(\bar{x})|\le\sup_{|y|=1}\Big|\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}u(x+z)\,\mathrm{d}\mu_s^N(z)-\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}u(\bar{x}+z)\,\mathrm{d}\mu_s^N(z)\Big|\tag{5.2}$$

For any $|y|=1$, we may write:

$$\begin{aligned}&\Big|\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}u(x+z)\,\mathrm{d}\mu_s^N(z)-\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}u(\bar{x}+z)\,\mathrm{d}\mu_s^N(z)\Big|\\&\quad=\frac{C(N,s)}{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty})}\cdot\Big|\int_{x+T_{\mathbf{p}}^{\varepsilon,\infty}(y)}\frac{u(z)}{|x-z|^{N+2s}}\,\mathrm{d}z-\int_{\bar{x}+T_{\mathbf{p}}^{\varepsilon,\infty}(y)}\frac{u(z)}{|\bar{x}-z|^{N+2s}}\,\mathrm{d}z\Big|\\&\quad\le\frac{C(N,s)\|u\|_{L^\infty}}{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty})}\cdot\Big(\int_{(x+T_{\mathbf{p}}^{\varepsilon,\infty}(y))\cap(\bar{x}+T_{\mathbf{p}}^{\varepsilon,\infty}(y))}\Big|\frac{1}{|x-z|^{N+2s}}-\frac{1}{|\bar{x}-z|^{N+2s}}\Big|\,\mathrm{d}z\\&\qquad+\int_{(x+T_{\mathbf{p}}^{\varepsilon,\infty}(y))\setminus(\bar{x}+T_{\mathbf{p}}^{\varepsilon,\infty}(y))}\frac{1}{|x-z|^{N+2s}}\,\mathrm{d}z+\int_{(\bar{x}+T_{\mathbf{p}}^{\varepsilon,\infty}(y))\setminus(x+T_{\mathbf{p}}^{\varepsilon,\infty}(y))}\frac{1}{|\bar{x}-z|^{N+2s}}\,\mathrm{d}z\Big).\end{aligned}$$

We now estimate the three last integral terms above. Since $|\bar{x}-z| > \frac{1}{2}|x-z|$ whenever $|x-z|>2$, it follows that for any $r\geq 2$ there holds:

$$\begin{aligned} \int_{(x+T_{\mathbf{p}}^{\varepsilon,\infty}(y))\cap(\bar{x}+T_{\mathbf{p}}^{\varepsilon,\infty}(y))} & \Big|\frac{1}{|x-z|^{N+2s}} - \frac{1}{|\bar{x}-z|^{N+2s}}\Big| \,\mathrm{d}z \\ & \leq \int_{x+T_{\mathbf{p}}^{r,\infty}} \frac{1+2^{N+2s}}{|x-z|^{N+2s}}\,\mathrm{d}z + \int_{x+T_{\mathbf{p}}^{\varepsilon,r}} \frac{N+2s}{\varepsilon^{N+1+2s}}|x-\bar{x}|\,\mathrm{d}z \\ & \leq \frac{1+2^{N+2s}}{2s}|A_{\mathbf{p}}|\cdot r^{-2s} + \frac{(N+2s)r^N}{N\varepsilon^{N+1+2s}}|A_{\mathbf{p}}|\cdot|x-\bar{x}|, \end{aligned} \tag{5.3}$$

where we also used that: $\big||x-z|^{-N-2s}-|\bar{x}-z|^{-N-2s}\big| \leq \frac{N+2s}{\varepsilon^{N+1+2s}}|x-\bar{x}|$ for $z\in(x+T_{\mathbf{p}}^{\varepsilon,\infty}(y))\cap(\bar{x}+T_{\mathbf{p}}^{\varepsilon,\infty}(y))$. On the other hand, we have:

$$\begin{aligned} \int_{(x+T_{\mathbf{p}}^{\varepsilon,\infty}(y))\setminus(\bar{x}+T_{\mathbf{p}}^{\varepsilon,\infty}(y))} & \frac{1}{|x-z|^{N+2s}}\,\mathrm{d}z \\ & \leq \frac{|A_{\mathbf{p}}|}{2s}r^{-2s} + \big|T_{\mathbf{p}}^{\varepsilon,r}(y)\setminus((\bar{x}-x)+T_{\mathbf{p}}^{\varepsilon,\infty}(y))\big|\cdot\frac{1}{\varepsilon^{N+2s}} \end{aligned} \tag{5.4}$$

Further, it easily follows that:

$$\sup_{|y|=1}\big|T_{\mathbf{p}}^{\varepsilon,r}(y)\setminus((\bar{x}-x)+T_{\mathbf{p}}^{\varepsilon,\infty}(y))\big| \leq \sup_{|z|<|x-\bar{x}|}\big|T_{\mathbf{p}}^{\varepsilon,r}\setminus(z+T_{\mathbf{p}}^{\varepsilon,\infty})\big| \leq \big|(\partial T_{\mathbf{p}}^{\varepsilon,r})+B_{|x-\bar{x}|}\big|.$$

In conclusion (5.2) becomes, in view of (5.3) and (5.4):

$$\begin{aligned} \big|f(x)-f(\bar{x})\big| \leq \frac{C(N,s)\|u\|_{L^\infty}}{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty})}\cdot\Big(&\frac{3+2^{N+2s}}{2s}|A_{\mathbf{p}}|\cdot r^{-2s} + \frac{(N+2s)r^N}{N\varepsilon^{N+1+2s}}|A_{\mathbf{p}}|\cdot|x-\bar{x}| \\ & + \big|(\partial T_{\mathbf{p}}^{\varepsilon,r})+B_{|x-\bar{x}|}\big|\cdot\frac{2}{\varepsilon^{N+2s}}\Big). \end{aligned}$$

It is clear that by taking r large and then $|x-\bar{x}|$ appropriately small, the right hand side above can be bounded by any $\delta>0$. This proves the claimed uniform continuity. ■

Proof of Theorem 5.1

Define $v_0 \equiv \inf F$ and set $v_n \doteq (S_\varepsilon)^n v_0$, where the operator S_ε is as in (5.1). Each function $v_n:\mathbb{R}^N\to\mathbb{R}$ is continuous in $\mathcal{D}$ by Lemma 5.2 and hence Borel in $\mathbb{R}^N$. The sequence $\{v_n\}_{n=1}^\infty$ is uniformly bounded: $\|v_n\|_{L^\infty}\leq\|F\|_{L^\infty}$ and nondecreasing because $v_0\leq v_1$, and S_ε is order-preserving. Thus, $\{v_n\}_{n=1}^\infty$ has a pointwise limit $v:\mathbb{R}^N\to\mathbb{R}$ which is bounded Borel and obeys the same bound: $\|v\|_{L^\infty}\leq\|F\|_{L^\infty}$.

We now show that one can take $u_\varepsilon^F = v$. Indeed, for every $x\in\mathcal{D}$ there holds:

$$\begin{aligned} |v_{n+1}(x)-S_\varepsilon v(x)| = |S_\varepsilon v_n(x)-S_\varepsilon v(x)| & \leq \sup_{|y|=1} \fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)} |(v_n-v)(x+z)|\,\mathrm{d}\mu_s^N(z) \\ & \leq \frac{1}{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty})}\int_{\mathbb{R}^N\setminus B_\varepsilon} |(v_n-v)(x+z)|\,\mathrm{d}\mu_s^N(z) \ \to 0 \quad \text{ as } n\to\infty, \end{aligned}$$

by the monotone convergence theorem. We thus obtain: $v=S_\varepsilon v$ on $\mathcal{D}$, as claimed.

To prove uniqueness, assume that $v,\bar{v}$ are two bounded, Borel solutions of $(\mathrm{DPP})_\varepsilon$. Clearly $v=\bar{v}$ on $\mathbb{R}^N\setminus\mathcal{D}$ and denote $M\doteq\sup_{\mathcal{D}}|v-\bar{v}|$. For every $x\in\mathcal{D}$ we observe that:

$$|(v-\bar{v})(x)| = |\mathcal{A}_\varepsilon v(x)-\mathcal{A}_\varepsilon\bar{v}(x)| \leq \sup_{|y|=1}\fint_{x+T_{\mathbf{p}}^{\varepsilon,\infty}(y)} |(v-\bar{v})(z)|\,\mathrm{d}\mu_s^N(z) \leq M\cdot\frac{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\mathrm{diam}\mathcal{D}})}{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty})}.$$

Hence: $M \leq M \cdot \frac{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\mathrm{diam}\mathcal{D}})}{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty})}$, so there must be $M = 0$ and thus $v = \bar{v}$ in $\mathcal{D}$.

Finally, the claimed monotonicity of the solution operator to $(\mathrm{DPP})_\varepsilon$ follows from the monotonicity of S_ε. The proof is done. ■

6. Convergence to viscosity solutions of $\Delta_{\mathbf{p}}^s u = 0$.

In this section we will identify the limits of solutions to the non-local dynamic programming principle $(\mathrm{DPP})_\varepsilon$ in the vanishing removed singularity radius $\varepsilon \to 0$, as viscosity solutions to the homogeneous Dirichlet problem for $\Delta_{\mathbf{p}}^s$.

Theorem 6.1. *Let $F : \mathbb{R}^N \to \mathbb{R}$ be uniformly continuous and bounded, and let $\mathcal{D}$ be an open, bounded subset of $\mathbb{R}^N$. Assume that $\{u_\varepsilon\}_{\varepsilon\in J}$ is a sequence of solutions to $(\mathrm{DPP})_\varepsilon$ which converges as $\varepsilon \to 0$, $\varepsilon \in J$ uniformly, to some continuous limit function $u \in C(\mathbb{R}^N)$. Then for every $x \in \mathcal{D}$, $r > 0$ and every $\phi \in C^2(\mathbb{R}^N)$ such that $\phi(x) = u(x)$ and $\nabla\phi(x) \neq 0$, we have:*

(i) if $\phi > u$ on $\bar{B}_r(x) \setminus \{x\}$, then $\mathcal{L}_{s,\mathbf{p}}[\tilde{\phi}](x) \geq 0$,
(ii) if $\phi < u$ on $\bar{B}_r(x) \setminus \{x\}$, then $\mathcal{L}_{s,\mathbf{p}}[\tilde{\phi}](x) \leq 0$,

where we denoted: $\tilde{\phi} = \mathbb{1}_{\bar{B}_r(x)} \cdot \phi + \mathbb{1}_{\mathbb{R}^N \setminus \bar{B}_r(x)} \cdot u$.

Remark 6.2. Recall by Remark 2.6 that $\mathcal{L}_{s,\mathbf{p}}$ differs from $\Delta_{\mathbf{p}}^s$ only by a multiplicative constant, depending on $N, s, \mathbf{p}$. It is also clear that u as in Theorem 6.1 satisfies $u = F$ in $\mathbb{R}^N \setminus \mathcal{D}$. A function u satisfying only the one-sided comparison with test functions (i) (rather than both conditions (i) and (ii)) is called a viscosity subsolution to the non-local Dirichlet problem:

$$\Delta_{\mathbf{p}}^s u = 0 \quad \text{in } \mathcal{D}, \qquad u = F \quad \text{in } \mathbb{R}^N \setminus \mathcal{D}. \tag{6.1}$$

When (i) is replaced by (ii), then u is called a viscosity supersolution. Satisfaction of both conditions is referred to u being a viscosity solution of (6.1). See also [2, Definition 4.4].

Proof of Theorem 6.1

1. Let ϕ, r and $x \in \mathcal{D}$ be as indicated. We will show that (ii) holds, while the property (i) can be deduced by a symmetric argument.

For all $j \in \mathbb{N}$ such that $\bar{B}_{1/j}(x) \subset \mathcal{D}$, define $\varepsilon_j > 0$ by requesting that:

$$\|u_\varepsilon - u\|_{L^\infty} \leq \frac{1}{2} \min_{\bar{B}_r(x)\setminus B_{1/j}(x)} (u - \phi) \qquad \text{for all } \ \varepsilon \leq \varepsilon_j, \ \varepsilon \in J.$$

Without loss of generality, the sequence $\{\varepsilon_j\}_{j\to\infty}$ is decreasing to 0. Let $\{x_\varepsilon \in \mathcal{D}\}_{\varepsilon\in J}$ be a sequence with the property that:

$$(u_\varepsilon - \phi)(x_\varepsilon) = \min_{\bar{B}_{1/j}(x)} (u_\varepsilon - \phi) \qquad \text{and} \qquad x_\varepsilon \in \bar{B}_{1/j}(x) \ \text{ for all } \ \varepsilon \in (\varepsilon_{j+1}, \varepsilon_j] \cap J.$$

Then, for all $\bar{x} \in \bar{B}_r(x) \setminus B_{1/j}(x)$ we have:

$$\begin{aligned}(u_\varepsilon - \phi)(\bar{x}) &\geq (u - \phi)(\bar{x}) - \frac{1}{2} \min_{\bar{B}_r(x)\setminus B_{1/j}(x)} (u - \phi) \geq \frac{1}{2} \min_{\bar{B}_r(x)\setminus B_{1/j}(x)} (u - \phi) \\ &\geq (u_\varepsilon - u)(x) = (u_\varepsilon - \phi)(x) \geq (u_\varepsilon - \phi)(x_\varepsilon).\end{aligned}$$

This implies:

$$(u_\varepsilon - \phi)(x_\varepsilon) = \min_{\bar{B}_r(x)} (u_\varepsilon - \phi) \ \text{ for all } \ \varepsilon \in J \qquad \text{and} \qquad x_\varepsilon \to x \ \text{ as } \ \varepsilon \to 0, \ \varepsilon \in J. \tag{6.2}$$

2. Since u_ε satisfies $(\mathrm{DPP})_\varepsilon$, it follows that:

$$\begin{aligned}\mathcal{A}_\varepsilon\tilde\phi(x_\varepsilon)-\tilde\phi(x_\varepsilon) &= \big(\mathcal{A}_\varepsilon\tilde\phi(x_\varepsilon)-\tilde\phi(x_\varepsilon)\big)-\big(\mathcal{A}_\varepsilon u_\varepsilon(x_\varepsilon)-u_\varepsilon(x_\varepsilon)\big)\\ &\leq \sup_{|y|=1}\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}\tilde\phi(x_\varepsilon+z)-u_\varepsilon(x_\varepsilon+z)-\phi(x_\varepsilon)+u_\varepsilon(x_\varepsilon)\,\mathrm{d}\mu_s^N(z).\end{aligned}$$

We fix $|y|=1$ and estimate the integral above. By (6.2) it follows that $\tilde\phi(x_\varepsilon+z)-u_\varepsilon(x_\varepsilon+z)\leq \phi(x_\varepsilon)-u_\varepsilon(x_\varepsilon)$ whenever $x_\varepsilon+z\in\bar B_r(x)$, so the said integral is bounded by:

$$\begin{aligned}&\frac{1}{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty})}\int_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)\setminus\bar B_r(x-x_\varepsilon)}u(x_\varepsilon+z)-u_\varepsilon(x_\varepsilon+z)-\phi(x_\varepsilon)+u_\varepsilon(x_\varepsilon)\,\mathrm{d}\mu_s^N(z)\\ &\qquad\leq\frac{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty}(y)\setminus\bar B_r(x-x_\varepsilon))}{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty})}\cdot\big(2\|u-u_\varepsilon\|_{L^\infty}+|u(x_\varepsilon)-\phi(x_\varepsilon)|\big).\end{aligned}$$

Hence we get:

$$\begin{aligned}\mathcal{A}_\varepsilon\tilde\phi(x_\varepsilon)-\tilde\phi(x_\varepsilon)&\leq\frac{\mu_s^N(T_{\mathbf{p}}^{r/2,\infty})}{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty})}\cdot\big(2\|u-u_\varepsilon\|_{L^\infty}+|u(x_\varepsilon)-u(x)|+|\phi(x_\varepsilon)-\phi(x)|\big)\\ &= o(\varepsilon^{2s})\quad\text{ as }\ \varepsilon\to 0.\end{aligned}\tag{6.3}$$

3. In this step, we show that:

$$\big|\mathcal{L}_{s,\mathbf{p}}^\varepsilon[\tilde\phi](x_\varepsilon)-\mathcal{L}_{s,\mathbf{p}}[\tilde\phi](x_\varepsilon)\big|=o(1)\quad\text{ as }\ \varepsilon\to 0.\tag{6.4}$$

The argument relies on verifying the proof of Lemma 3.3. With the parallel notation $m=\big|\frac{p_x^\varepsilon}{|p_x^\varepsilon|}-y_\delta^\varepsilon\big|$, where $p_x^\varepsilon=\nabla\phi(x_\varepsilon)\neq 0$, and where the unit vector y_δ^ε is an almost maximizer of the function $y\mapsto\int_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}\tilde\phi(x_\varepsilon+z)\,\mathrm{d}\mu_s^N(z)$, we get the following replacement of (3.5):

$$\begin{aligned}\delta &\geq\int_{T_{\mathbf{p}}^{\varepsilon,r-|x_\varepsilon-x|}(\frac{p_x^\varepsilon}{|p_x^\varepsilon|})}\phi(x_\varepsilon+z)-\phi\big(x_\varepsilon+R_{y_\delta^\varepsilon,\frac{p_x^\varepsilon}{|p_x^\varepsilon|}}z\big)\,\mathrm{d}\mu_s^N(z)\\ &\quad-\omega_u(m)\cdot\int_{T_{\mathbf{p}}^{r+|x_\varepsilon-x|,\infty}}(1+|z|)\,\mathrm{d}\mu_s^N(z)-2\|\tilde\phi\|_{L^\infty}\cdot\mu_s^N\big(T_{\mathbf{p}}^{r-|x_\varepsilon-x|,r+|x_\varepsilon-x|}\big)\\ &\geq\Big\langle p_x^\varepsilon,\big(Id_N-R_{y_\delta^\varepsilon,\frac{p_x^\varepsilon}{|p_x^\varepsilon|}}\big)\int_{T_{\mathbf{p}}^{\varepsilon,r-|x_\varepsilon-x|}(\frac{p_x^\varepsilon}{|p_x^\varepsilon|})}z\,\mathrm{d}\mu_s^N(z)\Big\rangle-O(1)m-O(1)\omega_u(m)-O(1)|x_\varepsilon-x|\\ &\geq\frac{m^2|p_x^\varepsilon|}{2}\int_{T_{\mathbf{p}}^{\varepsilon,r/2}}\langle z,e_1\rangle\,\mathrm{d}\mu_s^N(z)-O(1),\end{aligned}$$

where $O(1)$ depends on $N,s,\mathbf{p},r$ and $\|\tilde\phi\|_{L^\infty}$, $\|\nabla^2\phi\|_{L^\infty(\bar B_r(x))}$. Consequently:

$$m^2\leq\frac{O(1)}{|p_x^\varepsilon|\int_{T_{\mathbf{p}}^{\varepsilon,r/2}}\langle z,e_1\rangle\,\mathrm{d}\mu_s^N(z)}\leq\frac{O(1)}{|p_x^\varepsilon|}\varepsilon^{2s-1}.\tag{6.5}$$

Further, as in (3.8) we get:

$$\Big|\mathcal{L}_{s,\mathbf{p}}^\varepsilon[\tilde\phi](x_\varepsilon)-\int_{T_{\mathbf{p}}^{\varepsilon,\infty}(\frac{p_x^\varepsilon}{|p_x^\varepsilon|})}L_{\tilde\phi}(x_\varepsilon,z,z)\,\mathrm{d}\mu_s^N(z)\Big|\leq\delta+J_1+J_2+J_3,$$

where:

$$\begin{aligned} J_1 &\le \Big|\int_{T_{\mathbf{p}}^{\varepsilon, r-|x_\varepsilon - x|}(\frac{p_x^\varepsilon}{|p_x^\varepsilon|})} \phi(x_\varepsilon+z)-\phi\big(x_\varepsilon+R_{y_\delta^\varepsilon,\frac{p_x^\varepsilon}{|p_x^\varepsilon|}}z\big)+\phi(x_\varepsilon-z)-\phi\big(x_\varepsilon-R_{y_\delta^\varepsilon,\frac{p_x^\varepsilon}{|p_x^\varepsilon|}}z\big)\,\mathrm{d}\mu_s^N(z)\Big| \\ &\le \int_{T_{\mathbf{p}}^{\varepsilon, r-|x_\varepsilon - x|}(\frac{p_x^\varepsilon}{|p_x^\varepsilon|})} \Big|\big\langle \nabla\phi(x_\varepsilon+z)-\nabla\phi(x_\varepsilon-x), \big(Id_N - R_{y_\delta^\varepsilon,\frac{p_x^\varepsilon}{|p_x^\varepsilon|}}\big)z\big\rangle\Big|\,\mathrm{d}\mu_s^N(z) + O(1)m^2 \\ &\le O(1)m, \end{aligned}$$

$$\begin{aligned} J_2 &\le \Big|\int_{T_{\mathbf{p}}^{\varepsilon, r+|x_\varepsilon - x|}(\frac{p_x^\varepsilon}{|p_x^\varepsilon|})} u(x_\varepsilon+z)-u\big(x_\varepsilon+R_{y_\delta^\varepsilon,\frac{p_x^\varepsilon}{|p_x^\varepsilon|}}z\big)+u(x_\varepsilon-z)-u\big(x_\varepsilon-R_{y_\delta^\varepsilon,\frac{p_x^\varepsilon}{|p_x^\varepsilon|}}z\big)\,\mathrm{d}\mu_s^N(z)\Big| \\ &\le O(1)\omega_u(m), \\ J_3 &\le \int_{T_{\mathbf{p}}^{r-|x_\varepsilon - x|, r+|x_\varepsilon-x|}} 4\|\tilde\phi\|_{L^\infty}\,\mathrm{d}\mu_s^N(z) \le O(1)|x_\varepsilon - x|. \end{aligned}$$

After passing $\delta\to 0$ and recalling (6.5), we conclude:

$$\Big|\mathcal{L}_{s,\mathbf{p}}^\varepsilon[\tilde\phi](x_\varepsilon) - \int_{T_{\mathbf{p}}^{\varepsilon,\infty}(\frac{p_x^\varepsilon}{|p_x^\varepsilon|})} L_{\tilde\phi}(x_\varepsilon,z,z)\,\mathrm{d}\mu_s^N(z)\Big| \le O(1)\big(m+\omega_u(m)+|x_\varepsilon-x|\big) = o(1) \quad \text{as } \varepsilon\to 0.$$

The above implies (6.4), in view of: $\big|\int_{T_{\mathbf{p}}^{0,\varepsilon}(\frac{p_x^\varepsilon}{|p_x^\varepsilon|})} L_{\tilde\phi}(x_\varepsilon,z,z)\,\mathrm{d}\mu_s^N(z)\big| = O(1)\varepsilon^{2-2s}$.

4. Recall that by Theorem 3.1 we also directly have:

$$\Big|\mathcal{L}_{s,\mathbf{p}}^\varepsilon[\tilde\phi](x) - \mathcal{L}_{s,\mathbf{p}}[\tilde\phi](x)\Big| \le o(1) \quad \text{as } \varepsilon\to 0,$$

because $\tilde\phi=\phi$ on $\bar B_r(x)$ has regularity C^2, while $\tilde\phi = u$ on $\mathbb{R}^N\setminus \bar B_r(x)$ is uniformly continuous. Together with (6.4), this yields:

$$\begin{aligned} \Big|\big(\mathcal{A}_\varepsilon\tilde\phi(x_\varepsilon)-\tilde\phi(x_\varepsilon)\big) - \big(\mathcal{A}_\varepsilon\tilde\phi(x)-\tilde\phi(x)\big)\Big| &= \frac{1}{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty})}\Big|\mathcal{L}_{s,\mathbf{p}}^\varepsilon[\tilde\phi](x_\varepsilon)-\mathcal{L}_{s,\mathbf{p}}^\varepsilon[\tilde\phi](x)\Big| \\ &= \frac{1}{\mu_s^N(T_{\mathbf{p}}^{\varepsilon,\infty})}\Big(\Big|\mathcal{L}_{s,\mathbf{p}}[\tilde\phi](x_\varepsilon)-\mathcal{L}_{s,\mathbf{p}}[\tilde\phi](x)\Big| + o(1)\Big) \quad \text{as } \varepsilon\to 0. \end{aligned} \tag{6.6}$$

We now denote $R \doteq R_{\frac{p_x^\varepsilon}{|p_x^\varepsilon|},\frac{p_x}{|p_x|}}$ and estimate:

$$\begin{aligned} &\Big|\mathcal{L}_{s,\mathbf{p}}[\tilde\phi](x_\varepsilon)-\mathcal{L}_{s,\mathbf{p}}[\tilde\phi](x)\Big| \\ &\le \int_{T_{\mathbf{p}}^{0,\infty}(\frac{p_x}{|p_x|})} \Big|\big(\tilde\phi(x_\varepsilon+Rz)+\tilde\phi(x_\varepsilon-Rz)-2\tilde\phi(x_\varepsilon)\big) - \big(\tilde\phi(x+z)+\tilde\phi(x-z)-2\phi(x)\big)\Big|\,\mathrm{d}\mu_s^N(z) \\ &\le \frac12\int_{T_{\mathbf{p}}^{0,r}} |z|^2\,\mathrm{d}\mu_s^N(z)\cdot \sup_{z\in B_r(x)}\Big|\big(\nabla\phi(x_\varepsilon+Rz)+\nabla\phi(x_\varepsilon-Rz)\big)-\big(\nabla\phi(x+Rz)+\nabla\phi(x-Rz)\big)\Big| \\ &\quad + \int_{T_{\mathbf{p}}^{r+|x_\varepsilon-x|,\infty}} 2\omega_u(|x_\varepsilon-x|)+2\omega_u(|Id_N-R|)\cdot(1+|z|)\,\mathrm{d}\mu_s^N(z) \\ &\quad + \int_{T_{\mathbf{p}}^{r-|x_\varepsilon-x|,r+|x_\varepsilon-x|}} 6\|\tilde\phi\|_{L^\infty}\,\mathrm{d}\mu_s^N(z) \\ &\le O(1)\cdot\Big(|x_\varepsilon-x|+\omega_u(|x_\varepsilon-x|)+\omega_u\big(\Big|\frac{p_x^\varepsilon}{|p_x^\varepsilon|}-\frac{p_x}{|p_x|}\Big|\big)\Big) \le o(1) \quad \text{as } \varepsilon\to 0. \end{aligned}$$

At this point, combining (6.6) with (6.3) results in:

$$\mathcal{A}_\varepsilon\tilde\phi(x) - \tilde\phi(x) \leq o(\varepsilon^{2s}) \quad \text{ as } \varepsilon\to 0.$$

On the other hand, Theorem 3.1 implies that:

$$\mathcal{A}_\varepsilon\tilde\phi(x) - \tilde\phi(x) = \frac{s}{C(N,s)|A_{\mathbf{p}}|}\varepsilon^{2s}\cdot\mathcal{L}_{s,\mathbf{p}}[\tilde\phi](x) + o(\varepsilon^{2s}).$$

The above two asymptotic statements directly yield $\mathcal{L}_{s,\mathbf{p}}[\tilde\phi](x)\leq 0$, as claimed. ■

7. The non-local Tug-of-War game with noise

In this section, we develop the basic probability setting related to the equation $(\mathrm{DPP})_\varepsilon$.

1. Consider the probability space $(T_{\mathbf{p}}^{1,\infty}, \mathcal{B}, \frac{1}{\mu_s^N(T_{\mathbf{p}}^{1,\infty})}\mu_s^N)$ equipped with the standard Borel σ-algebra and the normalised μ_s^N measure, and define $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$ as the product space with the counting measure on the discrete set $\{1,2\}$. In particular, for every $B\in\mathcal{B}$, we have:

$$\mathbb{P}_1\big(B\times\{1,2\}\big) = \frac{2s}{|A_{\mathbf{p}}|}\int_B \frac{1}{|z|^{N+2s}}\,\mathrm{d}z.$$

Further, the countable product of $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$ is denoted by $(\Omega,\mathcal{F},\mathbb{P})$, where:

$$\Omega = (\Omega_1)^{\mathbb{N}} = \big\{\omega = \{(z_i,s_i)\}_{i=1}^\infty;\ z_i\in T_{\mathbf{p}}^{1,\infty},\ s_i\in\{1,2\}\ \text{ for all } i\in\mathbb{N}\big\}.$$

For each $n\in\mathbb{N}$, the probability space $(\Omega_n,\mathcal{F}_n,\mathbb{P}_n)$ is the product of n copies of $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$ and the σ-algebra $\mathcal{F}_n$ is identified with the sub-σ-algebra of $\mathcal{F}$, consisting of sets $A\times\prod_{i=n+1}^\infty\Omega_1$ for all $A\in\mathcal{F}_n$. Then $\{\mathcal{F}_n\}_{n=0}^\infty$ where $\mathcal{F}_0=\{\emptyset,\Omega\}$, is a filtration of $\mathcal{F}$.

2. Given are two families of functions $\sigma_I = \{\sigma_I^n\}_{n=0}^\infty$ and $\sigma_{II}=\{\sigma_{II}^n\}_{n=0}^\infty$, defined on the corresponding spaces of "finite histories" $H_n = \mathbb{R}^N\times(\mathbb{R}^N\times\Omega_1)^n$:

$$\sigma_I^n, \sigma_{II}^n : H_n \to \{y\in\mathbb{R}^N;\ |y|=1\},$$

assumed to be measurable with respect to the (target) Borel σ-algebra and the (domain) product σ-algebra on H_n. For every $x\in\mathbb{R}^N$ and $\varepsilon\in(0,1)$ we recursively define:

$$\big\{X_n^{\varepsilon,x,\sigma_I,\sigma_{II}}:\Omega\to\mathbb{R}^N\big\}_{n=0}^\infty.$$

For simplicity of notation, we often suppress some of the superscripts $\varepsilon, x, \sigma_I, \sigma_{II}$ and write X_n instead of $X_n^{\varepsilon,x,\sigma_I,\sigma_{II}}$, if no ambiguity arises. Recall that $R_{\tilde y,y}\in SO(N)$ is as in Definition 2.1. We put:

$$\begin{aligned} &X_0\equiv x,\\ &X_n\big((z_1,s_1),\ldots,(z_n,s_n)\big) \doteq X_{n-1} + \left\{\begin{array}{ll} \varepsilon R_{\sigma_I^{n-1},e_1} z_n & \text{for } s_n = 1\\ \varepsilon R_{\sigma_{II}^{n-1},e_1} z_n & \text{for } s_n=2.\end{array}\right.\end{aligned} \tag{7.1}$$

In this "game", each of the two players chooses (deterministically) a direction y, according to their "strategy" σ_I and σ_{II}. These choices are activated by the value of the equally probable outcomes: $s_n=1$ activates σ_I and $s_n=2$ activates σ_{II}. The position X_{n-1} is then advanced by a shift $\varepsilon Rz\in T_{\mathbf{p}}^{\varepsilon,\infty}(y)$, randomly in z according to the normalised measure μ_s^N on $T_{\mathbf{p}}^{1,\infty}$.

3. Given an open, bounded domain $\mathcal{D}\subset\mathbb{R}^N$, define further the $\mathcal{F}$-measurable random variable: $\tau^{\varepsilon,x,\sigma_I,\sigma_{II}}:\Omega\to\mathbb{N}\cup\{+\infty\}$ by:

$$\tau(\omega)\doteq\min\big\{n\geq 1;\ X_n\not\in\mathcal{D}\big\}.$$

We observe that τ is finite $\mathbb{P}$-a.e., making it a stopping time. Indeed, since $\mathbb{P}_1(T_{\mathbf{p}}^{\mathrm{diam}\mathcal{D},\infty}\times\{1,2\})>0$ it follows that $\mathbb{P}\big(\omega;\ \exists i\ z_i\in T_{\mathbf{p}}^{\mathrm{diam}\mathcal{D},\infty}\big)=1$, and on this event $\tau<\infty$.

Let $F:\mathbb{R}^N\to\mathbb{R}$ be a given bounded, uniformly continuous function. In our "game", the first "player" collects from his opponent the payoff, given by the data F at the stopping position. The incentive of the collecting "player" to maximize the outcome and of the disbursing "player" to minimize it, leads to the definition of the two game values in:

$$\begin{aligned} u_I^\varepsilon(x) &= \sup_{\sigma_I}\inf_{\sigma_{II}}\mathbb{E}\Big[F\circ\big(X^{\varepsilon,x,\sigma_I,\sigma_{II}}\big)_{\tau^{\varepsilon,x,\sigma_I,\sigma_{II}}}\Big],\\ u_{II}^\varepsilon(x) &= \inf_{\sigma_{II}}\sup_{\sigma_I}\mathbb{E}\Big[F\circ\big(X^{\varepsilon,x,\sigma_I,\sigma_{II}}\big)_{\tau^{\varepsilon,x,\sigma_I,\sigma_{II}}}\Big]. \end{aligned}\tag{7.2}$$

It is clear that u_I^ε, u_{II}^ε depend only on the values of F on $\mathbb{R}^N\setminus\mathcal{D}$. We now show that both game values coincide with the unique solution to the dynamic programming principle $(\mathrm{DPP})_\varepsilon$ modeled on the non-local asymptotic expansion in Theorem 3.1.

Lemma 7.1. *For each $\varepsilon\ll 1$ we have $u_I^\varepsilon=u_{II}^\varepsilon=u_\varepsilon$, where u_ε is the unique bounded, Borel solution to* $(\mathrm{DPP})_\varepsilon$.

Proof. **1.** We will show that $u_{II}^\varepsilon\le u_\varepsilon$, while the inequality $u_\varepsilon\le u_I^\varepsilon$ can be proved by a symmetric argument and $u_I^\varepsilon\le u_{II}^\varepsilon$ is always valid. Fix $x\in\mathbb{R}^N$ and $\varepsilon,\delta>0$. We choose a strategy $\sigma_{II,0}=\{\sigma_{II,0}^n(X_n)\}_{n=0}^\infty$ satisfying:

$$\inf_{|y|=1}\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}u_\varepsilon(x+z)\,\mathrm{d}\mu_s^N(z)+\frac{\delta}{2^{n+1}}\ge\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(\sigma_{II,0}^n)}u_\varepsilon(x+z)\,\mathrm{d}\mu_s^N(z).\tag{7.3}$$

The fact that such Borel-regular strategy exists follows from Lemma 5.2. Indeed, let $\{B(x_i,\xi)\}_{i=1}^\infty$ be a locally finite covering of $\mathbb{R}^N$, where:

$$\Big|\inf_{|y|=1}\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}u_\varepsilon(x+z)\,\mathrm{d}\mu_s^N(z)-\inf_{|y|=1}\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}u_\varepsilon(\bar x+z)\,\mathrm{d}\mu_s^N(z)\Big|\le\frac{\delta}{2^{n+2}}\qquad\text{for all }\ |x-\bar x|<\xi.$$

For each $i\in\mathbb{N}$ there exists then $|y_i|=1$ with the property: $\big|\inf_{|y|=1}\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}u_\varepsilon(x_i+z)\,\mathrm{d}\mu_s^N(z)-\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y_i)}u_\varepsilon(x_i+z)\,\mathrm{d}\mu_s^N(z)\big|<\frac{\delta}{2^{n+2}}$. We hence define:

$$\sigma_{II,0}^n(x)=y_i\qquad\text{for all }\ x\in B(x_i,\xi)\setminus\bigcup_{j=1}^{i-1}B(x_j,\xi).$$

2. Fix $x\in\mathcal{D}$ and a strategy σ_I. Consider the sequence of random variables:

$$M_n\doteq u_\varepsilon\circ X_{n\wedge\tau}^{\varepsilon,x,\sigma_I,\sigma_{II,0}}+\frac{\delta}{2^n}.$$

We now check that $\{M_n\}_{n=1}^\infty$ is a supermartingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$. On the event $n>\tau$ there clearly holds $\mathbb{E}(M_n|\mathcal{F}_{n-1})=M_{n-1}$. On the other hand, on the event $n\le\tau$ we have $X_{n-1}\in\mathcal{D}$, so the property $(\mathrm{DPP})_\varepsilon$ and (7.3) imply that:

$$\begin{aligned}&\mathbb{E}(M_n\mid\mathcal{F}_{n-1})-M_{n-1}\\ &=\frac12\Big(\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(\sigma_I^n)}u_\varepsilon(X_{n-1}+z)\,\mathrm{d}\mu_s^N(z)+\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(\sigma_{II,0}^n)}u_\varepsilon(X_{n-1}+z)\,\mathrm{d}\mu_s^N(z)\Big)-\frac{\delta}{2^{n+1}}-u_\varepsilon(X_{n-1})\\ &\le\mathcal{A}_\varepsilon u_\varepsilon(X_{n-1})-u_\varepsilon(X_{n-1})=0.\end{aligned}$$

Using Doob's optional stopping theorem, we arrive at:

$$u_\varepsilon(x)+\delta=\mathbb{E}[M_0]\geq\mathbb{E}[M_\tau]=\frac{\delta}{2^\tau}+\mathbb{E}\big[F\circ X_\tau^{\varepsilon,x,\sigma_I,\sigma_{II,0}}\big]\geq\frac{\delta}{2^\tau}+\inf_{\sigma_{II}}\mathbb{E}\big[F\circ X_\tau^{\varepsilon,x,\sigma_I,\sigma_{II}}\big].$$

This yields: $u_\varepsilon(x)\geq\delta\geq u_{II}^\varepsilon(x)$, concluding the proof in view of δ being arbitrary. ■

8. Auxiliary estimates for the barrier function

The purpose of this section is to show the first boundary regularity estimate for the game process $\{X_n\}_{n=0}^\infty$, towards establishing our main asymptotic equicontinuity result.

Theorem 8.1. *Let $\mathcal{D}\subset\mathbb{R}^N$ be an open, bounded domain satisfying the external cone condition. Namely, assume that there exists a finite cone C such that for each $x\in\partial\mathcal{D}$ there holds: $x+S_xC\subset\mathbb{R}^N\setminus\mathcal{D}$, for some rotation $S_x\in SO(N)$. Then:*

$$\begin{split}\forall\delta>0\quad\exists\hat\delta<\delta,\ \hat\varepsilon>0\quad\forall\varepsilon<\hat\varepsilon,\ x\in\partial\mathcal{D},\ x_0\in B_{\hat\delta}(x)\cap\mathcal{D}\\ \exists\sigma_{I,0}\quad\forall\sigma_{II}\quad\mathbb{P}\big(\exists n<\tau\quad X_n^{\varepsilon,x_0,\sigma_{I,0},\sigma_{II}}\not\in B_\delta(x)\big)\leq\bar\theta,\end{split}\tag{8.1}$$

with a constant $\bar\theta<1$ depending only on $N,\mathbf{p},s$ and the cone C.

The proof will rely on a suitable barrier functions, introduced in [2]. Namely, consider the uniformly continuous and bounded $f_t:\mathbb{R}^N\to\mathbb{R}$, where for each $t>0$ we define:

$$f_t(x)=\min\big\{2^t,|x|^{-t}\big\}.$$

We start by observing a refinement of [2, Lemma 3.10]:

Proposition 8.2. *There exists $t_0\gg1$ depending on $N,\mathbf{p},s$, such that for all $t\geq t_0$ and all $|x|\geq1$ there holds:*

$$\mathcal{L}_{s,\mathbf{p}}[f_t](x)\geq C|x|^{-2s-t},$$

with a constant C depending on $N,\mathbf{p},s$ but not on t or x.

Proof. Observe that $\mathcal{L}_{s,\mathbf{p}}[f_t](x)=\mathcal{L}_{s,\mathbf{p}}[f_t](|x|e_1)$ by rotational invariance. It hence suffices to estimate, after changing variables:

$$\begin{split}\int_{T_{\mathbf{p}}^{0,\infty}}\frac{L_{f_t}(|x|e_1,z,z)}{|z|^{N+2s}}\,\mathrm{d}z&=|x|^{-2s}\int_{T_{\mathbf{p}}^{0,\infty}}\frac{f_t(|x|(e_1+z))+f_t(|x|(e_1-z))-2f_t(|x|e_1)}{|z|^{N+2s}}\,\mathrm{d}z\\&\geq|x|^{-2s-t}\int_{T_{\mathbf{p}}^{0,\infty}}\frac{f_t(e_1+z)+f_t(e_1-z)-2f_t(e_1)}{|z|^{N+2s}}\,\mathrm{d}z,\end{split}\tag{8.2}$$

where in the last step above we used that $f_t(|x|z)\geq|x|^{-t}f_t(z)$ which can be easily checked directly. Further, $L_{f_t}(e_1,z,z)\geq-2$ for all z, while for $|z|\leq\frac12$ we have:

$$\begin{split}L_{f_t}(e_1,z,z)&=|1+|z|^2+2\langle e_1,z\rangle|^{-t/2}+|1+|z|^2-2\langle e_1,z\rangle|^{-t/2}-2\\&\geq2(1+|z|^2)^{-t/2}+\frac t2\big(\frac t2+1\big)(2\langle e_1,z\rangle)^2\big(1+|z|^2\big)^{-t/2-2}-2\\&\geq2\big(1-\frac t2|z|^2\big)+\frac t2(t+2)\langle e_1,z\rangle^2\big(1-\big(\frac t2+2\big)|z|^2\big)^{-t/2-2}-2\\&=\frac t2(t+2)\langle e_1,z\rangle^2-t|z|^2-\frac t4(t+2)(t+4)\langle e_1,z\rangle^2|z|^2,\end{split}$$

by Taylor's expansion and since $(1+|z|^2)^{-\alpha} \geq 1-\alpha|z|^2$ whenever $\alpha>0$. Thus (8.2) becomes:

$$\begin{aligned}\int_{T_{\mathbf{p}}^{0,\infty}} \frac{L_{f_t}(\rho e_1,z,z)}{|z|^{N+2s}}\,\mathrm{d}z &\geq |x|^{-2s-t}\Big(\int_{T_{\mathbf{p}}^{0,r}} \frac{L_{f_t}(e_1,z,z)}{|z|^{N+2s}}\,\mathrm{d}z - \int_{T_{\mathbf{p}}^{r,\infty}} \frac{2}{|z|^{N+2s}}\,\mathrm{d}z\Big) \\ = |x|^{-2s-t}|A_{\mathbf{p}}|\cdot\Big(\frac{t}{2}(t+2)\frac{\mathbf{p}-1}{N+\mathbf{p}-2}\cdot\frac{r^{2-2s}}{2-2s} &- t\frac{r^{2-2s}}{2-2s} \\ -\frac{t}{4}(t+2)(t+4)\frac{\mathbf{p}-1}{N+\mathbf{p}-2}\cdot\frac{r^{4-2s}}{4-2s} - \frac{r^{-2s}}{s}\Big) &\doteq |x|^{-2s-t}|A_{\mathbf{p}}|\cdot I_{N,\mathbf{p},s,t,r}\end{aligned}$$

by recalling Lemma 2.5 (i) and where we fixed some appropriate $r<\frac{1}{2}$. We now estimate the quantity $I_{N,\mathbf{p},s,t,r}$. When $\frac{t+2}{2}\cdot\frac{\mathbf{p}-1}{N+\mathbf{p}-2}\geq 2$, then the first two terms in $I_{N,\mathbf{p},s,t,r}$ are bounded from below by: $\frac{t}{4}(t+2)\frac{\mathbf{p}-1}{N+\mathbf{p}-2}\cdot\frac{r^{2-2s}}{2-2s}$. Further, when $(t+4)\frac{r^2}{4-2s}\leq\frac{1}{2(2-2s)}$, then the first three terms in $I_{N,\mathbf{p},s,t,r}$ are bounded from below by: $\frac{t}{8}(t+2)\frac{\mathbf{p}-1}{N+\mathbf{p}-2}\cdot\frac{r^{2-2s}}{2-2s}$. Finally, when $\frac{t}{8}(t+2)\frac{\mathbf{p}-1}{N+\mathbf{p}-2}\cdot\frac{r^2}{2-2s}\geq\frac{2}{s}$, then we have:

$$I_{N,\mathbf{p},s,t,r}\geq\frac{r^{-2s}}{s}\geq\frac{1}{s}\Big(\frac{2-2s}{2-s}(t+4)\Big)^s.$$

It is clear that the above listed conditions, namely:

$$\frac{t+2}{2}\cdot\frac{\mathbf{p}-1}{N+\mathbf{p}-2}\geq 2 \quad\text{ and }\quad \exists r<\frac{1}{2}:\quad \frac{16(2-2s)(N+\mathbf{p}-2)}{s(\mathbf{p}-1)}\cdot\frac{1}{t(t+2)}\leq r^2\leq\frac{2-s}{2-2s}\cdot\frac{1}{t+4}$$

are compatible for sufficiently large $t\geq t_0(N,\mathbf{p},s)$. The proof is done. ■

Corollary 8.3. *Let $t_0\gg 1$ be as Proposition 8.2. For every $t\geq t_0$ and $R>1$ there exists $\varepsilon_0>0$, depending on $N,\mathbf{p},s,t,R$ such that:*

$$\mathcal{A}_\varepsilon f_t(x)\geq f_t(x)+C\varepsilon^{2s}R^{-2s-t}\qquad\text{for all }\ |x|\in[1,R],\ \varepsilon<\varepsilon_0,$$

with a constant C depending only on $N,\mathbf{p},s$.

Proof. We apply Theorem 3.1 with $r_x\in\big(\frac{1}{4},\frac{1}{2}\big)$ and note that $C_x\leq C(N,t)$ and $|p_x|\geq tR^{-t-1}$ whenever $|x|\in[1,R]$. In view of Proposition 8.2 it follows that:

$$\mathcal{A}_\varepsilon f_t(x)\geq f_t(x)+C_{N,\mathbf{p},s}\varepsilon^{2s}R^{-2s-t}-C_{N,s,t}\varepsilon^2-C_{N,s,t}\varepsilon^{2s}\big(m_\varepsilon+\omega_{f_t}(m_\varepsilon)\big).$$

Recalling Remark 3.2 (i) we see that: $m_\varepsilon\leq C_{N,\mathbf{p},s,t}\varepsilon^{s-\frac{1}{2}}R^{t+1}$ and $\omega_{f_t}(m_\varepsilon)\leq C_t m_\varepsilon$. Hence:

$$\mathcal{A}_\varepsilon f_t(x)\geq f_t(x)+C_{N,\mathbf{p},s}\varepsilon^{2s}\big(R^{-2s-t}-C_{N,s,t}\varepsilon^{2-2s}-C_{N,\mathbf{p},s,t}\varepsilon^{s-\frac{1}{2}}R^{t+1}\big),$$

and so the result follows for $C_{N,s,t}\varepsilon^{2-2s}+C_{N,\mathbf{p},s,t}\varepsilon^{s-\frac{1}{2}}R^{t+1}\leq\frac{1}{2}R^{-2s-t}$. ■

Towards the proof of Theorem 8.1, we note that:

Proposition 8.4. *Given $R>1$ and $t\geq t_0$, $\varepsilon<\varepsilon_0$ as in Proposition 8.2 and Corollary 8.3, let $v_\varepsilon:\mathbb{R}^N\to\mathbb{R}$ be the unique bounded, Borel solution to the problem:*

$$v_\varepsilon(x)=\begin{cases}\mathcal{A}_\varepsilon v_\varepsilon(x) & \text{for }\ |x|\in(1,R)\\ f_t(x) & \text{for }\ |x|\notin(1,R).\end{cases}$$

Then we have:

(i) $v_\varepsilon\geq f_t$ *in* $\mathbb{R}^N$,

(ii) For every $\tilde R\in(1,R)$ *exists* $\theta_{\tilde R,R}<1$ *depending only on* $R,\tilde R,N,\mathbf{p},s$ *such that for all* $|x|\in[1,\tilde R]$ *and* $\varepsilon<\varepsilon_0$ *there holds:*

$$\exists\sigma_{I,0}\quad\forall\sigma_{II}\qquad \mathbb{P}\big(|X_\tau^{\varepsilon,x,\sigma_{I,0},\sigma_{II}}|\geq R\big)\leq\theta_{\tilde R,R},$$

where τ *denotes the first exit time from the annulus* $B_R(0)\setminus\bar B_1(0)$.

(iii) For a given $r>0$, *let* $\bar\tau=\min\{i\geq 0;\ |X_i|\not\in(r,rR^2)\}$. *Then, for every* $|x|\in[r,rR]$ *and* $\varepsilon<r\varepsilon_0$ *there holds, with a constant* $\theta_R<1$ *depending only on* $R,N,\mathbf{p},s$:

$$\exists\sigma_{I,0}\quad\forall\sigma_{II}\qquad \mathbb{P}\big(|X_{\bar\tau}^{\varepsilon,x,\sigma_{I,0},\sigma_{II}}|\geq rR^2\big)\leq\theta_R.$$

Proof. **1.** To show (i), observe that by Corollary 8.3 we have, for all $|x|\in(1,R)$:

$$\begin{aligned}v_\varepsilon(x)-f_t(x)&=\big(\mathcal{A}_\varepsilon v_\varepsilon(x)-\mathcal{A}_\varepsilon f_t(x)\big)+\big(\mathcal{A}_\varepsilon f_t(x)-f_t(x)\big)\\&\geq\inf_{|y|=1}\fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)}(v_\varepsilon-f_t)(x+z)\,\mathrm{d}\mu_s^N(z)+C_{N,\mathbf{p},s}\varepsilon^{2s}R^{-2s-t}\\&\geq\inf_{\mathbb{R}^N}(v_\varepsilon-f_t)+C_{N,\mathbf{p},s}\varepsilon^{2s}R^{-2s-t}.\end{aligned}\tag{8.3}$$

Assume that $M_\varepsilon\doteq\inf_{\mathcal{D}}(v_\varepsilon-f_t)<0$, in which case there also holds: $M_\varepsilon=\inf_{\mathbb{R}^N}(v_\varepsilon-f_t)$. Let $\{x_n\}_{n=1}^\infty$ me a minimizing sequence in $\mathcal{D}$. Applying (8.3) at each x_n and passing to the limit $n\to\infty$, it follows that: $M_\varepsilon>M_\varepsilon$, which is a contradiction.

2. To show (ii), fix $t=t_0$ and recall that (i) implies:

$$0\leq v_\varepsilon(x)-f_t(x)=\sup_{\sigma_I}\inf_{\sigma_{II}}\mathbb{E}\big[f_t\circ X_\tau^{\varepsilon,x,\sigma_I,\sigma_{II}}-f_t(x)\big].$$

Since $\frac{\tilde R^{-t}-R^{-t}}{2}>0$, it follows that there exists $\sigma_{I,0}$ such that for all σ_{II} there holds:

$$\begin{aligned}-\frac{\tilde R^{-t}-R^{-t}}{2}&\leq\mathbb{E}\big[f_t\circ X_\tau^{\varepsilon,x,\sigma_{I,0},\sigma_{II}}-f_t(x)\big]\\&=\int_{\{|X_\tau|\geq R\}}f_t(X_\tau)-f_t(x)\,\mathrm{d}\mathbb{P}+\int_{\{|X_\tau|<\leq 1\}}f_t(X_\tau)-f_t(x)\,\mathrm{d}\mathbb{P}\\&\leq\mathbb{P}(|X_\tau|\geq R)\big(R^{-t}-\tilde R^{-t}\big)+(1-\mathbb{P}(|X_\tau|\geq R))\big(2^t-\tilde R^{-t}\big)\\&=\mathbb{P}(|X_\tau|\geq R)\big(R^{-t}-2^t\big)+2^t-\tilde R^{-t}.\end{aligned}$$

Consequently, we obtain: $\mathbb{P}(|X_\tau|\geq R)\leq\frac{\frac12(\tilde R^{-t}-R^{-t})+2^t-\tilde R^{-t}}{2^t-R^{-t}}=\frac{2^t-\frac12(\tilde R^{-t}+R^{-t})}{2^t-R^{-t}}\doteq\theta_{\tilde R,R}<1$.

The statement in (iii) follows by scaling invariance after applying (ii) to $R<R^2$ in place of $\tilde R<R$, so that $\theta_R\doteq\theta_{\tilde R,R}$. This ends the proof. ∎

We finally are ready to give:

Proof of Theorem 8.1

The cone condition implies existence of $d>1$ and $\bar r>0$ such that for all $r<\bar r$ there is a ball $B_r(\bar x)\subset\mathbb{R}^N\setminus\mathcal{D}$, centered at $\bar x$ with $|x-\bar x|=rd$. Define $R=2d-1$, so that $x\in B_{rR}(\bar x)$.

Given $\delta>0$, let $r<\bar r$ be such that: $\delta\geq rR^2+rd=r\big(R^2+\frac{R+1}{2}\big)$. Letting $\hat\delta=rd$ we get:

$$B_{\hat\delta}(x)\subset B_{rR}(\bar x)\setminus\bar B_r(x)\qquad\text{and}\qquad B_{rR^2}(\bar x)\subset B_\delta(x).\tag{8.4}$$

Fix $x_0\in B_{\hat\delta}(x)\cap\mathcal{D}$ and $\varepsilon<r\varepsilon_0$, where ε_0 is as in Proposition 8.4 (iii). Denote by $\bar\tau$ the exit time from the annulus $B_{rR^2}(\bar x)\setminus\bar B_r(x)$. Then, there exists $\sigma_{I,0}$ such that for all σ_{II} there holds:

$$\mathbb{P}\big(\exists n<\tau\quad X_n^{\varepsilon,x_0,\sigma_{I,0},\sigma_{II}}\not\in B_\delta(x)\big)\leq\mathbb{P}\big(X_{\bar\tau}\not\in B_{rR^2}(\bar x)\big)\leq\theta_R\doteq\bar\theta.$$

The first inequality above follows from (8.4), while the second inequality is a direct consequence of Proposition 8.4 (iii). ■

9. Approximate equicontinuity of solutions to $(\mathrm{DPP})_\varepsilon$

In this section, we assume that the open, bounded domain $\mathcal{D} \subset \mathbb{R}^N$ satisfies the external cone condition. Our goal is to show that the family $\{u_\varepsilon\}_{\varepsilon\to 0}$ of solutions to $(\mathrm{DPP})_\varepsilon$ with a given bounded, uniformly continuous $F : \mathbb{R}^N \to \mathbb{R}$ is then approximately equicontinuous, namely:

$$\forall \xi > 0 \quad \exists \hat\varepsilon, \delta > 0 \quad \forall \varepsilon \in (0, \hat\varepsilon) \quad \forall x, \bar x \in \mathbb{R}^N \qquad |x - \bar x| < \delta \implies |u_\varepsilon(x) - u_\varepsilon(\bar x)| \le \xi. \tag{9.1}$$

Together with the uniform boundedness of the family $\{u_\varepsilon\}_{\varepsilon\to 0}$, the above condition yields, via the Ascoli-Arzelà theorem, that every sequence in the said family has a further subsequence, converging uniformly as $\varepsilon \to 0$ to some continuous limit function.

Lemma 9.1. *Condition (9.1) is implied by the following weaker equicontinuity statement:*

$$\begin{aligned} &\forall \xi > 0 \quad \exists \hat\varepsilon, \delta > 0 \quad \forall \varepsilon \in (0, \hat\varepsilon) \quad \forall x \in \mathcal{D},\ \bar x \in \partial\mathcal{D} \\ &\qquad\qquad\qquad\qquad\qquad |x - \bar x| < \delta \implies |u_\varepsilon(x) - u_\varepsilon(\bar x)| \le \xi. \end{aligned} \tag{9.2}$$

Proof. Since $u_\varepsilon = F$ on $\mathbb{R}^N \setminus \mathcal{D}$, we get (9.1) for $x, \bar x \notin \mathcal{D}$ in view of the uniform continuity of F. By (9.2), it suffices to consider the case $x, \bar x \in \mathcal{D}$. Fix $\xi > 0$ and choose $\hat\varepsilon, \hat\delta > 0$ such that:

$$\varepsilon \in (0, \hat\varepsilon),\ z \in \mathbb{R}^N \setminus \mathcal{D}^{\hat\delta},\ |w| \le \hat\delta \implies |u_\varepsilon(z) - u_\varepsilon(z + w)| \le \xi,$$

where we denoted the inner set:

$$\mathcal{D}^{\hat\delta} = \big\{x \in \mathcal{D};\ \mathrm{dist}(x, \mathbb{R}^N \setminus \mathcal{D}) > \hat\delta\big\}.$$

Fix $x, \bar x \in \mathcal{D}$ such that $|x - \bar x| < \frac{\hat\delta}{2}$ and consider the function $\bar u_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ given by:

$$\bar u_\varepsilon(z) \doteq u_\varepsilon\big(z - (x - \bar x)\big) + \xi.$$

Observe that $\bar u_\varepsilon$ solves $(\mathrm{DPP})_\varepsilon$ on $\mathcal{D}^{\hat\delta}$, and subject to its own external data $\bar u_\varepsilon$ on $\mathbb{R}^N \setminus \mathcal{D}^{\hat\delta}$, as:

$$\begin{aligned} \mathcal{A}_\varepsilon \bar u_\varepsilon(z) &= \frac{1}{2}\Big(\inf_{|y|=1} + \sup_{|y|=1}\Big) \fint_{T_{\mathbf{p}}^{\varepsilon,\infty}(y)} u_\varepsilon\big(z - (x - \bar x) + \hat z\big) + \xi \,\mathrm{d}\mu_s^N(\hat z) \\ &= u_\varepsilon(z - (x - \bar x)) + \xi = \bar u_\varepsilon(z) \qquad \text{for all } \ z \in \mathcal{D}^{\hat\delta}. \end{aligned}$$

On the other hand, u_ε solves the same problem (with its own external data). Since:

$$u_\varepsilon(z) - \bar u_\varepsilon(z) = u_\varepsilon(z) - u_\varepsilon(z - (x - \bar x)) - \xi \le 0 \qquad \text{for all } \ z \notin \mathcal{D}^{\hat\delta},$$

the monotonicity of the solution operator to $(\mathrm{DPP})_\varepsilon$ (see Theorem 5.1) implies that $u_\varepsilon \le \bar u_\varepsilon$ in $\mathbb{R}^N$, so in particular we get: $u_\varepsilon(x) \le u_\varepsilon(\bar x) + \xi$. The reverse inequality $u_\varepsilon(x) \ge u_\varepsilon(\bar x) - \xi$ can be shown by a symmetric argument. The proof is done. ■

We now replace condition (9.2) by the boundary game regularity condition in the spirit of condition (8.1). More precisely, we say that $x \in \partial\mathcal{D}$ is game-regular when:

$$\forall \xi, \delta > 0 \quad \exists \hat\delta, \hat\varepsilon > 0 \quad \forall \varepsilon < \hat\varepsilon,\ \bar x \in B_{\hat\delta}(x) \cap \mathcal{D} \quad \exists \sigma_{I,0} \quad \forall \sigma_{II} \quad \mathbb{P}\big(X_\tau^{\varepsilon, x, \sigma_{I,0}, \sigma_{II}} \notin B_\delta(x)\big) \le \xi. \tag{9.3}$$

Lemma 9.2. *If every boundary point $x \in \partial\mathcal{D}$ satisfies (9.3), then (9.2) holds for every bounded, uniformly continuous data function $F : \mathbb{R}^N \to \mathbb{R}$.*

Proof. Fix $\xi > 0$ and choose $\delta > 0$ such that:

$$|F(x) - F(\bar{x})| \leq \frac{\xi}{3} \qquad \text{for all } \ |x - \bar{x}| < \delta.$$

By (9.3) there exists $\hat{\delta}, \hat{\varepsilon} > 0$ so that for all $\varepsilon < \hat{\varepsilon}$, $x \in \partial\mathcal{D}$ and $\bar{x} \in B_{\hat{\delta}}(x)$ there exists $\sigma_{I,0}$ with:

$$\mathbb{P}\big(X_\tau^{\varepsilon,x,\sigma_{I,0},\sigma_{II}} \notin B_\delta(x)\big) \leq \frac{\xi}{1 + 6\|F\|_{L^\infty}} \qquad \text{for all } \ \sigma_{II}.$$

Taking $\varepsilon, x, \bar{x}$ as indicated, we obtain:

$$\begin{aligned} u_\varepsilon(\bar{x}) - u_\varepsilon(x) = u_I^\varepsilon(\bar{x}) - F(x) &\geq \inf_{\sigma_{II}} \mathbb{E}\big[F \circ X_\tau^{\varepsilon,x,\sigma_{I,0},\sigma_{II}} - F(x)\big] \\ &\geq \mathbb{E}\big[F \circ X_\tau^{\varepsilon,x,\sigma_{I,0},\sigma_{II,0}} - F(x)\big] - \frac{\xi}{3} \\ &= \int_{\{X_\tau \in B_\delta(x)\}} F(X_\tau) - F(x)\,\mathrm{d}\mathbb{P} + \int_{\{X_\tau \notin B_\delta(x)\}} F(X_\tau) - F(x)\,\mathrm{d}\mathbb{P} - \frac{\xi}{3} \\ &\geq -2\|F\|_{L^\infty} \cdot \mathbb{P}(X_\tau \notin B_\delta(x)) - \frac{\xi}{3} - \frac{\xi}{3} \geq -\xi, \end{aligned}$$

where we used an almost-infimizing strategy $\sigma_{II,0}$. The inequality $u_\varepsilon(\bar{x}) - u_\varepsilon(x) \leq \xi$ follows by a symmetric argument. This ends the proof. ∎

Proposition 9.3. *Fix $\delta > 0$, $k \geq 2$, $\varepsilon < \frac{\delta}{k}$ and $|x| < \frac{\delta}{k}$. Then there holds:*

$$\forall \sigma_I,\ \sigma_{II} \qquad \mathbb{P}\big(|X_{\bar{\bar{\tau}}}^{\varepsilon,x,\sigma_I,\sigma_{II}}| \geq \delta\big) \leq \Big(\frac{2}{k-1}\Big)^{2s} \doteq a_k,$$

where we defined the stopping time: $\bar{\bar{\tau}} \doteq \min\big\{i \geq 0;\ |X_i| \geq \frac{\delta}{k}\big\}$.

Proof. Let $\delta, k, \varepsilon, x$ be as in the statement of the result. It follows that:

$$\begin{aligned} &\mathbb{P}\big(|X_{\bar{\bar{\tau}}}| \geq \delta\big) \\ &\leq \sup\Big\{\frac{\mu_s^N\big((x + T_{\mathbf{p}}^{\varepsilon,\infty}(y)) \setminus B_\delta(0)\big) + \mu_s^N\big((x + T_{\mathbf{p}}^{\varepsilon,\infty}(\bar{y})) \setminus B_\delta(0)\big)}{\mu_s^N\big((x + T_{\mathbf{p}}^{\varepsilon,\infty}(y)) \setminus B_{\delta/k}(0)\big) + \mu_s^N\big((x + T_{\mathbf{p}}^{\varepsilon,\infty}(\bar{y})) \setminus B_{\delta/k}(0)\big)};\ |x| < \frac{\delta}{k},\ |y| = |\bar{y}| = 1\Big\} \\ &\leq \sup\Big\{\frac{\mu_s^N\big((x + T_{\mathbf{p}}^{\varepsilon,\infty}(y)) \setminus B_\delta(0)\big)}{\mu_s^N\big((x + T_{\mathbf{p}}^{\varepsilon,\infty}(y)) \setminus B_{\delta/k}(0)\big)};\ |x| < \frac{\delta}{k},\ |y| = 1\Big\}, \end{aligned}$$

where we used the fact that $\frac{\alpha_1+\beta_1}{\alpha_2+\beta_2} \leq \max\big\{\frac{\alpha_1}{\alpha_2}, \frac{\beta_1}{\beta_2}\big\}$. Further, denoting:

$$a = \inf_{|x|<\delta/k} \mathrm{dist}(x, \mathbb{R}^N \setminus B_\delta(0)) = \delta - \frac{\delta}{k}, \qquad b = \sup_{|x|<\delta/k} \mathrm{dist}(x, \mathbb{R}^N \setminus B_{\delta/k}(0)) = 2\frac{\delta}{k},$$

leads to:

$$\mathbb{P}\big(|X_{\bar{\bar{\tau}}}| \geq \delta\big) \leq \frac{\sup\Big\{\mu_s^N\big((x + T_{\mathbf{p}}^{\varepsilon,\infty}(y)) \setminus B_\delta(0)\big);\ |x| < \frac{\delta}{k},\ |y| = 1\Big\}}{\inf\Big\{\mu_s^N\big((x + T_{\mathbf{p}}^{\varepsilon,\infty}(y)) \setminus B_\delta(0)\big);\ |x| < \frac{\delta}{k},\ |y| = 1\Big\}} \leq \frac{\mu_s^N(T_{\mathbf{p}}^{a,\infty})}{\mu_s^N(T_{\mathbf{p}}^{b,\infty})}.$$

Since $\mu_s^N(T_{\mathbf{p}}^{a,\infty}) = \frac{C(N,s)|A_{\mathbf{p}}|}{2sa^{2s}}$, we obtain that $\mathbb{P}\big(|X_{\bar{\bar{\tau}}}| \geq \delta\big) \leq \big(\frac{b}{a}\big)^{2s}$, as claimed. ∎

Here is the main result of this section:

Theorem 9.4. *Let $\mathcal{D} \subset \mathbb{R}^N$ be an open, bounded domain satisfying the external cone condition. Then (9.3) holds for every $x \in \partial\mathcal{D}$.*

Proof. **1** Fix $\xi, \delta > 0$ and $x \in \partial\mathcal{D}$. Without loss of generality $x = 0$. Fix $k_0 \geq 4$ such that $a_{k_0} = \big(\frac{2}{k_0-1}\big)^{2s} < \frac{\xi}{2}$. It follows by Proposition 9.3 that for all $\varepsilon < \frac{\delta}{k_0}$ we have:

$$\begin{aligned}\mathbb{P}\big(X_\tau \notin B_\delta(0)\big) &= \mathbb{P}\Big(\big\{|X_\tau| \geq \delta\big\} \cap \big\{\exists n < \tau \quad |X_n| \in \big[\frac{\delta}{k}, \delta\big)\big\}\Big) \\ &\qquad + \mathbb{P}\Big(\big\{|X_\tau| \geq \delta\big\} \cap \big\{\ \nexists n < \tau \quad |X_n| \in \big[\frac{\delta}{k}, \delta\big)\big\}\Big) \\ &\leq \mathbb{P}\Big(\exists n < \tau \quad |X_n| \geq \frac{\delta}{k_0}\Big) + \frac{\xi}{2}.\end{aligned} \tag{9.4}$$

Denote: $\varepsilon_0 = \delta_1 = \frac{\delta}{k_0}$. We now show that:

$$\exists \hat\delta < \delta_1,\ \hat\varepsilon < \varepsilon_0 \quad \forall \varepsilon < \hat\varepsilon,\ |x_0| < \hat\delta \quad \exists \sigma_{I,0} \quad \forall \sigma_{II} \qquad \mathbb{P}\big(\exists n < \tau \quad |X_n| \geq \delta_1\big) \leq \frac{\xi}{2}. \tag{9.5}$$

Together with (9.4), (9.5) will establish the result.

2. By Proposition 9.3, there exists $k \geq k_0$ such that $\bar\theta + a_k < 1$. Let $m \geq 2$ satisfy:

$$\big(\bar\theta + a_k\big)^m \leq \frac{\xi}{2}, \tag{9.6}$$

where $\bar\theta$ is as in (8.1). We now define $\{\varepsilon_i\}_{i=1}^m$, $\{\delta_i\}_{i=2}^m$, by applying Theorem 8.1 to δ_i in place of δ, recursively in:

$$\varepsilon_i = \min\{\varepsilon_{i-1}, \hat\varepsilon(\delta_i)\}, \quad \delta_i = \frac{\hat\delta(\delta_{i-1})}{k}. \tag{9.7}$$

We also set:

$$\hat\varepsilon \doteq \min\{\varepsilon_m, \frac{\delta_m}{k}\}, \quad \hat\delta \doteq \hat\delta(\delta_m), \qquad \text{and} \qquad \kappa_i \doteq \min\{j \geq 0;\ |X_j| \geq \delta_i\} \quad \text{for all } \ i = 1 \dots m.$$

Given $\varepsilon < \hat\varepsilon$ and $|x_0| < \hat\delta$, define the strategy $\sigma_{I,0}$ as follows:

- For $j < \kappa_m$, we utilize the strategy $\sigma_{I,0,m}$ from (8.1), chosen for the starting point x_0:
$$\sigma_{I,0}^j\big(x_0, (x_1, z_1, s_1), \dots, (x_j, z_j, s_j)\big) = \sigma_{I,0,m}^j\big(x_0, (x_1, z_1, s_1), \dots, (x_j, z_j, s_j)\big).$$
- If $|X_{\kappa_m}| \geq k\delta_m$, we keep the definition above for all $j \geq \kappa_m$.
- If $|X_{\kappa_m}| \in [\delta_m, k\delta_m)$, then for $j \in [\kappa_m, \kappa_{m-1})$ we utilize the strategy $\sigma_{I,0,m-1}$ from (8.1), chosen for the starting point X_{κ_m}:
$$\sigma_{I,0}^j\big(x_0, (x_1, z_1, s_1), \dots, (x_j, z_j, s_j)\big) = \sigma_{I,0,m-1}^{j-\kappa_m}\big(x_{\kappa_m}, (x_{\kappa_m+1}, z_{\kappa_m+1}, s_{\kappa_m+1}), \dots, (x_j, z_j, s_j)\big).$$
- Continue in this fashion, concatenating the strategies $\sigma_{I,0,i}$ for the remaining indices $i = m-2, \dots, 1$. Each $\sigma_{I,0,i}$ is chosen from (8.1) for the starting point $X_{\kappa_{i+1}}$.

By several applications of (8.1) and Proposition 9.3, it follows that:

$$\begin{aligned}&\mathbb{P}\big(\exists n < \tau \quad |X_n^{\varepsilon, x_0, \sigma_{I,0}, \sigma_{II}}| \geq \delta_1\big) = \mathbb{P}\big(\kappa_1 < \tau\big) \\ &\qquad = \mathbb{P}\big(\kappa_2 < \kappa_1 < \tau \text{ and } |X_{\kappa_2}| \in [\delta_2, 4\delta_2)\big) + \mathbb{P}\big(\kappa_2 \leq \kappa_1 < \tau \text{ and } |X_{\kappa_2}| \geq 4\delta_2\big) \\ &\qquad \leq \bar\theta \cdot \mathbb{P}\big(\kappa_2 < \tau \text{ and } |X_{\kappa_2}| \in [\delta_2, 4\delta_2)\big) + \mathbb{P}\big(|X_{\kappa_2}| \geq 4\delta_2\big) \cdot \mathbb{P}\big(\kappa_2 < \tau\big) \\ &\qquad \leq \big(\bar\theta + a_k\big)\mathbb{P}(\kappa_2 < \tau).\end{aligned}$$

An iteration of the above argument and one final application of (8.1) together with (9.6) yield:

$$\mathbb{P}\big(\exists n < \tau \quad |X_n| \geq \delta_1\big) \leq \big(\bar\theta + a_k\big)^{m-1}\mathbb{P}(\kappa_m < \tau) \leq \big(\bar\theta + a_k\big)^m \leq \frac{\xi}{2}.$$

Thus (9.5) has been verified. ■

By Lemma 9.2, Lemma 9.1 and invoking the Ascoli-Arzelà theorem, we immediately get, in view of Theorem 6.1 and Remark 6.2:

Corollary 9.5. *Let $\mathcal{D} \subset \mathbb{R}^N$ be an open, bounded domain satisfying the external cone condition. For a given bounded, uniformly continuous data function $F : \mathbb{R}^N \to \mathbb{R}$, consider the family $\{u_\varepsilon\}_{\varepsilon\to 0}$ of solutions to $(\mathrm{DPP})_\varepsilon$. Then, every sequence $\{u_\varepsilon\}_{\varepsilon\in J,\varepsilon\to 0}$ has a further subsequence that converges uniformly as $\varepsilon \to 0$, to a viscosity solution of (6.1).*

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M. Lewicka: University of Pittsburgh, Department of Mathematics, 139 University Place, Pittsburgh, PA 15260

E-mail address: `lewicka@pitt.edu`