ISOMETRIC IMMERSIONS AND APPLICATIONS

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Abstract. We provide an introduction to the old-standing problem of isometric immersions. We combine a historical account of its multifaceted advances, which have fascinated geometers and analysts alike, with some of the applications in the mathematical physics and mathematical materials science, old and new.

1. The isometric immersion problem

The concept of a *Riemannian manifold* (M^d, g), an abstract d-dimensional manifold with a metric structure, was first formulated by Bernhard Riemann in 1868 to generalize the classical objects such as curves and surfaces in $\mathbb{R}^3$. A manifold is a topological space that locally resembles Euclidean space near each point: in M (we write M^d only to emphasize the dimension) each point $p \in M$ has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^d$. Induced by that homeomorphism, there is a tangent space T_pM, namely a d-dimensional vector space that is further equipped with a positive-definite inner product g_p. The family $g = \{g_p\}_{p\in M}$ is called a *Riemannian metric* on M. For details, we recommend consulting the textbook [2] and references listed therein.

All Euclidean spaces are manifolds and, endowed with the standard Euclidean inner product, are Riemannian manifolds. The simplest nontrivial $d = 2$ - dimensional example is the unit sphere $\mathbb{S}^2$ consisting of all unit vectors in $\mathbb{R}^3$. At each $p \in \mathbb{S}^2$, the tangent space $T_p\mathbb{S}^2$ is the hyperplane in $\mathbb{R}^3$ perpendicular to the unit vector p. The standard inner product in $\mathbb{R}^3$ restricted to $T_p\mathbb{S}^2$ induces then a Riemannian metric on $\mathbb{S}^2$, called the round metric. Naturally, there arises the question of whether any abstract (M^d, g) can be identified, in a similar manner, as a submanifold of some Euclidean space $\mathbb{R}^n$ with its induced metric. This is the *isometric embedding question*, which has assumed a position of fundamental conceptual importance in Differential Geometry.

The metric requirement can be expressed in terms of partial differential equations (PDEs). Consider a special case: let B_1 be the unit ball in $\mathbb{R}^d$, regarded as a d-dimensional manifold. The given inner product g_p can be identified with a $d \times d$ symmetric, positive definite matrix at each point $p \in B_1$, so that g is a function from B_1 to $\mathbb{R}^{d\times d}_{\mathrm{sym},>}$. Isometrically immersing (B_1, g) in some Euclidean space $\mathbb{R}^n$ means that there exists $u = (u^1, \dots, u^n) : B_1 \to \mathbb{R}^n$ such that the induced metric agrees with g at each point:

$$(\nabla u)^T \nabla u = g \quad \text{in } B_1. \tag{1}$$

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Above, we used the matrix notation for the transpose and product. We call the map u in (1) an *isometric embedding* or *immersion* according to whether it is injective or not. In the general setting, and when globally defined, both sides of (1) become second order covariant tensors. We now briefly review several important aspects of isometric embeddings and immersions of Riemannian manifolds in the Euclidean space.

2. General isometric embeddings and immersions of Riemannian manifolds

2.1. The analytic solutions. Let us examine (1) closely. First, we notice that it has n unknowns $\{u^i\}_{i=1}^n$ and there are $s_d \doteq d(d+1)/2$ equations, corresponding to the entries in $d \times d$ symmetric matrices. The integer s_d is called the *Janet dimension*. As a rule in solving PDEs, the number of unknowns should be bigger than or equal to the number of equations, otherwise, solutions are not expected to exist in general. Hence we require that $n \geq s_d$. In 1873, Ludwig Schlaefli conjectured that every d-dimensional smooth Riemannian manifold admits a *smooth local isometric embedding* in $\mathbb{R}^{s_d}$. It was more than 50 years later that an affirmative answer was given in the analytic case by Maurice Janet (for $d = 2$) and Élie Cartan (for $d \geq 3$).

2.2. The smooth case and the Nash-Moser iteration. Second, (1) is classified as a first-order nonlinear system of PDEs, the order of derivatives being 1 and the derivatives of u appearing quadratically (nonlinearly) in the system. In general, nonlinear PDEs are solved by Newton's method, an iteration process generating an actual solution from an approximate one. Here, the difficulty arises from the loss of derivatives at each iteration.

To explain this clearly, recall how Newton's method is used to find a root of a single-variable function. Let f be a scalar-valued twice continuously differentiable function on $\mathbb{R}$. Pick an initial guess $t_0 \in \mathbb{R}$ for a root of f, meaning that $f(t_0)$ is small. To find $t_1 = t_0 + s_1$ such that $f(t_1)$ is smaller, we use Taylor's theorem to have:

$$f(t_1) = f(t_0) + f'(t_0)s_1 + \frac{1}{2}f''(t_0 + \xi s_1)s_1^2 \quad \text{for some } \xi \in (0,1).$$

It is natural to choose s_1 such that $f(t_0) + f'(t_0)s_1 = 0$ and hence:

$$s_1 = -\frac{f(t_0)}{f'(t_0)}, \qquad f(t_1) = \frac{1}{2}f''(t_0 + \xi s_1)s_1^2.$$

We see that $s_1 = t_1 - t_0$ is linear in $f(t_0)$ and so $f(t_1)$ is quadratic in $f(t_0)$. We can iterate this process and obtain a sequence $\{t_l\}_{l\to\infty}$ inductively by:

$$t_l \doteq t_{l-1} + s_l, \qquad s_l \doteq -\frac{f(t_{l-1})}{f'(t_{l-1})}, \qquad f(t_l) = \frac{1}{2}f''(t_{l-1} + \xi_l s_l)s_l^2. \tag{2}$$

We next need to prove that $\{t_l\}_{l\to\infty}$ converges to some t_∞ and that $\{f(t_l)\}_{l\to\infty}$ converges to 0. As a consequence, $f(t_\infty) = 0$ resulting in a root of f at t_∞. An important condition in the convergence proof is given by:

$$|f'(t_l)| \geq c \quad \text{for all } l \geq 0,$$

for some positive constant c, independent of l. With (2.2) and an appropriate condition on f'', we get that indeed $\{s_l\}_{l\to\infty}$ converges to zero fast and so $f(t_l) \to 0$.

We now apply the same idea to (1), equivalent to finding roots of:

$$G[u] = (\nabla u)^T \nabla u - g.$$

We first take a $u_0 : B_1 \to \mathbb{R}^n$ such that $G[u_0]$ is small, write $u_1 = u_0 + v_1$ and compute:

$$G[u_1] = G[u_0] + 2\,\mathrm{sym}\big((\nabla u_0)^T \nabla v_1\big) + (\nabla v_1)^T \nabla v_1,$$

where we arranged the expression in the right-hand side according to the powers of (the derivatives of) v_1. As in the scalar case, we would like to choose v_1 such that:

$$G[u_0] + (\nabla u_0)^T \nabla v_1 + (\nabla v_1)^T \nabla u_0 = 0 \quad \text{in } B_1. \tag{3}$$

This is a first-order linear system of PDEs that can be solved by imposing appropriate boundary values. However, there is a serious issue in the iteration process.

In studying PDEs or systems thereof, we need to introduce appropriate spaces of functions and equip them with appropriate norms to form Banach spaces. The process outlined above is performed in Banach spaces to generate a "root" of the functional G. An important feature of (3) is that although a linear combination of derivatives of v_1 is governed by $G[u_0]$, it is v_1, not ∇v_1, that is determined by $G[u_0]$. More generally, derivatives of $G[u_0]$ of a certain order determine only derivatives of v_1 up to the same order. Hence, in the inductive version of (3) and a counterpart of (2):

$$u_l \doteq u_{l-1} + v_l, \qquad G[u_{l-1}] + 2\,\mathrm{sym}\big((\nabla u_{l-1})^T \nabla v_l\big) = 0, \qquad G[u_l] = (\nabla v_l)^T \nabla v_l, \tag{4}$$

each step in the iteration contributes a loss of derivatives. If u_0 is, say, 100 times continuously differentiable then $G[u_0]$, as it involves ∇u_0, is 99 times continuously differentiable and thus v_1 is also 99 times differentiable and so is u_1. Then, u_{100} will be continuous only and $G[u_{100}]$ does not make sense. As a consequence, the iteration process is terminated.

In an outstanding paper published in 1956, John Nash introduced an important technique of *smoothing operators* to compensate for the aforementioned loss of derivatives. He proved that any smooth d-dimensional Riemannian manifold admits a (global) smooth isometric embedding in $\mathbb{R}^n$, for $n = 3s_d + 4d$ in the compact case and $n = (d+1)(3s_d+4d)$ in the general case. Specifically for (4), Nash replaced u_{l-1} by a function with a better regularity to regain the lost derivative, solved for v_l, and modified the iteration process accordingly. His technique proved to be extremely useful in nonlinear PDEs, and it is now known as the *hard implicit function theorem*, or the *Nash-Moser iteration.*

2.3. **Quest for the smallest dimension *n*.** Following Nash, one naturally looks for the smallest dimension of the ambient space. In 1970, Mikhael Gromov and Vladimir Rokhlin, and independently John Greene, proved that any d-dimensional smooth Riemannian manifold admits a local smooth isometric embedding in $\mathbb{R}^{s_d+d}$. The proof is based on Nash's iteration scheme. In his book [1], Gromov studied various problems related to the isometric embedding of Riemannian manifolds. He proved that $n = s_d + 2d + 3$ is enough for the compact case. Then in 1989, Matthias Günther vastly simplified Nash's original proof: by rewriting the differential equations cleverly, he was able to employ the contraction mapping principle, instead of the Nash-Moser iteration, to construct solutions. Günther also improved the dimension of the target space to $n = \max\{s_d + 2d, s_d + d + 5\}$. It is still not clear whether this is the best possible result.

For $d = 2$ better results are available. According to Gromov and Günther, any compact 2-dimensional smooth Riemannian manifold can be isometrically embedded in $\mathbb{R}^{10}$ smoothly, whereas the local version in $\mathbb{R}^4$ is due to Eduard Poznyak in 1973. The case of $d = 2$ and $n = s_2 = 3$ will be discussed in a separate section below.

On the other hand, the case $d \geq 3$, $n = s_d$ is sharply different from the 2-dimensional case in which there is only one curvature function. This function determines the type of the Darboux equation associated to (1), whereas for $d \geq 3$, the role of various curvatures is not clear. In 1983, Robert Bryant, Phillip Griffiths and Dean Yang studied the characteristic varieties associated with differential systems for the isometric embedding in $\mathbb{R}^{s_d}$ of smooth (M^d, g). They proved that these varieties are never empty if $d \geq 3$, implying, in particular, that the governing systems are never elliptic, no matter what assumptions are put on curvatures. They also proved that the characteristic varieties are smooth for $d = 4$ and not smooth for $d = 6, 10, 14, \ldots$. In 2012, Qing Han and Marcus Khuri extended that result to any $d \geq 5$ under an additional "smallness" assumption.

For $d = 3$, Bryant, Griffiths and Yang in 1983 classified the type of differential systems for the isometric embedding by its curvature functions. Here, an important quantity is the signature of the curvature tensor viewed as a symmetric linear operator acting on the space of 2-forms. In particular, any smooth 3-dimensional Riemannian manifold admits a smooth local isometric embedding in $\mathbb{R}^6$ if the signature is different from $(0, 0)$ and $(0, 1)$. In 1989, Yusuke Nakamura and Yota Maeda showed the same existence result if the curvature tensors are not zero; their key argument was the local existence of solutions to differential systems of principal type. In 2018, Chen, Clelland, Slemrod, Wang and Yang provided an alternative proof using strongly symmetric positive systems.

2.4. **Non-smooth immersions.** On the other end of the spectrum, there are results showing that isometric immersions have completely different qualitative behaviours at low and high regularity. Nash in 1954 and Nicolaas Kuiper in 1955 proved the existence of a global C^1 isometric embedding of d-dimensional Riemannian manifolds in $\mathbb{R}^{d+1}$. These are not mere existence statements, as their results in fact show that every *short immersion* (or embedding), i.e. $u : B_1 \to \mathbb{R}^{d+1}$ for which condition (1) is replaced by:

$$(\nabla u)^T \nabla u < g \quad \text{in } B_1, \tag{5}$$

can be uniformly approximated by C^1-regular actual solutions (immersions or embeddings) to (1). The inequality above is understood pointwise, in the sense of matrices. This abundance of solutions, usually referred to as *flexibility results*, is typical in applications of Gromov's *h-principle* in which a PDE is replaced by a partial differential relation (a differential inclusion) whose solutions are then modified through an iteration technique called *convex integration* to produce a nearby solution of the underlying PDE.

In 1965, Yuri Borisov used this approach to Hölder - regular solutions and announced that flexibility holds with regularity $C^{1,\alpha}$ for any $\alpha < \frac{1}{1+2s_d}$ and analytic g on B_1. He subsequently gave full details of the proof in 2004 for dimension $d = 2$ and $\alpha < \frac{1}{13}$. In 2012 Sergio Conti, Camillo De Lellis and Laszlo Szekelyhidi validated the original Borisov's statement in case of C^2 metrics on d-dimensional balls, and in case of compact Riemannian manifolds (M^d, g) with $\alpha < \frac{1}{1+2s_d(d+1)}$. The same results and exponent bounds also hold for any target ambient dimension $n > d$. However, in 1978 Anders Källen proved that any C^β metric, with $\beta < 2$, allows for flexibility up to exponent $\frac{\beta}{2}$, provided that n is sufficiently large. Recently, the second author of this paper proved flexibility of the related *Monge-Ampere system*, which is the linearization of the isometric immersion problem (1) around $g = \mathrm{Id}_d$, and in which any C^1 - regular subsolution can be uniformly approximated by $C^{1,\alpha}$ exact solutions, for any $\alpha < \frac{1}{1+2s_d/(n-d)}$, in agreement with the fully nonlinear case at $n = d+1$ and the Kallen result when $n \to \infty$.

Dependence of the flexibility threshold exponents on s_d reflects the technical limitation of the method rather than the absolute lack of flexibility beyond those thresholds. In the proofs, the symmetric, positive definite "defect" $\mathcal{D} = g - (\nabla u)^T \nabla u$ is decomposed into a linear combination of s_d rank-one defects with nonnegative coefficients. Each of these "primitive defects" is then cancelled by adding to u a small but fast oscillating perturbation, which however causes increase of the second derivative of u by the factor of the oscillation frequency. The ultimate Hölder regularity of the approximating immersion interpolates between the controlled C^1 norms of the adjusted u and the blow-up rate of the C^2 norm, dictated by the number of these one-dimensional adjustments. In case of higher codimension when $n > d+1$, several primitive defects may be cancelled at once, reducing the blow-up rate of second derivatives and thus improving the regularity exponent α. In the same vein, if the number of primitive defects in the decomposition of $\mathcal{D}$ could be lowered, for example by an appropriate change of variables, then flexibility would hold with higher α. This observation is precisely behind the improved regularity statements for $d = 2$ - dimensional problems, listed in the next section.

3. Isometric embeddings of surfaces

3.1. **Local isometric embedding of surfaces in $\mathbb{R}^3$.** We now give an overview of the question of isometrically embedding a 2-dimensional Riemannian manifold in $\mathbb{R}^3$. There are basically two methods to study the local case. The first one, already known to Jean Darboux in 1894, restates the problem equivalently as finding a local solution of a nonlinear equation of the *Monge-Ampère type*. Specifically, let g be a C^r-metric on a simply connected $\Omega \subset \mathbb{R}^2$ for some $r \in [2, \infty]$. If there exists $u \in C^s(\omega, \mathbb{R})$, solving:

$$\det(\nabla_g^2 u) = K(\det g)(1 - |\nabla_g u|^2), \tag{6}$$

with $|\nabla_g u| < 1$ for some $s \in [2, r]$, then (Ω, g) admits a C^s-isometric immersion in $\mathbb{R}^3$. The equation (6) is now called the *Darboux equation*; and its type is determined by the sign of the Gauss curvature K of g: elliptic if K is positive, hyperbolic if K is negative, and degenerate if K vanishes. Remarkably, even today, the local solvability of the Darboux equation in the general case is not covered by any known theory of PDEs.

A different method to study the local isometric embedding of surfaces in $\mathbb{R}^3$ relies on the classical theory of surfaces asserting that such immersion exists provided the solvability of the *Gauss-Codazzi system*. Namely, let $\{\Gamma^i_{jk}\}_{i,j,k=1,2}$ be the Christoffel symbols of the given metric g, and K its Gauss curvature. Then the coefficients of the second fundamental form $I\!I = Ldx_1^2 + 2Mdx_1dx_2 + Ndx_2^2$ satisfy:

$$\begin{aligned} \partial_2 L - \partial_1 M &= \Gamma^1_{12} L + (\Gamma^2_{12} - \Gamma^1_{11}) M - \Gamma^2_{11} N, \\ \partial_2 M - \partial_1 N &= \Gamma^1_{22} L + (\Gamma^2_{22} - \Gamma^1_{21}) M - \Gamma^2_{21} N, \\ LN - M^2 &= K(g_{11}g_{22} - g_{12}^2). \end{aligned} \tag{7}$$

We note in passing that the first attempt to establish the local isometric embedding of surfaces in $\mathbb{R}^3$ was neither through (6) nor (7): in 1908, Hans Levi solved the case of surfaces with negative curvature by using the equations of virtual asymptotes.

It was several decades later that (6) attracted attention of those interested in the isometric embedding. In the early 1950s, Philip Hartman and Aurel Wintner studied (6) with $K \neq 0$ and proved existence of its local solution. The case when K vanishes did not give way to the efforts of mathematicians for a long time. In 1985 and 1986, Chang-Shou Lin made important breakthroughs, establishing existence in a neighborhood of $p \in \Omega$ such that $K(p) = 0$ and $dK(p) \neq 0$ (in 2005 Han gave an alternative proof of this result), or p such that $K \geq 0$ in the whole neighbourhood. Later, in 1987, Gen Nakamura covered the case of $K(p) = 0$, $dK(p) = 0$ and $\operatorname{Hess} K(p) < 0$. For the case of nonpositive K, Jia-Xing Hong in 1991 also proved the existence of a sufficiently smooth local isometric embedding in a neighborhood of p if $K = h\varphi^{2m}$, where h is a negative function and φ is a function with $\varphi(p) = 0$ and $d\varphi(p) \neq 0$. In 2010, Han and Khuri proved existence of the smooth local isometric embedding near p if K changes its sign only along two smooth curves intersecting transversely at p. All these results are based on careful studies of the Darboux equation.

In 2003, Han, Hong, and Lin studied (7) and proved the local isometric embedding for a large class of metrics with nonpositive Gauss curvature K, for which directional derivative has a simple structure for its zero set. This gives the results of Nakamura and Hong as special cases. On the other hand, Aleksei Pogorelov in 1972 constructed a $C^{2,1}$ metric g on $B_1 \subset \mathbb{R}^2$ with a sign-changing K such that (B_r, g) cannot be realized as a C^2 surface in $\mathbb{R}^3$ for any $r > 0$. Nikolai Nadirashvili and Yu Yuan in 2008 constructed a $C^{2,1}$ metric g on B_1 of $K \geq 0$, with no C^2 local isometric embedding in $\mathbb{R}^3$.

3.2. **Global isometric embedding of surfaces in $\mathbb{R}^3$.** In 1916, Hermann Weyl posed the following problem: does every smooth metric on $\mathbb{S}^2$ with positive Gauss curvature admit a smooth isometric embedding in $\mathbb{R}^3$? The first attempt to solve the problem was made by Weyl himself, through the continuity method and the a priori estimates up to the second derivatives. Twenty years later, Hans Lewy solved the problem for analytic metrics g. In 1953, Luis Nirenberg gave a complete solution under the very mild hypothesis that g is C^4. The result was extended to the C^3 case by Erhard Heinz in 1962. In a completely different approach, Aleksandr Alexandroff in 1942 obtained a generalized solution to Weyl's problem as a limit of polyhedra. Further, it is known from the work of Pogorelov in the 1950s that closed C^1 surfaces with positive Gauss curvature and bounded extrinsic curvature are convex, and that closed convex surfaces are *rigid* in the sense that their isometric immersions are unique up to rigid motions. In 1994 and 1995, Pei-Feng Guan and Yanyan Li, and Hong and Zuily independently generalized Nirenberg's result for metrics on $\mathbb{S}^2$ with nonnegative Gauss curvature.

The investigation of the isometric immersion of metrics with negative curvature goes back to David Hilbert. He proved in 1901 that the full hyperbolic plane cannot be isometrically immersed in $\mathbb{R}^3$. The next natural step is to extend such a result to complete surfaces whose Gauss curvature is bounded above by a negative constant. The final solution to this problem was obtained by Nikolai Efimov in 1963: he proved that any complete negatively curved smooth surface does not admit a C^2 isometric immersion in $\mathbb{R}^3$ if its Gauss curvature is bounded away from zero. In the years following, Efimov extended his result in several ways. Before the 1970s, the study of negatively curved surfaces was largely directed at the nonexistence of isometric immersions in $\mathbb{R}^3$. In the 1980s, Shing-Tung Yau proposed to find a sufficient condition for complete negatively curved surfaces to be isometrically immersed in $\mathbb{R}^3$. In 1993, Hong identified such condition in terms of the Gauss curvature decaying at a certain rate at infinity. His discussion was based on a differential system equivalent to the Gauss-Codazzi system (7).

3.3. **Non-smooth immersions of surfaces.** Extension of the rigidity of Weyl's problem to Hölder-regular $C^{1,\alpha}$ isometric immersions, is originally due to Borisov in the late 1950s, who proved that for $\alpha > \frac{2}{3}$, the image of a surface with positive Gauss curvature has bounded extrinsic curvature. Hence, if $(\mathbb{S}^2, g)$ is a Riemannian manifold with $K > 0$ then $u(\mathbb{S}^2)$ is the boundary of a bounded convex set that is unique up to rigid motions of $\mathbb{R}^3$, provided that $u \in C^{1,\alpha}$ with $\alpha > \frac{2}{3}$. In particular, if K is constant then $u(\mathbb{S}^2) = \frac{1}{\sqrt{K}}\mathbb{S}^2$. In 2012 Conti, De Lellis and Szekelyhidi provided a direct analytic proof of these results, based on a clever use of the commutator estimate.

As we have mentioned before, flexibility for isometric immersions of surfaces in $\mathbb{R}^3$ has been proved by Conti, De Lellis and Szekelyhidi, up to the regularity exponent $\frac{1}{7}$. This results has been improved by De Lellis, Dominik Inauen and Szekelyhidi in 2018 where they proved that any short immersion (or embedding) of a 2-dimensional Riemannian manifold (B_1, g) into $\mathbb{R}^3$, can be uniformly approximated by a sequence of $C^{1,\alpha}$ isometric immersions (embeddings) for any $\alpha < \frac{1}{5}$. Their key argument relies on the fact that every two-dimensional metric is locally conformally equivalent to the Euclidean metric $g = \mathrm{Id}_2$. This means that a positive definite defect $\mathcal{D} = g - (\nabla u)^T \nabla u$ may be, by a change of variables, reduced to the diagonal form, which decomposes into two primitive defects rather than three, resulting in a lower rate of blow-up of the second derivatives in the Nash-Kuiper iteration scheme and subsequently higher regularity of the immersions derived in the limiting process. The same statement, albeit at the linearized level, has been recently used by the second author of this paper, to show density in the space of continuous functions on $\bar{\Omega} \subset \mathbb{R}^2$, of the set of weak solutions $u \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^n)$ with any $\alpha < \frac{1}{1+4/(n-2)}$, to the following equation with a given right hand side f:

$$\partial_{11}u \cdot \partial_{22}u - |\partial_{12}u|^2 = f \quad \text{in } \Omega.$$

For the codimension $n - 2 = 1$, this generalizes the prior density result for the *Monge-Ampère equation* and its weak $C^{1,\alpha}$ solutions at $\alpha < \frac{1}{5}$, due to Wentao Cao and Szekelyhidi. The parallel rigidity statements are likewise available when $\alpha > \frac{2}{3}$.

Regarding the flexibility vs rigidity in the regularity interval $[\frac{1}{5}, \frac{2}{3}]$, Gromov conjectured that the actual threshold occurs sharply at $\alpha = \frac{1}{2}$. This is supported by the work of De Lellis and Inauen in 2020 in which they proved that for any $\alpha < \frac{1}{2}$, an appropriate convex integration construction yields $C^{1,\alpha}$ isometric immersions of a spherical cup, whose Levi-Civita connection differs from the standard one, whereas any such immersion with regularity $\alpha > \frac{1}{2}$ must necessarily induce the compatible Levi-Civita connection.

4. Applications to general relativity

Quasi-local mass in general relativity is a notion associated with closed spacelike 2-surfaces in a 4-dimensional spacetime. Its purpose is to evaluate the amount of matter and gravitational energy contained within the surface, and can potentially be used to detect the formation of black holes. In this section, we briefly discuss an application of Weyl's embedding problem to the quasi-local masses. For more information, see [4].

Consider a smooth, orientable, compact Riemannian manifold (Ω^3, g), with connected boundary Σ of positive Gaussian curvature. According to Weyl's embedding theorem, Σ may be uniquely (up to rigid motions) isometrically embedded into $\mathbb{R}^3$. This embedding induces the mean curvature H_0, which, in general, differs from the mean curvature H of Σ as a submanifold of Ω. In 1992, based on a Hamilton-Jacobi analysis of the Einstein-Hilbert action, David Brown and James York defined the quasi-local mass of Σ to be:

$$m_{BY}(\Sigma) = \frac{1}{8\pi}\Big(\int_\Sigma H_0 \, \mathrm{d}\sigma - \int_\Sigma H \, \mathrm{d}\sigma\Big).$$

A fundamental result concerning m_{BY} was established by Yuguang Shi and Luen-Fai Tam in 2002. Namely, they showed that if H is positive and the scalar curvature of g is nonnegative, then $m_{BY}(\Sigma)$ is nonnegative and it vanishes if and only if (Ω, g) isometrically embeds into $\mathbb{R}^3$. From a geometric perspective, this result may be interpreted as a comparison theorem for compact manifolds of nonnegative scalar curvature.

Despite this beautiful result, the Brown-York definition has several deficiencies, most notably that it is not 'gauge independent' when considered in a spacetime context. This motivated Chiu-Chu Melissa Liu and Shing-Tung Yau in 2003, and then Mu-Tao Wang and Yau in 2009 to each define more general notions of quasi-local mass which satisfy a range of desirable properties. Like m_{BY}, both of these masses also employ the Weyl embedding theorem, and are consequently restricted to surfaces Σ which are topologically 2-spheres. It should be noted that Wang-Yau utilize the theorem to produce isometric embeddings into Minkowski space, even if the Guassian curvature changes sign. Recently, another quasi-local mass in this family was proposed by Aghil Alaee, Khuri and Yau, which allows for surfaces of higher genus and also requires their embedding into Minkowski space. A natural question then arises that would have important implications: which closed surfaces admit isometric embeddings into Minkowski space? This problem gives rise to an underdetermined system of equations, and thus one may guess that there are no obstructions. However, even for the torus, the problem remains open.

5. Applications to the mathematical materials science

When the ambient and intrinsic dimensions agree, $n = d$, the problem (1) is linked with the satisfaction of the orientation preservation by $u : B_1 \to \mathbb{R}^d$, expressed as:

$$\det \nabla u > 0 \quad \text{in } \ B_1. \tag{8}$$

Under this condition, a sufficient and necessary condition for the local solvability of (1) is the vanishing of the Riemannian curvature of g, which also guarantees that the solution u is smooth and unique up to rigid motions. On the other hand, without the restriction (8), there always exists a Lipschitz continuous u constructed by convex integration, which indeed changes orientation in any neighbourhood of any point at which g has non-zero curvature. The set of such Lipschitz immesions is dense in the set of short immersions, similarly to other h-principle statements that we have listed before.

In the former context, it is natural to pose the quantitative question: what is the infimum of the averaged pointwise deficit of u from being an orientation-preserving isometric immersion of g? This deficit is measured by the following *non-Euclidean energy* on a domain $\Omega \subset \mathbb{R}^d$ with respect to the Riemannian manifold (Ω, g):

$$\mathcal{E}_g(u) = \int_\Omega \operatorname{dist}^2\big((\nabla u) g^{-1/2}, \mathrm{SO}(d)\big)\ \mathrm{d}x. \tag{9}$$

Above, $\mathrm{SO}(d)$ denotes the special orthogonal group, namely rotations in $\mathbb{R}^d$, and $\operatorname{dist}(\cdot,\cdot)$ is the distance in the space of matrices $\mathbb{R}^{d\times d}$. By the polar decomposition theorem, an equidimensional u satisfies (1) and (8) if and only if $\nabla u \in \mathrm{SO}(d) g^{1/2}$ in Ω, which happens precisely when $\mathcal{E}_g(u) = 0$. The follow-up questions now are:

(i) Can one quantify the infimum of $\mathcal{E}_g$ in relation to g and Ω?
(ii) What is the structure of minimizers to (9), if they exist?
(iii) In the limit of Ω becoming $(d-1)$-dimensional, what are the asymptotic properties of the energies $\mathcal{E}_g$ and their minimizers in relation to the curvatures of g?

5.1. **Connection to calculus of variations and elasticity.** The field of calculus of variations originally centered around minimization problems for integral functionals of the general form below, in which $W : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ is a given *energy density*, and where u may be subject to various constraints, for example the boundary conditions:

$$\mathcal{E}(u) = \int_\Omega W\big(x, u(x), \nabla u(x)\big)\ \mathrm{d}x \quad \text{for } \ u : \mathbb{R}^d \supset \Omega \to \mathbb{R}^n, \tag{10}$$

The systematic study of existence of minimizers to (10), their uniqueness and qualitative properties, began with Leonhard Euler and Johann Bernoulli in the XVIIth century and progressed due to seminal contributions by Charles Morrey and Ennio De Giorgi in the XXth century. These questions are strongly tied to the convexity, in the ∇u variable in W, in turn implying the so-called sequential lower semicontinuity of $\mathcal{E}$; a condition necessary to conclude that the minimizing sequences to (10) accumulate at the minimizers. This is the *direct method of calculus of variations*, which however does not apply to the functional in (9), precisely due to its lack of convexity.

An example of an important class of problems of the form (10) is the basic variational model pertaining to the *nonlinear elastic energy of prestressed bodies*:

$$\mathcal{E}_g(u) = \int_\Omega W\big((\nabla u)g^{-1/2}\big)\ \mathrm{d}x \quad \text{ for } \ u : \mathbb{R}^3 \supset \Omega \to \mathbb{R}^3, \tag{11}$$

where $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is the given energy density, carrying the elastic moduli of the physical material whose referential configuration is Ω, and satisfying the necessary physically-relevant conditions (frame invariance and the zero-penalty for all rigid motions). The theory of elasticity is one of the most important subfields of continuum mechanics. It studies materials which are capable of undergoing large deformations, due to the distribution of *local stresses and displacements*, and resulting from the application of mechanical or thermal loads. The model (11) postulates formation of a target Riemannian metric g and the induced multiplicative decomposition of the deformation gradient ∇u into the elastic part $(\nabla u)g^{1/2}$, and the inelastic part $g^{1/2}$ responsible for the morphogenesis. The form of g is dictated by the material's response to pH, humidity, temperature, growth hormone distribution and other stimuli, and it is specific to each problem.

The functional in (11) corresponds to a range of hyperelastic energies approximating the behavior of a large class of elastomeric materials, and it is consistent with the microscopic derivations based on statistical mechanics. It reduces (via a change of variables) to the classical nonlinear three-dimensional elasticity, for g with vanishing Riemannian curvature, which occurs precisely when $\min \mathcal{E}_g = 0$. It can be proved that in the opposite case, i.e. for a non-Euclidean g, the infimum of $\mathcal{E}_g$ in the absence of any forces or boundary conditions remains strictly positive, pointing to the existence of residual strain.

For domains Ω that are *thin films*, one considers a family of problems:

$$\mathcal{E}^h_g(u^h) = \int_{\Omega^h} W\big((\nabla u^h)g^{-1/2}\big)\ \mathrm{d}x \quad \text{ for } \ u : \Omega^h \to \mathbb{R}^3, \tag{12}$$

parametrised by the small thickness h of $\Omega^h = \omega \times (-h/2, h/2)$. The task is now to determine the asymptotic limit of their minimizations as $h \to 0$, rather than to minimize $\mathcal{E}^h_g$ for each particular h. This can be achieved using the method of Γ-*convergence*, to identify the "singular limit" energy $\mathcal{I}_g$ characterized by the property that the minimizers and minimum values of (12) converge to the minimizers and the minimum values of $\mathcal{I}_g$. In general, one expects that $\inf \mathcal{E}^h_g \simeq h^\gamma$ as $h \to 0$ with the optimal scaling exponent γ determined from g and the appropriate curvatures of g contributing to the form of $\mathcal{I}_g$.

5.2. **Dimension reduction of thin prestressed films.** Using flexibiliy of the isometric immersions of Riemannian manifolds (M^2, g) into $\mathbb{R}^3$ at Hölder regularity below $C^{1,1/5}$, one can show that $\inf \mathcal{E}_g^h \leq Ch^\beta$ for any $\beta < \frac{2}{3}$. On the other hand, having any $\beta \geq 2$ automatically implies (in fact, it is equivalent to) that the restriction $g(\cdot, 0)_{2\times 2}$ of the metric g to the midplate $\omega \subset \mathbb{R}^2$ has an isometric immersion into $\mathbb{R}^3$ with Sobolev regularity H^2 (i.e. with its second order derivatives square integrable).

β	constraint / regularity	asymptotic expansion	limiting energy $\mathcal{I}_g$
2	$y \in H^2(\omega, \mathbb{R}^3)$ $(\nabla y)^T \nabla y = g(\cdot, 0)_{2\times 2}$	$y(\cdot)$ $\{3d: \ y + x_3 b\}$	$\|(\nabla y)^T \nabla b - \frac{1}{2}\partial_3 g(\cdot, 0)_{2\times 2}\|^2$ $\big[\partial_1 y, \partial_2 y, b\big] \in \mathrm{SO}(3) g(\cdot, 0)^{1/2}$
4	$\mathcal{R}_{12,cd}(\cdot, 0) = 0$ $(V, w^h) \in H^2 \times H^1(\omega, \mathbb{R}^3)$ $\mathrm{sym}\big((\nabla y_0)^T \nabla V\big) = 0,$ $\mathrm{sym}\big((\nabla y_0)^T \nabla w^h\big) \to S$	$y_0 + hV + \ h^2 w^h$	$\|\frac{1}{2}(\nabla V)^T \nabla V + S + \frac{1}{24}(\nabla b_1)^T \nabla b_1$ $-\frac{1}{48}\partial_{33} g(\cdot, 0)_{2\times 2}\|^2$ $+\|(\nabla y_0)^T \nabla p + (\nabla V)^T \nabla b_1\|^2$ $+\|\big[\mathcal{R}_{i3,j3}(\cdot, 0)\big]_{i,j=1,2}\|^2$
6 $\vdots$	$\mathcal{R}_{ab,cd}(\cdot, 0) = 0$ $V \in H^2(\omega, \mathbb{R}^3)$ $\mathrm{sym}\big((\nabla y_0)^T \nabla V\big) = 0$	$y_0 + h^2 V$	$\|(\nabla y_0)^T \nabla p + (\nabla V)^T \nabla b_1 + \alpha\big[\partial_3 \mathcal{R}\big]\|^2$ $+\|\mathbb{P}_{\mathcal{S}_{y_0}^\perp}\big[\partial_3 \mathcal{R}\big]\|^2 + \|\mathbb{P}_{\mathcal{S}_{y_0}}\big[\partial_3 \mathcal{R}\big]\|^2$
$2k$ $\vdots$	$\mathcal{R}_{ab,cd}(\cdot, 0) = 0$ $\big[\partial_3^{(i)} \mathcal{R}\big](\cdot, 0) = 0 \ \ \forall i \leq n-3$ $V \in H^2(\omega, \mathbb{R}^3)$ $\mathrm{sym}\big((\nabla y_0)^T \nabla V\big) = 0$	$y_0 + h^{k-1} V$ $\{3d: \ y_0 + \sum_{i=1}^{k-1} \frac{x_3^i}{i!} b_i$ $+h^{k-1} V + h^{k-1} x_3 p\}$	$\|(\nabla y_0)^T \nabla p + (\nabla V)^T \nabla b_1$ $+\alpha\big[\partial_3^{(k-2)} \mathcal{R}\big]\|^2 + \|\mathbb{P}_{\mathcal{S}_{y_0}^\perp}\big[\partial_3^{(k-2)} \mathcal{R}\big]\|^2$ $+\|\mathbb{P}_{\mathcal{S}_{y_0}}\big[\partial_3^{(k-2)} R\big]\|^2$

FIGURE 1. The first column gathers equivalent conditions for the scaling $\inf \mathcal{E}_g^h \sim Ch^\beta$, in terms of the Riemann curvatures $\mathcal{R}_{ab,cd}$ of g. The second column provides the asymptotic expansion of the minimizing sequence. The third column gives the form of the Γ-limits in this infinite hierarchy.

One can show that the only viable scaling exponents in: $\inf \mathcal{E}_g^h \sim Ch^\beta$ in the regime $\beta \geq 2$, are the even powers $\beta = 2k$. For each k, the complete set of results is available: the conditions/constraints on the curvatures of g equivalent to the indicated scaling; the necessary asymptotic expansion of the minimizing sequence u^h in terms of the transversal (thin direction) variable x_3 and the regularity of the fields present in that expansion; the limiting energy $\mathcal{I}_g$ given in terms of these fields and the unconstrained curvatures. That *hierarchy of dimensionally reduced energies* is schematically presented in Figure 1.

Parallel general results can be derived in the abstract setting of Riemannian manifolds, for more general dimensions of the midplate $d > 2$ and the ambient space $n > d$, and also for g replaced by the film's thickness - depending prestrain, incompatible only through a perturbation of order which is a power of h. While the systematic description of the singular limits associated with the exponents $\beta \in (\frac{2}{3}, 2)$ is not yet available, there are a number of illustrative examples of the emerging patterns and the corresponding scalings in that range. For this discussion we refer to [3] and the references therein.

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