FULL FLEXIBILITY OF ISOMETRIC IMMERSIONS OF METRICS WITH LOW HÖLDER REGULARITY IN POZNYAK THEOREM'S DIMENSION

MARTA LEWICKA

ABSTRACT. A classical result by Poznyak asserts that any smooth 2-dimensional Riemannian metric g, posed on the closure of a simply connected domain $\omega \subset \mathbb{R}^2$, has a smooth isometric immersion into $\mathbb{R}^4$. Using techniques of convex integration, we prove that for any 2-dimensional $g \in \mathcal{C}^{r,\beta}$, an isometric immersion of regularity $\mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^4)$ for any $\alpha < \min\{\frac{r+\beta}{2}, 1\}$, may be found arbitrarily close to any short immersion. The fact that this result's regularity reaches $\mathcal{C}^{1,1-}$ for $g \in \mathcal{C}^2$, which is referred to as "full flexibility", should be contrasted with: (i) the regularity $\mathcal{C}^{1,1/3-}$ achieved in [10] for isometric immersions into $\mathbb{R}^3$ and the lack of flexibility (rigidity) of such isometric immersions with regularity $\mathcal{C}^{1,2/3+}$ proved in [1]-[5], [16]; (ii) the regularity $\mathcal{C}^{1,1-}$ obtained in [30] for isometric immersions into higher codimensional space $\mathbb{R}^8$; and (iii) the regularity $\mathcal{C}^{1,\frac{1}{1+d(d+1)/k}-}$ achieved in [32] in the general case of d-dimensional metrics and $(d+k)$-dimensional immersions for the closely related Monge-Ampère system.

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1. Introduction

As mentioned in Nash's fundamental work "The imbedding problem for Riemannian manifolds" [38]: "apparently rigidity disappears completely when the imbedding space has enough dimensions". The purpose of this paper is to show that for 2-dimensional metrics, rigidity indeed disappears completely, already in $\mathbb{R}^4$. More precisely, we prove:

Theorem 1.1. *Let $g \in \mathcal{C}^{r,\beta}(\bar\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym},>})$ be defined on the closure of an open set $\omega \subset \mathbb{R}^2$ diffeomorphic to B_1, for some $r+\beta>0$. Then, for every $u \in \mathcal{C}^1(\bar\omega, \mathbb{R}^4)$ satisfying:*

$$(\nabla u)^T \nabla u > 0 \quad \textit{and} \quad g - (\nabla u)^T\nabla u > 0 \quad \textit{in } \bar\omega,$$

for every $\epsilon > 0$, and for every regularity exponent α in:

$$0 < \alpha < \min\Big\{\frac{r+\beta}{2}, 1\Big\},$$

there exists $\tilde u \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^4)$ such that:

$$\|\tilde u - u\|_0 \leq \epsilon \quad \textit{and} \quad (\nabla \tilde u)^T \nabla \tilde u = g \quad \textit{in } \bar\omega.$$

To position our result in regards to other known works, we make the following observations:

- The metric regularity $\mathcal{C}^2$ is critical [38, 29, 19, 21]; since the assumed shortness condition can be easily achieved by taking an arbitrary immersion u and scaling it by a small constant, our result implies that 2-dimensional $\mathcal{C}^2$ metrics always have an isometric immersion of any regularity up to $\mathcal{C}^{1,1}$, i.e. just below $\mathcal{C}^2$, already in dimension $n=4$.
- As is known in various contexts [15, 24, 22, 40, 5, 16], rigidity does not disappear for isometric immersions of 2-dimensional metrics into $\mathbb{R}^3$; we show that no rigidity statement is possible for isometric immersions into $\mathbb{R}^4$.
- It is known [10] that any short immersion of a 2-dimensional $\mathcal{C}^2$ metric g can be approximated by $\mathcal{C}^{1,\alpha}$ isometric immersions into $\mathbb{R}^3$, for any $\alpha < 1/3$; we assert the same statement up to regularity $\mathcal{C}^{1,1}$ already at the next target dimension $n=4$.
- Dimension $n=4$ is the target dimension in Poznyak's theorem [41]: the disk $B_1 \subset \mathbb{R}^2$ with arbitrary $\mathcal{C}^\infty$ metric can be isometrically $\mathcal{C}^\infty$-immersed into $\mathbb{R}^4$. On the other hand [30], for n large enough there exists a $\mathcal{C}^{1,\alpha}$ isometric immersion of any $g \in \mathcal{C}^{r,\beta}$ where $r+\beta<2$, with any $\alpha < \frac{r+\beta}{2}$. We reach the same statements in dimension 4, in the context of flexibility, rather than themere existence of a single isometric immersion.
- The same result as ours has been as proved [27] for the Monge-Ampère system; now we treat the fully nonlinear case of (1.1).

These observations are made precise in the historical account of the existence, flexibility and rigidity of isometric immersions in subsections below.

1.1. Classical existence and flexibility results.

The question whether a Riemannian manifold of dimension d may be isometrically immersed in a Euclidean space of dimension n, is, in local coordinates, equivalent to solving the first order system of partial differential equations:

$$\begin{aligned} &(\nabla u)^T \nabla u = g \quad \text{in } \ \omega \subset \mathbb{R}^d, \\ &\text{for } \ u : \omega \to \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where $g : \omega \to \mathbb{R}^{d\times d}_{\mathrm{sym},>}$ is a given first fundamental form i.e. the Riemannian metric of the manifold written in these coordinates. The exact answer to this question and to the question on the minimal dimension n, depend on g and on how smooth u is required to be. In the analytic case, the Janet-Cartan-Burstin theorem [25, 14, 8] gives the affirmative resolution of

Schlaefli's conjecture [42], stating that any d-dimensional analytic g has a local analytic isometric immersion u into $\mathbb{R}^{s_d}$. Here, s_d is the Janet dimension, corresponding to the number of equations in (1.1) and the number of independent entries in $d \times d$ symmetric matrices:

$$s_d \doteq \frac{d(d+1)}{2}. \tag{1.2}$$

In this theorem $n = s_d$ cannot be replaced by $s_d - 1$. For $d = 2$, we have $s_d = 3$. The crucial part of the Janet-Burstin proof consists in showing that a local analytic isometric immersion that is "free", always exists, in the minimal dimension $d + s_d$, the freeness of an immersion meaning that d vectors of its first derivatives and s_d of second derivatives are linearly independent at each $x \in \omega$. Invoking this result at $d-1$ is then supplemented by extending the hence produced analytic immersion by one extra dimension that is missing from $(d-1) + s_{d-1} = s_d - 1$ to s_d, via an application of the Cauchy-Kowalewsky theorem.

In the non-analytic case and for general metrics of regularity at least $\mathcal{C}^2$, an isometric immersion into any $\mathbb{R}^n$ cannot be of regularity better than the regularity of the metric. This fact was first noticed by Hartman and Wintner [23]: given a 2-dimensional Riemannian metric $g \in \mathcal{C}^{r,\beta}$ with $r \geq 2$ and $\beta \in [0, 1]$, its Gaussian curvature κ can be computed from the second fundamental form of an immersion $u \in \mathcal{C}^l$, which gives $\kappa \in \mathcal{C}^{l-2}$ (for l not necessarily an integer). On the other hand, κ is an intrinsic quantity and as such, can be computed directly from g, whereas $\kappa \in \mathcal{C}^{r-2,\beta}$. One can explicitly construct a metric whose κ is no smoother than $\mathcal{C}^{r-2,\beta}$, hence in general there must be: $l \leq r + \beta$. In dimension $d > 2$, the same argument applies to sectional curvatures and can be found in [29]. The fact that the regularity exponent $l = r + \beta$ of an isometric immersion can be achieved, is based on the following ideas of Nash.

For the integer regularity $r > 2$, the Nash theorem [38] states that every $\mathcal{C}^r$ metric has a $\mathcal{C}^r$ isometric immersion into some $\mathbb{R}^n$. A partition of unity argument reduces this theorem to the local case in (1.1), in which n can be taken as $3s_d + 4d$, the dimension later decreased by Gromov [19] to $s_d + 2d + 3$, and by Günter (with a different proof) [21] to $\max\{s_d + 2d, s_d + d + 5\}$. For $d = 2$, both these numbers equal 10. It is worth noting that the discussed dimensions do not depend on the differentiability exponent r. The Hölder regular case has been studied by Jacobowitz, who proved [29] that every $\mathcal{C}^{r,\beta}$ Riemannian metric with $r \geq 2$ and $\beta \in (0, 1]$ has a $\mathcal{C}^{r,\beta}$ isometric immersion into some $\mathbb{R}^n$. Combining Nash's and Jacobowitz's results yields immersability of any $\mathcal{C}^l$ metric where $l > 2$ is not necessarily an integer, with the immersion of the same regularity. The key argument of their proofs relies on an implicit function theorem (later generalized to what is called the Nash-Moser implicit function theorem) showing that for a large class of analytic metrics, any $\mathcal{C}^l$ metric in their vicinity (depending on l), possesses a $\mathcal{C}^l$ isometric immersion. Such analytic metrics are shown to be dense in $\mathcal{C}^0$, due to the earlier construction by Nash producing $\mathcal{C}^1$ isometries, that we now describe. We note in passing that for $r \geq 2$, a $\mathcal{C}^r$ Riemannian manifold of dimension d has a local $\mathcal{C}^r$ isometric immersion into $\mathbb{R}^d$ if an only if its Riemann curvature tensor vanishes at each x.

In case of low immersion regularity $\mathcal{C}^1$, the Nash theorem in [37] states that if a $\mathcal{C}^0$ Riemannian manifold of dimension d has a short immersion into $\mathbb{R}^{d+2}$, then it has a nearby $\mathcal{C}^1$ isometric immersion (into $\mathbb{R}^{d+2}$). In coordinates, the shortness of $u \in \mathcal{C}^1(\bar{\omega}, \mathbb{R}^n)$ is expressed by:

$$g - (\nabla u)^T \nabla u > 0 \quad \text{in } \bar{\omega},$$

i.e. by requiring that the matrix in left hand side be strictly positive definite for every $x \in \bar{\omega}$. This condition can be achieved by scaling an arbitrary immersion by a small positive number. Nash's theorem is proved via iterative modifications of an initial short immersion u by a cascade of highly oscillatory perturbations, with sum of their amplitudes small, and with the property that each of them replaces the specific (rank-one) portion of the defect:

$$\mathcal{D}(g,u) \doteq g - (\nabla u)^T \nabla u, \tag{1.3}$$

by a much smaller (in the $\mathcal{C}^0$ norm) contribution. After decreasing $\mathcal{D}$ in this manner, the new defect is computed and the construction is repeated in, what is nowadays referred to as, the Nash-Kuiper iteration scheme. The nearby (to the given short immersion u) immersion $\tilde{u}$ is obtained in the limit of such modifications. The perturbations, called Nash's spirals, each of them constituting a single Step in this convex integration algorithm, utilize 2 codimensions, and we exhibit a version of this construction in Lemma 2.4, later used in the proof of Theorem 4.1. At the end of [37], Nash conjectured that the condition $n \geq d+2$ needed to achieve the modifications, can be weakened to $n \geq d+1$. A proof of this assertion was subsequently provided by Kuiper [31] via another perturbation definition, which indeed utilizes only one codimension. We exhibit a version of such Kuiper's corrugation in Lemma 2.5, constituting a convex integration Step in the proof of Theorem 1.1. To reassume, the Nash-Kuiper theorem demonstrates that any short immersion $u \in \mathcal{C}^1(\bar{\omega}, \mathbb{R}^{d+1})$ can be uniformly approximated, with arbitrary precision, by an isometric immersion $\tilde{u} \in \mathcal{C}^1(\bar{\omega}, \mathbb{R}^{d+1})$, for any continuous, d-dimensional g, yielding the abundance, or flexibility, of $\mathcal{C}^1$ solutions to (1.1) with $n = d+1$.

1.2. **The dimension $d = 2$ and rigidity results.** For $d = 2$ we have $s_d = 3$, so an analytic 2-dimensional Riemannian metric always has a local analytic isometric immersion into $\mathbb{R}^3$. Further, the Gromov and Günter exponents equal 10, so any $\mathcal{C}^r$ metric possesses a $\mathcal{C}^r$ isometric immersion into $\mathbb{R}^{10}$, for $r > 2$. One way of reading our Theorem 1.1 is that, at the critical regularity $r = 2$, a $\mathcal{C}^2$ metric always has an isometric immersion of any regularity just below $\mathcal{C}^2$ (i.e., $\mathcal{C}^{1,\alpha}$ for any $\alpha < 1$), already in dimension $n = 4$. This is precisely the target dimension in Poznyak's theorem: as shown in [41], the disk $B_1 \subset \mathbb{R}^2$ with arbitrary $\mathcal{C}^\infty$ metric can be isometrically $\mathcal{C}^\infty$-immersed in $\mathbb{R}^4$. For completeness, we present a proof of this result in the Appendix section 9. In fact, the dimension $n = 4$ cannot be lowered for this general statement, because of an example in [41] of a smooth metric on a disk, for which there is no $\mathcal{C}^2$ isometric immersion into $\mathbb{R}^3$. In [20] a more striking example has been put forward, of an analytic metric of positive Gaussian curvature that is not induced by any $\mathcal{C}^2$ immersion into $\mathbb{R}^3$ and keeps this property under any $\mathcal{C}^2$-small perturbation of the metric. We also note that $s_2 + 2 = 5$, so below this dimension, a free immersion of any 2-dimensional Riemannian metric cannot exist.

The target dimension $n = 3$ is different. Firstly, according to the classical rigidity theorems due to Cohn-Vossen [15] and Herglotz [24], in the resolution of the Weyl problem [44], a $\mathcal{C}^2$ isometric immersion of $\mathbb{S}^2$ into $\mathbb{R}^3$ must be a rigid motion, i.e. a composition of a rotation and a translation. Second, any $\mathcal{C}^2$ surface with positive Gauss's curvature must be locally convex; for generalizations of this statement we refer to [22], and to [39, Chapter II], [40, Chapter IX] where the requirement of the $\mathcal{C}^2$ regularity is replaced by the requirement that the immersion is $\mathcal{C}^1$ and that the measure on $\mathbb{S}^2$ induced by the Gauss map has bounded variation. Third, using geometric arguments, the same rigidity statement has been proved by Borisov in a series of papers [1, 2, 3, 4, 5], for $\mathcal{C}^{1,\alpha}$ isometric immersions at $\alpha > 2/3$, of $\mathcal{C}^2$ metrics, with a simpler analytic proof of this result obtained in [16]. In particular, we see that $2/3$ is an upper bound on the range of Hölder exponents that can be reached using convex integration for isometric immersions of 2-dimensional metrics into $\mathbb{R}^3$. To the contrary, our Theorem 1.1 shows that no such rigidity statement is possible for the isometric immersions already into $\mathbb{R}^4$.

1.3. **Recent results on flexibility.** For the codimension-one case, i.e. when $n = d + 1$ in (1.1), it already transpires from the Nash-Kuiper construction, that flexibility should hold not only in $\mathcal{C}^1$, but also in $\mathcal{C}^{1,\alpha}$ provided that $\alpha > 0$ is sufficiently small. A first precise study of this type is due to Borisov, who announced in [6] and provided a proof in case of $d = 2$ in [7], that for any analytic g, the Nash-Kuiper theorem extends to local isometric immersions with regularity $\mathcal{C}^{1,\alpha}$ and $\alpha < \frac{1}{1+2s_d}$. For arbitrary dimension d, this statement was proved by Conti, De Lellis and Szekelyhidi in [16]. For the special case $d = 2$, the hence derived flexibility exponent $\frac{1}{1+2s_2} = 1/7$ was improved to 1/5 in [17], capitalizing on the conformal equivalence of 2-dimensional Riemannian metrics with the Euclidean metric, thus reducing the number of primitive (rank-one) defects in the decomposition of $\mathcal{D}$ in (1.3) from $s_2 = 3$ to 2.

The reasoning behind all the so-far Hölder exponent improvements in flexibility results, follows precisely this line: if one can construct a modification of a short immersion u to another immersion, as done in [37, 31], to the effect that $\mathcal{D}$ decreases by a factor of σ^S, for any large σ and some power S, at the expense of having $\nabla^2 u$ increase by a factor of σ^J, for some power J, then the Nash-Kuiper iteration scheme (see our Theorem 1.3) is capable of producing, in the limit, an isometric immersion $\tilde{u} \in \mathcal{C}^{1,\alpha}$ with any prescribed regularity in the range:

$$0 < \alpha < \frac{1}{1 + 2J/S}, \tag{1.4}$$

further restricted only by the regularity of g when below $\mathcal{C}^2$ (see formula (1.13)). Now, a rule of thumb is that $\nabla^2 u$ increases by one power of σ already at the single application of Kuiper's corrugation Step, so in case of a single codimension available, and since in general one requires s_d of such Steps to decrease the defect by one power of σ, corresponding to s_d rank-one components in $\mathcal{D}$, formula (1.4) with $J = s_d$ and $S = 1$ implies the Borisov-Conti-De Lellis-Szekelyhidi exponent $\frac{1}{1+2s_d}$. Similarly, for $d = 2$ we get after a conformal change of variable that $J = 2$ and $S = 1$, yielding the exponent 1/5 as in [17]. In a recent paper [10], Cao, Hirch and Inauen utilized an auxiliary construction which transfers exactly d rank-one primitive defects onto the remaining $s_d - d$ ones, hence allowing for $J = s_d - d$ and $S = 1$, and resulting in flexibility up to $\frac{1}{1+2(s_d-d)} = \frac{1}{1+d^2-d}$ Hölder regularity in the first derivatives. For $d = 2$ this implies that any short immersion of a 2-dimensional $\mathcal{C}^2$ metric g can be approximated by $\mathcal{C}^{1,\alpha}$ isometric immersions into $\mathbb{R}^3$, for any $\alpha < 1/3$. In this context, our Theorem 1.1 asserts the same statement up to regularity $\mathcal{C}^{1,1}$ already at the next target dimension $n = 4$.

As it is clear from the proofs, allowing for a higher codimension $n - d > 1$ only increases the regularity of the approximating isometric immersions. From our technical viewpoint, this is linked to the increase of $\nabla^2 u$ by only one power of σ despite several applications of Kuiper's corrugations in multiple Steps, as long as these are done in different codimensions. This implies the increase of S relative to J or, the decrease of the ratio J/S and therefore the increase of α. Without paying attention to such detailed considerations, but applying the convex integration construction with Nash's spirals as Steps, Källen proved in [30] that for $n \geq 6(d+1)(s_d+1)+2d$, there exists a $\mathcal{C}^{1,\alpha}$ isometric immersion of any $g \in \mathcal{C}^{r,\beta}$ where $r + \beta < 2$, for any $\alpha < \frac{r+\beta}{2}$. Our proof of Theorem 1.1 combines the insights from [30, 10, 16] with some new tricks, to reach exactly the Källen exponent already in dimension 4, in the context of flexibility rather than the mere existence of a single isometric immersion. Some aspects of the analysis leading to our result, was developed in the parallel context of the Monge-Ampère system, discussed below.

1.4. **The Monge-Ampère system.** The Monge-Ampère system was introduced in [32] as a higher-dimensional version of the classical Monge-Ampère equation:

$$\begin{aligned} & \det \nabla^2 v = f \quad \text{ in } \ \omega \subset \mathbb{R}^2, \\ & \text{for } \ v : \omega \to \mathbb{R}, \end{aligned} \tag{1.5}$$

in relation to its interpretation as the prescription, to the leading order terms, of the Gaussian curvature of a shallow surface given as the graph of v. In the same vein, prescription of the full Riemann curvature tensor $[R_{ij,st}]_{i,j,s,t:1\dots d}$ of a d-dimensional shallow manifold, reads:

$$\begin{aligned} & \mathfrak{Det}\, \nabla^2 v \doteq \big[\langle \partial^2_{is} v, \partial^2_{jt} v\rangle - \langle \partial^2_{it} v, \partial^2_{js} v\rangle\big]_{i,j,s,t:1\dots d} = F \quad \text{ in } \ \omega \subset \mathbb{R}^d, \\ & \text{for } \ v : \omega \to \mathbb{R}^k, \end{aligned} \tag{1.6}$$

where $F : \omega \to \mathbb{R}^{d^4}$ is a given field. Indeed, the Riemann curvatures of the family of immersions $u_\epsilon : \omega \to \mathbb{R}^{d+k}$ parametrized by $\epsilon \to 0$ in $u_\epsilon(x) = (x, \epsilon v(x))$, are calculated as:

$$R_{ij,st}\big((\nabla u_\epsilon)^T \nabla u_\epsilon)\big) = R_{ij,st}\big(\mathrm{Id}_d + \epsilon^2 (\nabla v)^T \nabla v\big) = \epsilon^2 (\mathfrak{Det}\, \nabla^2 v)_{ij,st} + o(\epsilon^2),$$

similarly to the formula for Gauss's curvature κ in case $d = 2$, $k = 1$:

$$\kappa((\nabla u_\epsilon)^T \nabla u_\epsilon) = \kappa(\mathrm{Id}_2 + \epsilon^2 \nabla v \otimes \nabla v) = \frac{\epsilon^2 \det \nabla^2 v}{(1 + \epsilon^2 |\nabla v|^2)^2} = \epsilon^2 \det \nabla^2 v + o(\epsilon^2).$$

Relying on the special structure of $\mathfrak{Det}\, \nabla^2$, we call $v \in H^1(\omega, \mathbb{R}^k)$ a weak solution to (1.6), if the following identity holds in the sense of distributions:

$$-\frac{1}{2} \mathfrak{C}^2((\nabla v)^T \nabla v) = F \quad \text{ in } \ \omega. \tag{1.7}$$

Here, $\mathfrak{C}^2$ is a second-order differential operator, which in dimension $d = 2$ reduces, up to symmetries, to taking $\operatorname{curl}\operatorname{curl}$ of a $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$- valued matrix field, whereas for $k = 1$, the identity (1.7) becomes exactly the weak formulation of (1.5) as studied in [36]:

$$-\frac{1}{2} \operatorname{curl}\operatorname{curl} (\nabla v \otimes \nabla v) = f.$$

The expression in the left hand side appeared in [28], called therein the very weak Hessian. It can be observed that the system (1.7), when posed on a contractible $\omega \subset \mathbb{R}^d$, reduces to:

$$\begin{aligned} & \frac{1}{2} (\nabla v)^T \nabla v + \operatorname{sym} \nabla w = A \\ & \text{for } \ v : \omega \to \mathbb{R}^k, \quad w : \omega \to \mathbb{R}^d, \end{aligned} \tag{1.8}$$

where $A : \omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ satisfies the compatibility condition: $-\mathfrak{C}^2(A) = F$, viable under appropriate symmetry, algebraic and differential identities assumptions on F. The system (1.8) is called the von Kárman system, in relation to the von Kárman stretching content in the theory of elasticity, where for $d = 2, k = 1$ the fields v, w are interpreted, respectively, as the out-of-plane and in-plane displacements of the midsurface ω of a thin elastic plate [35].

1.5. **Flexibility of the Monge-Ampère system.** The system (1.8) encodes the agreement, to the leading order, between the family of Riemannian metrics $g_\epsilon = \mathrm{Id}_d + 2\epsilon^2 A$ and the induced metrics of the augmented immersions $\bar{u}_\epsilon(x) = (x + \epsilon^2 w(x), \epsilon v(x))$, thus revealing a close connection between (1.6) and (1.1) at $n = d + k$. For this reason, one expects flexibility results for (1.8) of the same type as for (1.1). Indeed, given any short displacement pair $(v,w)\in\mathcal{C}^1(\bar\omega,\mathbb{R}^{d+k})$, where the shortness means positive definiteness of the defect $\mathcal{D}$ in:

$$\mathcal{D}(A,v,w)\doteq A-\big(\frac{1}{2}(\nabla v)^T\nabla v+\mathrm{sym}\nabla w\big)>0 \quad \text{in } \bar\omega,$$

and given any $\epsilon>0$, there exists a displacement pair $(\tilde v,\tilde w)\in\mathcal{C}^{1,\alpha}(\bar\omega,\mathbb{R}^{d+k})$ satisfying:

$$\|\tilde v-v\|_0+\|\tilde w-w\|_0\leq\epsilon \quad \text{and} \quad \mathcal{D}(A,\tilde v,\tilde w)=0 \quad \text{in } \bar\omega.$$

The fields $(\tilde v,\tilde w)$ are obtained in the limit of a version of the Nash-Kuiper iteration scheme with α in the range specified in (1.4), where S is the decay rate of $\mathcal{D}$ and J is the blow-up rate of $\nabla^2 v$ resulting from a single convex integration Stage construction. In [32], we proposed that a Stage consist of the least common multiple $lcm(s_d,k)$ Steps, each of them applying a version of Kuiper's corrugation in v and a matching perturbation in w, to the effect of replacing, as before, one rank-one primitive component of the defect by higher order error terms. Since $\mathcal{D}$ contains s_d such components, each time the counter of the number of Steps goes over a multiple of s_d, the defect goes down by one factor of σ, while each time this Step counter goes over a multiple of k, the $\nabla^2 v$ goes up, also by one factor of σ. Hence, after $lcm(s_d,k)$ of Steps, one reads the relative blow-up/decay ratio $J/S=s_d/k$, and therefore the flexibility up to the exponent $\frac{1}{1+2s_d/k}$. This exponent coincides with $\frac{1}{1+2s_d}$ from [16] for $k=1$, and for $d=2,k=1$ with the exponent $1/7$ that has been previously obtained in [36] for the Monge-Ampère equation.

In [32] we also exhibited an alternative convex integration construction for (1.6), based on superposing s_d Nash's spirals in a single Step, applicable for $k\geq 2s_d$ codimensions. We showed how the Källen iteration technique from [30] allows for the cancellation of arbitrarily high order defects all at once, thus leading to $J=1$ with S arbitrarily large, and ultimately yielding the flexibility exponent 1, always for $k\geq 2s_d$.

In the special case of $d=2$, the insight from [17] was used to improve the exponent $1/7$ to $1/5$ for the equation (1.5), and in [33] to $\frac{1}{1+4/k}$ for the system (1.6) and a general codimension k. In [9], the exponent $1/5$ was further increased to $1/3$ by introducing another version of the basic 2-dimensional defect decomposition with sharper bounds that distinguish between differentiation in the slow and fast variables. That idea preceded the construction in [10] in which certain components of $\mathcal{D}$ are initially transferred on other components, thus effectively making the spacial directions that they correspond to in the entries of $\mathcal{D}$, fast- or slow-like. In [34], the construction from [9] and the Källen-Nash spirals approach from [32] led to the flexibility exponent $3/7$ at $k=2$, the exponent $7/15$ at $k=3$ (from the general formula $\frac{2^k-1}{2^{k+1}-1}$), and the full flexibility with regularity of the solutions to (1.8), up to $\mathcal{C}^{1,1}$, for $k\geq 4$.

In [26] we studied the Monge-Ampère system at $d=2,k=3$ and proved its flexibility up to regularity $\mathcal{C}^{1,1-1/\sqrt{5}}$. That was the first result in which the obtained Hölder exponent $1-\frac{1}{\sqrt{5}}$ was larger than $1/2$, but not contained in a result of full flexibility up to $\mathcal{C}^{1,1}$. The new technique in [26] was based on the following observation. In all previous works, a Stage consisted of consecutive modifications of (v,w) by the oscillatory perturbations in different codimension directions, while keeping track of the magnitudes of the defect $\mathcal{D}$ and the partial derivatives $\partial_{ij}v^q$. These Steps were carried out until all of the components $\partial_{ij}v^q$ had the same order, at which point we read the blowup rate J of $\nabla^2 v$ and the corresponding decay rate S of $\mathcal{D}$. Presently, we were able to construct a perturbation cascade of Kuiper's corrugations, relative to the progression of frequencies which never allows for the same order in all $\partial_{ij}v^q$ at once. Since new corrugations are built on the previous ones in a staggered manner, then including larger and larger number of Steps in a single Stage (later iterated by the Nash-Kuiper algorithm) keeps decreasing J/S. A particular progression of frequencies, based on the Fibonacci sequence, yielded the regularity exponent $1-\frac{1}{\sqrt{5}}$ in [26]. The same construction, combined with the transfer of the defect portions as in [10] and another progression of frequencies, allowed in [27] for concluding the full flexibility up to $\mathcal{C}^{1,1}$ for the Monge-Ampère system in dimension $d=2$ and codimension $k=2$. Our present paper achieves the same result as [27], now for the fully nonlinear system (1.1), at the expense of a more complex construction that we describe below.

1.6. **The main technical contributions of this paper and an outline of proofs.** Our main contribution is given in terms of estimates gathered in the Stage theorem, whose iteration via the Nash-Kuiper algorithm ultimately provides the proof of Theorem 1.1:

Theorem 1.2. [STAGE] *Let $g\in\mathcal{C}^{r,\beta}(\bar\omega,\mathbb{R}^{2\times 2}_{\mathrm{sym},>})$ be defined on the closure of an open set $\omega\subset\mathbb{R}^2$ diffeomorphic to B_1, for some regularity exponents:*

$$0<r+\beta\leq 2. \tag{1.9}$$

Fix $\underline{\gamma}>0$ and integers $N,K\geq 4$. Then, there exists $\underline{\delta}\in(0,1)$ and $\underline{\sigma}>1$, depending only on $\underline{\gamma},\omega,g,N,K$, such that the following holds. Given any $u\in\mathcal{C}^2(\bar\omega,\mathbb{R}^4)$ and any δ,μ,σ such that:

$$\delta\leq\underline{\delta},\qquad \mu\delta^{1/2}\geq 1,\qquad \sigma\geq\underline{\sigma},\qquad \sigma^{3N+3}\delta\leq 1, \tag*{$(1.10)_1$}$$

$$\frac{1}{2\underline{\gamma}}\mathrm{Id}_2\leq(\nabla u)^T\nabla u\leq 2\underline{\gamma}\mathrm{Id}_2\quad \text{ in } \bar\omega, \tag*{$(1.10)_2$}$$

$$\|\mathcal{D}(g-\delta H_0,u)\|_0\leq\frac{r_0}{4}\delta\quad\text{ and }\quad \|u\|_2\leq\delta^{1/2}\mu, \tag*{$(1.10)_3$}$$

where r_0,H_0 are independent quantities in Lemma 2.2, there exists $\tilde u\in\mathcal{C}^2(\bar\omega,\mathbb{R}^4)$ satisfying:

$$\|\tilde u-u\|_1\leq C\delta^{1/2},\qquad \|\tilde u\|_2\leq C\mu\delta^{1/2}\sigma^{2K+N}, \tag*{$(1.11)_1$}$$

$$\Big\|\mathcal{D}\Big(g-\frac{\delta}{\sigma^{NK}}H_0,\tilde u\Big)\Big\|_0\leq\frac{r_0}{5}\frac{\delta}{\sigma^{NK}}+\frac{\|g\|_{r,\beta}}{\mu^{r+\beta}}, \tag*{$(1.11)_2$}$$

with constants C depending only on $\underline{\gamma},\omega,g,N,K$.

The above yields the decay rate $S_{K,N}=KN$ of $\mathcal{D}$ and the blow-up rate $J_{K,N}=2K+N$ of $\nabla^2 u$, where the quotient of these rates becomes arbitrarily small for large K,N:

$$\lim_{K\to\infty}\lim_{N\to\infty}\frac{J_{K,N}}{S_{K,N}}=\lim_{K\to\infty}\lim_{N\to\infty}\frac{2K+N}{KN}=0.$$

The Hölder regularity of the limiting immersion deduced from iterating the Stage, depends only on the aforementioned quotient and the regularity of g, through the formula in (1.13). Theorem 1.1 results hence in view of the following:

Theorem 1.3. [NASH-KUIPER'S ITERATION] *Let $\underline{u}\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^4)$ be an immersion, defined on the closure of an open set $\omega\subset\mathbb{R}^2$ diffeomorphic to B_1, together with $g\in\mathcal{C}^{r,\beta}(\bar\omega,\mathbb{R}^{2\times 2}_{\mathrm{sym},>})$ for some regularity exponents in (1.9). Assume that:*

$$\mathcal{D}(g,\underline{u})=g-(\nabla\underline{u})^T\nabla\underline{u}>0\qquad\text{on } \bar\omega.$$

Assume further that for some $S,J,p>0$ and for some parameters:

$$\underline{\delta}\in(0,1),\quad \underline{\gamma}>1,\quad \underline{\sigma}>1$$

depending only on $\omega, \underline{u}, g, S, J, p$ and with $\underline{\gamma}$ possibly larger than $\underline{\gamma}$ in Theorem 4.1 and $\underline{\delta}$ possibly smaller than $\underline{\delta}$ in Theorem 4.1, the following holds:

$$\left[\begin{array}{l} \text{Given any } u \in \mathcal{C}^2(\bar\omega, \mathbb{R}^4) \text{ and any } \delta, \mu, \sigma \text{ such that:} \\ \qquad \delta \leq \underline{\delta}, \qquad \mu\delta^{1/2} \geq 1, \qquad \sigma \geq \underline{\sigma}, \qquad \sigma^p \delta^{1/2} \leq 1, \\ \qquad \dfrac{1}{2\underline{\gamma}} \leq (\nabla u)^T \nabla u \leq 2\underline{\gamma}\mathrm{Id}_2 \quad \text{in } \bar\omega, \\ \qquad \|\mathcal{D}(g - \delta H_0, u)\|_0 \leq \dfrac{r_0}{4}\delta \quad \text{and} \quad \|u\|_2 \leq \delta^{1/2}\mu, \\ \text{with } r_0, H_0 \text{ as in Lemma 2.2, there exists } \tilde u \in \mathcal{C}^2(\bar\omega, \mathbb{R}^4) \text{ satisfying:} \\ \qquad \|\tilde u - u\|_1 \leq C\delta^{1/2}, \qquad \|\tilde u\|_2 \leq C\mu\delta^{1/2}\sigma^J, \\ \qquad \Big\|\mathcal{D}\Big(g - \dfrac{\delta}{\sigma^S} H_0, \tilde u\Big)\Big\|_0 \leq \dfrac{r_0}{5}\dfrac{\delta}{\sigma^S} + \dfrac{\|g\|_{r,\beta}}{\mu^{r+\beta}}, \\ \text{with constants } C \text{ depending only on } \omega, \underline{u}, g, S, J, p. \end{array}\right] \tag{1.12}$$

Then, for every $\epsilon > 0$ and every exponent α in the range:

$$0 < \alpha < \min\Big\{\frac{r+\beta}{2}, \frac{1}{1+2J/S}\Big\}, \tag{1.13}$$

there exists an immersion $\bar u \in \mathcal{C}^{1,\alpha}(\bar\omega, \mathbb{R}^4)$ such that:

$$\|\bar u - \underline{u}\|_0 \leq \epsilon \quad \text{and} \quad \mathcal{D}(g, \bar u) = 0 \quad \text{in } \bar\omega.$$

A proof of Theorem 1.3 will be given in section 8. It relies on iterating Theorem 1.2 with a particular progression of δ, μ and σ. For the base of the induction, one necessitates the initial Stage which decreases the large but positive definite initial defect $\mathcal{D}(g, \underline{u})$ to a much smaller defect that satisfies the first condition in $(1.10)_3$, tied with specific scaling laws on the initial increase of $\nabla^2 u$. A self-contained proof of that initial Stage is presented in Theorem 4.1, via a construction based on Nash's spirals Step in Lemma 2.4.

We now sketch the proof of Theorem 1.2.

1. The new field $\tilde u = u_K$ is constructed from u as the final, K-th immersion in the K-tuple $\{u_k\}_{k=1}^K$. The initial immersion u_0 is a mollified version of u at the given length scale of the order of $\mu_0 = \mu$, and we set $\delta_0 = \delta$. This immersion induces two unit normal vector fields $(E^1_{u_0}, E^2_{u_0})$, perpendicular to ∇u_0 and to each other. Only $(E^1_{u_0}, E^2_{u_0})$ are defined independently, while each consequtive normal frame $(E^1_{u_{k+1}}, E^2_{u_{k+1}})$ is constructed from the previous $(E^1_{u_k}, E^2_{u_k})$, see Section 3. The intermediate defect bounds and frequencies $\{\delta_k, \mu_k\}_{k=1}^K$ are set to satisfy:

$$\mu_1 = \mu_0\sigma^{N+2}, \quad \mu_{k+1} = \mu_k\sigma^{N/2+2}, \qquad \delta_{k+1} = \frac{\delta_k}{\sigma^N}, \tag{1.14}$$

(see Figure 1) and the following inductive estimates are proved to hold, for all $k = 0 \dots K-1$ and all m up to a sufficiently large (but finite) number:

$$\begin{aligned} &\|u_{k+1} - u_k\|_1 \leq C\delta_k^{1/2}, \\ &\|\nabla^{(m+1)}(u_{k+1} - u_k)\|_0 + \sum_{i=1}^2 \|\nabla^{(m)}(E^i_{u_{k+1}} - E^i_{u_k})\|_0 \leq C\delta_k^{1/2}\mu_{k+1}^m, \\ &\|\nabla^{(m)}\mathcal{D}(g_0 - \delta_{k+1}H_0, u_{k+1})\|_0 \leq \frac{\delta_k}{\sigma^N}\mu_{k+1}^m = \delta_{k+1}\mu_{k+1}^m. \end{aligned} \tag{1.15}$$

Hence, at the counter $k=K$, the recursion (1.15) yields:

$$\delta_K = \frac{\delta}{\sigma^{KN}} \quad \text{ and } \quad \delta_{K-1}^{1/2}\mu_K = \delta^{1/2}\mu\sigma^{2K+N}$$

implying the claimed bounds $(1.11)_1$, $(1.11)_2$ in view of (1.15). The additional term in the right hand side of $(1.11)_2$ is due to the fact that in (1.15) we calculate the defect with respect to the mollified version g_0 of g rather than g itself, while the prefactor $r_0/5$ is secured by obtaining a slightly larger decay $\delta_k/\sigma^{N+1/2}$ rather than δ_k/σ^N, and taking σ large. The necessity of this prefactor as well as the need for estimating the defect relative to a fixed matrix H_0, follow from the validity of a decomposition of a symmetric matrix $H\in\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ as a linear combination of three rank-one matrices with nonnegative and uniformly bounded coefficients $\{a_i\in(\frac{1}{2},\frac{3}{2})\}_{i=1}^3$:

$$H = \sum_{i=1}^3 a_i^2 \eta_i\otimes\eta_i, \tag{1.16}$$

not for all H, but in the vicinity of, say, $H_0=\sum_{i=1}^3\eta_i\otimes\eta_i$ (see Lemma 2.2).

2. The core of proof of (1.15) relies on obtaining specific bounds on the components of $\nabla^2 u_k$:

$$\begin{aligned} &\|\nabla^{(m)}\langle\partial_{ij}u_k, E^1_{u_k}\rangle\|_0 \leq C\frac{\delta_{k-1}^{1/2}}{\sigma^{N/2}}\mu_k^{m+1}, \qquad \|\nabla^{(m)}\langle\partial_{11}u_k, E^2_{u_k}\rangle\|_0 \leq C\frac{\delta_{k-1}^{1/2}}{\sigma^{N}}\mu_k^{m+1},\\ &\|\nabla^{(m)}\langle\partial_{12}u_k, E^2_{u_k}\rangle\|_0 \leq C\frac{\delta_{k-1}^{1/2}}{\sigma^{N/2}}\mu_k^{m+1}, \qquad \|\nabla^{(m)}\langle\partial_{22}u_k, E^2_{u_k}\rangle\|_0 \leq C\delta_{k-1}^{1/2}\mu_k^{m+1}. \end{aligned} \tag{1.17}$$

The above bounds can be anticipated from the next definition. Namely, u_{k+1} is constructed from u_k by passing through two extra intermediate immersion fields in:

$$\begin{aligned} U &= u_k + \frac{\Gamma(\lambda\langle x,\eta_1\rangle)}{\lambda}a_1E^1_{u_k} + T_{u_k}\Big(\frac{\bar\Gamma(\lambda\langle x,\eta_1\rangle)}{\lambda}a_1^2\eta_1 + W\Big),\\ \bar U &= U + \frac{\Gamma(\kappa\langle x,\eta_2\rangle)}{\kappa}a_2E^1_{U} + T_{U}\Big(\frac{\bar\Gamma(\kappa\langle x,\eta_2\rangle)}{\kappa}a_2^2\eta_2 + \bar W\Big),\\ u_{k+1} &= \bar U + \frac{\Gamma(\mu_{k+1}\langle x,\eta_3\rangle)}{\mu_{k+1}}bE^2_{\bar U} + T_{\bar U}\Big(\frac{\bar\Gamma(\mu_{k+1}\langle x,\eta_3\rangle)}{\mu_{k+1}}b^2\eta_3 + \bar{\bar W}\Big), \end{aligned} \tag{1.18}$$

where the frequencies (see Figure 2) are:

$$\lambda = \mu_k\sigma, \quad \kappa = \mu_k\sigma^2.$$

The coefficients $\{a_i\}_{i=1}^3$, b are obtained from decomposing the defect via (1.19), where b contains a_3 and other components, specified later. As a first attempt, think of applying (1.16) to:

$$H = \mathcal{D}(g_0 - \delta_{k+1}H_0, u_k).$$

Since that decomposition is linear, the induction assumption implies $a_i, b\sim\delta_k^{1/2}$. The definition (1.18) is consistent with the single Kuiper's corrugation construction (see Lemma 2.5), in which the first term is always aligned with the chosen normal direction (here, E^1 in U and $\bar U$, and E^2 in u_{k+1}) to the current immersion (here u_k, then U, then $\bar U$), while the second term carries a tangential component via the basis of the tangent vectors to u (the tangent frame) given in $T_u \doteq (\nabla u)((\nabla u)^T\nabla u)^{-1}$ and a chosen 2-dimensional perturbation field W.

To deduce (1.17), one uses (1.18) and the induction assumption. Neglecting the tangential components and gathering only the highest order powers in frequencies, we have:

$$\begin{aligned}\|\nabla^{(m)}\langle\partial_{ij}u_{k+1},E^1_{u_{k+1}}\rangle\|_0 &\sim \|\nabla^{(m)}\langle\partial_{ij}u_k,E^1_{u_k}\rangle\|_0+\|a_1\|_0\lambda^{m+1}+\|a_2\|_0\kappa^{m+1}\\ &\leq C\frac{\delta_{k-1}^{1/2}}{\sigma^{N/2}}\mu_k^{m+1}+C\delta_k^{1/2}\frac{\mu_{k+1}^{m+1}}{\sigma^{N/2}}\mu_{k+1}^{m+1}=C\frac{\delta_k^{1/2}}{\sigma^{N/2}}\mu_{k+1}^{m+1},\\ \|\nabla^{(m)}\langle\partial_{22}u_{k+1},E^2_{u_{k+1}}\rangle\|_0 &\sim \|\nabla^{(m)}\langle\partial_{22}u_k,E^2_{u_k}\rangle\|_0+\|b\|_0\mu_{k+1}^{m+1}\\ &\leq C\delta_{k-1}^{1/2}\mu_k^{m+1}+C\delta_k^{1/2}\mu_{k+1}^{m+1}\leq C\delta_k^{1/2}\mu_{k+1}^{m+1},\end{aligned}$$

in view of the assumed increase rates: from δ_k to δ_{k+1}, and from μ_k to λ to κ to μ_{k+1}. For the remaining components we use that $\eta_2=e_2$, so that the bounds (1.17) are closed in:

$$\begin{aligned}\|\nabla^{(m)}\langle\partial_{11}u_{k+1},E^2_{u_{k+1}}\rangle\|_0 &\sim \|\nabla^{(m)}\langle\partial_{11}u_k,E^2_{u_k}\rangle\|_0+\|b\|_0\mu_{k+1}^{m-1}\\ &\leq C\frac{\delta_{k-1}^{1/2}}{\sigma^{N}}\mu_k^{m+1}+C\delta_k^{1/2}\mu_{k+1}^{m-1}\sim C\frac{\delta_k^{1/2}}{\sigma^{N}}\mu_{k+1}^{m+1},\\ \|\nabla^{(m)}\langle\partial_{12}u_{k+1},E^2_{u_{k+1}}\rangle\|_0 &\sim \|\nabla^{(m)}\langle\partial_{12}u_k,E^2_{u_k}\rangle\|_0+\|b\|_0\mu_{k+1}^{m}\leq C\frac{\delta_k^{1/2}}{\sigma^{N/2}}\mu_{k+1}^{m+1}.\end{aligned}$$

3. We now trace the changes of the defect: from that corresponding to u_k, to U then to $\bar{U}$. The departing point in our construction is the decomposition (1.16), which needs to be administered under the Källen iteration technique (see Theorem 6.1). This technique allows to decompose the defect together with a specific portion of the future error, namely the non-oscillatory portion which cannot be handled otherwise. A new error $\mathcal{F}$ is hence introduced, however it can be made arbitrarily small. This is the key outcome of Källen's iteration:

$$\begin{aligned}&\mathcal{D}(g_0-\delta_{k+1}H_0,u_k)-\frac{1}{\lambda^2}\nabla a_1\otimes\nabla a_1-\frac{1}{\kappa^2}\nabla a_2\otimes\nabla a_2=\sum_{i=1}^3 a_i^2\eta_i\otimes\eta_i+\mathcal{F},\\ &\text{where}\quad \|\nabla^m\mathcal{F}\|_0\leq C\frac{\delta_k}{\sigma^N}\mu^m.\end{aligned}\tag{1.19}$$

Now, Kuiper's corrugation in U in (1.18), has the purpose of cancelling both $a_1^2\eta_1\otimes\eta_1$ and $\frac{1}{\lambda^2}\nabla a_1\otimes\nabla a_1$ from $\mathcal{D}(g_0-\delta_{k+1}H_0,u_k)$ above, and it does so, because (see Lemma 2.5):

$$\begin{aligned}&(\nabla U)^T\nabla U-(\nabla u_k)^T\nabla u_k=a_1^2\eta_1\otimes\eta_1+\frac{1}{\lambda^2}\nabla a_1\otimes\nabla a_1\\ &+\frac{\Gamma(\lambda\langle x,\eta_1\rangle)^2-1}{\lambda^2}S_1+\frac{\Gamma\Gamma'(\lambda\langle x,\eta_1\rangle)}{\lambda}S_2+\frac{\Gamma(\lambda\langle x,\eta_1\rangle)}{\lambda}S_3+\frac{\bar{\Gamma}(\lambda\langle x,\eta_1\rangle)}{\lambda}S_4\\ &+\mathrm{sym}\nabla W+\mathcal{R},\end{aligned}\tag{1.20}$$

where $\mathcal{R}$ is a residual error term, arbitrarily small in the sense given below, while the quantities S_1,S_2,S_3,S_4 define the leading order errors. The field W is such that it cancels all entries of these principal four errors apart from their $e_2\otimes e_2$ components, at the expense of a much smaller new error $\mathcal{G}$. This can be done precisely because the oscillation profiles Γ^2-1, $\Gamma\Gamma'$, Γ and $\bar{\Gamma}$ have mean zero on the period (see Lemma 5.1). The main idea is based on the decomposition:

$$\begin{aligned}\Gamma(\lambda x_1)H&=\Gamma(\lambda x_1)\mathrm{sym}((H_{11},2H_{12})\otimes e_1)+\Gamma(\lambda x_1)H_{22}e_2\otimes e_2\\ &=\mathrm{sym}\nabla\Big(\frac{\Gamma_1(\lambda x_1)}{\lambda}(H_{11},2H_{12})\Big)-\frac{\Gamma_1(\lambda x_1)}{\lambda}\mathrm{sym}\nabla(H_{11},2H_{12})+\Gamma(\lambda x_1)H_{22}e_2\otimes e_2,\end{aligned}\tag{1.21}$$

where $\Gamma_1'=\Gamma$,

that is iterated N times, decreasing the second term in its right hand side to the order $1/\lambda^N$. Of course, the primitives Γ_N given iteratively by the formula $(\frac{d}{dt})^{(N)}\Gamma_N = \Gamma$, remain bounded provided that their initial oscillatory profile Γ has mean zero on its period. In conclusion, our first intermediate defect is composed of:

$$\begin{aligned} &\mathcal{D}(g-\delta_{k+1}H_0, U) = \sum_{i=2}^3 a_i^2\eta_i\otimes\eta_i + \frac{1}{\kappa^2}\nabla a_2\otimes\nabla a_2 + \mathcal{F}+\mathcal{G}+\mathcal{R} - Ge_2\otimes e_2,\\ &\text{where:}\quad \|\nabla^{(m)}\mathcal{F}\|_0 \sim C\frac{\delta_k}{\sigma^N}\lambda^m \quad \text{from Källen's iteration}\\ &\qquad\qquad \|\nabla^{(m)}\mathcal{G}\|_0 \sim C\frac{\delta_k}{(\lambda/\mu_k)^N}\lambda^m \leq C\frac{\delta_k}{\sigma^N}\lambda^m \quad \text{from oscillatory decomposition}\\ &\qquad\qquad \|\nabla^{(m)}\mathcal{R}\|_0 \leq C\delta_k^{3/2}\sigma^{N/2}\lambda^m \leq C\frac{\delta_k}{\sigma^N}\lambda^m \quad \text{from the last assumption in } (1.10)_1. \end{aligned} \tag{1.22}$$

The bound on the derivatives of $\mathcal{G}$ is due to the fact that S_1, S_2, S_3, S_4 oscillate with frequency μ_k and $\lambda/\mu_k = \sigma$. A similar reasoning applies to the second intermediate immersion $\bar{U}$ in (1.18), where the second Kuiper corrugation is used to cancel $a_2^2\eta_2\otimes\eta_2$ and $\frac{1}{\kappa^2}\nabla a_2\otimes\nabla a_2$ from $\mathcal{D}(g_0-\delta_{k+1}H_0,U)$, while $\bar{W}$ is defined to cancel (see Lemma 5.3) all but the $e_2\otimes e_2$ entries of:

$$\frac{\Gamma(\kappa\langle x,\eta_2\rangle)^2-1}{\kappa^2}\bar{S}_1 + \frac{\Gamma\Gamma'(\kappa\langle x,\eta_2\rangle)}{\kappa}\bar{S}_2 + \frac{\Gamma(\kappa\langle x,\eta_2\rangle)}{\kappa}\bar{S}_3 + \frac{\bar{\Gamma}(\kappa\langle x,\eta_2\rangle)}{\kappa}\bar{S}_4,$$

derived as the four principal errors terms (see Lemma 2.5) in $(\nabla\bar{U})^T\nabla\bar{U} - (\nabla U)^T\nabla U$, similarly as in (1.20). Hence, the second intermediate defect is composed of:

$$\begin{aligned} &\mathcal{D}(g-\delta_{k+1}H_0, \bar{U}) = \big(a_3^2 - G - \bar{G}\big)\eta_3\otimes\eta_3 + \mathcal{E} + \bar{\mathcal{G}} + \bar{\mathcal{R}}\\ &\text{where:}\quad \|\nabla^{(m)}\mathcal{E}\|_0 \sim C\frac{\delta_k}{\sigma^N}\lambda^m \quad \text{from } (1.22)\\ &\qquad\qquad \|\nabla^{(m)}\bar{\mathcal{G}}\|_0 \sim C\frac{\delta_k}{(\kappa/\lambda)^N}\kappa^m \leq C\frac{\delta_k}{\sigma^N}\kappa^m \quad \text{from oscillatory decomposition}\\ &\qquad\qquad \|\nabla^{(m)}\bar{\mathcal{R}}\|_0 \leq C\delta_k^{3/2}\sigma^{N/2}\kappa^m \leq C\frac{\delta_k}{\sigma^N}\kappa^m \quad \text{from the last assumption in } (1.10)_1. \end{aligned} \tag{1.23}$$

The bound on the derivatives of $\bar{\mathcal{G}}$ is due to the fact that $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$, $\bar{S}_4$ oscillate with frequency λ and $\kappa/\lambda = \sigma$. From the bounds on $G, \bar{G}$ one can correctly define the augmented amplitude:

$$b^2 = a_3^2 - G - \bar{G}$$

that obeys: $b^2 \sim \delta_k$ and $\|\nabla^{(m)}b^2\|_0 \leq C\frac{\delta_k}{\sigma}\kappa^m$.

4. We now analyze the change of the defect from that corresponding to the immersion $\bar{U}$ to the final u_{k+1}. The third Kuiper's corrugation in (1.18) is used to cancel the term $b^2e_2\otimes e_2$ in $\mathcal{D}(g-\delta_{k+1}H_0,\bar{U})$ and note that it is the first time that we use the second codimension normal field E^2. The field $\bar{\bar{W}}$ is defined by applying the oscillatory decomposition in Lemma 5.3, to cancel all but the $e_1\otimes e_1$ entries of the following principal errors in Lemma 2.5:

$$\frac{\Gamma\Gamma'(\mu_{k+1}\langle x,\eta_3\rangle)}{\mu_{k+1}}\bar{\bar{S}}_2 + \frac{\Gamma(\mu_{k+1}\langle x,\eta_3\rangle)}{\mu_{k+1}}\bar{\bar{S}}_3 + \frac{\bar{\Gamma}(\mu_{k+1}\langle x,\eta_3\rangle)}{\mu_{k+1}}\bar{\bar{S}}_4,$$

with only two iterations of (1.21), rather than N as before. This is precisely why we necessitate $\mu_{k+1}/\kappa \sim \sigma^{N/2}$ large, whereas $\lambda/\mu_k = \sigma$ and $\kappa/\lambda = \sigma$ has sufficed. The contribution corresponding to $\bar{\bar{S}}_1 = \nabla b \otimes \nabla b$ is estimated directly in the second bound below, due to b and hence $\bar{\bar{S}}_1$ oscillating with frequency κ. It follows that:

$$\mathcal{D}(g - \delta_{k+1}H_0, u_{k+1}) = \bar{\bar{G}}e_1 \otimes e_1 + \mathcal{E} + \bar{\mathcal{E}} + \frac{\Gamma(\mu_{k+1}\langle x, \eta_3\rangle)^2}{\mu_{k+1}^2}\nabla b \otimes \nabla b + \bar{\bar{\mathcal{R}}} + \bar{\bar{\mathcal{G}}},$$

$$\begin{aligned} \text{where:}\quad & \|\nabla^{(m)}(\mathcal{E} + \bar{\mathcal{E}})\|_0 \sim C\frac{\delta_k}{\sigma^N}\kappa^m \qquad \text{from (1.22), (1.23)}, \\ & \Big\|\nabla^{(m)}\Big(\frac{\Gamma(\mu_{k+1}\langle x, \eta_3\rangle)^2}{\mu_{k+1}^2}\nabla b \otimes \nabla b\Big)\Big\|_0 \leq C\frac{\delta_k}{(\mu_{k+1}/\kappa)^2}\mu_{k+1}^m \leq C\frac{\delta_k}{\sigma^N}\mu_{k+1}^m, \\ & \|\nabla^{(m)}\bar{\bar{\mathcal{G}}}\|_0 \sim \frac{\delta_{k-1}}{(\mu_{k+1}/\kappa)^3}\mu_{k+1}^m = \frac{\delta_k\sigma^{N/2}}{(\mu_{k+1}/\kappa)^3}\mu_{k+1}^m \leq C\frac{\delta_k}{\sigma^N}\mu_{k+1}^m \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{from oscillatory decomposition}, \\ & \|\nabla^{(m)}\bar{\bar{\mathcal{R}}}\|_0 \leq C\delta_k^{3/2}\sigma^{N/2}\mu_{k+1}^m \leq C\frac{\delta_k}{\sigma^N}\mu_{k+1}^m \qquad \text{by last assumption in } (1.10)_1. \end{aligned} \tag{1.24}$$

The third bound above follows in virtue of (1.17), since the worst term in $\bar{\bar{\mathcal{G}}}$ is:

$$\bar{\bar{\mathcal{G}}} \sim \frac{\Gamma(\mu_{k+1}\langle x, \eta_3\rangle)}{\mu_{k+1}^3}\nabla^2\bar{\bar{S}} \qquad \text{where} \qquad \bar{\bar{S}} \sim b(\nabla\bar{U})^T\nabla E_{\bar{U}}^2.$$

In conclusion, the final assertion of the inductive bound (1.15), namely:

$$\|\nabla^{(m)}\mathcal{D}(g_0 - \delta_{k+1}H_0, u_{k+1})\|_0 \sim \frac{\delta_k}{\sigma^N}\mu_{k+1}^m,$$

follows in view of (1.24), if we are able to check that:

$$\|\nabla^{(m)}\bar{\bar{G}}\|_0 \sim \frac{\delta_k}{\sigma^N}\mu_{k+1}^m. \tag{1.25}$$

This indeed holds, and can be seen from the exact formulas on the $e_1 \otimes e_1$ residual components in the decomposition parallel to that in (1.21), given in Lemma 5.2:

$$\begin{aligned} & \bar{\bar{G}} \sim \sum_{i=0}^{1}\frac{\Gamma_i(\mu_{k+1}\langle x, \eta_3\rangle)}{\mu_{k+1}^{i+1}}P_i(\bar{\bar{S}}_2 + \bar{\bar{S}}_3 + \bar{\bar{S}}_4) \qquad \text{where} \quad \bar{\bar{S}}_2 \sim b\,\mathrm{sym}(\nabla b \otimes \eta_3), \\ & \text{and} \quad \bar{\bar{S}}_3 \sim b\,\mathrm{sym}\big((\nabla\bar{U})^T\nabla E_{\bar{U}}^2)\big), \quad \bar{\bar{S}}_4 \sim \mathrm{sym}\big((\nabla\bar{U})^T\nabla(b^2T_{\bar{U}}\eta_3)\big), \\ & \text{with} \quad P_0(H) = H_{11}, \quad P_1(H) = \partial_1 H_{12}. \end{aligned}$$

In particular, only components $(\bar{\bar{S}}_i)_{11}$ and $\partial_1(\bar{\bar{S}}_i)_{12}$ enter in the formula on $\bar{\bar{G}}$. We prove (1.25) using all the information in (1.17) as well as the relation between κ and μ_{k+1}.

This ends the sketch of proof of Theorem 1.2. We close by a few extra points:

- The distinction between definitions of μ_1 and μ_k at $k > 1$ in (1.14) is needed for the proof of the inductive bounds on $\langle\partial_{11}u_k, E_{u_k}^2\rangle$ in (1.17), separately at the induction base $k = 1$ and then at the induction step.
- Along the proof, we need to keep track of the evolution of the orthonormal frame (E^1, E^2) and the tangent frame T to the surfaces given by immersions $u_k, U, \bar{U}$, and also to have all these immersions satisfy the uniform condition $(1.10)_2$.
- We provide bounds on any order of derivatives $\nabla^{(m)}$ of various fields, however only a finite number of them is relevant (in the proofs we specify how many, for completeness). Hence all constants C in our estimates are uniform, though depending on N (which

refers to the decay rate of the defect passing from u_k to u_{k+1}), and K (which stands for the number of triple Steps in a Stage).

- Several error quantities are estimated by means of higher powers of δ (like $\delta^{3/2}$ or δ^2). All such quantities are bounded by a quotient of δ and the relative frequency σ raised to a convenient power at hand, because of the last assumption in $(1.10)_1$.

Remark 1.4. The same result as in Theorem 1.1, remains valid in any target space $\mathbb{R}^n$ replacing $\mathbb{R}^4$, for $n \geq 4$. Our proofs only require existence of and the propagation bounds on two normal vector fields, which are guaranteed by having $n-2 \geq 2$. The same statement as in Theorem 1.2 also holds and one concludes the final result by a version of Theorem 1.3.

1.7. **Notation.** By $\mathbb{R}^{k\times k}_{\mathrm{sym}}$ and we denote the space of symmetric $k \times k$ matrices, and $\mathbb{R}^{k\times k}_{\mathrm{sym},>}$ is the cone of such matrices that are additionally positive definite. The space of Hölder continuous vector fields $\mathcal{C}^{m,\beta}(\bar\omega, \mathbb{R}^k)$ where $m \geq 0$, $\beta \in [0,1]$ consists of restrictions of all $v \in \mathcal{C}^{m,\beta}(\mathbb{R}^2, \mathbb{R}^k)$ to the closure $\bar\omega$ of an open, bounded set $\omega \subset \mathbb{R}^2$. The $\mathcal{C}^m(\bar\omega, \mathbb{R}^k)$ norm of such restriction is denoted by $\|v\|_m$, while its Hölder norm in $\mathcal{C}^{m,\beta}(\bar\omega, \mathbb{R}^k)$ is $\|v\|_{m,\beta}$. Below we gather other notation that is recurring in our paper, specifying the first time when it appears.

$\mathcal{D}(g,u) = g - (\nabla u)^T\nabla u$**:** the defect, i.e. the difference between the given metric and the immersion metric, see (1.3),

r_0**:** the radius assuring positivity of coefficients in the metric decomposition, see Lemma 2.2.

$\eta_1 = e_1, \eta_2 = \frac{e_1+e_2}{\sqrt{2}}, \eta_3 = e_2$**:** the unit vectors yielding the primitive defect decomposition, see Lemma 2.2.

$H_0 = \sum_{i=1}^3 \eta_i \otimes \eta_i$**:** the referential (constant) metric whose all decomposition coefficients equal 1, see Lemma 2.2.

$\{\bar a_i\}_{i=1}^3$**:** the linear projections in the decomposition of the defect, see Lemma 2.2.

$\Gamma, \bar\Gamma$**:** the leading oscillatory profiles in the given convex integration Step constructions, see Lemmas 2.4 and 2.5.

E_u^1, E_u^2**:** the orthonormal pair of normal vectors fields (the normal frame) to the surface given by immersion u, see Lemma 2.4.

$T_u = (\nabla u)((\nabla u)^T\nabla u)^{-1}$**:** the tangent frame to the surface $u(\bar\omega)$ given by the immersion u, see Lemma 3.2.

$\mathcal{R}, \mathcal{R}_1, \mathcal{R}_2$**:** the secondary error terms in the single Step constructions, see Lemmas 2.4, 2.5.

$\underline{u}$**:** the referential smooth immersion with positive definite defect, see Theorem 1.3.

$\underline{\gamma}$**:** the referential immersion constant after an application of the initial Stage; all immersions in the inductive stages obey $1/(2\underline{\gamma})\mathrm{Id}_2 \leq (\nabla u)^T\nabla u \leq 2\underline{\gamma}\mathrm{Id}_2$, see Lemma 4.1 and Theorem 1.2.

By C we denote a universal constant that may change from line to line of the proof, where it depends on the specified parameters while being independent from all the induction counters.

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2. The defect decomposition lemmas and two Step constructions

This section contains some preliminary technical lemmas that serve as the building blocks of the following proofs. We first gather the convolution and commutator estimates from [16]:

Lemma 2.1. *Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{R})$ be a standard mollifier that is nonnegative, radially symmetric, supported on the unit ball $B(0,1) \subset \mathbb{R}^2$ and such that $\int_{\mathbb{R}^2} \phi \,\mathrm{d}x = 1$. Denote:*

$$\phi_l(x) = \frac{1}{l^2}\phi(\frac{x}{l}) \quad \text{for all } \ l \in (0,1], \ x \in \mathbb{R}^2.$$

Then, for every $f, g \in \mathcal{C}^0(\mathbb{R}^2, \mathbb{R})$, every $m \geq 0$ and $\beta \in (0,1)$, there holds:

$$\|\nabla^{(m)}(f * \phi_l)\|_0 \leq \frac{C}{l^m}\|f\|_0, \tag{2.1$_1$}$$

$$\|f - f * \phi_l\|_0 \leq \min\big\{l^2\|\nabla^2 f\|_0, l^{1+\beta}\|\nabla f\|_{0,\beta}, l\|\nabla f\|_0, l^\beta\|f\|_{0,\beta}\big\}, \tag{2.1$_2$}$$

$$\|\nabla^{(m)}\big((fg) * \phi_l - (f * \phi_l)(g * \phi_l)\big)\|_0 \leq Cl^{2-m}\|\nabla f\|_0\|\nabla g\|_0, \tag{2.1$_3$}$$

with constants $C > 0$ depending only on m.

The next two lemmas provide the decomposition of symmetric matrices into linear combinations of rank-one "primitive matrices". The first result is self-evident, proved for the general dimensionality $d \geq 2$ in [16, Lemma 5.2], while the second one is a combination of the local decomposition with a partition of unity - type statement from [43, Lemma 3.3]:

Lemma 2.2. *There exist $r_0 \in (0,1)$ and linear maps $\{\bar{a}_i : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}\}_{i=1}^3$ such that, denoting:*

$$\eta_1 = e_1, \quad \eta_2 = \frac{e_1 + e_2}{\sqrt{2}}, \quad \eta_3 = e_2 \quad \text{and} \quad H_0 = \sum_{i=1}^3 \eta_i \otimes \eta_i,$$

for all $H \in B(H_0, r_0) \subset \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ there holds:

$$H = \sum_{i=1}^3 \bar{a}_i(H)\eta_i \otimes \eta_i \quad \text{and} \quad |\bar{a}_i(H) - 1| \leq \frac{1}{2} \ \text{ for all } \ i = 1 \dots 3.$$

Lemma 2.3. *There exists an integer N_0 and a sequence of unit vectors $\{\eta_i \in \mathbb{R}^2\}_{i=1}^\infty$ together with a sequence of nonnegative functions $\{\varphi_i \in \mathcal{C}_c^\infty(\mathbb{R}^{2\times 2}_{\mathrm{sym},>}, \mathbb{R})\}_{i=1}^\infty$, such that:*

$$H = \sum_{i=1}^\infty \varphi_i(H)^2 \eta_i \otimes \eta_i \quad \text{for all } \ H \in \mathbb{R}^{2\times 2}_{\mathrm{sym},>},$$

and that:

(i) at most N_0 terms in the above sum are nonzero, for each H,
(ii) every compact set $\mathcal{K} \subset \mathbb{R}^{2\times 2}_{\mathrm{sym},>}$ induces a finite set of indices $J(\mathcal{K}) \subset \mathbb{N}$, such that $\varphi_i(H) = 0$ for all $H \in \mathcal{K}$ and all $i \notin J(\mathcal{K})$.

The remaining two lemmas in this section provide two types of Step construction in the convex integration algorithm. The given immersion is modified by adding oscillatory perturbations, with the scope of cancelling a single term in the pull back of the Euclidean metric via the immersion, of the rank-one type featured in the decomposition Lemma 2.2. The first construction, referred to as Nash's spiral, necessitates two normal vector fields:

Lemma 2.4. [STEP: NASH'S SPIRAL] *Let $u \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^4)$ be an immersion and let $E_u^1, E_u^2 \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^4)$ be a given orthonormal pair of its normal vector fields, so that:*

$$(\nabla u)^T E_u^1 = (\nabla u)^T E_u^2 = 0, \quad |E_u^1| = |E_u^2| = 1, \quad \langle E_u^1, E_u^2 \rangle = 0 \quad \text{in } \mathbb{R}^2.$$

For a given unit vector $\eta \in \mathbb{R}^2$ we set $t = \langle x, \eta \rangle$ and denote:

$$\Gamma(t) = \sin t, \quad \bar{\Gamma}(t) = \cos t.$$

Then, for every $\lambda > 0$ and $a \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$, the vector field $\tilde{u} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^4)$ given by the formula:

$$\tilde{u}(x) = u(x) + \frac{\Gamma(\lambda t)}{\lambda} a(x) E_u^1(x) + \frac{\bar{\Gamma}(\lambda t)}{\lambda} a(x) E_u^2(x),$$

satisfies the following identity:

$$(\nabla \tilde{u})^T \nabla \tilde{u} - (\nabla u)^T \nabla u = a^2 \eta \otimes \eta + \mathcal{R},$$

with the error term $\mathcal{R}$ in:

$$\begin{aligned} \mathcal{R} &= \mathcal{R}(\lambda, a, \eta, \nabla u, E_u^1, E_u^2) \\ &= \Big\{ 2\frac{\Gamma(\lambda t)}{\lambda} a \,\mathrm{sym}((\nabla u)^T \nabla E_u^1) + 2\frac{\bar{\Gamma}(\lambda t)}{\lambda} a \,\mathrm{sym}((\nabla u)^T \nabla E_u^2) + \frac{2}{\lambda} a^2 \mathrm{sym}\big((\nabla E_u^2)^T E_u^1 \otimes \eta\big) \Big\} \\ &\quad + \Big\{ \frac{\Gamma(\lambda t)^2}{\lambda^2} a^2 (\nabla E_u^1)^T \nabla E_u^1 + \frac{\bar{\Gamma}(\lambda t)^2}{\lambda^2} a^2 (\nabla E_u^2)^T \nabla E_u^2 \\ &\qquad + 2\frac{\Gamma(\lambda t)\bar{\Gamma}(\lambda t)}{\lambda^2} a^2 \,\mathrm{sym}((\nabla E_u^1)^T \nabla E_u^2) + \frac{1}{\lambda^2} \nabla a \otimes \nabla a \Big\}. \end{aligned}$$

Proof. We first calculate the gradient of the modified immersion:

$$\begin{aligned} \nabla \tilde{u} = \nabla u &+ \Gamma'(\lambda t) a E_u^1 \otimes \eta + \frac{\Gamma(\lambda t)}{\lambda} \big(a \nabla E_u^1 + E_u^1 \otimes \nabla a \big) \\ &+ \bar{\Gamma}'(\lambda t) a E_u^2 \otimes \eta + \frac{\bar{\Gamma}(\lambda t)}{\lambda} \big(a \nabla E_u^2 + E_u^2 \otimes \nabla a \big). \end{aligned}$$

This leads to the following formula, where we suppress the argument λt in Γ and $\bar{\Gamma}$ and order the terms according to powers of λ:

$$\begin{aligned} &(\nabla \tilde{u})^T \nabla \tilde{u} - (\nabla u)^T \nabla u \\ &= \Big\{ (\Gamma')^2 a^2 \eta \otimes \eta + (\bar{\Gamma}')^2 a^2 \eta \otimes \eta \Big\} \\ &\quad + \Big\{ 2\frac{\Gamma}{\lambda} a \,\mathrm{sym}\big((\nabla u)^T \nabla E_u^1\big) + 2\frac{\bar{\Gamma}}{\lambda} a \,\mathrm{sym}\big((\nabla u)^T \nabla E_u^2\big) + 2\frac{\Gamma\Gamma'}{\lambda} a \,\mathrm{sym}(\nabla a \otimes \eta) \\ &\qquad + 2\frac{\Gamma'\bar{\Gamma}}{\lambda} a^2 \mathrm{sym}\big((\nabla E_u^2)^T E_u^1 \otimes \eta\big) + 2\frac{\Gamma\bar{\Gamma}'}{\lambda} a^2 \mathrm{sym}\big((\nabla E_u^1)^T E_u^2 \otimes \eta\big) \\ &\qquad + 2\frac{\bar{\Gamma}\bar{\Gamma}'}{\lambda} a \,\mathrm{sym}(\nabla a \otimes \eta) \Big\} \\ &\quad + \Big\{ \frac{\Gamma^2}{\lambda^2} a^2 (\nabla E_u^1)^T \nabla E_u^1 + \frac{\Gamma^2}{\lambda^2} \nabla a \otimes \nabla a + 2\frac{\Gamma\bar{\Gamma}}{\lambda^2} a^2 \mathrm{sym}\big((\nabla E_u^1)^T \nabla E_u^2\big) \\ &\qquad + \frac{\bar{\Gamma}^2}{\lambda^2} a^2 (\nabla E_u^2)^T \nabla E_u^2 + \frac{\bar{\Gamma}^2}{\lambda^2} \nabla a \otimes \nabla a \Big\}. \end{aligned}$$

We now observe that the sum of the two terms in the first parentheses equals $a^2 \eta \otimes \eta$ because:

$$(\Gamma')^2 + (\bar{\Gamma}')^2 = 1,$$

whereas the remaining terms are exactly $\mathcal{R}$. This is because the third and the sixth term in the second parentheses cancel out due to $\Gamma\Gamma' + \bar\Gamma\bar\Gamma' = 0$, and the fourth and the fifth term in there combine due to $\Gamma'\bar\Gamma - \Gamma\bar\Gamma' = 1$. Also, the second and the fifth term in the third parentheses combine, due to $\Gamma^2 + \bar\Gamma^2 = 1$. The proof is done. ■

In the second Step construction, referred to as Kuiper's corrugation, the oscillatory modification is added only in one normal direction and it is augmented by a matching tangential perturbation, a construction inspired by [31]. A similar formula appeared in [10, Lemma 3.1] and for the Monge-Ampère system in [32, 36]. Note the additional tangential perturbation component w, that is crucial for our proofs and that will be chosen based on the oscillatory defect decomposition in section 5, as was done in [10, 27]. Of course, the error terms in (2.2) are now more complicated, and we split them into four groups: the non-oscillatory term $\frac{1}{\lambda^2}\nabla a\otimes\nabla a$, then the four principal oscillatory terms, the term $\mathrm{sym}\nabla w$, and the residual terms $\mathcal{R}_1, \mathcal{R}_2$:

Lemma 2.5. [STEP: KUIPER'S CORRUGATION] *Let $u \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^4)$ be an immersion and let $E_u \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^4)$ be a given unit normal vector field to u, so that:*

$$(\nabla u)^T E_u = 0, \quad |E_u| = 1 \quad \textit{in } \mathbb{R}^2.$$

For a given unit vector $\eta \in \mathbb{R}^2$ we set $t = \langle x, \eta\rangle$ and denote:

$$\Gamma(t) = \sqrt{2}\sin t, \quad \bar\Gamma(t) = -\frac{1}{4}\sin(2t).$$

Then, for every $\lambda > 0$, $a \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ and $w \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$, the vector field $\tilde u \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^4)$ in:

$$\begin{aligned}&\tilde u(x) = u(x) + \frac{\Gamma(\lambda t)}{\lambda} a(x) E_u(x) + T_u(x)\Big(\frac{\bar\Gamma(\lambda t)}{\lambda} a(x)^2\eta + w(x)\Big),\\ &\textit{where: } T_u = (\nabla u)\big((\nabla u)^T\nabla u\big)^{-1} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^{4\times 2}),\end{aligned}$$

satisfies the following identity:

$$\begin{aligned}(\nabla\tilde u)^T\nabla\tilde u - (\nabla u)^T\nabla u = {} & a^2\eta\otimes\eta + \frac{1}{\lambda^2}\nabla a\otimes\nabla a\\ & + \frac{\Gamma(\lambda t)^2 - 1}{\lambda^2} S_1 + \frac{\Gamma(\lambda t)\Gamma'(\lambda t)}{\lambda} S_2 + \frac{\Gamma(\lambda t)}{\lambda} S_3 + \frac{\bar\Gamma(\lambda t)}{\lambda} S_4\\ & + 2\mathrm{sym}\nabla w + \mathcal{R}_1 + \mathcal{R}_2,\end{aligned} \tag{2.2}$$

with the leading order error terms given via:

$$\begin{aligned}&S_1 = \nabla a\otimes\nabla a, && S_2 = 2a\,\mathrm{sym}(\nabla a\otimes\eta),\\ &S_3 = 2a\,\mathrm{sym}\big((\nabla u)^T\nabla E_u\big), && S_4 = 2\,\mathrm{sym}\big((\nabla u)^T\nabla(a^2T_u\eta)\big),\end{aligned}$$

the first residual error term given in:

$$\begin{aligned}\mathcal{R}_1 &= \mathcal{R}_1(\lambda, a, \eta, \nabla u, E_u) = (\bar\Gamma')^2 a^4|T_u\eta|^2\eta\otimes\eta\\ &+ \Big\{2\frac{\Gamma'(\lambda t)\bar\Gamma(\lambda t)}{\lambda} a\,\mathrm{sym}\big((\eta\otimes E_u)\nabla(a^2T_u\eta)\big) + 2\frac{\Gamma(\lambda t)\bar\Gamma'(\lambda t)}{\lambda} a^3\,\mathrm{sym}\big((\eta\otimes\eta)(T_u)^T\nabla E_u\big)\\ &\qquad + 2\frac{\bar\Gamma(\lambda t)\bar\Gamma'(\lambda t)}{\lambda} a^2\mathrm{sym}\big((\eta\otimes\eta)(T_u)^T\nabla(a^2T_u\eta)\big)\Big\}\\ &+ \Big\{\frac{\Gamma(\lambda t)^2}{\lambda^2} a^2(\nabla E_u)^T\nabla E_u + 2\frac{\Gamma(\lambda t)\bar\Gamma(\lambda t)}{\lambda^2}\mathrm{sym}\big(\nabla(aE_u)^T\nabla(a^2T_u\eta)\big)\\ &\qquad + \frac{\bar\Gamma(\lambda t)^2}{\lambda^2}\nabla(a^2T_u\eta)^T\nabla(a^2T_u\eta)\Big\},\end{aligned}$$

and the second residual error term involving w:

$$\begin{aligned}\mathcal{R}_2 &= \mathcal{R}_2(\lambda, a, \eta, \nabla u, E_u, w)\\ &= \Big\{2\,\mathrm{sym}\big((\nabla u)^T\big[(\partial_1 T_u)w, (\partial_2 T_u)w\big]\big) + 2\Gamma'(\lambda t)a\,\mathrm{sym}\big((\eta\otimes E_u)\nabla(T_u w)\big)\\ &\qquad + 2\bar\Gamma'(\lambda t)\,a^2\mathrm{sym}\big((\eta\otimes\eta)(T_u)^T\nabla(T_u w)\big) + \nabla(T_u w)^T\nabla(T_u w)\Big\}\\ &\quad + \Big\{2\frac{\Gamma(\lambda t)}{\lambda}\mathrm{sym}\big(\nabla(aE_u)^T\nabla(T_u w)\big) + 2\frac{\bar\Gamma(\lambda t)}{\lambda}\mathrm{sym}\big(\nabla(a^2T_u\eta)^T\nabla(T_u w)\big)\Big\}.\end{aligned}$$

Proof. We start by calculating the gradient of the modified immersion:

$$\begin{aligned}\nabla\tilde u &= \nabla u + a\Gamma'(\lambda t)E_u\otimes\eta + a^2\bar\Gamma'(\lambda t)T_u\eta\otimes\eta + \nabla(T_u w)\\ &\quad + \frac{\Gamma(\lambda t)}{\lambda}\big(a\nabla E_u + E_u\otimes\nabla a\big) + \frac{\bar\Gamma(\lambda t)}{\lambda}\nabla(a^2T_u\eta).\end{aligned}$$

We obtain the following formula, where we suppress the argument λt in Γ and $\bar\Gamma$, and order the terms according to powers of λ and their independence / dependence on w:

$$\begin{aligned}&(\nabla\tilde u)^T\nabla\tilde u - (\nabla u)^T\nabla u\\ &= \Big\{a^2(\Gamma')^2\eta\otimes\eta + 2a^2\bar\Gamma'(\nabla u)^T T_u\eta\otimes\eta + (\bar\Gamma')^2a^4(\eta\otimes\eta)(T_u)^T T_u(\eta\otimes\eta)\Big\}\\ &\quad + \Big\{2\frac{\Gamma}{\lambda}a\,\mathrm{sym}\big((\nabla u)^T\nabla E_u\big) + 2\frac{\bar\Gamma}{\lambda}\mathrm{sym}\big((\nabla u)^T\nabla(a^2T_u\eta)\big) + 2\frac{\Gamma\Gamma'}{\lambda}a\,\mathrm{sym}\big(\nabla a\otimes\eta\big)\\ &\qquad + 2\frac{\Gamma'\bar\Gamma}{\lambda}a\,\mathrm{sym}\big((\eta\otimes E_u)\nabla(a^2T_u\eta)\big) + 2\frac{\Gamma\bar\Gamma'}{\lambda}a^3\,\mathrm{sym}\big((\eta\otimes\eta)(T_u)^T\nabla E_u)\big)\\ &\qquad + 2\frac{\bar\Gamma\bar\Gamma'}{\lambda}a^2\,\mathrm{sym}\big((\eta\otimes\eta)(T_u)^T\nabla(a^2T_u\eta)\big)\Big\}\\ &\quad + \Big\{\frac{\Gamma^2}{\lambda^2}a^2(\nabla E_u)^T\nabla E_u + \frac{\Gamma^2}{\lambda^2}\nabla a\otimes\nabla a + 2\frac{\Gamma\bar\Gamma}{\lambda^2}\mathrm{sym}\big(\nabla(aE_u)^T\nabla(a^2T_u\eta)\big)\\ &\qquad + \frac{\bar\Gamma^2}{\lambda^2}\nabla(a^2T_u\eta)^T\nabla(a^2T_u\eta)\Big\}\\ &\quad + \Big\{2\,\mathrm{sym}\big((\nabla u)^T\nabla(T_u w)\big) + 2\Gamma'a\,\mathrm{sym}\big((\eta\otimes E_u)\nabla(T_u w)\big)\\ &\qquad + 2\bar\Gamma'a^2\mathrm{sym}\big((\eta\otimes\eta)(T_u)^T\nabla(T_u w)\big) + \nabla(T_u w)^T\nabla(T_u w)\Big\}\\ &\quad + \Big\{2\frac{\Gamma}{\lambda}\mathrm{sym}\big(\nabla(aE_u)^T\nabla(T_u w)\big) + 2\frac{\bar\Gamma}{\lambda}\mathrm{sym}\big(\nabla(a^2T_u\eta)^T\nabla(T_u w)\big)\}.\end{aligned}$$

This precisely yields the claim, upon noticing that first two terms in the first parentheses sum to $a^2\eta\otimes\eta$ because $(\nabla u)^T T_u = \mathrm{Id}_2$ and because:

$$(\Gamma')^2 + 2\bar\Gamma' = 1,$$

while the third term in the first parentheses can be rewritten as $(\bar\Gamma')^2a^4|T_u\eta|^2\eta\otimes\eta$. Finally, the first term in the fourth parentheses equals:

$$2\mathrm{sym}\nabla w + 2\mathrm{sym}\big((\nabla u)^T\big[(\partial_1 T_u)w, (\partial_2 T_u)w\big]\big).$$

The proof is done. ∎

3. Propagation of normal vectors lemmas

In this section, we gather some lemmas on properties satisfied by all the immersions inductively constructed along our convex integration algorithm. The first observation is standard:

Lemma 3.1. *Let $u \in \mathcal{C}^1(\bar\omega, \mathbb{R}^4)$, defined on the closure of an open, bounded set $\omega \subset \mathbb{R}^2$, satisfy:*

$$\frac{1}{\gamma}\mathrm{Id}_2 \leq (\nabla u)^T \nabla u \leq \gamma \mathrm{Id}_2 \quad \text{in } \bar\omega, \tag{3.1}$$

for some $\gamma > 1$. Then there holds:

$$\|\nabla u\|_0 \leq (2\gamma)^{1/2} \quad \text{and} \quad \frac{1}{\gamma^2} \leq \det((\nabla u)^T\nabla u) \leq \gamma^2 \text{ in } \bar\omega.$$

Proof. The first assertion holds as $|\partial_i u(x)|^2 = \langle \nabla u(x)^T(\nabla u(x))e_i, e_i\rangle \leq \gamma$ for $i = 1, 2$. For the second assertion, note that both eigenvalues of the symmetric matrix $(\nabla u)^T\nabla u$ are within the interval $[1/\gamma, \gamma]$ for all $x \in \bar\omega$, by (3.1). This implies the stated bound on the determinant. ■

The next lemma gathers bounds on the tangent frame of an immersion:

Lemma 3.2. *Let $u \in \mathcal{C}^{k+2}(\bar\omega, \mathbb{R}^4)$, defined on the closure of an open, bounded set $\omega \subset \mathbb{R}^2$, satisfy (3.1) for some $\gamma > 1$. Assume additionally that:*

$$\|\nabla^{(m)}\nabla^{(2)}u\|_0 \leq \bar C \mu^{m+1} A \quad \text{for all } m = 0\dots k,$$

with some constants $\mu > 1$, $A < 1$ and $\bar C > 1$. Then, the following bounds are valid for the tangent field $T_u = (\nabla u)\big((\nabla u)^T\nabla u\big)^{-1} \in \mathcal{C}^{k+1}(\bar\omega, \mathbb{R}^{4\times 2})$:

$$\|T_u\| \leq C \quad \text{and} \quad \|\nabla^{(m)}T_u\|_0 \leq C\mu^m A \quad \text{for all } m = 1\dots k+1,$$

where C depends only on γ, $\bar C$ and k, but not on μ, A.

Proof. Write $T_u = \frac{1}{\det((\nabla u)^T\nabla u)}(\nabla u)\,\mathrm{cof}\big((\nabla u)^T\nabla u\big)$, whereupon Lemma 3.1 implies the first assertion, with C depending on γ. For the second assertion, estimate first:

$$\begin{aligned}\|\nabla^{(m)}\det((\nabla u)^T\nabla u)\|_0 &\leq C\sum_{p+q+t+s=m}\|\nabla^{(p+1)}u\|_0\|\nabla^{(q+1)}u\|_0\|\nabla^{(s+1)}u\|_0\|\nabla^{(t+1)}u\|_0 \\ &\leq C\mu^m A \quad \text{for all } m = 1\dots k+1,\end{aligned}$$

because at least one of the exponents p, q, s, t must be positive, say $p \geq 1$, so that $p+1 \geq 2$ and we can use the assumption bound, whereas other terms are estimated as in $\|\nabla^{(q+1)}u\|_0 \leq C\mu^q$ in view of Lemma 3.1 as $A < 1$. An application of Faà di Bruno's formula yields:

$$\begin{aligned}&\Big\|\nabla^{(m)}\Big(\frac{1}{\det((\nabla u)^T\nabla u)}\Big)\Big\|_0 \\ &\leq C\sum_{p_1+2p_2+\dots mp_m=m}\Big\|\det((\nabla u)^T\nabla u)^{-1-(p_1+\dots+p_m)}\prod_{j=1}^m|\nabla^{(j)}\det((\nabla u)^T\nabla u)|^{p_j}\Big\|_0 \\ &\leq C\mu^m A \quad \text{for all } m = 1\dots k+1.\end{aligned}$$

In conclusion, and by a similar argument, we get:

$$\begin{aligned}\|\nabla^{(m)}T_u\|_0 &\leq C\sum_{p+q+t+s=m}\Big\|\nabla^{(p)}\Big(\frac{1}{\det((\nabla u)^T\nabla u)}\Big)\Big\|_0\|\nabla^{(q+1)}u\|_0\|\nabla^{(s+1)}u\|_0\|\nabla^{(t+1)}u\|_0 \\ &\leq C\mu^m A \quad \text{for all } m = 1\dots k+1.\end{aligned}$$

This completes the proof. ■

Concerning existence of the normal frame, the following has been proved in [11, Lemma 3.5]:

Lemma 3.3. *Let $u \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^4)$, defined on the closure of $\omega \subset \mathbb{R}^2$ diffeomorphic to B_1, satisfy (3.1) with some $\gamma > 1$. Fix $N \geq 1$. Then, there exist a normal frame $E_u^1, E_u^2 \in \mathcal{C}^N(\bar\omega, \mathbb{R}^4)$:*

$$(\nabla u)^T E_u^1 = (\nabla u)^T E_u^2 = 0, \quad |E_u^1| = |E_u^2| = 1, \quad \langle E_u^1, E_u^2\rangle = 0 \quad \text{in } \bar\omega, \tag{3.2}$$

obeying the bounds:

$$\|\nabla^{(m)} E_u^i\|_0 \leq C\big(1 + \|\nabla u\|_m\big) \quad \text{for all } m = 1 \dots N, \ i = 1, 2, \tag{3.3}$$

where C depends only on ω, γ and N.

We now sketch the argument; it starts with a local construction on $\omega = B_1$, by first fixing an orthonormal frame $\xi^1, \xi^2 \in \mathbb{R}^4$ to $\nabla u(0)$, then defining the smooth fields:

$$\nu_u^i(x) = \big(\mathrm{Id}_4 - (\nabla u(x))(\nabla u(x)^T(\nabla u(x)))^{-1}\nabla u(x)^T\big)\xi^i \quad \text{for } i = 1, 2,$$

which form a basis of the orthogonal complement of $\mathrm{span}\{\partial_1 u(x), \partial_2 u(x)\}$ and can be Gramm-Schmidt orthonormalized to E_u^1, E_u^2 satisfying (3.2), (3.1), in a sufficiently small neighbourhood $\bar B_r$ of 0. Then, the key ingredient in the proof is to show that such local frame can be extended on $\bar B_{r+\delta}$ and obey the same bounds, with $\delta > 0$ that depends only on γ, N. The construction is explicit, via a partition of unity argument. ∎

The final lemma of this section provides the key construction and estimates on the propagation of normal vectors from a given, to a nearby immersion:

Lemma 3.4. *Let $u \in \mathcal{C}^{k+2}(\bar\omega, \mathbb{R}^4)$ be defined on the closure of an open, bounded set $\omega \subset \mathbb{R}^2$, and satisfy (3.1) for some $\gamma > 1$. Let $E_u^1, E_u^2 \in \mathcal{C}^{k+1}(\bar\omega, \mathbb{R}^4)$ satisfy (3.2), and fix $v \in \mathcal{C}^{k+2}(\bar\omega, \mathbb{R}^4)$.*

(i) There exists $\rho \in (0,1)$ depending only on γ, such that if $\|\nabla v - \nabla u\|_0 \leq \rho$ then a normal frame $E_v^1, E_v^2 \in \mathcal{C}^{k+1}(\bar\omega, \mathbb{R}^4)$ to v, namely:

$$(\nabla v)^T E_v^1 = (\nabla v)^T E_v^2 = 0 \quad \text{and} \quad |E_v^1| = |E_v^2| = 1 \quad \text{and} \quad \langle E_v^1, E_v^2\rangle = 0 \quad \text{in } \bar\omega,$$

can be defined via the following formulas:

$$\begin{aligned} & E_v^1 = \frac{\nu_v^1}{|\nu_v^1|}, \qquad E_v^2 = \frac{\nu_v^2 - \langle \nu_v^2, E_v^1\rangle E_v^1}{|\nu_v^2 - \langle \nu_v^2, E_v^1\rangle E_v^1|}, \\ & \text{where } \nu_v^i = \big(\mathrm{Id}_4 - T_v(\nabla v - \nabla u)^T\big)E_u^i \quad \text{for } i = 1, 2 \\ & \text{and} \quad T_v = (\nabla v)((\nabla v)^T \nabla v)^{-1}. \end{aligned} \tag{3.4}$$

(ii) If, in addition to $\|\nabla v - \nabla u\|_0 \leq \rho$, there holds:

$$\|\nabla^{(m)}(\nabla v - \nabla u)\|_0 \leq \bar C \mu^m A \quad \text{for } m = 0 \dots k+1,$$
$$\|\nabla^{(m)}\nabla^{(2)} u\|_0 \leq \bar C \mu^{m+1}, \quad \|\nabla^{(m+1)} E_u^i\|_0 \leq \bar C \mu^{m+1} \quad \text{for } m = 0 \dots k, \ i = 1, 2,$$

with some constants $\mu > 1, A < 1$ and $\bar C > 1$, then E_v^1, E_v^2 given in (3.4) satisfy:

$$\|\nabla^{(m)}(E_v^i - E_u^i)\|_0 \leq C\mu^m A \quad \text{for all } m = 0 \dots k+1, \ i = 1, 2,$$

where C depends only on $\gamma, \bar C$ and k, but not on μ, A.

Proof. **1.** Observe that, by the first bound in Lemma 3.1:

$$\|(\nabla v)^T \nabla v - (\nabla u)^T \nabla u\|_0 \leq \|\nabla v - \nabla u\|_0 (\|\nabla u\|_0 + \|\nabla v\|_0) \leq \|\nabla v - \nabla u\|_0 (2^{3/2}\gamma^{1/2} + 1),$$

implying for small ρ that:

$$\frac{1}{2\gamma} \leq (\nabla v)^T \nabla v \leq 2\gamma \mathrm{Id}_2 \quad \text{in } \bar\omega. \tag{3.5}$$

Then, $T_v \in \mathcal{C}^{k+1}(\bar\omega, \mathbb{R}^{4\times 2})$ is well defined and ν_v^i are normal to ∇v, as $(\nabla v)^T T_v = \mathrm{Id}_2$ and:

$$(\nabla v)^T \nu_v^i = \big((\nabla v)^T - (\nabla v - \nabla u)^T\big) E_u^i = (\nabla u)^T E_u^i = 0 \quad \text{ for } i = 1, 2.$$

Further, applying Lemma 3.2 to bound $\|T_v\|_0$ by a constant that only depends on γ, we get:

$$\|\nu_v^i - E_u^i\|_0 \leq \|T_v\|_0 \|\nabla v - \nabla u\|_0 \leq C\|\nabla v - \nabla u\|_0 \leq \frac{1}{2} \tag{3.6}$$

for ρ sufficiently small, so in particular:

$$|\nu_v^i(x)| \geq |E_u^i(x)| - \frac{1}{2} \geq \frac{1}{2} \quad \text{ for all } \ x \in \bar\omega, \ i = 1, 2$$

implying that E_v^1 in (3.4) is well defined. Next:

$$\|\langle \nu_v^1, \nu_v^2\rangle\|_0 \leq 2\|T_v\|_0 \|\nabla v - \nabla u\|_0 + \|T_v\|_0^2 \|\nabla v - \nabla u\|_0^2 \leq \frac{1}{8},$$

for ρ sufficiently small, implying the well definiteness of E_v^2 because:

$$|\nu_v^2(x) - \langle \nu_v^2(x), E_v^1(x)\rangle E_v^1(x)| \geq \frac{1}{2} - \frac{|\langle \nu_v^2(x), \nu_v^1(x)\rangle|}{|\nu_v^1(x)|} \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \quad \text{ for all } \ x \in \bar\omega.$$

This justifies all the claims in (i), in view of the final property:

$$\langle E_v^1, E_v^2\rangle = \frac{\langle E_v^1, \nu_v^2\rangle - \langle \nu_v^2, E_v^1\rangle}{|\nu_v^2 - \langle \nu_v^2, E_v^1\rangle E_v^1|} = 0.$$

2. Recall that v obeys (3.5) and also, from the assumptions we get: $\|\nabla^{(m)}\nabla^{(2)} v\|_0 \leq 2\bar C \mu^{m+1}$ for all $m = 0 \ldots k$. Hence, Lemma 3.2 implies:

$$\|\nabla^{(m)} T_v\|_0 \leq C\mu^m \quad \text{ for all } m = 0 \ldots k+1. \tag{3.7}$$

Consequently, for all $m = 0 \ldots k+1$:

$$\|\nabla^{(m)}(\nu_v^i - E_u^i)\|_0 \leq C \sum_{p+q+t=m} \|\nabla^{(p)} T_v\|_0 \|\nabla^{(q)}(\nabla u - \nabla v)\|_0 \|\nabla^{(t)} E_u^i\|_0 \leq C\mu^m A, \tag{3.8}$$

and further:

$$\|\nabla^{(m)} \nu_v^i\|_0 \leq C\mu^m \quad \text{ for all } m = 0 \ldots k+1. \tag{3.9}$$

We write:

$$E_v^1 - E_u^1 = f(\nu_v^1) - f(E_u^1) = \Big(\int_0^1 \nabla f(t\nu_v^1 + (1+t)E_u^1)\, \mathrm{d}t\Big)(\nu_v^1 - E_u^1)$$

$$\text{where } f(z) = \frac{z}{|z|} \quad \text{ with } \quad \partial_i f^j(z) = \frac{\delta_{ij}|z|^2 - z_i z_j}{|z|^3} \quad \text{ for all } z \in \mathbb{R}^4 \setminus \{0\}, \ i, j = 1 \ldots 4.$$

For each fixed $t \in (0,1)$, we now bound $\|\nabla_x^{(m)} \partial_i f^j(t(\nu_v^1 - E_u^1) + E_u^1)\|_0$. Firstly, by (3.8):

$$\|\nabla_x^{(m)}(t(\nu_v^1 - E_u^1) + E_u^1)\|_0 \leq C\mu^m \quad \text{ for all } m = 0 \ldots k+1,$$

and also the same estimate is valid for the derivatives of $|t(\nu_v^1 - E_u^1) + E_u^1|^2$. Secondly, since $|t(\nu_v^1 - E_u^1) + E_u^1|$ is lower bounded by 1/2 by (3.6), the above yields, by Faà di Bruno's formula:

$$\begin{aligned}
&\big\|\nabla_x^m\big(\frac{1}{|t(\nu_v^1 - E_u^1) + E_u^1|^3}\big)\big\|_0 = \big\|\nabla_x^m\big(|t(\nu_v^1 - E_u^1) + E_u^1|^2\big)^{-3/2}\big\|_0 \\
&\leq C \sum_{p_1 + 2p_2 + \ldots mp_m = m} \Big\||t(\nu_v^1 - E_u^1) + E_u^1|^{2(-3/2 - (p_1 + \ldots + p_m))} \prod_{j=1}^m \big|\nabla_x^{(j)}|t(\nu_v^1 - E_u^1) + E_u^1|^2\big|\Big\|_0 \\
&\leq C\mu^m \quad \text{ for all } m = 1 \ldots k+1.
\end{aligned}$$

It hence follows that for all $m = 0 \dots k+1$, independently of $t \in (0,1)$:

$$\begin{aligned}&\|\nabla_x^{(m)}\nabla f\big(t(\nu_v^1 - E_u^1) + E_u^1\big)\|_0\\ &\leq C\sum_{p+q=m}\|\nabla_x^{(p)}|t(\nu_v^1 - E_u^1)+E_u^1|^2\|_0\big\|\nabla_x^q\big(\frac{1}{|t(\nu_v^1 - E_u^1)+E_u^1|^3}\big)\big\|_0 \leq C\mu^m,\end{aligned}$$

which implies, through (3.8), as claimed in the lemma:

$$\begin{aligned}&\|\nabla^{(m)}(E_v^1 - E_u^1)\| \leq C\sum_{p+q=m}\big\|\int_0^1 \nabla_x^{(p)}\nabla f\big(t(\nu_v^1 - E_u^1)+E_u^1\big)\;\mathrm{d}t\big\|_0 \|\nabla^{(q)}(\nu_v^1 - E_u^1)\|_0\\ &\leq C\mu^m A \quad \text{ for all } m = 0\dots k+1.\end{aligned} \tag{3.10}$$

In particular, the above yields:

$$\|\nabla^{(m)}E_v^1\| \leq C\mu^m \quad \text{ for all } m = 0\dots k+1. \tag{3.11}$$

3. It remains to prove the bound as in (3.10) for $E_v^2 - E_u^2$. To this end, observe that:

$$\begin{aligned}\langle \nu_v^2, E_v^1\rangle &= \frac{1}{|\nu_v^1|}\big\langle E_u^2 - T_v(\nabla v - \nabla u)^T E_u^2, E_u^1 - T_v(\nabla v - \nabla u)^T E_u^1\big\rangle\\ &= \frac{1}{|\nu_v^1|}\Big(-2\big\langle \mathrm{sym}(T_v(\nabla v - \nabla u)^T)E_u^1, E_u^2\big\rangle + \big\langle T_v(\nabla v - \nabla u)^T E_u^1, T_v(\nabla v - \nabla u)^T E_u^2\big\rangle\Big).\end{aligned}$$

Since $|\nu_v^1|$ is lower bounded by $1/2$ in view of (3.6), the estimate (3.9) implies:

$$\big\|\nabla^{(m)}\big(\frac{1}{|\nu_v^1|}\big)\big\|_0 \leq C\mu^m \quad \text{ for all } m = 0\dots k+1,$$

via the application of Faà di Bruno's formula, as before. The bounds on the derivatives of T_v in (3.7) and the assumed bounds on the derivatives of $\nabla v - \nabla u$ and E_u^i now result in:

$$\|\nabla^{(m)}\langle \nu_v^2, E_v^1\rangle\|_0 \leq C\mu^m A \quad \text{ for all } m = 0\dots k+1. \tag{3.12}$$

Denote:

$$\tilde\nu_v^2 = \nu_v^2 - \langle \nu_v^2, E_v^1\rangle E_v^1 \tag{3.13}$$

and observe that by (3.8), (3.11), (3.12):

$$\begin{aligned}&\|\nabla^{(m)}(\tilde\nu_v^2 - E_u^2)\|_0 \leq \|\nabla^{(m)}(\nu_v^2 - E_u^2)\|_0 + \|\nabla^{(m)}\big(\langle \nu_v^2, E_v^1\rangle E_v^1\big)\|_0\\ &\leq C\mu^m A + C\sum_{p+q=m}\|\nabla^{(p)}\langle \nu_v^2, E_v^1\rangle\|_0\|\nabla^{(q)}E_v^1\|_0 \leq C\mu^m A \quad \text{ for all } m = 0\dots k+1.\end{aligned}$$

As in the previous step, we now write:

$$E_v^2 - E_u^2 = f(\tilde\nu_v^2) - f(E_u^2),$$

and recall that by (3.13), (3.6) and the following two bounds in step 1:

$$\begin{aligned}&|t(\tilde\nu_v^2(x) - E_u^2(x)) + E_u^2(x)| \geq |E_u^2(x)| - \|\nu_v^2 - E_u^2\|_0 - \|\langle \nu_v^2, E_v^1\rangle\|_0\\ &\geq 1 - \frac{1}{2} - \frac{1}{4} \geq \frac{1}{4} \quad \text{ for all } x \in \bar\omega.\end{aligned}$$

The above lower bound allows for the same estimates leading to (3.10), likewise yield:

$$\begin{aligned}&\|\nabla^{(m)}(E_v^2 - E_u^2)\| \leq C\sum_{p+q=m}\big\|\int_0^1 \nabla_x^{(p)}\nabla f\big(t(\tilde\nu_v^2 - E_u^2)+E_u^2\big)\;\mathrm{d}t\big\|_0 \|\nabla^{(q)}(\tilde\nu_v^2 - E_u^2)\|_0\\ &\leq C\mu^m A \quad \text{ for all } m = 0\dots k+1.\end{aligned}$$

This ends the proof. ∎

4. The initial Stage in the Nash-Kuiper scheme

In this section, we show how to reduce the given positive definite defect $\mathcal{D}$ to $\tilde{\mathcal{D}}$ that is arbitrarily small, at the expense of increasing the second derivatives of the modified immersion by a specific power of the defect's decrease. This information was not needed in the convex integration analysis of the Monge-Ampère system in papers [36, 32, 33, 34, 26, 27], where only the decrease of $\mathcal{D}$ mattered in the initial Stage. Similar statements were put forward in [9, proof of Theorem 1.1], [13, Proposition 3.2]. Here, we provide a self-contained proof of the assertions that we use in our future arguments, applying the Step construction in Lemma 2.4.

Theorem 4.1. [INITIAL STAGE] *Let $\underline{u} \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^4)$ be an immersion, defined on the closure of an open set $\omega \subset \mathbb{R}^2$ diffeomorphic to B_1, together with a metric $g \in \mathcal{C}^{r,\beta}(\bar\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym},>})$ where:*

$$0 < r + \beta \leq 2.$$

Assume that:

$$\mathcal{D}(g, \underline{u}) \doteq g - (\nabla \underline{u})^T \nabla \underline{u} > 0 \qquad \text{on } \bar\omega.$$

Then, there exist $\underline{\gamma} > 1$, $\underline{\delta} \in (0,1)$ and $\underline{\tau} \geq 1 + \frac{1}{r+\beta}$, depending only on $\omega, \underline{u}$ and g, such that the following holds. For every $\delta \in (0, \underline{\delta})$ there exists $u \in \mathcal{C}^2(\bar\omega, \mathbb{R}^4)$ such that:

$$\|u - \underline{u}\|_0 \leq C\delta, \quad \|\nabla(u - \underline{u})\|_0 \leq C, \quad \|\nabla^2(u - \underline{u})\|_0 \leq \frac{C}{\delta^{\underline{\tau}}}, \tag{4.1$_1$}$$

$$\|\mathcal{D}(g - \delta H_0, u)\|_0 \leq \frac{r_0}{4}\delta, \tag{4.1$_2$}$$

$$\frac{1}{\underline{\gamma}}\mathrm{Id}_2 \leq (\nabla u)^T \nabla u \leq \underline{\gamma}\mathrm{Id}_2 \qquad \text{in } \bar\omega. \tag{4.1$_3$}$$

with r_0 as in Lemma 2.2 and with constants C depending only on ω, $\underline{u}$ and g.

Proof. **1.** For a sufficiently small $l \in (0,1)$, in dependence of only $\omega, \underline{u}, g$, we define the mollifications $g_l = g * \phi_l \in \mathcal{C}^\infty(\bar\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym},>})$ as in Lemma 2.1. Fix $\delta \in (0, \underline{\delta})$. We write:

$$\mathcal{D}(g_l - \delta H_0, \underline{u}) = \mathcal{D}(g, \underline{u}) + (g_l - g) - \delta H_0.$$

Recall, from $(2.1)_2$, $(2.1)_1$, the estimate:

$$\|g_l - g\|_0 \leq l^{r+\beta}\|g\|_{r,\beta}, \quad \|\nabla^{(m)} g_l\|_0 \leq C_m \frac{1}{l^m}\|g\|_0, \tag{4.2}$$

where C_m depends only on m and ω. Taking l and $\underline{\delta}$ sufficiently small guarantees hence that:

$$\mathcal{D}(g_l - \delta H_0, \underline{u}) \geq \frac{1}{2}\mathcal{D}(g, \underline{u}) \quad \text{in } \bar\omega \quad \text{ and } \quad \|\mathcal{D}(g_l - \delta H_0, \underline{u})\|_0 \leq 2\|\mathcal{D}(g, \underline{u})\|_0 \tag{4.3}$$

and that the image of $\bar\omega$ through $\mathcal{D}(g_l - \delta H_0, \underline{u})$ is contained in a compact region $\mathcal{K} \subset \mathbb{R}^{2\times 2}_{\mathrm{sym},>}$ depending only on $\underline{u}, g$. By Lemma 2.3, there exists a finite set of N indices, depending only on $\mathcal{K}$, for which the decomposition of $\mathcal{D}(g_l - \delta H_0, \underline{u})$ into primitive rank-one metrics is active:

$$\mathcal{D}(g_l - \delta H_0, \underline{u}) = \sum_{i=1}^N a_i^2 \eta_i \otimes \eta_i \qquad \text{on } \bar\omega,$$

where $\{a_i \in \mathcal{C}^\infty(\bar\omega, \mathbb{R})\}_{i=1}^N$ are given by $a_i(x) = \varphi_i\Big(\mathcal{D}(g_l - \delta H_0, \underline{u})(x)\Big)$ for all $x \in \bar\omega$.

Above, the coefficient functions $\{\varphi_i \in \mathcal{C}_c^\infty(\mathbb{R}^{2\times 2}_{\mathrm{sym},>}, \mathbb{R})\}_{i=1}^N$ and the unit vectors $\{\eta_i \in \mathbb{R}^2\}_{i=1}^N$ are as in Lemma 2.3. By Faà di Bruno's formula and the second bound in (4.2), we obtain:

$$\begin{aligned} \|\nabla^{(m)} a_i\|_0 &\leq C \sum_{p_1+2p_2+\dots mp_m=m} \|\nabla^{(p_1+\dots+p_m)}\varphi_i\|_0 \prod_{j=1}^m \|\nabla^{(j)}\mathcal{D}(g_l - \delta H_0, \underline{u})\|^{p_j} \\ &\leq C \sum_{p_1+\dots mp_m=m} \prod_{j=1}^m \Big(\frac{\|g\|_0}{l^j} + 1\Big)^{p_j} \leq \frac{C}{l^m}, \end{aligned}$$

valid for $m = 1\dots N-1$, with C that depends only on $N, \omega, \underline{u}$, hence on $\omega, \underline{u}, g$. In conclusion:

$$\|\nabla^{(m)} a_i\|_0 \leq \frac{C}{l^m} \quad \text{ for all } m = 0\dots N-1,\ i = 1\dots N. \tag{4.4}$$

We set $u_0 = \underline{u}$ and inductively define the N-tuple of immersions $\{u_i \in \mathcal{C}^{N-i+2}(\bar\omega, \mathbb{R}^4)\}_{i=1}^N$ according to Lemma 2.4, namely:

$$u_{i+1} = u_i + \frac{a_{i+1}}{\lambda_{i+1}}\Gamma(\lambda_{i+1}t_{\eta_{i+1}})E^1_{u_i} + \frac{a_{i+1}}{\lambda_{i+1}}\bar\Gamma(\lambda_{i+1}t_{\eta_{i+1}})E^2_{u_i} \quad \text{ for } i = 0\dots N-1,$$

where we denote $t_{\eta_{i+1}} = \langle x, \eta_{i+1}\rangle$ and where the orthonormal normal vector fields $\{E^1_{u_i}, E^2_{u_i} \in \mathcal{C}^{N-i+1}(\bar\omega, \mathbb{R}^4)\}_{i=0}^{N-1}$ are defined through Lemma 3.3. The frequencies λ are defined as follows:

$$\lambda_i = \frac{1}{(\epsilon l)^i} \quad \text{ for all } i = 0\dots N, \tag{4.5}$$

where $\epsilon \in (0,1)$ is sufficiently small, in function of $\omega, \underline{u}, g$.

2. Let $\gamma > 1$ be such that:

$$\frac{1}{\gamma}\mathrm{Id}_2 \leq (\nabla\underline{u})^T\nabla\underline{u} \leq \gamma\mathrm{Id}_2 \quad \text{ in } \bar\omega.$$

We will show that, provided ϵ in (4.5) is sufficiently small, there holds:

$$\begin{aligned} &\frac{1}{2\gamma}\mathrm{Id}_2 \leq (\nabla u_i)^T\nabla u_i \leq \big(\gamma + 2\|\mathcal{D}(g,\underline{u})\|_0 + 1\big)\mathrm{Id}_2 \ \text{ in } \bar\omega \quad \text{ for } i = 0\dots N, && (4.6)_1 \\ &\|\nabla^{(m)} u_i\|_0 \leq C\lambda_i^{m-1} \qquad \text{ for } i = 0\dots N-1,\ m = 1\dots N-i+2, && (4.6)_2 \\ &\|\nabla^{(m)} E^j_{u_i}\|_0 \leq C\lambda_i^m \qquad \text{ for } i = 0\dots N-1,\ m = 0\dots N-i+1,\ j = 1,2, && (4.6)_3 \\ &\|\nabla^{(m)}(u_{i+1} - u_i)\|_0 \leq C\lambda_{i+1}^{m-1} \qquad \text{ for } i = 0\dots N-1,\ m = 0\dots N-i+1. && (4.6)_4 \end{aligned}$$

Firstly, $(4.6)_1$ and $(4.6)_2$ are trivially true at $i = 0$ where $\lambda_0 = 1$. Secondly, $(4.6)_1$ and $(4.6)_2$ directly imply $(4.6)_3$ by Lemma 3.3. Thirdly, $(4.6)_3$ implies $(4.6)_4$, because:

$$\begin{aligned} \|\nabla^{(m)}(u_{i+1} - u_i)\|_0 &\leq C \sum_{p+q+t=m} \lambda_{i+1}^{p-1}\|\nabla^{(q)} a_{i+1}\|_0\big(\|\nabla^{(t)} E^1_{u_i}\|_0 + \|\nabla^{(t)} E^2_{u_i}\|_0\big) \\ &\leq C \sum_{p+q+t=m} \lambda_{i+1}^{p-1}\frac{\lambda_i^t}{l^q} \leq C\lambda_{i+1}^{m-1} \sum_{p+q+t=m} \frac{1}{(l\lambda_{i+1})^q} \leq C\lambda_{i+1}^{m-1}, \end{aligned}$$

by (4.4) and as $l\lambda_{i+1} \geq l\lambda_1 = \frac{1}{\epsilon} > 1$. Fourthly, $(4.6)_2$, $(4.6)_4$ at i counter imply $(4.6)_2$ at $i+1$:

$$\begin{aligned} \|\nabla^{(m)} u_{i+1}\|_0 &\leq \|\nabla^{(m)} u_i\|_0 + \|\nabla^{(m)}(u_{i+1} - u_i)\|_0 \leq C(\lambda_i^{m-1} + \lambda_{i+1}^{m-1}) \\ &\leq C\lambda_{i+1}^{m-1} \quad \text{ for } m = 1\dots N-i+1. \end{aligned}$$

We now fix $i = 0 \dots N-1$ and show that the validity of $(4.6)_2$, $(4.6)_3$ for all $j = 0 \dots i$ implies $(4.6)_1$ at $i+1$. To this end, we recall the form of the error $\mathcal{R}$ in Lemma 2.4 and estimate:

$$\begin{aligned}
&\|(\nabla u_{j+1})^T\nabla u_{j+1} - (\nabla u_j)^T\nabla u_j - a_{j+1}^2\eta_{j+1}\otimes\eta_{j+1}\|_0 = \|\mathcal{R}\|_0 \\
&\leq C\Big(\frac{\|a_{j+1}\|_0}{\lambda_{j+1}}\|\nabla^{(2)}u_j\|_0 + \frac{\|a_{j+1}\|_0^2}{\lambda_{j+1}}\|\nabla E_{u_j}\|_0 + \frac{\|a_{j+1}\|_0^2}{\lambda_{j+1}^2}\|\nabla E_{u_j}\|_0^2 + \frac{\|\nabla a_{j+1}\|_0^2}{\lambda_{j+1}^2}\Big) \\
&\leq C\Big(\frac{\lambda_j}{\lambda_{j+1}} + \frac{\lambda_j^2}{\lambda_{j+1}^2} + \frac{1}{(\lambda_{j+1}l)^2}\Big) \leq C\big(\epsilon l + (\epsilon l)^2 + \epsilon\big) \leq C\epsilon \qquad \text{for all } j = 0\dots i.
\end{aligned}$$

Consequently:

$$\|(\nabla u_{i+1})^T\nabla u_{i+1} - (\nabla u_0)^T\nabla u_0 - \sum_{j=1}^{i+1} a_j^2\eta_j\otimes\eta_j\|_0 \leq C\epsilon, \tag{4.7}$$

and we see that taking ϵ sufficiently small yields in $\bar\omega$:

$$\begin{aligned}
&(\nabla u_{i+1})^T\nabla u_{i+1} \geq (\nabla u_0)^T\nabla u_0 + \sum_{j=1}^{i+1} a_j^2\eta_j\otimes\eta_j - C\epsilon\mathrm{Id}_2 \geq \frac{1}{\gamma}\mathrm{Id}_2 - C\epsilon\mathrm{Id}_2 \geq \frac{1}{2\gamma}\mathrm{Id}_2, \\
&(\nabla u_{i+1})^T\nabla u_{i+1} \leq (\nabla u_0)^T\nabla u_0 + \sum_{j=1}^{N} a_j^2\eta_j\otimes\eta_j + C\epsilon\mathrm{Id}_2 \\
&\qquad\qquad \leq \gamma\mathrm{Id}_2 + \|\mathcal{D}(g_l - \delta H_0, \underline{u})\|_0\mathrm{Id}_2 + C\epsilon\mathrm{Id}_2 \leq \big(\gamma + 2\|\mathcal{D}(g,\underline{u})\|_0 + 1\big)\mathrm{Id}_2,
\end{aligned}$$

by (4.3). This ends the proof of $(4.6)_1$ and of all the inductive estimates.

3. We declare $u = u_N$ and from $(4.6)_4$ deduce that:

$$\|\nabla^{(m)}(u - \underline{u})\|_0 \leq \sum_{i=0}^{N-1}\|\nabla^{(m)}(u_{i+1} - u_i)\|_0 \leq C\sum_{i=0}^{N-1}\lambda_{i+1}^{m-1} \qquad \text{for } m = 0,1,2. \tag{4.8}$$

On the other hand, from (4.2) and (4.7):

$$\begin{aligned}
&\|\mathcal{D}(g - \delta H_0, u)\|_0 \leq \|\mathcal{D}(g_l - \delta H_0, u_N)\|_0 + \|g - g_l\|_0 \\
&= \Big\|\sum_{j=1}^{N} a_j^2\eta_j\otimes\eta_j - \big((\nabla u_N)^T\nabla u_N - (\nabla u_0)^T\nabla u_0\big)\Big\|_0 + \|g - g_l\|_0 \leq \bar C(\epsilon + l^{r+\beta}),
\end{aligned}$$

where $\bar C > 1$ depends only on $\omega, \underline{u}, g$. We hence take:

$$\epsilon = \frac{r_0}{8\bar C}\delta, \qquad l^{r+\beta} = \frac{r_0}{8\bar C}\delta$$

which is consistent with the requirements of smallness of ϵ and l provided that $\underline{\delta}$ is small. This implies $(4.1)_2$, whereas $(4.1)_3$ follows from $(4.6)_1$ with

$$\underline{\gamma} = \max\{2\gamma, \gamma + 2\|\mathcal{D}(g,\underline{u})\|_0 + 1\}.$$

Finally, (4.8) yields:

$$\begin{aligned}
&\|u - \underline{u}\|_0 \leq \frac{C}{\lambda_1} \leq C\epsilon \leq C\delta \quad \text{ and } \quad \|\nabla(u - \underline{u})\|_0 \leq C, \\
&\|\nabla^{(2)}(u - \underline{u})\|_0 \leq C\lambda_N = \frac{C}{(\epsilon l)^N} \leq C\delta^{-(1+\frac{1}{r+\beta})N}.
\end{aligned}$$

This implies $(4.1)_1$ with $\underline{\tau} = (1 + \frac{1}{r+\beta})N \geq 1 + \frac{1}{r+\beta}$, as claimed. The proof is done. ■

5. The oscillatory defect decomposition lemmas

The three results in this section provide a symmetric matrix field decomposition, complementary to that in Lemma 2.2. It removes a further symmetric gradient from the given oscillatory component of a defect at hand, and reduces it to a defect of higher order in the frequency, plus another term that agrees in the frequency yet has a lower dimensionality rank. The lemmas below are the explicit versions of the argument called "integration by parts" in [10]. The precise formulas on the decomposition coefficients (5.3), (5.5), (5.7) are of crucial importance in closing the estimates in Theorem 1.2. First, we quote from [27, Lemma 2.4, Corollary 2.5]:

Lemma 5.1. *Given $H \in \mathcal{C}^{k+1}(\mathbb{R}^2, \mathbb{R}^{2\times 2}_{\rm sym})$, $\lambda > 0$, and $\Gamma_0 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, we have the decomposition:*

$$\begin{aligned} \frac{\Gamma_0(\lambda x_1)}{\lambda} H = & (-1)^{k+1} \frac{\Gamma_{k+1}(\lambda x_1)}{\lambda^{k+2}} \mathrm{sym}\nabla L_k^{\eta_1} \\ & + \mathrm{sym}\nabla\Big(\sum_{i=0}^k (-1)^i \frac{\Gamma_{i+1}(\lambda x_1)}{\lambda^{i+2}} L_i^{\eta_1}\Big) + \Big(\sum_{i=0}^k (-1)^i \frac{\Gamma_i(\lambda x_1)}{\lambda^{i+1}} P_i^{\eta_1}\Big) e_2 \otimes e_2 \end{aligned} \tag{5.1}$$

where the functions $\Gamma_i \in \mathcal{C}^i(\mathbb{R}, \mathbb{R})$ satisfy the recursive definition:

$$\Gamma'_{i+1} = \Gamma_i \quad \textit{for all } \; i = 0 \dots k, \tag{5.2}$$

while $L_i^{\eta_1} \in \mathcal{C}^{k+1-i}(\mathbb{R}^2, \mathbb{R}^2)$ and $P_i^{\eta_1} \in \mathcal{C}^{k+1-i}(\mathbb{R}^2, \mathbb{R})$ are given in:

$$\begin{aligned} & L_0^{\eta_1} = (H_{11}, 2H_{12}), \qquad P_0^{\eta_1} = H_{22}, \\ & \left.\begin{aligned} & L_i^{\eta_1} = (\partial_1^{(i)} H_{11}, 2\partial_1^{(i)} H_{12} + i\partial_1^{(i-1)}\partial_2 H_{11}), \\ & P_i^{\eta_1} = 2\partial_1^{(i-1)}\partial_2 H_{12} + (i-1)\partial_1^{(i-2)}\partial_2^{(2)} H_{11} \end{aligned}\right\} \quad \textit{for all } \; i = 1 \dots k. \end{aligned} \tag{5.3}$$

Lemma 5.2. *Let H, λ, Γ_0 be as in Lemma 5.1 and $\{\Gamma_i \in \mathcal{C}^i(\mathbb{R}, \mathbb{R})\}_{i=1}^{k+1}$ as in (5.2). Then:*

$$\begin{aligned} \frac{\Gamma_0(\lambda x_2)}{\lambda} H = & (-1)^{k+1} \frac{\Gamma_{k+1}(\lambda x_2)}{\lambda^{k+2}} \mathrm{sym}\nabla L_k^{\eta_3} \\ & + \mathrm{sym}\nabla\Big(\sum_{i=0}^k (-1)^i \frac{\Gamma_{i+1}(\lambda x_2)}{\lambda^{i+2}} L_i^{\eta_3}\Big) + \Big(\sum_{i=0}^k (-1)^i \frac{\Gamma_i(\lambda x_2)}{\lambda^{i+1}} P_i^{\eta_3}\Big) e_1 \otimes e_1, \end{aligned} \tag{5.4}$$

with $L_i^{\eta_3} \in \mathcal{C}^{k+1-i}(\mathbb{R}^2, \mathbb{R}^2)$, $P_i^{\eta_3} \in \mathcal{C}^{k+1-i}(\mathbb{R}^2, \mathbb{R})$ given in:

$$\begin{aligned} & L_0^{\eta_3} = (2H_{12}, H_{22}), \qquad P_0^{\eta_3} = H_{11}, \\ & \left.\begin{aligned} & L_i^{\eta_3} = (2\partial_2^{(i)} H_{12} + i\partial_1\partial_2^{(i-1)} H_{22}, \partial_2^{(i)} H_{22}), \\ & P_i^{\eta_3} = 2\partial_1\partial_2^{(i-1)} H_{12} + (i-1)\partial_1^{(2)}\partial_2^{(i-2)} H_{22} \end{aligned}\right\} \quad \textit{for all } \; i = 1 \dots k. \end{aligned} \tag{5.5}$$

There likewise holds, with respect to the oscillations in the $\eta_2 = \frac{e_1+e_2}{\sqrt{2}}$ spatial direction and the residue accumulating in the $e_2 \otimes e_2$ component of the matrix field, as in Lemma 5.1:

Lemma 5.3. *Let H, λ, Γ_0 be as in Lemma 5.1 and $\{\Gamma_i \in \mathcal{C}^i(\mathbb{R}, \mathbb{R})\}_{i=1}^{k+1}$ as in (5.2). Denote:*

$$t = \langle x, \eta_2 \rangle.$$

Then, we have:

$$\begin{aligned}\frac{\Gamma_0(\lambda t)}{\lambda}H = {} & (-1)^{k+1}\frac{\Gamma_{k+1}(\lambda t)}{\lambda^{k+2}}\mathrm{sym}\nabla L_k^{\eta_2} \\ & + \mathrm{sym}\nabla\Big(\sum_{i=0}^{k}(-1)^i\frac{\Gamma_{i+1}(\lambda t)}{\lambda^{i+2}}L_i^{\eta_2}\Big) + \Big(\sum_{i=0}^{k}(-1)^i\frac{\Gamma_i(\lambda t)}{\lambda^{i+1}}P_i^{\eta_2}\Big)e_2\otimes e_2,\end{aligned} \tag{5.6}$$

with $L_i^{\eta_2}\in\mathcal{C}^{k+1-i}(\mathbb{R}^2,\mathbb{R}^2)$, $P_i^{\eta_2}\in\mathcal{C}^{k+1-i}(\mathbb{R}^2,\mathbb{R})$ *given in:*

$$\begin{aligned} & L_0^{\eta_2} = \sqrt{2}(H_{11}, 2H_{12}-H_{11}), \qquad P_0^{\eta_2} = H_{22}-2H_{12}+H_{11}, \\ & \left.\begin{aligned} & L_i^{\eta_2} = 2^{(i+1)/2}\big(\partial_1^{(i)}H_{11}, 2\partial_1^{(i)}H_{12} + i\partial_1^{(i-1)}\partial_2 H_{11} - (i+1)\partial_i^{(i)}H_{11}\big), \\ & P_i^{\eta_2} = 2^{i/2}\big(2\partial_1^{(i-1)}\partial_2 H_{12} - 2\partial_1^{(i)}H_{12} + (i-1)\partial_1^{(i-2)}\partial_2^{(2)}H_{11} \\ & \qquad -2i\partial_1^{(i-1)}\partial_2 H_{11} + (i+1)\partial_i^{(i)}H_{11}\big) \end{aligned}\right\} \textit{for}\ \ i = 1\dots k.\end{aligned} \tag{5.7}$$

Proof. **1.** Observe that $\mathrm{sym}(L_0^{\eta_2}\otimes\eta_2) = H - P_0^{\eta_2}e_2\otimes e_2$ and that:

$$\mathrm{sym}\nabla\Big(\frac{\Gamma(\lambda t)}{\lambda^2}L_0^{\eta_2}\Big) = \frac{\Gamma(\lambda t)}{\lambda^2}\mathrm{sym}\nabla L_0^{\eta_2} + \frac{\Gamma'(\lambda t)}{\lambda}\mathrm{sym}(L_0^{\eta_2}\otimes\eta_2).$$

Applying the above with $\Gamma=\Gamma_1$ so that $\Gamma'=\Gamma_0$, we get (5.6) at $k=0$:

$$\frac{\Gamma_0(\lambda t)}{\lambda}H = -\frac{\Gamma_1(\lambda t)}{\lambda^2}\,\mathrm{sym}\nabla L_0^{\eta_2} + \mathrm{sym}\nabla\Big(\frac{\Gamma_1(\lambda t)}{\lambda^2}L_0^{\eta_2}\Big) + \frac{\Gamma_0(\lambda t)}{\lambda}P_0^{\eta_2}e_2\otimes e_2. \tag{5.8}$$

2. The proof of (5.6) is carried out by induction on k. Assume that it holds at some $k\geq 0$ and apply (5.8) to $H=\mathrm{sym}\nabla L_k^{\eta_2}$ and Γ_0 replaced by Γ_{k+1}, to get:

$$\begin{aligned}\frac{\Gamma_{k+1}(\lambda t)}{\lambda}\,\mathrm{sym}\nabla L_k = {} & -\frac{\Gamma_{k+2}(\lambda t)}{\lambda^2}\,\mathrm{sym}\nabla L_{k+1} \\ & + \mathrm{sym}\nabla\Big(\frac{\Gamma_{k+2}(\lambda t)}{\lambda^2}L_{k+1}\Big) + \frac{\Gamma_{k+1}(\lambda t)}{\lambda}P_{k+1}e_2\otimes e_2,\end{aligned} \tag{5.9}$$

where, in virtue of the definitions (5.7):

$$\begin{aligned} L_{k+1} & = \sqrt{2}(\partial_1(L_k^{\eta_2})_1, \partial_1(L_k^{\eta_2})_2 + \partial_2(L_k^{\eta_2})_1 - \partial_1(L_k^{\eta_2})_1 \\ & = 2^{(k+2)/2}\big(\partial_1^{(k+1)}H_{11}, 2\partial_1^{(k+1)}H_{12} + (k+1)\partial_1^{(k)}\partial_2 H_{11} - (k+2)\partial_i^{(k+1)}H_{11}\big) = L_{k+1}^{\eta_2}, \\ P_{k+1} & = \partial_2(L_k^{\eta_2})_2 - \partial_1(L_k^{\eta_2})_2 - \partial_2(L_k^{\eta_2})_1 + \partial_1(L_k^{\eta_2})_1 \\ & = 2^{(k+1)/2}\big(2\partial_1^{(k)}\partial_2 H_{12} - 2\partial_1^{(k+1)}H_{12} + k\partial_1^{(k-1)}\partial_2^{(2)}H_{11} \\ & \qquad - 2(k+1)\partial_1^{(k)}\partial_2 H_{11} + (k+2)\partial_i^{(k+1)}H_{11}\big) = P_{k+1}^{\eta_2}.\end{aligned}$$

Introduce now (5.9) into (5.6) to obtain:

$$\begin{aligned}\frac{\Gamma_0(\lambda t)}{\lambda}H &= \frac{(-1)^{k+1}}{\lambda^{k+1}}\Big(-\frac{\Gamma_{k+2}(\lambda t)}{\lambda^2}\,\mathrm{sym}\nabla L_{k+1}^{\eta_2} \\ &\qquad\qquad + \mathrm{sym}\nabla\Big(\frac{\Gamma_{k+2}(\lambda t)}{\lambda^2}L_{k+1}^{\eta_2}\Big) + \frac{\Gamma_{k+1}(\lambda t)}{\lambda}P_{k+1}^{\eta_2}e_2\otimes e_2\Big) \\ &\quad + \mathrm{sym}\nabla\Big(\sum_{i=0}^{k}(-1)^i\frac{\Gamma_{i+1}(\lambda t)}{\lambda^{i+2}}L_i^{\eta_2}\Big) + \Big(\sum_{i=0}^{k}(-1)^i\frac{\Gamma_i(\lambda t)}{\lambda^{i+1}}P_i^{\eta_2}\Big)e_2\otimes e_2, \\ &= (-1)^{k+2}\frac{\Gamma_{k+2}(\lambda t)}{\lambda^{k+3}}\mathrm{sym}\nabla L_{k+1}^{\eta_2} \\ &\quad + \mathrm{sym}\nabla\Big(\sum_{i=0}^{k+1}(-1)^i\frac{\Gamma_{i+1}(\lambda t)}{\lambda^{i+2}}L_i^{\eta_2}\Big) + \Big(\sum_{i=0}^{k+1}(-1)^i\frac{\Gamma_i(\lambda t)}{\lambda^{i+1}}P_i^{\eta_2}\Big)e_2\otimes e_2,\end{aligned}$$

which is exactly (5.6) at $k+1$, as claimed. ■

6. The Källén iteration technique

In this section, we carry out a version of Källén's iteration, with the purpose of canceling the non-oscillatory portion of the defect term $\frac{1}{\lambda^2}\nabla a\otimes\nabla a$ in Lemma 2.5. The remaining portion:

$$\frac{\Gamma(\lambda t)^2-1}{\lambda^2}S_1 = -\frac{\cos(2\lambda t)}{\lambda^2}\nabla a\otimes\nabla a,$$

where we note that $\cos(2\lambda t)$ has mean zero on its period, will be canceled in the leading order via lemmas in section 5. The matrix field H in the statement below should be thought of as the scaled defect $\mathcal{D}$. Similar result appeared in [27, Proposition 3.1], with only one extra term of the type $\frac{1}{\lambda^2}\nabla a\otimes\nabla a$, while [10, Lemma 2.2] featured more absorbed terms, as below.

Lemma 6.1. *Let $H\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^{2\times2}_{\mathrm{sym}})$ be defined on the closure of an open, bounded set $\omega\subset\mathbb{R}^2$, and let $M,N\geq 1$ be two integers. Assume that:*

$$\|H-H_0\|_0\leq\frac{r_0}{2}\quad\text{and}\quad\|\nabla^{(m)}H\|_0\leq\bar C\mu^m\quad\text{for all } m=1\dots M+N, \tag{6.1}$$

for some given $\mu,\bar C>1$ and with r_0 as in Lemma 2.2. Then, there exists $\underline{\sigma}>2$ depending only on M,N such that the following holds. Given the constants $\kappa>\lambda>\mu$ satisfying $\lambda/\mu\geq\underline{\sigma}$, there exist $\{a_i\in\mathcal{C}^\infty(\bar\omega,\mathbb{R})\}_{i=1}^3$ such that, writing:

$$H=\sum_{i=1}^3 a_i^2\eta_i\otimes\eta_i+\frac{1}{\lambda^2}\nabla a_1\otimes\nabla a_1+\frac{1}{\kappa^2}\nabla a_2\otimes\nabla a_2+\mathcal{F},$$

there hold the estimates:

$$\begin{aligned}&\frac12\leq a_i^2\leq\frac32 \quad\text{and}\quad \frac12\leq a_i\leq\frac32 \quad\text{in } \bar\omega \quad\text{for } i=1\dots3,\\ &\|\nabla^{(m)}a_i^2\|_0\leq C\mu^m \quad\text{and}\quad \|\nabla^{(m)}a_i\|_0\leq C\mu^m \quad\text{for } m=1\dots M+1,\ i=1\dots3,\\ &\|\nabla^{(m)}\mathcal{F}\|_0\leq C\frac{\mu^m}{(\lambda/\mu)^{2N}} \quad\text{for } m=0\dots M.\end{aligned}\tag{6.2}$$

The constant C above depends only on $M,N,\bar C$.

Proof. **1.** We will inductively define the triples of coefficients $\{a_{i,j} \in \mathcal{C}^\infty(\bar\omega, \mathbb{R})\}_{j=0}^N$, $i = 1\dots 3$, by setting $a_{i,0} \equiv 0$ and further utilizing Lemma 2.2 and the linear maps $\bar a_i$ in there:

$$\begin{aligned} & a_{i,j} = \big(\bar a_i(H - \mathcal{E}_{j-1})\big)^{1/2} \qquad \text{for } j = 1\dots N,\ i = 1\dots 3, \\ & \text{where} \quad \mathcal{E}_j = \frac{1}{\lambda^2}\nabla a_{1,j} \otimes \nabla a_{1,j} + \frac{1}{\kappa^2}\nabla a_{2,j}\otimes\nabla a_{2,j} \quad \text{for } j = 1\dots N. \end{aligned} \tag{6.3}$$

This means that, with fixed unit vectors $\{\eta_i\}_{i=1}^3$:

$$\sum_{i=1}^3 a_{i,j}^2 \eta_i\otimes\eta_i = H - \mathcal{E}_{j-1} \qquad \text{for } j = 1\dots N,$$

and we also denote:

$$\mathcal{F}_j = H - \sum_{i=1}^3 a_{i,j}^2\eta_i\otimes\eta_i - \frac{1}{\lambda^2}\nabla a_{1,j}\otimes\nabla a_{1,j} - \frac{1}{\kappa^2}\nabla a_{2,j}\otimes\nabla a_{2,j} = \mathcal{E}_{j-1} - \mathcal{E}_j.$$

We will show that the above decomposition is well posed and that it yields:

$$a_i = a_{i,N} \quad \text{for } i = 1\dots 3 \quad \text{and} \quad \mathcal{F} = \mathcal{F}_N \tag{6.4}$$

with the desired properties (6.2). To this end, we will inductively show for all $j = 1\dots N$:

$$\frac12 \le a_{i,j}^2 \le \frac32 \qquad \text{and} \qquad \frac12 \le a_{i,j} \le \frac32 \quad \text{in } \bar\omega \quad \text{for } i = 1\dots 3, \tag{6.5$_1$}$$

$$\begin{aligned} & \|\nabla^{(m)} a_{i,j}^2\|_0 \le C\mu^m \quad \text{and} \quad \|\nabla^{(m)} a_{i,j}\|_0 \le C\mu^m \\ & \qquad\qquad\qquad\qquad \text{for } m = 1\dots M+N-j+1,\ i = 1\dots 3, \end{aligned} \tag{6.5$_2$}$$

$$\|\nabla^{(m)}\mathcal{F}_j\|_0 \le C\frac{\mu^m}{(\lambda/\mu)^{2j}} \qquad \text{for } m = 0\dots M+N-j, \tag{6.5$_3$}$$

with C that depends only on M, N.

2. We analyze the induction base at $j = 1$. By the first condition in (6.1), all $a_{i,1}^2 = \bar a_i(H)$ for $i = 1\dots 3$ are well-defined and $(6.5)_1$ holds by Lemma 2.2. Further:

$$\|\nabla^{(m)} a_{i,1}^2\|_0 \le C\|\nabla^{(m)} H\|_0 \le C\mu^m \qquad \text{for } m = 1\dots M+N,\ i = 1\dots 3,$$

by the second condition in (6.1). Observe that the first conditions in $(6.5)_1$, $(6.5)_2$ always imply the respective second conditions in there, because:

$$\begin{aligned} & |a_{i,j} - 1| = \frac{|a_{i,j}^2 - 1|}{a_{i,j}+1} \le |a_{i,j}^2 - 1| \le \frac12 \quad \text{in } \bar\omega,\ j = 1\dots N,\ i = 1\dots 3, \\ & \|\nabla^{(m)} a_{i,j}\|_0 \le C \sum_{p_1+2p_2+\dots mp_m = m} \Big\| a_{i,j}^{2(1/2 - p_1 - \dots p_m)} \prod_{t=1}^m \big|\nabla^{(t)} a_{i,j}^2\big|^{p_t}\Big\|_0 \le C\mu^m, \\ & \text{for } j = 1\dots N,\ m = 1\dots M+N-j+1,\ i = 1\dots 3, \end{aligned}$$

by an application of Faá di Bruno's formula. This proves, in particular, $(6.5)_1$, $(6.5)_2$ at $j = 1$, whereas $(6.5)_3$ holds by:

$$\begin{aligned} \|\nabla^{(m)}\mathcal{F}_1\|_0 & \le \frac{C}{\lambda^2}\sum_{p+q=m}\big(\|\nabla^{(p+1)}a_{1,1}\|_0\|\nabla^{(q+1)}a_{1,1}\|_0 + \|\nabla^{(p+1)}a_{2,1}\|_0\|\nabla^{(q+1)}a_{2,1}\|_0\big) \\ & \le C\frac{\mu^m}{(\lambda/\mu)^2}. \end{aligned}$$

This ends the justification of the induction base. We additionally note that applying Faá di Bruno's formula to the inverse rather than the square root, $(6.5)_2$ yields:

$$\begin{aligned}
&\|\nabla^{(m)}\Big(\frac{1}{a_{i,j}+a_{i,j-1}}\Big)\|_0 \\
&\leq C \sum_{p_1+2p_2+\dots mp_m=m} \Big\|(a_{i,j}+a_{i,j-1})^{-1-p_1-\dots p_m}\prod_{t=1}^m |\nabla^{(t)}(a_{i,j}+a_{i,j-1})|^{p_t}\Big\|_0 \\
&\leq C\mu^m \quad \text{for } j=1\dots N,\ m=0\dots M+N-j+1,\ i=1\dots 3.
\end{aligned} \tag{6.6}$$

3. We now exhibit the induction step. Assume that $(6.5)_1$-$(6.5)_3$ have been proved up to some counter $j=1\dots N-1$. To check that $\{a_{i,j+1}\}_{i=1}^3$ are well defined, we need that $\|(H-\mathcal{E}_j)-H_0\|_0 \leq r_0$, which indeed holds with $\underline{\sigma}$ large, because:

$$\begin{aligned}
\|(H-\mathcal{E}_j)-H_0\|_0 &\leq \|H-H_0\|_0 + \sum_{t=1}^j \|\mathcal{E}_t - \mathcal{E}_{t-1}\|_0 \leq \frac{r_0}{2} + \sum_{t=1}^j \|\mathcal{F}_t\|_0 \\
&\leq \frac{r_0}{2} + \sum_{t=1}^\infty \frac{1}{(\lambda/\mu)^{2t}} \leq \frac{r_0}{2} + \frac{C}{(\lambda/\mu)^2},
\end{aligned}$$

where we used that $1-\frac{1}{(\lambda/\mu)^2} \geq 2$ valid from $(\lambda/\mu)^2 \geq \underline{\sigma}^2 \geq 2$. This proves $(6.5)_1$ at the counter $j+1$. We similarly get $(6.5)_2$ from:

$$\begin{aligned}
\|\nabla^{(m)} a_{i,j+1}^2\|_0 &\leq C\big(\|\nabla^{(m)}H\|_0 + \|\nabla^{(m)}\mathcal{E}_j\|_0\big) \leq C\Big(\mu^m + \sum_{t=1}^j \|\nabla^{(m)}\mathcal{F}_t\|_0\Big) \\
&\leq C\Big(\mu^m + \frac{\mu^m}{(\lambda/\mu)^2)}\Big) \leq C\mu^m.
\end{aligned}$$

It remains to show $(6.5)_3$ at $j+1$. We write:

$$\begin{aligned}
&\mathcal{F}_{j+1} = \mathcal{E}_j - \mathcal{E}_{j+1} \\
&= \frac{1}{\bar\lambda^2}\big(\nabla a_{1,j}\otimes\nabla a_{1,j} - \nabla a_{1,j+1}\otimes\nabla a_{1,j+1}\big) + \frac{1}{\bar\kappa^2}\big(\nabla a_{2,j}\otimes\nabla a_{2,j} - \nabla a_{2,j+1}\otimes\nabla a_{2,j+1}\big) \\
&= \frac{1}{\bar\lambda^2}\big(\nabla(a_{1,j}-a_{1,j+1})\otimes\nabla a_{1,j} + \nabla a_{1,j+1}\otimes\nabla(a_{1,j}-a_{1,j+1})\big) \\
&\quad + \frac{1}{\bar\kappa^2}\big(\nabla(a_{2,j}-a_{2,j+1})\otimes\nabla a_{2,j} + \nabla a_{2,j+1}\otimes\nabla(a_{2,j}-a_{2,j+1})\big)
\end{aligned} \tag{6.7}$$

and recall that $\mathcal{F}_j = (H-\mathcal{E}_j)-(H-\mathcal{E}_{j-1}) = \sum_{i=1}^3 \big(a_{i,j+1}^2 - a_{i,j}^2\big)\eta_i\otimes\eta_i$. This yields:

$$a_{i,j+1}^2 - a_{i,j}^2 = \bar a_i(\mathcal{F}_j) \qquad \text{for } i=1\dots 3$$

and further, by $(6.5)_3$:

$$\begin{aligned}
&\|\nabla^{(m)}(a_{i,j+1}^2 - a_{i,j}^2)\|_0 \leq C\|\nabla^{(m)}\mathcal{F}_j\|_0 \leq C\frac{\mu^m}{(\lambda/\mu)^{2j}} \\
&\text{for all } m=0\dots M+N-j,\ i=1\dots 3.
\end{aligned}$$

In view of (6.6), we thus obtain:

$$\begin{aligned}&\|\nabla^{(m)}(a_{i,j+1}-a_{i,j})\|_0 = \Big\|\nabla^{(m)}\big(\frac{a_{i,j+1}^2-a_{i,j}^2}{a_{i,j+1}-a_{i,j}}\big)\Big\|_0\\ &\leq C\sum_{p+q=m}\|\nabla^{(p)}(a_{i,j+1}^2-a_{i,j}^2)\|_0\|\nabla^{(q)}\Big(\frac{1}{a_{i,j}+a_{i,j-1}}\Big)\|_0 \leq C\sum_{p+q=m}\frac{\mu^p}{(\lambda/\mu)^{2j}}\mu^q\\ &\leq C\frac{\mu^m}{(\lambda/\mu)^{2j}} \qquad \text{for all } m=0\dots M+N-j,\ i=1\dots 3.\end{aligned}$$

In conclusion, by (6.7):

$$\begin{aligned}\|\nabla^{(m)}\mathcal{F}_{j+1}\|_0 &\leq C\sum_{p+q=m}\frac{1}{\lambda^2}\frac{\mu^{p+1}}{(\lambda/\mu)^{2j}}\mu^{q+1}\\ &\leq C\frac{\mu^m}{(\lambda/\mu)^{2(j+1)}} \qquad \text{for all } m=0\dots M+N-(j+1).\end{aligned}$$

This ends the proof of all the inductive estimates $(6.5)_1$ - $(6.5)_3$ and therefore of the desired bounds (6.2) for (6.4). The proof is done. ■

7. Stage construction: the proof of Theorem 1.2

This section is devoted to our main construction, following its sketch in subsection 1.6.

Proof of Theorem 1.2

1. (Mollification and initial bounds) Fix $g, \underline{\gamma}, N, K$ as in the statement of the theorem, and δ, μ, σ, u satisfying $(1.10)_1$-$(1.10)_3$. All the bounds in the course of the proof below will be valid under the assumption that $\underline{\delta}$ is sufficiently small and $\underline{\sigma}$ sufficiently large, in function of $\underline{\gamma}, \omega, g, N, K$. We define the following initial parameters and the mollified fields:

$$\begin{aligned}&\mu_0=\mu, \quad \delta_0=\delta, \quad l=\frac{1}{C\mu},\\ &u_0=u*\phi_l\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^4), \quad g_0=g*\phi_l\in\mathcal{C}^\infty(\bar\omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}}),\end{aligned}$$

where $C>1$ above is an independent constant, assuring that:

$$\|(\nabla u_0)^T\nabla u_0-\big((\nabla u)^T\nabla u\big)*\phi_l\|_0\leq C_0 l^2\|\nabla^2 u\|_0^2\leq\frac{C_0}{C^2\mu^2}\delta\mu^2\leq\frac{r_0}{12}\delta_0, \tag{7.1}$$

which follows from the second assumption in $(1.10)_3$ and by applying $(2.1)_3$ to $f=g=\nabla u$. Further application of Lemma 2.1 yields:

$$\|g_0-g\|_0\leq l^{r+\beta}\|g\|_{r,\beta}\leq\frac{\|g\|_{r,\beta}}{\mu_0^{r+\beta}}, \tag*{$(7.2)_1$}$$

$$\|u_0-u\|_1\leq l\|u\|_2\leq\delta_0^{1/2}, \tag*{$(7.2)_2$}$$

$$\|\nabla^{(m)}\nabla^{(2)}u_0\|_0\leq\frac{C}{l^m}\|\nabla^2 u\|_0\leq C\mu_0^{m+1}\delta_0^{1/2} \quad \text{for all } m=0\dots(3N+6)K. \tag*{$(7.2)_3$}$$

In particular, u_0 is an immersion, because by Lemma 3.1:

$$\begin{aligned}\|(\nabla u_0)^T\nabla u_0-(\nabla u)^T\nabla u\|_0 &\leq\|\nabla(u_0-u)\|_0\big(\|\nabla u_0\|_0+\|\nabla u\|_0\big)\\ &\leq\|\nabla(u_0-u)\|_0\big(2\|\nabla u\|_0+\|\nabla(u_0-u)\|_0\big)\leq\delta_0^{1/2}\big(2(4\gamma)^{1/2}+\delta_0^{1/2}\big),\end{aligned}$$

so when $\delta_0 \leq \underline{\delta}$ is sufficiently small, we obtain:

$$\frac{1}{3\underline{\gamma}}\mathrm{Id}_2 \leq (\nabla u_0)^T\nabla u_0 \leq 3\underline{\gamma}\mathrm{Id}_2 \quad \text{in } \bar\omega. \tag{7.3}$$

Consequently, we may apply Lemma 3.2 to u_0 and μ_0 with $A = \delta_0^{1/2}$ and obtain the following bound for the tangent field $T_{u_0} = (\nabla u_0)\big((\nabla u_0)^T\nabla u_0\big)^{-1}$:

$$\|T_{u_0}\|_0 \leq C, \qquad \|\nabla^{(m)}T_{u_0}\|_0 \leq C\delta_0^{1/2}\mu_0^m \quad \text{for all } m = 1\dots(3N+6)K-1. \tag{7.4}$$

Also, Lemma 3.3 implies existence of a normal frame $E^1_{u_0}, E^2_{u_0} \in \mathcal{C}^{(3N+6)K+1}(\bar\omega,\mathbb{R}^4)$, namely:

$$(\nabla u_0)^T E^i_{u_0} = 0, \quad |E^i_{u_0}| = 1 \quad \text{for } i=1,2 \quad \text{and} \quad \langle E^1_{u_0}, E^2_{u_0}\rangle = 0 \quad \text{in } \bar\omega,$$

obeying the bounds:

$$\begin{split} & \|\nabla^{(m)}E^i_{u_0}\|_0 \leq C(1+\|\nabla u_0\|_m) \leq C\delta_0^{1/2}\mu_0^m \\ & \text{for all } m = 1\dots(3N+6)K+1,\ i=1,2, \end{split} \tag{7.5}$$

where we used the second condition in $(1.10)_1$ and $(7.2)_3$. In particular, we get:

$$\begin{split} \|\nabla^{(m)}\big((\nabla u_0)^T\nabla E^i_{u_0}\big)\|_0 & \leq C\sum_{p+q=m}\|\nabla^{(p+1)}u_0\|_0\|\nabla^{(q+1)}E^i_{u_0}\|_0 \\ & \leq C\sum_{p+q=m}\mu_0^p\mu_0^{q+1}\delta_0^{1/2} \leq C\delta_0^{1/2}\mu_0^{m+1} \quad \text{for all } m=0\dots(3N+6)K,\ i=1,2. \end{split} \tag{7.6}$$

Finally, writing:

$$\mathcal{D}(g_0-\delta_0H_0,u_0) = \mathcal{D}(g-\delta_0H_0,u)*\phi_l + \Big((\nabla u_0)^T\nabla u_0 - \big((\nabla u)^T\nabla u\big)*\phi_l\Big),$$

we get in view of (7.1), $(1.10)_3$ and Lemma 2.1:

$$\begin{split} \|\mathcal{D}(g_0-\delta_0H_0,u_0)\|_0 & \leq \|\mathcal{D}(g-\delta_0H_0,u)\|_0 + \frac{r_0}{12}\delta_0 \leq \frac{r_0}{4}\delta_0 + \frac{r_0}{12}\delta_0 = \frac{r_0}{3}\delta_0, \\ \|\nabla^{(m)}\mathcal{D}(g_0-\delta_0H_0,u_0)\|_0 & \leq \frac{C}{l^m}\|\mathcal{D}(g-\delta_0H_0,u)\|_0 + \frac{C}{l^{m-2}}\|\nabla^2 u\|_0^2 \\ & \leq C\delta_0\mu_0^m \quad \text{for all } m=1\dots(3N+6)K+1. \end{split} \tag{7.7}$$

2. (Setting up the induction) In the course of the proof, we will define the immersions $\{u_k \in \mathcal{C}^{2+(3N+6)(K-k)}(\bar\omega,\mathbb{R}^4)\}_{k=1}^K$ and their normal vectors $E^1_{u_k}, E^2_{u_k} \in \mathcal{C}^{1+(3N+6)(K-k)}(\bar\omega,\mathbb{R}^4)$:

$$(\nabla u_k)^T E^i_{u_k} = 0, \quad |E^i_{u_k}| = 1 \quad \text{for } i=1,2 \quad \text{and} \quad \langle E^1_{u_k}, E^2_{u_k}\rangle = 0 \quad \text{in } \bar\omega,$$

with respect to the increasing progression of frequencies $\{\mu_k\}_{k=1}^N$ and the decreasing progression of interpolating defect magnitudes $\{\delta_k\}_{k=1}^K$ given in:

$$\mu_k = \mu_0\sigma^{N+2}\sigma^{(\frac{N}{2}+2)(k-1)} = \mu_0\sigma^{2k+\frac{N}{2}(k+1)} \quad \text{and} \quad \delta_k = \frac{\delta_0}{\sigma^{kN}} \quad \text{for } k=1\dots K. \tag{7.8}$$

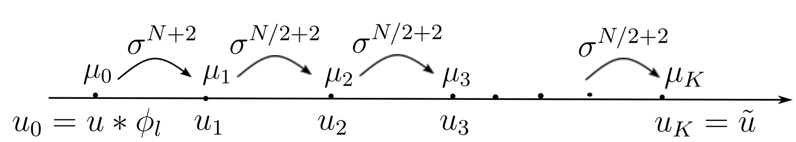

FIGURE 1. Progression of the principal frequencies in (7.8).

We note in passing that $\{\delta_k^{1/2}\mu_{k+1}\}_{k=0}^{K-1}$ is an increasing K-tuple of numbers bigger than 1, as:

$$\frac{\delta_k^{1/2}\mu_{k+1}}{\delta_{k-1}^{1/2}\mu_k} \geq \frac{\sigma^{\frac{N}{2}+2}}{\sigma^{\frac{N}{2}}} = \sigma^2 \geq 1 \quad \text{for } k = 1 \dots K-1. \tag{7.9}$$

We will inductively prove the satisfaction of the following properties, valid for all $k = 1 \dots K$:

$$\frac{1}{3\underline{\gamma}}\mathrm{Id}_2 \leq (\nabla u_k)^T \nabla u_k \leq 3\underline{\gamma}\mathrm{Id}_2 \quad \text{in } \bar{\omega}, \tag*{(7.10)$_1$}$$

$$\|u_k - u_{k-1}\|_1 \leq C\delta_{k-1}^{1/2}, \tag*{(7.10)$_2$}$$

$$\|\nabla^{(m+1)}(u_k - u_{k-1})\|_0 \leq C\delta_{k-1}^{1/2}\mu_k^m \quad \text{for all } m = 0\dots 1+(3N+6)(K-k), \tag*{(7.10)$_3$}$$

$$\|\nabla^{(m)}(E^i_{u_k} - E^i_{u_{k-1}})\|_0 \leq C\delta_{k-1}^{1/2}\mu_k^m \quad \text{for all } m = 0\dots 1+(3N+6)(K-k),\ i = 1,2, \tag*{(7.10)$_4$}$$

$$\|\nabla^{(m)}\mathcal{D}(g_0 - \delta_k H_0, u_k)\|_0 \leq \frac{\delta_{k-1}}{\sigma^{N+1/2}}\mu_k^m \quad \text{for all } m = 0\dots 1+(3N+6)(K-k), \tag*{(7.10)$_5$}$$

together with the following specific bounds on the components of the second derivatives:

$$\|\nabla^{(m)}\langle\partial_{ij}u_k, E^1_{u_k}\rangle\|_0 \leq C\frac{\delta_{k-1}^{1/2}}{\sigma^{N/2}}\mu_k^{m+1} \quad \text{for } m = 0\dots(3N+6)(K-k),\ i,j = 1\dots 2, \tag*{(7.11)$_1$}$$

$$\|\nabla^{(m)}\langle\partial_{11}u_k, E^2_{u_k}\rangle\|_0 \leq C\frac{\delta_{k-1}^{1/2}}{\sigma^{N}}\mu_k^{m+1} \quad \text{for } m = 0\dots(3N+6)(K-k), \tag*{(7.11)$_2$}$$

$$\|\nabla^{(m)}\langle\partial_{12}u_k, E^2_{u_k}\rangle\|_0 \leq C\frac{\delta_{k-1}^{1/2}}{\sigma^{N/2}}\mu_k^{m+1} \quad \text{for } m = 0\dots(3N+6)(K-k), \tag*{(7.11)$_3$}$$

$$\|\nabla^{(m)}\langle\partial_{22}u_k, E^2_{u_k}\rangle\|_0 \leq C\delta_{k-1}^{1/2}\mu_k^{m+1} \quad \text{for } m = 0\dots(3N+6)(K-k). \tag*{(7.11)$_4$}$$

The above will yield the desired estimates by setting:

$$\tilde{u} = u_K \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^4)$$

Indeed, the bounds in $(1.11)_1$ follow from $(7.2)_2$, $(7.10)_2$ and then from $(7.2)_3$, $(7.10)_3$, (7.9):

$$\begin{aligned}
\|u_K - u\|_1 &\leq \|u_0 - u\|_1 + \sum_{k=0}^{K-1}\|u_{k+1} - u_k\|_1 \leq C\sum_{k=0}^{K-1}\delta_k^{1/2} \leq C\delta_0^{1/2}, \\
\|u_K\|_2 &\leq \|u_0\|_2 + \sum_{k=0}^{K-1}\|u_{k+1} - u_k\|_2 \leq C\delta_0^{1/2}\mu_0 + C\sum_{k=0}^{K-1}\delta_k^{1/2}\mu_{k+1} \leq C\delta_{K-1}^{1/2}\mu_K \\
&= C\mu_0\delta_0^{1/2}\sigma^{2K+\frac{N}{2}(K+1)-\frac{N}{2}(K-1)} = C\mu_0\delta_0^{1/2}\sigma^{2K+N},
\end{aligned}$$

in view of the definition of frequencies and defect measures in (7.8), whereas $(1.11)_2$ follows from $(7.2)_1$ and $(7.10)_5$, by setting $\sigma \geq \underline{\sigma}$ sufficiently large:

$$\begin{aligned}
&\|\mathcal{D}(g - \delta_K H_0, u_K)\|_0 \leq \|g_0 - g\|_0 + \|\mathcal{D}(g_0 - \delta_K H_0, u_K)\|_0 \\
&\leq \frac{\|g\|_{r+\beta}}{\mu_0^{r+\beta}} + \frac{\delta_{K-1}}{\sigma^{N+1/2}} = \frac{\|g\|_{r+\beta}}{\mu_0^{r+\beta}} + \frac{1}{\sigma^{N(K-1)}}\frac{\delta_0}{\sigma^{N+1/2}} = \frac{\|g\|_{r+\beta}}{\mu_0^{r+\beta}} + \frac{r_0}{5}\frac{\delta_0}{\sigma^{NK}}.
\end{aligned}$$

3. (Decomposition of the k-th defect) We now fix the counter:

$$k = 0\dots K-1$$

and carry out the construction of u_{k+1}. We either have $k=0$ and the estimates $(7.2)_2$-(7.7) in step 1, or we have $k=1\dots K-1$ and then we assume $(7.10)_1$-$(7.11)_4$. We proceed by applying the Källen decomposition in Theorem 6.1 to the following parameters:

$$H=\frac{\mathcal{D}(g_0-\delta_{k+1}H_0,u_k)}{\delta_k},\quad \mu=\mu_k,\quad \lambda=\mu_k\sigma,\quad \kappa=\mu_k\sigma^2.$$

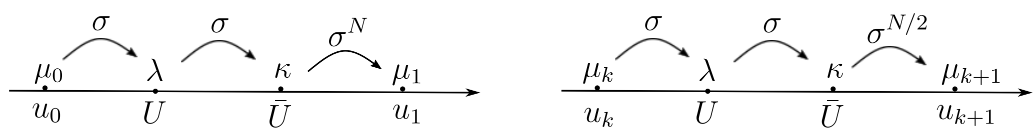

Figure 2. Progression of frequencies and the corresponding intermediate fields $U,\bar U$ in the three corrugation modification from u_k to u_{k+1}. Note the distinction of cases $k=0$ and $k=1\dots K-1$.

We need to verify the two assumptions in (6.1). For the first one, we write:

$$\|H-H_0\|_0=\frac{\|\mathcal{D}(g_0-\delta_{k+1}H_0,u_k)-\delta_kH_0\|_0}{\delta_k}\leq\frac{\|\mathcal{D}(g_0-\delta_kH_0,u_k)\|_0}{\delta_k}+\frac{\delta_{k+1}}{\delta_k}|H_0|\leq\frac{r_0}{2},$$

which is valid provided that:

$$\|\mathcal{D}(g_0-\delta_kH_0,u_k)\|_0\leq\frac{r_0}{3}\delta_k\quad\text{ and }\quad\frac{\delta_{k+1}}{\delta_k}|H_0|\leq\frac{r_0}{6}.$$

It is clear that the second bound above holds by (7.8) if we set $\sigma\geq\underline{\sigma}$ sufficiently large. The first bound holds at $k=0$ by (7.7), whereas for $k=1\dots K-1$ it holds by $(7.10)_5$ for large $\underline{\sigma}$:

$$\|\mathcal{D}(g_0-\delta_kH_0,u_k)\|_0\leq\frac{\delta_{k-1}}{\sigma^{N+1/2}}=\frac{\delta_k}{\sigma^{1/2}}\leq\frac{r_0}{4}\delta_k.$$

We now verify the second assumption in (6.1):

$$\|\nabla^{(m)}H\|_0=\frac{\|\nabla^{(m)}\mathcal{D}(g_0-\delta_kH_0,u_k\|_0)}{\delta_k}\leq C\mu_k^m\quad\text{ for all } m=1\dots1+(3N+6)(K-k),$$

following for $k=0$ from (7.7) and for $k=1\dots K-1$ from $(7.10)_5$. Assuring that $\underline{\sigma}$ is larger than $\underline{\sigma}$ required in Theorem 6.1, it yields the decomposition:

$$\mathcal{D}(g_0-\delta_{k+1}H_0,u_k)=\sum_{i=1}^3a_i^2\eta_i\otimes\eta_i+\frac{1}{\lambda^2}\nabla a_1\otimes\nabla a_1+\frac{1}{\kappa^2}\nabla a_2\otimes\nabla a_2+\mathcal{F},\tag{7.12}$$

with the bounds resulting from (6.2):

$$\begin{aligned}&\frac12\delta_k\leq a_i^2\leq\frac32\delta_k\quad\text{ and }\quad\frac12\delta_k^{1/2}\leq a_i\leq\frac32\delta_k^{1/2}\quad\text{ in }\bar\omega\quad\text{ for } i=1\dots3,\\&\|\nabla^{(m)}a_i^2\|_0\leq C\delta_k\mu_k^m\quad\text{ and }\quad\|\nabla^{(m)}a_i\|_0\leq C\delta_k^{1/2}\mu_k^m\\&\qquad\qquad\qquad\text{ for }\ m=1\dots2+(3N+6)(K-k)-N,\ i=1\dots3,\\&\|\nabla^{(m)}\mathcal{F}\|_0\leq C\frac{\delta_k}{\sigma^{2N}}\mu_k^m\quad\text{ for }\ m=0\dots1+(3N+6)(K-k)-N.\end{aligned}\tag{7.13}$$

We end this step by gathering bounds including $(7.2)_3$, (7.4) and (7.5):

$$\|\nabla u_k\|_0 \leq (6\underline{\gamma})^{1/2} \tag{7.14$_1$}$$

$$\|\nabla^{(m)}\nabla^{(2)}u_k\|_0 \leq C\delta_{k-1}^{1/2}\mu_k^{m+1} = C\delta_k^{1/2}\mu_k^{m+1}\sigma^{N/2} \quad \text{for all } m = 0\ldots(3N+6)(K-k), \tag{7.14$_2$}$$

$$\|T_{u_k}\|_0 \leq C \quad \text{and} \quad \|\nabla^{(m)}T_{u_k}\|_0 \leq C\mu_k^m\delta_k^{1/2}\sigma^{N/2}$$
$$\text{for all } m = 1\ldots 1+(3N+6)(K-k), \tag{7.14$_3$}$$

$$\|\nabla^{(m)}E_{u_k}^i\|_0 \leq C\mu_k^m\delta_k^{1/2}\sigma^{N/2} \quad \text{for all } m = 1\ldots 1+(3N+6)(K-k),\ i=1,2. \tag{7.14$_4$}$$

Indeed, $(7.14)_1$ follows by Lemma 3.1 and (7.3) and $(7.10)_1$. Then $(7.14)_2$ follows by $(7.10)_3$, (7.9) and (7.8) when $k \geq 1$, whereas its final bound follows directly by $(7.2)_3$ for $k = 0$. Similarly, $(7.14)_3$ follows from $(7.10)_4$. Applying Lemma 3.2 with $A = \delta_k^{1/2}\sigma^{N/2} \leq \delta_0^{1/2}\sigma^{N/2} \leq 1$ in view of the last assumption in $(1.10)_1$, yields $(7.14)_3$.

4. (The first corrugation) We define the intermediary field $U \in \mathcal{C}^{1+(3N+6)(K-k)-2N}(\bar{\omega}, \mathbb{R}^4)$:

$$U = u_k + \frac{\Gamma(\lambda x_1)}{\lambda}a_1E_{u_k}^1 + T_{u_k}\Big(\frac{\bar{\Gamma}(\lambda x_1)}{\lambda}a_1^2e_1 + W\Big), \tag{7.15}$$

in accordance with Lemma 2.5, where Γ, $\bar{\Gamma}$ are the oscillatory profiles in there, namely with:

$$\Gamma(t) = \sqrt{2}\sin t, \qquad \bar{\Gamma}(t) = -\frac{1}{4}\sin(2t) \quad \text{and} \quad \bar{\bar{\Gamma}}(t) \doteq \Gamma(t)^2 - 1 = -\cos(2t). \tag{7.16}$$

The oscillation direction in (7.15) is set to $\eta = \eta_1 = e_1$, the frequency is:

$$\lambda = \mu_k\sigma,$$

and the tangential correction W is given recalling the definition (5.3) and the recursion (5.2):

$$\begin{aligned}-2W =& \sum_{i=0}^N(-1)^i\frac{\bar{\bar{\Gamma}}_{i+1}(\lambda x_1)}{\lambda^{i+3}}L_i^{\eta_1}(S_1) + \sum_{i=0}^N(-1)^i\frac{(\Gamma'\Gamma)_{i+1}(\lambda x_1)}{\lambda^{i+2}}L_i^{\eta_1}(S_2)\\ &+ \sum_{i=0}^N(-1)^i\frac{\Gamma_{i+1}(\lambda x_1)}{\lambda^{i+2}}L_i^{\eta_1}(S_3) + \sum_{i=0}^N(-1)^i\frac{\bar{\Gamma}_{i+1}(\lambda x_1)}{\lambda^{i+2}}L_i^{\eta_1}(S_4).\end{aligned}$$

Above, the fields $\{S_i\}_{i=1}^4$ are as in Lemma 2.5, namely:

$$\begin{aligned}&S_1 = \nabla a_1 \otimes \nabla a_1, && S_2 = 2a_1\mathrm{sym}(\nabla a_1 \otimes e_1),\\ &S_3 = 2a_1\mathrm{sym}\big((\nabla u_k)^T\nabla E_{u_k}^1\big), && S_4 = 2\,\mathrm{sym}\big((\nabla u_k)^T\nabla(a_1^2T_{u_k}e_1)\big).\end{aligned} \tag{7.17}$$

By (5.1) in Lemma 5.1 we see that:

$$-2\,\mathrm{sym}\nabla W = \frac{\bar{\bar{\Gamma}}(\lambda x_1)}{\lambda^2}S_1 + \frac{(\Gamma'\Gamma)(\lambda x_1)}{\lambda}S_2 + \frac{\Gamma(\lambda x_1)}{\lambda}S_3 + \frac{\bar{\Gamma}(\lambda x_1)}{\lambda}S_4 - \mathcal{G} - Ge_2\otimes e_2, \tag{7.18}$$

with the following formulas:

$$\begin{aligned}\mathcal{G} = {} & (-1)^{N+1}\frac{\bar{\bar{\Gamma}}_{N+1}(\lambda x_1)}{\lambda^{N+3}}\mathrm{sym}\nabla L_N^{\eta_1}(S_1) + (-1)^{N+1}\frac{(\Gamma'\Gamma)_{N+1}(\lambda x_1)}{\lambda^{N+2}}\mathrm{sym}\nabla L_N^{\eta_1}(S_2)\\ & + (-1)^{N+1}\frac{\Gamma_{N+1}(\lambda x_1)}{\lambda^{N+2}}\mathrm{sym}\nabla L_N^{\eta_1}(S_3) + (-1)^{N+1}\frac{\bar{\Gamma}_{N+1}(\lambda x_1)}{\lambda^{N+2}}\mathrm{sym}\nabla L_N^{\eta_1}(S_4),\\ G = {} & \sum_{i=0}^{N}(-1)^i\frac{\bar{\bar{\Gamma}}_i(\lambda x_1)}{\lambda^{i+2}}P_i^{\eta_1}(S_1) + \sum_{i=0}^{N}(-1)^i\frac{(\Gamma'\Gamma)_i(\lambda x_1)}{\lambda^{i+1}}P_i^{\eta_1}(S_2)\\ & + \sum_{i=0}^{N}(-1)^i\frac{\Gamma_i(\lambda x_1)}{\lambda^{i+1}}P_i^{\eta_1}(S_3) + \sum_{i=0}^{N}(-1)^i\frac{\bar{\Gamma}_i(\lambda x_1)}{\lambda^{i+1}}P_i^{\eta_1}(S_4).\end{aligned}$$

By (7.12), (2.2) and (7.18) we now obtain:

$$\begin{aligned}\mathcal{D}(g_0 - \delta_{k+1}H_0, U) & = \mathcal{D}(g_0 - \delta_{k+1}H_0, u_k) - \Big((\nabla U)^T\nabla U - (\nabla u_k)^T\nabla u_k\Big)\\ & = \sum_{i=2}^{3} a_i^2\eta_i\otimes\eta_i + \frac{1}{\kappa^2}\nabla a_2\otimes\nabla a_2 + \mathcal{F} - \mathcal{R}_1 - \mathcal{R}_2 - \mathcal{G} - Ge_2\otimes e_2,\end{aligned}\tag{7.19}$$

where the primary and the W-related error terms $\mathcal{R}_1$ and $\mathcal{R}_2$ are as in Lemma 2.5.

5. (Bounds on the errors in first corrugation defect) We now estimate all terms in the decomposition (7.19), together with their derivatives. We first observe, by (7.13), $(7.14)_3$:

$$\begin{aligned}\|\nabla^{(m+1)}(a_1^2T_{u_k}e_1)\|_0 \leq C\sum_{p+q=m+1}\delta_k\mu_k^p\mu_k^q = C\delta_k\mu_k^{m+1}\\ \text{for all } m = 0\dots 1 + (3N+6)(K-k) - N.\end{aligned}\tag{7.20}$$

Consequently, recalling additionally $(7.14)_4$, we get that each term in $\mathcal{R}_1$ is bounded by:

$$\|\nabla^{(m)}\mathcal{R}_1\|_0 \leq C\delta_k^{3/2}\lambda^m\sigma^{N/2} \leq \frac{\delta_k}{\sigma^{N+1}}\lambda^m \quad \text{for all } m = 0\dots 1 + (3N+6)(K-k) - N. \tag{7.21}$$

where the worst term, responsible for the first bound above is $\frac{\Gamma^2}{\lambda^2}a_1^2(\nabla E_{u_k}^1)^T\nabla E_{u_k}^1$, and where the final inequality is due to $C\sigma^{3N/2+1}\delta_k^{1/2} \leq \sigma^{3(N+1)/2}\delta_0^{1/2} \leq 1$, from the last assumption in $(1.10)_1$. Before treating $\mathcal{R}_2$, we need to find the bounds on W. To this end, we estimate:

$$\begin{aligned}& \|\nabla^{(m)}S_1\|_0 \leq C\delta_k\mu_k^{m+2},\\ & \|\nabla^{(m)}S_2\|_0 \leq C\delta_k\mu_k^{m+1},\\ & \|\nabla^{(m)}S_3\|_0 \leq C\sum_{p+q=m}\|\nabla^{(p)}a_1\|_0\|\nabla^{(q)}((\nabla u_k)^T\nabla E_{u_k}^1)\|_0\\ & \qquad\qquad \leq C\sum_{p+q=m}\delta_k^{1/2}\mu_k^p\delta_k^{1/2}\mu_k^{q+1} \leq C\delta_k\mu_k^{m+1},\\ & \|\nabla^{(m)}S_4\|_0 \leq C\delta_k\mu_k^{m+1} \quad \text{for all } m = 0\dots 1 + (3N+6)(K-k) - N,\end{aligned}\tag{7.22}$$

where the first two bounds result from (7.13), the third bound from (7.6) and $(7.11)_1$, and the fourth from $(7.14)_2$ and (7.20). We now read from (5.3) that:

$$\|\nabla^{(m)}L_i^{\eta_1}(S)\|_0 \leq C_{m+i}\|\nabla^{(m+i)}S\|_0 \quad \text{for all } m, i \geq 0, \tag{7.23}$$

recall (7.17), and observe that all the primitives of the functions $\bar{\bar{\Gamma}}, (\Gamma'\Gamma), \Gamma, \bar\Gamma$ used in the definition of W are bounded, being periodic with mean zero on the period. This yields that:

$$\begin{aligned} \|\nabla^{(m)}W\|_0 &\leq C\sum_{i=0}^N \sum_{p+q=m} \Big(\lambda^{p-i-3}\delta_k\mu_k^{q+i+2} + \lambda^{p-i-2}\delta_k\mu_k^{q+i+1}\Big) \\ &\leq C\delta_k\lambda^{m-1} \qquad \text{for all } m = 0\ldots 1+(3N+6)(K-k)-2N. \end{aligned} \tag{7.24}$$

We are now ready to bound the terms in $\mathcal{R}_2$. Observe first that, by (7.13), $(7.14)_3$, $(7.14)_4$:

$$\begin{aligned} &\|\nabla^{(m+1)}(T_{u_k}W)\|_0 \leq C\delta_k\lambda^m && \text{for } m = 0\ldots(3N+6)(K-k)-2N, \\ &\|\nabla^{(m+1)}(a_1E^1_{u_k})\|_0 \leq C\delta_k^{1/2}\mu_k^{m+1} && \text{for } m = 0\ldots 1+(3N+6)(K-k)-N, \end{aligned}$$

resulting in all the terms in $\mathcal{R}_2$ obeying:

$$\|\nabla^{(m)}\mathcal{R}_2\|_0 \leq C\delta_k^{3/2}\sigma^{N/2}\lambda^m \leq \frac{\delta_k}{\sigma^{N+1}}\lambda^m \qquad \text{for all } m = 0\ldots(3N+6)(K-k)-2N, \tag{7.25}$$

where the worst term, responsible for the first bound above is $2\mathrm{sym}((\nabla u_k)^T[(\partial_1T_{u_k})W, (\partial_2T_{u_k})W])$, and the final inequality is due to having $C\sigma^{3N/2+1}\delta_k^{1/2} \leq \sigma^{3(N+1)/2}\delta_0^{1/2} \leq 1$ according to the last assumption in $(1.10)_1$ and for $\sigma \geq \underline{\sigma}$ large. For estimating $\mathcal{G}$, we again use (7.23), (7.22):

$$\begin{aligned} \|\nabla^{(m)}\mathcal{G}\|_0 &\leq C\sum_{p+q=m} \Big(\lambda^{p-N-3}\delta_k\mu_k^{q+N+3} + \lambda^{p-N-2}\delta_k\mu_k^{q+N+2}\Big) \\ &\leq C\frac{\delta_k}{(\lambda/\mu_k)^{N+2}}\lambda^m \leq \frac{\delta_k}{\sigma^{N+1}}\lambda^m \qquad \text{for all } m = 0\ldots(3N+6)(K-k)-2N. \end{aligned} \tag{7.26}$$

In conclusion, the bounds (7.13), (7.21), (7.25), (7.26) in (7.19) yield:

$$\begin{aligned} &\mathcal{D}(g_0 - \delta_{k+1}H_0, U) = a_2^2\eta_2\otimes\eta_2 + \frac{1}{\kappa^2}\nabla a_2\otimes\nabla a_2 + (a_3^2 - G)e_2\otimes e_2 + \mathcal{E} \\ &\text{where } \|\nabla^{(m)}\mathcal{E}\|_0 = \|\nabla^{(m)}(\mathcal{F} - \mathcal{R}_1 - \mathcal{R}_2 - \mathcal{G})\|_0 \leq 4\frac{\delta_k}{\sigma^{N+1}}\lambda^m \\ &\text{for all } m = 0\ldots(3N+6)(K-k)-2N. \end{aligned} \tag{7.27}$$

We conclude this step by estimating the derivatives of G. Reading from (5.3) that:

$$\|\nabla^{(m)}P_i^{\eta_1}(S)\|_0 \leq C_{m+i}\|\nabla^{(m+i)}S\|_0 \quad \text{for all } m, i \geq 0,$$

we get, in view of (7.22) and utilizing, as usual, that $\lambda/\mu_k = \sigma$:

$$\begin{aligned} \|\nabla^{(m)}G\|_0 &\leq C\sum_{i=0}^N \sum_{p+q=m} \Big(\lambda^{p-i-2}\delta_k\mu_k^{q+i+2} + \lambda^{p-i-1}\delta_k\mu_k^{q+i+1}\Big) \\ &\leq C\frac{\delta_k}{\sigma}\lambda^m \qquad \text{for all } m = 0\ldots 1+(3N+6)(K-k)-2N. \end{aligned} \tag{7.28}$$

6. (Propagation of bounds in the first corrugation) A rough bound without specifying to components, and using only (7.13), $(7.14)_3$, $(7.14)_4$, (7.24) yields the following:

$$\begin{aligned} &\|\nabla^{(m)}(U - u_k)\|_0 \leq C\sum_{p+q+t=m} \lambda^{p-1}\|\nabla^{(q)}a_1\|_0\|\nabla^{(t)}E^1_{u_k}\|_0 \\ &+ C\sum_{p+q+t=m} \lambda^{p-1}\|\nabla^{(q)}a_1^2\|_0\|\nabla^{(t)}T_{u_k}\|_0 + C\sum_{p+q=m}\|\nabla^{(p)}T_{u_k}\|_0\|\nabla^{(q)}W\|_0 \leq C\delta_k^{1/2}\lambda^{m-1}. \end{aligned} \tag{7.29}$$

Now, we can use Lemma 3.4 to $u = u_k$, $v = U$, $\mu = \lambda$, $A = \delta_k^{1/2}$ because:

$$\|\nabla U - \nabla u_k\|_0 \leq C\delta_k^{1/2} \leq C\underline{\delta}^{1/2} \leq \rho,$$

in virtue of (7.29) and upon taking $\underline{\delta}$ sufficiently small. Recalling further $(7.10)_1$, $(7.14)_2$, $(7.14)_4$, we obtain existence of the normal frame $E_U^1, E_U^2 \in \mathcal{C}^{(3N+6)(K-k)-2N}(\bar\omega, \mathbb{R}^4)$ in:

$$(\nabla U)^T E_U^i = 0, \quad |E_U^1| = 1 \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad \langle E_U^1, E_U^2\rangle = 0 \quad \text{in } \bar\omega,$$

satisfying the the second bound below, while the first bound is included in (7.29):

$$\begin{aligned} &\|\nabla^{(m+1)}(U - u_k)\|_0 \leq C\delta_k^{1/2}\lambda^m \quad \text{ and } \quad \|\nabla^{(m)}(E_U^i - E_{u_k}^i)\|_0 \leq C\delta_k^{1/2}\lambda^m \\ &\text{for all } m = 0 \ldots (3N+6)(K-k) - 2N, \ i = 1, 2. \end{aligned} \tag{7.30}$$

We also get, recalling $(7.2)_2$ and the induction assumptions $(7.10)_2$:

$$\|U - u\|_1 \leq \|U - u_k\|_1 + \|u_0 - u\|_0 + \sum_{i=0}^{k-1} \|u_{i+1} - u_i\|_0 \leq C\delta_0^{1/2}, \tag{7.31}$$

leading, as in step 1 and the proof of (7.3), to:

$$\frac{1}{3\underline{\gamma}}\mathrm{Id}_2 \leq (\nabla U)^T \nabla U \leq 3\underline{\gamma}\mathrm{Id}_2 \quad \text{ in } \bar\omega. \tag{7.32}$$

We now gather estimates similar to those in $(7.14)_1$ - $(7.14)_4$. Indeed, by the above and Lemma 3.1, we get the first estimate, while the second and the fourth follow from (7.30):

$$\|\nabla U\|_0 \leq (6\underline{\gamma})^{1/2} \tag*{$(7.33)_1$}$$

$$\|\nabla^{(m)}\nabla^{(2)} U\|_0 \leq C\delta_k^{1/2}\lambda^{m+1}\sigma^{N/2} \quad \text{for all } m = 0 \ldots (3N+6)(K-k) - 2N - 1, \tag*{$(7.33)_2$}$$

$$\begin{aligned} &\|T_U\|_0 \leq C \quad \text{ and } \quad \|\nabla^{(m)} T_U\|_0 \leq C\delta_k^{1/2}\lambda^m\sigma^{N/2} \\ &\qquad\qquad\qquad\qquad \text{for all } m = 1 \ldots (3N+6)(K-k) - 2N, \end{aligned} \tag*{$(7.33)_3$}$$

$$\|\nabla^{(m)} E_U^i\|_0 \leq C\delta_k^{1/2}\lambda^m\sigma^{N/2} \quad \text{for all } m = 1 \ldots (3N+6)(K-k) - 2N, \ i = 1, 2, \tag*{$(7.33)_4$}$$

and the estimates $(7.33)_3$ follow from Lemma 3.2. In the remaining part of this step, we refine the first bound in (7.30) to the components of $\nabla^2 U$. For any $i, j, s = 1 \ldots 2$ we write:

$$\langle \partial_{ij} U, E_U^s\rangle = \langle \partial_{ij} u_k, E_{u_k}^s\rangle + \langle \partial_{ij}(U - u_k), E_{u_k}^s\rangle + \langle \partial_{ij} U, (E_U^s - E_{u_k}^s)\rangle. \tag{7.34}$$

The first term in the right hand side above obeys the bounds in $(7.11)_1$-$(7.11)_2$ when $k \geq 1$ and (7.6) when $k = 0$. For the third term, we use (7.30) and $(7.33)_2$, $(7.33)_4$ and get:

$$\begin{aligned} &\|\nabla^{(m)}\langle \partial_{ij} U, (E_U^s - E_{u_k}^s)\rangle\|_0 \leq C\delta_k\lambda^{m+1}\sigma^{N/2} \\ &\text{for all } m = 0 \ldots (3N+6)(K-k) - 2N - 1, \ i, j, s = 1 \ldots 2. \end{aligned}$$

We now estimate the second term in the right hand side of (7.34), where by (7.30):

$$\begin{aligned} &\|\nabla^{(m)}\langle \partial_{ij}(U - u_k), E_{u_k}^s\rangle\|_0 \leq C\delta_k^{1/2}\lambda^{m+1} \\ &\text{for all } m = 0 \ldots (3N+6)(K-k) - 2N - 1, \ i, j, s = 1 \ldots 2. \end{aligned}$$

When $s = 2$, we use that $\langle E_{u_k}^1, E_{u_k}^2\rangle = 0$ and write:

$$\begin{aligned} \langle \partial_{ij}(U - u_k), E_{u_k}^2\rangle = {}& \partial_i\Big(\frac{\Gamma(\lambda x_1)}{\lambda}a_1\Big)\langle \partial_j E_{u_k}^1, E_{u_k}^2\rangle + \partial_j\Big(\frac{\Gamma(\lambda x_1)}{\lambda}a_1\Big)\langle \partial_i E_{u_k}^1, E_{u_k}^2\rangle \\ & + \frac{\Gamma(\lambda x_1)}{\lambda}a_1\langle \partial_{ij} E_{u_k}^1, E_{u_k}^2\rangle + \Big\langle \partial_{ij}\Big(T_{u_k}\Big(\frac{\bar\Gamma(\lambda x_1)}{\lambda}a_1^2 e_1 + W\Big)\Big), E_{u_k}^2\Big\rangle, \end{aligned}$$

which implies, by (7.13), $(7.14)_3$, $(7.14)_4$, (7.24):

$$\|\nabla^{(m)}\langle\partial_{ij}(U-u_k),E^2_{u_k}\rangle\|_0\le C\sum_{p+q+t=m}\Big\|\nabla^{(p+1)}\Big(\frac{\Gamma(\lambda x_1)}{\lambda}a_1\Big)\Big\|_0\|\nabla^{(q+1)}E^1_{u_k}\|_0\|\nabla^{(t)}E^2_{u_k}\|_0$$
$$+\sum_{p+q+t}\Big\|\nabla^{(p)}\Big(\frac{\Gamma(\lambda x_1)}{\lambda}a_1\Big)\Big\|_0\|\nabla^{(q+2)}E^1_{u_k}\|_0\|\nabla^{(t)}E^2_{u_k}\|_0$$
$$+C\sum_{p+q=m}\sum_{t+s=p+2}\|\nabla^{(t)}T_{u_k}\|_0\Big\|\nabla^{(s)}\Big(\frac{\bar\Gamma(\lambda x_1)}{\lambda}a_1^2e_1+W\Big)\Big\|_0\|\nabla^{(q)}E^2_{u_k}\|_0$$
$$\le C\delta_k\lambda^{m+1}\sigma^{N/2}\quad\text{for all } m=0\dots(3N+6)(K-k)-2N-1,\ i,j=1\dots2.$$

In conclusion, using $\delta_k^{1/2}\sigma^{N/2}\le 1$ by $(1.10)_1$, we obtain for all $m=0\dots(3N+6)(K-k)-2N-1$:

$$\|\nabla^{(m)}\langle\partial_{ij}U,E^1_U\rangle\|_0\le\|\nabla^{(m)}\langle\partial_{ij}u_k,E^1_{u_k}\rangle\|_0+C\delta_k^{1/2}\lambda^{m+1}, \tag{7.35$_1$}$$
$$\|\nabla^{(m)}\langle\partial_{ij}U,E^2_U\rangle\|_0\le\|\nabla^{(m)}\langle\partial_{ij}u_k,E^2_{u_k}\rangle\|_0+C\delta_k\lambda^{m+1}\sigma^{N/2}\quad\text{for } i,j=1\dots2. \tag{7.35$_2$}$$

In particular, since $(7.11)_1$ and (7.6) yield $\|\nabla^{(m)}\langle\partial_{ij}u_k,E^1_{u_k}\rangle\|_0\le C\delta_k^{1/2}\lambda^{m+1}$, we also get:

$$\|\nabla^{(m)}((\nabla U)^T\nabla E^1_U)\|_0\le C\delta_k^{1/2}\lambda^{m+1}\qquad\text{for all } m=0\dots(3N+6)(K-k)-2N-1. \tag{7.36}$$

7. (The second corrugation) We define $\bar U\in\mathcal{C}^{(3N+6)(K-k)-3N-1}(\bar\omega,\mathbb{R}^4)$:

$$\bar U=U+\frac{\Gamma(\kappa t)}{\kappa}a_2E^1_U+T_U\Big(\frac{\bar\Gamma(\kappa t)}{\kappa}a_2^2\eta_2+\bar W\Big), \tag{7.37}$$

that is the second intermediary field, in accordance with Lemma 2.5. We use the oscillatory profiles Γ, $\bar\Gamma$, $\bar{\bar\Gamma}$ in (7.16), the oscillation direction $\eta=\eta_2$, where we denote:

$$t=\langle x,\eta_2\rangle,$$

and the oscillation frequency in:

$$\kappa=\lambda\sigma=\mu_k\sigma^2.$$

The tangential correction $\bar W$ is given recalling the definition (5.7) and the recursion (5.2):

$$-2\bar W=\sum_{i=0}^N(-1)^i\frac{\bar{\bar\Gamma}_{i+1}(\kappa t)}{\kappa^{i+3}}L_i^{\eta_2}(\bar S_1)+\sum_{i=0}^N(-1)^i\frac{(\Gamma'\Gamma)_{i+1}(\kappa t)}{\kappa^{i+2}}L_i^{\eta_2}(\bar S_2)$$
$$+\sum_{i=0}^N(-1)^i\frac{\Gamma_{i+1}(\kappa t)}{\kappa^{i+2}}L_i^{\eta_2}(\bar S_3)+\sum_{i=0}^N(-1)^i\frac{\bar\Gamma_{i+1}(\kappa t)}{\kappa^{i+2}}L_i^{\eta_2}(\bar S_4),$$

with the fields $\{\bar S_i\}_{i=1}^4$ as in Lemma 2.5, namely:

$$\begin{aligned}&\bar S_1=\nabla a_2\otimes\nabla a_2, &&\bar S_2=2a_2\mathrm{sym}(\nabla a_2\otimes\eta_2),\\ &\bar S_3=2a_2\mathrm{sym}\big((\nabla U)^T\nabla E^1_U\big), &&\bar S_4=2\,\mathrm{sym}\big((\nabla U)^T\nabla(a_2^2T_U\eta_2)\big).\end{aligned} \tag{7.38}$$

By (5.6) in Lemma 5.3 we see that:

$$-2\,\mathrm{sym}\nabla\bar W=\frac{\bar{\bar\Gamma}(\kappa t)}{\kappa^2}\bar S_1+\frac{(\Gamma'\Gamma)(\kappa t)}{\kappa}\bar S_2+\frac{\Gamma(\kappa t)}{\kappa}\bar S_3+\frac{\bar\Gamma(\kappa t)}{\kappa}\bar S_4-\bar{\mathcal{G}}-\bar Ge_2\otimes e_2, \tag{7.39}$$

with the following formulas:

$$\begin{aligned}\bar{\mathcal{G}} = & (-1)^{N+1}\frac{\bar{\bar{\Gamma}}_{N+1}(\kappa t)}{\kappa^{N+3}}\mathrm{sym}\nabla L_N^{\eta_2}(\bar{S}_1) + (-1)^{N+1}\frac{(\Gamma'\Gamma)_{N+1}(\kappa t)}{\lambda^{N+2}}\mathrm{sym}\nabla L_N^{\eta_2}(\bar{S}_2) \\ & + (-1)^{N+1}\frac{\Gamma_{N+1}(\kappa t)}{\lambda^{N+2}}\mathrm{sym}\nabla L_N^{\eta_2}(\bar{S}_3) + (-1)^{N+1}\frac{\bar{\Gamma}_{N+1}(\kappa t)}{\kappa^{N+2}}\mathrm{sym}\nabla L_N^{\eta_2}(\bar{S}_4), \\ \bar{G} = & \sum_{i=0}^N (-1)^i\frac{\bar{\bar{\Gamma}}_i(\kappa t)}{\kappa^{i+2}}P_i^{\eta_2}(\bar{S}_1) + \sum_{i=0}^N (-1)^i\frac{(\Gamma'\Gamma)_i(\kappa t)}{\kappa^{i+1}}P_i^{\eta_2}(\bar{S}_2) \\ & + \sum_{i=0}^N (-1)^i\frac{\Gamma_i(\kappa t)}{\kappa^{i+1}}P_i^{\eta_2}(\bar{S}_3) + \sum_{i=0}^N (-1)^i\frac{\bar{\Gamma}_i(\kappa t)}{\kappa^{i+1}}P_i^{\eta_2}(\bar{S}_4).\end{aligned}$$

By (7.27), (2.2) and (7.39) we now obtain:

$$\begin{aligned}& \mathcal{D}(g_0 - \delta_{k+1}H_0, \bar{U}) = \mathcal{D}(g_0 - \delta_{k+1}H_0, U) - \big((\nabla\bar{U})^T\nabla\bar{U} - (\nabla U)^T\nabla U\big) \\ & = \big(a_3^2 - G - \bar{G}\big)e_2\otimes e_2 + \mathcal{E} - \bar{\mathcal{R}}_1 - \bar{\mathcal{R}}_2 - \bar{\mathcal{G}},\end{aligned} \tag{7.40}$$

where the primary and the $\bar{W}$-related error terms $\bar{\mathcal{R}}_1$ and $\bar{\mathcal{R}}_2$ are as in Lemma 2.5.

8. (Bounds on the errors in second corrugation defect) We now estimate, as in step 5, all terms in the decomposition (7.40). We first observe, by (7.13), $(7.33)_3$:

$$\begin{aligned}& \|\nabla^{(m+1)}(a_2^2 T_U\eta_2)\|_0 \leq C\sum_{p+q=m+1}\delta_k\mu_k^p\lambda^q = C\delta_k\lambda^{m+1} \\ & \text{for all } m = 0\ldots(3N+6)(K-k)-2N-1.\end{aligned} \tag{7.41}$$

Consequently, recalling additionally $(7.33)_4$, we get that each term in $\bar{\mathcal{R}}_1$ is bounded by:

$$\begin{aligned}& \|\nabla^{(m)}\bar{\mathcal{R}}_1\|_0 \leq C\delta_k^{3/2}\kappa^m\sigma^{N/2} \leq \frac{\delta_k}{\sigma^{N+1}}\kappa^m \\ & \text{for all } m = 0\ldots(3N+6)(K-k)-2N-2,\end{aligned} \tag{7.42}$$

where the worst term, responsible for the first bound is $\frac{\Gamma^2}{\kappa^2}a_2^2(\nabla E_U^1)^T\nabla E_U^1$, and where the final inequality follows by $C\sigma^{3N/2+1}\delta_k^{1/2} \leq \sigma^{3(N+1)/2}\delta_0^{1/2} \leq 1$ according to the last assumption in $(1.10)_1$ with $\underline{\sigma}$ sufficiently large. Before bounding $\bar{\mathcal{R}}_2$, we find the bounds on $\bar{W}$. Firstly:

$$\begin{aligned}& \|\nabla^{(m)}\bar{S}_1\|_0 \leq C\delta_k\mu_k^{m+2} \leq C\delta_k\lambda^{m+2}, \\ & \|\nabla^{(m)}\bar{S}_2\|_0 \leq C\delta_k\mu_k^{m+1} \leq C\delta_k\lambda^{m+1}, \\ & \|\nabla^{(m)}\bar{S}_3\|_0 \leq C\sum_{p+q=m}\|\nabla^{(p)}a_2\|_0\|\nabla^{(q)}((\nabla U)^T\nabla E_U^1)\|_0 \\ & \qquad \leq C\sum_{p+q=m}\delta_k^{1/2}\mu_k^p\delta_k^{1/2}\lambda^{q+1} \leq C\delta_k\lambda^{m+1}, \\ & \|\nabla^{(m)}\bar{S}_4\|_0 \leq C\delta_k\lambda^{m+1} \quad \text{ for all } m = 0\ldots(3N+6)(K-k)-2N-1,\end{aligned} \tag{7.43}$$

where the first two bounds result from (7.13), the third bound from (7.36), and the fourth from $(7.33)_1$, $(7.33)_2$ and (7.41). We now read from (5.7) that:

$$\|\nabla^{(m)}L_i^{\eta_2}(S)\|_0 \leq C_{m+i}\|\nabla^{(m+i)}S\|_0 \quad \text{for all } m, i \geq 0. \tag{7.44}$$

By (7.43), and recalling that all the primitives of $\bar{\bar{\Gamma}}, (\Gamma'\Gamma), \Gamma, \bar{\Gamma}$ in the definition of $\bar{W}$ are bounded, being periodic with mean zero on the period, we get:

$$\begin{aligned} \|\nabla^{(m)}\bar{W}\|_0 &\leq C\sum_{i=0}^{N}\sum_{p+q=m}\Big(\kappa^{p-i-3}\delta_k\lambda^{q+i+2} + \kappa^{p-i-2}\delta_k\lambda^{q+i+1}\Big) \\ &\leq C\delta_k\kappa^{m-1} \quad \text{ for all } m = 0\dots(3N+6)(K-k)-3N-1. \end{aligned} \tag{7.45}$$

We are now ready to bound the terms in $\bar{\mathcal{R}}_2$. Observe first that, by (7.13), $(7.33)_3$, $(7.33)_4$:

$$\begin{aligned} &\|\nabla^{(m+1)}(T_U\bar{W})\|_0 \leq C\delta_k\kappa^m && \text{for } m = 0\dots(3N+6)(K-k)-3N-2, \\ &\|\nabla^{(m+1)}(a_2E_U^1)\|_0 \leq C\delta_k^{1/2}\lambda^{m+1} && \text{for } m = 0\dots(3N+6)(K-k)-2N, \end{aligned}$$

resulting in all the terms in $\bar{\mathcal{R}}_2$ being bounded by:

$$\|\nabla^{(m)}\bar{\mathcal{R}}_2\|_0 \leq C\delta_k^{3/2}\kappa^m\sigma^{N/2} \leq \frac{\delta_k}{\sigma^{N+1}}\kappa^m \quad \text{ for } m = 0\dots(3N+6)(K-k)-3N-2, \tag{7.46}$$

where the worst term, responsible for the first bound above is $2\mathrm{sym}((\nabla U)^T[(\partial_1T_U)\bar{W}, (\partial_2T_U)\bar{W}])$, and the final inequality is due to having $C\sigma^{3N/2+1}\delta_k^{1/2} \leq \sigma^{3(N+1)/2}\delta_0^{1/2} \leq 1$ from the last assumption in $(1.10)_1$ and for $\sigma \geq \underline{\sigma}$ large. For estimating $\bar{\mathcal{G}}$, we again use (7.44), (7.43):

$$\begin{aligned} \|\nabla^{(m)}\bar{\mathcal{G}}\|_0 &\leq C\sum_{p+q=m}\Big(\kappa^{p-N-3}\delta_k\lambda^{q+N+3} + \kappa^{p-N-2}\delta_k\lambda^{q+N+2}\Big) \leq C\frac{\delta_k}{(\kappa/\lambda)^{N+2}}\kappa^m \\ &\leq \frac{\delta_k}{\sigma^{N+1}}\kappa^m \quad \text{ for all } m = 0\dots(3N+6)(K-k)-3N-2. \end{aligned} \tag{7.47}$$

Consequently, the bounds (7.27), (7.42), (7.46), (7.47) in (7.40) yield:

$$\begin{aligned} &\mathcal{D}(g_0 - \delta_{k+1}H_0, \bar{U}) = (a_3^2 - G - \bar{G})e_2\otimes e_2 + \bar{\mathcal{E}} \\ &\text{where } \|\nabla^{(m)}\bar{\mathcal{E}}\|_0 = \|\nabla^{(m)}(\mathcal{E} - \bar{\mathcal{R}}_1 - \bar{\mathcal{R}}_2 - \bar{\mathcal{G}})\|_0 \leq 7\frac{\delta_k}{\sigma^{N+1}}\lambda^m \\ &\text{for all } m = 0\dots(3N+6)(K-k)-3N-2. \end{aligned} \tag{7.48}$$

We conclude this step by estimating the current leading order defect $(a_3^2 - G - \bar{G})e_2\otimes e_2$. Reading from (5.7) that:

$$\|\nabla^{(m)}P_i^{\eta_2}(S)\|_0 \leq C_{m+i}\|\nabla^{(m+i)}S\|_0 \quad \text{for all } m, i \geq 0,$$

we get, in view of (7.43) and utilizing, as before, that $\kappa/\lambda = \sigma$:

$$\begin{aligned} &\|\nabla^{(m)}\bar{G}\|_0 \leq C\sum_{i=0}^{N}\sum_{p+q=m}\Big(\kappa^{p-i-2}\delta_k\lambda^{q+i+2} + \kappa^{p-i-1}\delta_k\lambda^{q+i+1}\Big) \leq C\frac{\delta_k}{\sigma}\kappa^m \\ &\text{hence} \quad \|\nabla^{(m)}(G+\bar{G})\|_0 \leq C\frac{\delta_k}{\sigma}\kappa^m \quad \text{ for all } m = 0\dots(3N+6)(K-k)-3N-1. \end{aligned} \tag{7.49}$$

By (7.13) it results that taking $\sigma \geq \underline{\sigma}$ sufficiently large, the new positive profile function $b \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R})$ is well defined, through:

$$b^2 = a_3^2 - G - \bar{G}.$$

We then obtain:

$$\begin{aligned} &\frac{1}{4}\delta_k \leq b^2 \leq \frac{7}{4}\delta_k \qquad \text{and} \quad \frac{1}{4}\delta_k^{1/2} \leq b \leq \frac{7}{4}\delta_k^{1/2} \quad \text{in } \bar\omega, \\ &\|\nabla^{(m)} b^2\|_0 \leq C\frac{\delta_k}{\sigma}\kappa^m \quad \text{and} \quad \|\nabla^{(m)} b\|_0 \leq C\frac{\delta_k^{1/2}}{\sigma}\kappa^m \\ &\qquad\qquad\qquad\qquad \text{for all } m = 1\dots(3N+6)(K-k)-3N-1, \end{aligned} \tag{7.50}$$

where the bound on the non-zero derivatives of b^2 is a direct consequence of (7.13) and (7.49), while the following bound on the derivatives of b is justified by an application of Faà di Bruno's formula, as in step 2 of the proof of Theorem 6.1.

9. (Propagation of bounds in the second corrugation) A rough bound without specifying to components, and using only (7.13), $(7.33)_3$, $(7.33)_4$, (7.45) yields the following:

$$\begin{aligned} \|\nabla^{(m)}(\bar U - U)\|_0 \leq C &\sum_{p+q+t=m} \kappa^{p-1}\|\nabla^{(q)} a_2\|_0 \|\nabla^{(t)} E_U^1\|_0 \\ + C &\sum_{p+q+t=m} \kappa^{p-1}\|\nabla^{(q)} a_2^2\|_0 \|\nabla^{(t)} T_U\|_0 + C\sum_{p+q=m} \|\nabla^{(p)} T_U\|_0 \|\nabla^{(q)} \bar W\|_0 \leq C\delta_k^{1/2}\kappa^{m-1}. \end{aligned} \tag{7.51}$$

We may now apply Lemma 3.4 with $u = U$, $v = \bar U$, $\mu = \kappa$, $A = \delta_k^{1/2}$ because $\|\nabla\bar U - \nabla U\|_0 \leq C\delta_k^{1/2} \leq C\underline{\delta}^{1/2} \leq \rho$ for $\underline{\delta}$ sufficiently small, in view of (7.51). Recalling further (7.32), $(7.33)_2$, $(7.33)_4$, we obtain existence of the unit normal frame $E_{\bar U}^1, E_{\bar U}^2 \in \mathcal{C}^{(3N+6)(K-k)-3N-2}(\bar\omega, \mathbb{R}^4)$ in:

$$(\nabla\bar U)^T E_{\bar U}^i = 0, \quad |E_{\bar U}^1| = 1 \quad \text{for} \quad i = 1,2 \quad \text{and} \quad \langle E_{\bar U}^1, E_{\bar U}^2 = 0 \quad \text{in } \bar\omega,$$

satisfying the the second bound below, with the first bound included in (7.51):

$$\begin{aligned} &\|\nabla^{(m+1)}(\bar U - U)\|_0 \leq C\delta_k^{1/2}\kappa^m \quad \text{and} \quad \|\nabla^{(m)}(E_{\bar U}^i - E_U^i)\|_0 \leq C\delta_k^{1/2}\kappa^m \\ &\text{for all } m = 0\dots(3N+6)(K-k)-3N-2,\ i = 1,2. \end{aligned} \tag{7.52}$$

We also note, recalling (7.31):

$$\|\bar U - u\|_1 \leq \|\bar U - U\|_1 + \|U - u\|_0 \leq C\delta_0^{1/2}, \tag{7.53}$$

which leads, as in step 1 and the proof of (7.3), to:

$$\frac{1}{3\underline{\gamma}}\mathrm{Id}_2 \leq (\nabla\bar U)^T\nabla\bar U \leq 3\underline{\gamma}\mathrm{Id}_2 \quad \text{in } \bar\omega. \tag{7.54}$$

We proceed to gather estimates similar to those in $(7.33)_1$-$(7.33)_4$. The first bound below follows from (7.54) and Lemma 3.1, while the second and the fourth bounds follow from (7.52):

$$\|\nabla\bar U\|_0 \leq (6\underline{\gamma})^{1/2} \tag*{$(7.55)_1$}$$

$$\|\nabla^{(m)}\nabla^{(2)}\bar U\|_0 \leq C\delta_k^{1/2}\kappa^{m+1}\sigma^{N/2} \quad \text{for all } m = 0\dots(3N+6)(K-k)-3N-3, \tag*{$(7.55)_2$}$$

$$\|T_{\bar U}\|_0 \leq C \quad \text{and} \quad \|\nabla^{(m)} T_{\bar U}\|_0 \leq C\delta_k^{1/2}\kappa^m\sigma^{N/2} \quad \text{for all } m = 1\dots(3N+6)(K-k)-3N-2, \tag*{$(7.55)_3$}$$

$$\|\nabla^{(m)} E_{\bar U}^i\|_0 \leq C\delta_k^{1/2}\kappa^m\sigma^{N/2} \quad \text{for all } m = 1\dots(3N+6)(K-k)-3N-2,\ i = 1,2. \tag*{$(7.55)_4$}$$

The estimate $(7.55)_3$ results from Lemma 3.2. Below, we refine the first bound in $(7.55)_2$ to the components of $\nabla^2\bar U$. For any $i,j,s = 1\dots 2$ we write:

$$\langle\partial_{ij}\bar U, E_{\bar U}^s\rangle = \langle\partial_{ij}U, E_U^s\rangle + \langle\partial_{ij}(\bar U - U), E_U^s\rangle + \langle\partial_{ij}\bar U, (E_{\bar U}^s - E_U^s)\rangle. \tag{7.56}$$

The first term in the right hand side above obeys the bounds in $(7.35)_1$-$(7.35)_2$, while for the third term, we use (7.52) and $(7.55)_2$, $(7.55)_4$ and get:

$$\begin{aligned}&\|\nabla^{(m)}\langle\partial_{ij}\bar{U}, (E^s_{\bar{U}} - E^s_U)\rangle\|_0 \leq C\delta_k\kappa^{m+1}\sigma^{N/2}\\ &\text{for } m = 0\dots(3N+6)(K-k) - 3N - 3,\ i,j,s = 1\dots 2.\end{aligned}$$

We now estimate the second term in the right hand side of (7.56), where by (7.52), $(7.33)_3$:

$$\begin{aligned}&\|\nabla^{(m)}\langle\partial_{ij}(\bar{U} - U), E^s_U\rangle\|_0 \leq C\delta_k^{1/2}\kappa^{m+1}\\ &\text{for } m = 0\dots(3N+6)(K-k) - 3N - 3,\ i,j,s = 1\dots 2.\end{aligned}$$

In the specific case of $s = 2$, we use that $\langle E^1_U, E^2_U\rangle = 0$ and write:

$$\begin{aligned}\langle\partial_{ij}(\bar{U} - U), E^2_U\rangle = {}& \partial_i\Big(\frac{\Gamma(\kappa t)}{\kappa}a_2\Big)\langle\partial_j E^1_U, E^2_U\rangle + \partial_j\Big(\frac{\Gamma(\kappa t)}{\kappa}a_2\Big)\langle\partial_i E^1_U, E^2_U\rangle\\ &+ \frac{\Gamma(\kappa t)}{\kappa}a_2\langle\partial_{ij}E^1_U, E^2_U\rangle + \Big\langle\partial_{ij}\Big(T_U\Big(\frac{\bar{\Gamma}(\kappa t)}{\kappa}a_2^2\eta_2 + \bar{W}\Big)\Big), E^2_U\Big\rangle,\end{aligned}$$

which implies, by (7.13), $(7.33)_3$, $(7.33)_4$, (7.45):

$$\begin{aligned}&\|\nabla^{(m)}\langle\partial_{ij}(\bar{U} - U), E^2_U\rangle\|_0 \leq C\sum_{p+q+t=m}\Big\|\nabla^{(p+1)}\Big(\frac{\Gamma(\kappa t)}{\kappa}a_2\Big)\Big\|_0\|\nabla^{(q+1)}E^1_U\|_0\|\nabla^{(t)}E^2_U\|_0\\ &+ \sum_{p+q+t}\Big\|\nabla^{(p)}\Big(\frac{\Gamma(\kappa t)}{\kappa}a_2\Big)\Big\|_0\|\nabla^{(q+2)}E^1_U\|_0\|\nabla^{(t)}E^2_U\|_0\\ &+ C\sum_{p+q=m}\ \sum_{t+s=p+2}\|\nabla^{(t)}T_U\|_0\Big\|\nabla^{(s)}\Big(\frac{\bar{\Gamma}(\kappa t)}{\kappa}a_2^2\eta_2 + \bar{W}\Big)\Big\|_0\|\nabla^{(q)}E^2_U\|_0\\ &\leq C\delta_k\kappa^{m+1}\sigma^{N/2}\quad\text{for all } m = 0\dots(3N+6)(K-k) - 3N - 3,\ i,j = 1\dots 2.\end{aligned}$$

Using $(7.35)_1$-$(7.35)_2$ and recalling that $\delta_k^{1/2}\sigma^{N/2} \leq \delta^{1/2}\sigma^{N/2} \leq 1$ by $(1.10)_1$, we obtain for $m = 0\dots(3N+6)(K-k) - 3N - 3$:

$$\|\nabla^{(m)}\langle\partial_{ij}\bar{U}, E^1_{\bar{U}}\rangle\|_0 \leq \|\nabla^{(m)}\langle\partial_{ij}u_k, E^1_{u_k}\rangle\|_0 + C\delta_k^{1/2}\kappa^{m+1}\quad\text{for } i,j = 1\dots 2, \tag{7.57$_1$}$$

$$\|\nabla^{(m)}\langle\partial_{ij}\bar{U}, E^2_{\bar{U}}\rangle\|_0 \leq \|\nabla^{(m)}\langle\partial_{ij}u_k, E^2_{u_k}\rangle\|_0 + C\delta_k\kappa^{m+1}\sigma^{N/2}. \tag{7.57$_2$}$$

In particular, $(7.11)_2$-$(7.11)_4$ and (7.6) yield: $\|\nabla^{(m)}\langle\partial_{ij}u_k, E^2_{u_k}\rangle\|_0 \leq C\delta_k^{1/2}\mu_k^{m+1}\sigma^{N/2}$, hence:

$$\begin{aligned}&\|\nabla^{(m)}((\nabla\bar{U})^T\nabla E^2_{\bar{U}})\|_0 \leq \|\nabla^{(m)}\big((\nabla u_k)^T\nabla E^2_{u_k}\big)\|_0 + C\delta_k\kappa^{m+1}\sigma^{N/2}\\ &\leq C\delta_k^{1/2}\mu_k^{m+1}\sigma^{N/2} \leq C\delta_k^{1/2}\kappa^{m+1}\sigma^{N/2-2}\\ &\text{for all } m = 0\dots(3N+6)(K-k) - 3N - 3,\end{aligned} \tag{7.58}$$

in view of $(1.10)_1$ and provided that $N \geq 4$.

10. (The third and final corrugation) We define $u_{k+1} \in \mathcal{C}^{(3N+6)(K-k)-3N-4}(\bar{\omega}, \mathbb{R}^4)$, whose regularity is as stipulated in the induction set-up, as $(3N+6)(K-k) - 3N - 4 = 2 + (3N+6)(K-(k+1))$. This is the final field in our triple-corrugation Stage:

$$u_{k+1} = \bar{U} + \frac{\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}bE^2_{\bar{U}} + T_{\bar{U}}\Big(\frac{\bar{\Gamma}(\mu_{k+1}x_2)}{\mu_{k+1}}b^2e_2 + \bar{\bar{W}}\Big), \tag{7.59}$$

in accordance with Lemma 2.5, where Γ, $\bar{\Gamma}$ are the oscillatory profiles in (7.16), the oscillation direction is set to $\eta = \eta_3 = e_2$, the oscillation frequency is μ_{k+1} given in (7.8) and the amplitude function b obeys the bounds in (7.50). The tangential correction $\bar{\bar{W}}$ is:

$$\begin{aligned} -2\bar{\bar{W}} = & \sum_{i=0}^{1}(-1)^i \frac{(\Gamma'\Gamma)_{i+1}(\mu_{k+1}x_2)}{\mu_{k+1}^{i+2}} L_i^{\eta_3}(\bar{\bar{S}}_2) \\ & + \sum_{i=0}^{1}(-1)^i \frac{\Gamma_{i+1}(\mu_{k+1}x_2)}{\mu_{k+1}^{i+2}} L_i^{\eta_3}(\bar{\bar{S}}_3) + \sum_{i=0}^{1}(-1)^i \frac{\bar{\Gamma}_{i+1}(\mu_{k+1}x_2)}{\mu_{k+1}^{i+2}} L_i^{\eta_3}(\bar{\bar{S}}_4), \end{aligned}$$

where we recall (5.5), (5.2) and where the fields $\{\bar{\bar{S}}_i\}_{i=2}^4$ are as in Lemma 2.5, namely:

$$\begin{aligned} & \bar{\bar{S}}_2 = 2b\,\mathrm{sym}(\nabla b \otimes e_2), \\ & \bar{\bar{S}}_3 = 2b\,\mathrm{sym}\big((\nabla \bar{U})^T \nabla E_{\bar{U}}^2\big), \qquad \bar{\bar{S}}_4 = 2\,\mathrm{sym}\big((\nabla \bar{U})^T \nabla (b^2 T_{\bar{U}} e_2)\big). \end{aligned} \tag{7.60}$$

By (5.4) in Lemma 5.2 we see that:

$$-2\,\mathrm{sym}\nabla\bar{\bar{W}} = \frac{(\Gamma'\Gamma)(\mu_{k+1}x_2)}{\mu_{k+1}}\bar{\bar{S}}_2 + \frac{\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}\bar{\bar{S}}_3 + \frac{\bar{\Gamma}(\mu_{k+1}x_2)}{\mu_{k+1}}\bar{\bar{S}}_4 - \bar{\bar{\mathcal{G}}} - \bar{\bar{G}}e_1 \otimes e_1, \tag{7.61}$$

with the following formulas:

$$\begin{aligned} \bar{\bar{\mathcal{G}}} = & \frac{(\Gamma'\Gamma)_2(\mu_{k+1}x_2)}{\mu_{k+1}^3}\mathrm{sym}\nabla L_1^{\eta_3}(\bar{\bar{S}}_2) + \frac{\Gamma_2(\mu_{k+1}x_2)}{\mu_{k+1}^3}\mathrm{sym}\nabla L_1^{\eta_3}(\bar{\bar{S}}_3) + \frac{\bar{\Gamma}_2(\mu_{k+1}x_2)}{\mu_{k+1}^3}\mathrm{sym}\nabla L_1^{\eta_3}(\bar{\bar{S}}_4), \\ \bar{\bar{G}} = & \sum_{i=0}^{1}(-1)^i \frac{(\Gamma'\Gamma)_i(\mu_{k+1}x_2)}{\mu_{k+1}^{i+1}} P_i^{\eta_3}(\bar{\bar{S}}_2) + \sum_{i=0}^{1}(-1)^i \frac{\Gamma_i(\mu_{k+1}x_2)}{\mu_{k+1}^{i+1}} P_i^{\eta_3}(\bar{\bar{S}}_3) \\ & + \sum_{i=0}^{1}(-1)^i \frac{\bar{\Gamma}_i(\mu_{k+1}x_2)}{\mu_{k+1}^{i+1}} P_i^{\eta_3}(\bar{\bar{S}}_4). \end{aligned}$$

By (7.48), (2.2) and (7.61) we now obtain:

$$\begin{aligned} & \mathcal{D}(g_0 - \delta_{k+1}H_0, u_{k+1}) = \mathcal{D}(g_0 - \delta_{k+1}H_0, \bar{U}) - \big((\nabla u_{k+1})^T \nabla u_{k+1} - (\nabla \bar{U})^T \nabla \bar{U}\big) \\ & = \bar{\mathcal{E}} - \frac{1 + \bar{\bar{\Gamma}}(\mu_{k+1}x_2)}{\mu_{k+1}^2}\nabla b \otimes \nabla b - \bar{\bar{\mathcal{R}}}_1 - \bar{\bar{\mathcal{R}}}_2 - \bar{\bar{\mathcal{G}}} - \bar{\bar{G}}e_1 \otimes e_1 \end{aligned} \tag{7.62}$$

where the primary and the $\bar{\bar{W}}$-related error terms $\bar{\bar{\mathcal{R}}}_1$ and $\bar{\bar{\mathcal{R}}}_2$ are as in Lemma 2.5.

11. (Bounds on the errors in third corrugation defect) We now estimate, as in steps 5 and 8, all terms in the final decomposition (7.62). Firstly, by (7.50):

$$\begin{aligned} & \Big\|\nabla^{(m)}\Big(\frac{1 + \bar{\bar{\Gamma}}(\mu_{k+1}x_2)}{\mu_{k+1}^2}\nabla b \otimes \nabla b\Big)\Big\|_0 \leq C \sum_{p+q=m} \mu_{k+1}^{p-2}\frac{\delta_k}{\sigma^2}\kappa^{q+2} \\ & \leq C\frac{\delta_k}{(\mu_{k+1}/\kappa)^2\sigma^2}\mu_{k+1}^m \leq \frac{\delta_k}{\sigma^{N+1}}\mu_{k+1}^m \qquad \text{for all } m = 0 \ldots (3N+6)(K-k) - 3N - 2, \end{aligned} \tag{7.63}$$

if only $\sigma \geq \underline{\sigma}$ is sufficiently large. Secondly, from (7.50), $(7.55)_3$:

$$\begin{aligned} & \|\nabla^{(m+1)}(b^2 T_{\bar{U}} e_2)\|_0 \leq C \sum_{p+q=m+1} \delta_k \kappa^p \kappa^q = C\delta_k \kappa^{m+1} \\ & \text{for all } m = 0 \ldots (3N+6)(K-k) - 3N - 3. \end{aligned} \tag{7.64}$$

Consequently, recalling additionally $(7.55)_4$, we get that each term in $\bar{\bar{\mathcal{R}}}_1$ is bounded by:

$$\begin{aligned}&\|\nabla^{(m)}\bar{\bar{\mathcal{R}}}_1\|_0 \leq C\delta_k^{3/2}\mu_{k+1}^m\sigma^{N/2} \leq \frac{\delta_k}{\sigma^{N+1}}\mu_{k+1}^m \\ &\text{for all } m = 0\dots(3N+6)(K-k)-3N-2,\end{aligned} \tag{7.65}$$

where the worst term, responsible for the first bound is $\frac{\Gamma(\mu_{k+1}x_2)^2}{\mu_{k+1}^2}b^2(\nabla E_{\bar{U}}^2)^T\nabla E_{\bar{U}}^2$, and where the final inequality follows by $C\sigma^{3N/2+1}\delta_k^{1/2} \leq \sigma^{3(N+1)/2}\delta_0^{1/2} \leq 1$ from the last assumption in $(1.10)_1$, for $\underline{\sigma}$ sufficiently large. Before bounding $\bar{\mathcal{R}}_2$, we find the bounds on $\bar{\bar{W}}$. We get:

$$\begin{aligned}&\|\nabla^{(m)}\bar{\bar{S}}_2\|_0 \leq C\delta_k\kappa^{m+1},\\ &\|\nabla^{(m)}\bar{\bar{S}}_3\|_0 \leq C\sum_{p+q=m}\|\nabla^{(p)}b\|_0\|\nabla^{(q)}((\nabla\bar{U})^T\nabla E_{\bar{U}}^2)\|_0\\ &\qquad\qquad \leq C\sum_{p+q=m}\delta_k^{1/2}\kappa^p\delta_k^{1/2}\kappa^{q+1}\sigma^{N/2-2} \leq C\delta_k\kappa^{m+1}\sigma^{N/2-2},\\ &\|\nabla^{(m)}\bar{\bar{S}}_4\|_0 \leq C\delta_k\kappa^{m+1} \quad \text{for all } m = 0\dots(3N+6)(K-k)-3N-3,\end{aligned} \tag{7.66}$$

where the first bound results from (7.50), the second bound from (7.58), and the third from $(7.55)_1$, $(7.55)_2$ and (7.64). We now read from (5.5) that:

$$\|\nabla^{(m)}L_i^{\eta_3}(S)\|_0 \leq C_{m+i}\|\nabla^{(m+i)}S\|_0 \quad \text{for all } m,i\geq 0, \tag{7.67}$$

which implies by recalling from (7.66) that $\|\nabla^{(m)}\bar{\bar{S}}_i\|_0 \leq C\delta_k\mu_{k+1}^{m+1}$:

$$\begin{aligned}&\|\nabla^{(m)}\bar{\bar{W}}\|_0 \leq C\sum_{i=0}^{1}\sum_{p+q=m}\mu_{k+1}^{p-i-2}\delta_k\mu_{k+1}^{q+i+1} \leq C\delta_k\mu_{k+1}^{m-1}\\ &\text{for all } m = 0\dots(3N+6)(K-k)-3N-4.\end{aligned} \tag{7.68}$$

We are now ready to bound the terms in $\bar{\bar{\mathcal{R}}}_2$. Observe first that, by (7.50), $(7.55)_3$, $(7.55)_4$:

$$\begin{aligned}&\|\nabla^{(m+1)}(T_{\bar{U}}\bar{\bar{W}})\|_0 \leq C\delta_k\mu_{k+1}^m \qquad \text{for } m = 0\dots(3N+6)(K-k)-3N-5,\\ &\|\nabla^{(m+1)}(bE_{\bar{U}}^2)\|_0 \leq C\delta_k^{1/2}\kappa^{m+1} \qquad \text{for } m = 0\dots(3N+6)(K-k)-3N-3,\end{aligned}$$

resulting in all the terms in $\bar{\bar{\mathcal{R}}}_2$ being bounded by:

$$\begin{aligned}&\|\nabla^{(m)}\bar{\bar{\mathcal{R}}}_2\|_0 \leq C\delta_k^{3/2}\mu_{k+1}^m\sigma^{N/2} \leq \frac{\delta_k}{\sigma^{N+1}}\mu_{k+1}^m\\ &\text{for all } m = 0\dots(3N+6)(K-k)-3N-5.\end{aligned} \tag{7.69}$$

The worst term, responsible for the first bound above is $2\mathrm{sym}((\nabla\bar{U})^T[(\partial_1T_{\bar{U}})\bar{\bar{W}},(\partial_2T_{\bar{U}})\bar{\bar{W}}])$, and the final inequality is due to having $C\sigma^{3N/2+1}\delta_k^{1/2} \leq 1$ according to the last assumption in $(1.10)_1$ for $\sigma \geq \underline{\sigma}$ large. To estimate $\bar{\bar{\mathcal{G}}}$, we use (7.67) with $i=1$, and (7.66):

$$\begin{aligned}&\|\nabla^{(m)}\bar{\bar{\mathcal{G}}}\|_0 \leq C\sum_{i=2}^{4}\sum_{p+q=m}\mu_{k+1}^{p-3}\|\nabla^{(q+2)}\bar{\bar{S}}_i\|_0 \leq C\sum_{p+q=m}\mu_{k+1}^{p-3}\delta_k\kappa^{q+3}\sigma^{N/2-2}\\ &= C\frac{\delta_k}{(\mu_{k+1}/\kappa)^3}\mu_{k+1}^m\sigma^{N/2-2} \leq C\frac{\delta_k}{\sigma^{3N/2}}\mu_{k+1}^m\sigma^{N/2-2} \leq \frac{\delta_k}{\sigma^{N+1}}\mu_{k+1}^m\\ &\text{for all } m = 0\dots(3N+6)(K-k)-3N-5,\end{aligned} \tag{7.70}$$

when $\sigma \geq \underline{\sigma}$ large. In conclusion, the bounds (7.48), (7.65), (7.69), (7.70) in (7.62) yield:

$$\begin{aligned} & \mathcal{D}(g_0 - \delta_{k+1}H_0, u_{k+1}) = -\bar{\bar{G}}e_1 \otimes e_1 + \bar{\bar{\mathcal{E}}} \\ & \text{where } \|\nabla^{(m)}\bar{\bar{\mathcal{E}}}\|_0 = \Big\|\nabla^{(m)}\Big(\bar{\mathcal{E}} - \frac{1-\bar{\bar{\Gamma}}(\mu_{k+1}x_2)}{\mu_{k+1}^2}\nabla b\otimes\nabla b - \bar{\bar{\mathcal{R}}}_1 - \bar{\bar{\mathcal{R}}}_2 - \bar{\bar{\mathcal{G}}}\Big)\Big\|_0 \\ & \qquad \leq 11\frac{\delta_k}{\sigma^{N+1}}\mu_{k+1}^m \quad \text{ for all } m = 0\dots(3N+6)(K-k)-3N-5. \end{aligned} \tag{7.71}$$

12. (Estimating the error term $\bar{\bar{G}}$) In order to estimate the derivatives of $\bar{\bar{G}}$, we write:

$$\bar{\bar{G}} = \bar{\bar{G}}_2 + \bar{\bar{G}}_3 + \bar{\bar{G}}_4,$$

and analyze the following three terms:

$$\begin{aligned} & \bar{\bar{G}}_2 = \sum_{i=0}^{1}(-1)^i\frac{(\Gamma'\Gamma)_i(\mu_{k+1}x_2)}{\mu_{k+1}^{i+1}}P_i^{\eta_3}(\bar{\bar{S}}_2), \qquad \bar{\bar{G}}_3 = \sum_{i=0}^{1}(-1)^i\frac{\Gamma_i(\mu_{k+1}x_2)}{\mu_{k+1}^{i+1}}P_i^{\eta_3}(\bar{\bar{S}}_3), \\ & \bar{\bar{G}}_4 = \sum_{i=0}^{1}(-1)^i\frac{\bar{\Gamma}_i(\mu_{k+1}x_2)}{\mu_{k+1}^{i+1}}P_i^{\eta_3}(\bar{\bar{S}}_4). \end{aligned}$$

We will use the formulas in (5.5) at $i = 0, 1$, namely:

$$P_0^{\eta_3}(S) = S_{11}, \qquad P_1^{\eta_3}(S) = 2\partial_1 S_{12}, \tag{7.72}$$

separately for S equal to $\bar{\bar{S}}_2, \bar{\bar{S}}_3$ and $\bar{\bar{S}}_4$. In case of $\bar{\bar{S}}_2$, there holds:

$$P_0^{\eta_3}(\bar{\bar{S}}_2) = 0, \qquad P_1^{\eta_3}(\bar{\bar{S}}_2) = 4\partial_1(b\partial_1 b),$$

so, we estimate directly from (7.50):

$$\begin{aligned} \|\nabla^{(m)}\bar{\bar{G}}_2\|_0 & \leq C\sum_{p+q=m}\mu_{k+1}^{p-2}\|\nabla^{(q+1)}(b\partial_1 b)\|_0 \leq C\sum_{p+q=m}\mu_{k+1}^{p-2}\frac{\delta_k}{\sigma}\kappa^{q+2} \\ & \leq C\frac{\delta_k}{\sigma^{N+1}}\mu_{k+1}^m \quad \text{ for all } m = 0\dots(3N+6)(K-k)-3N-4. \end{aligned} \tag{7.73}$$

In case of $\bar{\bar{S}}_4$, formulas (7.72) become:

$$P_0^{\eta_3}(\bar{\bar{S}}_4) = 2\langle\partial_1\bar{U}, \partial_1(b^2T_{\bar{U}}e_2)\rangle, \quad P_1^{\eta_3}(\bar{\bar{S}}_4) = 2\Big(\partial_1\langle\partial_1\bar{U}, \partial_2(b^2T_{\bar{U}}e_2)\rangle + \partial_1\langle\partial_2\bar{U}, \partial_1(b^2T_{\bar{U}}e_2)\rangle\Big).$$

We note in passing that $\langle\partial_i\bar{U}, T_{\bar{U}}e_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$, which yields that:

$$\begin{aligned} \big\langle\partial_i\bar{U}, \partial_j(b^2T_{\bar{U}}e_2)\big\rangle & = \partial_j(b^2)\big\langle\partial_i\bar{U}, T_{\bar{U}}e_2\big\rangle + b^2\big\langle\partial_i\bar{U}, \partial_j(T_{\bar{U}}e_2)\big\rangle \\ & = \partial_j(b^2)\delta_{i2} - b^2\big\langle\partial_{ij}\bar{U}, T_{\bar{U}}e_2\big\rangle \quad \text{ for all } i, j = 1\dots 2. \end{aligned}$$

Consequently, we get from (7.50), $(7.55)_2$, $(7.55)_3$:

$$\begin{aligned} \|\nabla^{(m)}P_0^{\eta_3}(\bar{\bar{S}}_4)\|_0 & = 2\big\|\nabla^{(m)}\big(b^2\big\langle\partial_{11}\bar{U}, T_{\bar{U}}e_2\big\rangle\big)\big\|_0 \\ & \leq C\sum_{p+q+t=m}\delta_k\kappa^p\delta_k^{1/2}\kappa^{q+1}\sigma^{N/2}\kappa^t \leq C\delta_k^{3/2}\kappa^{m+1}\sigma^{N/2}, \\ \|\nabla^{(m)}P_1^{\eta_3}(\bar{\bar{S}}_4)\|_0 & \leq C\big(\|\nabla^{(m+2)}b^2\|_0 + \|\nabla^{(m+1)}\big(b^2\big\langle\partial_{12}\bar{U}, T_{\bar{U}}e_2\big\rangle\big)\big\|_0\big) \\ & \leq C\frac{\delta_k}{\sigma}\kappa^{m+2} + C\delta_k^{3/2}\kappa^{m+2}\sigma^{N/2} \leq C\frac{\delta_k}{\sigma}\kappa^{m+2}, \\ & \text{for all } m = 0\dots(3N+6)(K-k)-3N-4, \end{aligned}$$

since $\delta_k^{1/2}\sigma^{N/2+1}\leq 1$ by the fourth assumption in $(1.10)_1$. We are ready to conclude that:

$$\begin{aligned} \|\nabla^{(m)}\bar{\bar{G}}_4\|_0 &\leq C\sum_{p+q=m}\mu_{k+1}^{p-1}\delta_k^{3/2}\sigma^{N/2}\kappa^{q+1}+C\sum_{p+q=m}\mu_{k+1}^{p-2}\frac{\delta_k}{\sigma}\kappa^{q+2} \\ &\leq C\delta_k^{3/2}\mu_{k+1}^m+C\frac{\delta_k}{\sigma^{N+1}}\mu_{k+1}^m\leq C\frac{\delta_k}{\sigma^{N+1}}\mu_{k+1}^m \\ &\text{for all } m=0\dots(3N+6)(K-k)-3N-4, \end{aligned} \tag{7.74}$$

valid provided that $\delta_k^{1/2}\sigma^{N+1}\leq 1$, as usual from $(1.10)_1$.

We now deal with the term $\bar{\bar{G}}_3$, where the induction assumptions $(7.11)_2$-$(7.11)_4$ will be used for the first time in a tight manner. Given $\bar{\bar{S}}_3$, from (7.72) and (7.60) we read:

$$P_0^{\eta_3}(\bar{\bar{S}}_3)=2b\langle\partial_{11}\bar{U},E_{\bar{U}}^2\rangle,\qquad P_1^{\eta_3}(\bar{\bar{S}}_2)=4\partial_1\big(b\langle\partial_{12}\bar{U},E_{\bar{U}}^2\rangle\big).$$

Observe, from $(7.11)_2$ and $(7.11)_3$, that:

$$\begin{aligned} &\|\nabla^{(m)}\langle\partial_{11}u_k,E_{u_k}^2\rangle\|_0\leq C\frac{\delta_{k-1}^{1/2}}{\sigma^N}\mu_k^{m+1}\leq C\frac{\delta_k^{1/2}}{\sigma^{N/2+2}}\kappa^{m+1}, \\ &\|\nabla^{(m)}\langle\partial_{12}u_k,E_{u_k}^2\rangle\|_0\leq C\frac{\delta_{k-1}^{1/2}}{\sigma^{N/2}}\mu_k^{m+1}\leq C\frac{\delta_k^{1/2}}{\sigma^2}\kappa^{m+1} \\ &\text{for all}\quad k=1\dots K-1,\quad m=0\dots(3N+6)(K-k). \end{aligned}$$

In view of $(7.57)_2$, this implies:

$$\begin{aligned} &\|\nabla^{(m)}\langle\partial_{11}\bar{U},E_{\bar{U}}^2\rangle\|_0\leq C\frac{\delta_k^{1/2}}{\sigma^{N/2+2}}\kappa^{m+1}+C\delta_k\kappa^{m+1}\sigma^{N/2}\leq C\frac{\delta_k^{1/2}}{\sigma^{N/2+2}}\kappa^{m+1}, \\ &\|\nabla^{(m)}\langle\partial_{12}\bar{U},E_{\bar{U}}^2\rangle\|_0\leq C\frac{\delta_k^{1/2}}{\sigma^2}\kappa^{m+1}+C\delta_k\kappa^{m+1}\sigma^{N/2}\leq C\frac{\delta_k^{1/2}}{\sigma^2}\kappa^{m+1} \\ &\text{for all}\quad k=1\dots K-1,\quad m=0\dots(3N+6)(K-k)-3N-3, \end{aligned}$$

where we have used that $\sigma_k^{1/2}\sigma^{N+2}\leq 1$ by $(1.10)_1$. Consequently, recalling (7.50), we see that:

$$\begin{aligned} &\|\nabla^{(m)}P_0^{\eta_3}(\bar{\bar{S}}_3)\|_0\leq C\|\nabla^{(m)}\big(b\langle\partial_{11}U,E_U^2\rangle\big)\|_0\leq C\frac{\delta_k}{\sigma^{N/2+2}}\kappa^{m+1}, \\ &\|\nabla^{(m)}P_1^{\eta_3}(\bar{\bar{S}}_3)\|_0\leq C\|\nabla^{(m+1)}\big(b\langle\partial_{12}U,E_U^2\rangle\big)\|_0\leq C\frac{\delta_k}{\sigma^2}\kappa^{m+2} \\ &\text{for all } m=0\dots(3N+6)(K-k)-3N-4,\ k=1\dots K-1. \end{aligned}$$

This leads to:

$$\begin{aligned} \|\nabla^{(m)}\bar{\bar{G}}_3\|_0 &\leq C\sum_{p+q=m}\mu_{k+1}^{p-1}\frac{\delta_k}{\sigma^{N/2+2}}\kappa^{q+1}+C\sum_{p+q=m}\mu_{k+1}^{p-2}\frac{\delta_k}{\sigma^2}\kappa^{q+2} \\ &\leq C\frac{\delta_k}{\sigma^{N+2}}\mu_{k+1}^m\qquad\text{for all } m=0\dots(3N+6)(K-k)-3N-4,\ k=1\dots K-1. \end{aligned} \tag{7.75}$$

Now, at $k=0$ we get by (7.6) and $(7.57)_2$:

$$\begin{aligned} \|\nabla^{(m)}\langle\partial_{ij}\bar{U},E_{\bar{U}}^2\rangle\|_0 &\leq\|\nabla^{(m)}((\nabla u_0)^T\nabla E_{u_0}^2)\|_0+C\delta_k\kappa^{m+1}\sigma^{N/2} \\ &\leq C\delta_0^{1/2}\mu_0^{m+1}+C\delta_k\kappa^{m+1}\sigma^{N/2}\leq C\frac{\delta_k^{1/2}}{\sigma^2}\kappa^{m+1} \\ &\text{for all } m=0\dots(3N+6)(K-k)-3N-3,\ i,j=1\dots 2,\ k=0, \end{aligned}$$

so we directly note the same bound as in (7.75):

$$\begin{aligned}&\|\nabla^{(m)}\bar{\bar{G}}_3\|_0 \leq C\sum_{p+q=m}\mu_{k+1}^{p-1}\frac{\delta_k}{\sigma^2}\kappa^{q+1} + C\sum_{p+q=m}\mu_{k+1}^{p-2}\frac{\delta_k}{\sigma^2}\kappa^{q+2} \leq C\frac{\delta_k}{\sigma^{N+2}}\mu_{k+1}^m\\ &\text{for all } m = 0\dots(3N+6)(K-k)-3N-4.\end{aligned}\tag{7.76}$$

In summary, by (7.73), (7.74), (7.75), (7.76) we get:

$$\|\nabla^{(m)}\bar{\bar{G}}\|_0 \leq C\frac{\delta_k}{\sigma^{N+1}}\mu_{k+1}^m \quad \text{for all } m = 0\dots(3N+6)(K-k)-3N-4,\tag{7.77}$$

which yields, upon recalling (7.71) and for σ is sufficiently large:

$$\begin{aligned}&\|\nabla^{(m)}\mathcal{D}(g_0-\delta_{k+1}H_0, u_{k+1})\|_0 \leq C\frac{\delta_k}{\sigma^{N+1}}\mu_{k+1}^m \leq \frac{\delta_k}{\sigma^{N+1/2}}\mu_{k+1}^m\\ &\text{for all } m = 0\dots(3N+6)(K-k)-3N-5.\end{aligned}\tag{7.78}$$

We note that the regularity exponent is consistent with the induction statement, as $(3N+6)(K-k)-3N-5 = 1+(3N+6)(K-(k+1))$.

13. (Propagation of bounds in the third corrugation and closing the bounds $(7.11)_1$, $(7.11)_4$) In the last two steps we will validate all inductive bounds $(7.10)_1$-$(7.11)_4$ at:

$$k+1 = 1\dots K.$$

Note that $(7.10)_5$ has already been shown in (7.78). Using (7.13), $(7.55)_3$, $(7.55)_4$, (7.68) yields:

$$\begin{aligned}\|\nabla^{(m)}(u_{k+1}-\bar{U})\|_0 &\leq C\sum_{p+q+t=m}\mu_{k+1}^{p-1}\|\nabla^{(q)}b\|_0\|\nabla^{(t)}E_{\bar{U}}^2\|_0\\ &+ C\sum_{p+q+t=m}\mu_{k+1}^{p-1}\|\nabla^{(q)}b^2\|_0\|\nabla^{(t)}T_{\bar{U}}\|_0 + C\sum_{p+q=m}\|\nabla^{(p)}T_{\bar{U}}\|_0\|\nabla^{(q)}\bar{\bar{W}}\|_0 \leq C\delta_k^{1/2}\mu_{k+1}^{m-1}.\end{aligned}\tag{7.79}$$

Hence, $(7.10)_2$ at $k+1$ follows directly from the above and (7.29), (7.51). Apply now Lemma 3.4 with $u=\bar{U}$, $v=u_{k+1}$, $\mu=\mu_{k+1}$, $A=\delta_k^{1/2}$. Indeed, in virtue of the bound displayed above, there holds $\|\nabla u_{k+1}-\nabla\bar{U}\|_0 \leq C\delta_k^{1/2} \leq C\underline{\delta}^{1/2} \leq \rho$ provided that $\underline{\delta}$ is sufficiently small. Recalling further (7.54), $(7.55)_2$, $(7.55)_4$, we obtain existence of the orthonormal frame $E_{u_{k+1}}^1, E_{u_{k+1}}^2 \in \mathcal{C}^{(3N+6)(K-k)-3N-5}(\bar{\omega},\mathbb{R}^4)$, namely:

$$(\nabla u_{k+1})^T E_{u_{k+1}}^i = 0, \quad |E_{u_{k+1}}^1| = 1 \quad \text{for} \quad i=1,2 \quad \text{and} \quad \langle E_{u_{k+1}}^1, E_{u_{k+1}}^2\rangle = 0 \quad \text{in } \bar{\omega},$$

satisfying the the second bound below, while the first bound is included in (7.79):

$$\begin{aligned}&\|\nabla^{(m+1)}(u_{k+1}-\bar{U})\|_0 \leq C\delta_k^{1/2}\mu_{k+1}^m\\ &\text{and} \quad \|\nabla^{(m)}(E_{u_{k+1}}^i - E_{\bar{U}}^i)\|_0 \leq C\delta_k^{1/2}\mu_{k+1}^m \quad \text{for } m = 0\dots(3N+6)(K-k)-3N-5.\end{aligned}\tag{7.80}$$

Together with (7.30) and (7.80), we immediately conclude $(7.10)_3$ and $(7.10)_4$ at the counter $k+1$. To show $(7.10)_1$ at $k+1$, recall (7.53) and write:

$$\|u_{k+1}-u\|_1 \leq \|u_{k+1}-\bar{U}\|_1 + \|U-u\|_0 \leq C\delta_0^{1/2},\tag{7.81}$$

which indeed leads, as in step 1 and the proof of (7.3), to:

$$\frac{1}{3\underline{\gamma}}\mathrm{Id}_2 \leq (\nabla u_{k+1})^T\nabla u_{k+1} \leq 3\underline{\gamma}\mathrm{Id}_2 \quad \text{in } \bar{\omega}.\tag{7.82}$$

It remains to show the coordinate-specific bounds $(7.11)_1$-$(7.11)_4$. Using Lemma 3.1 and (7.82), together with (7.80) in view of $(7.55)_2$, $(7.55)_4$, we observe that:

$$\|\nabla u_{k+1}\|_0 \leq (6\underline{\gamma})^{1/2} \tag{7.83$_1$}$$
$$\|\nabla^{(m)}\nabla^{(2)} u_{k+1}\|_0 \leq C\delta_k^{1/2}\mu_{k+1}^{m+1} \quad \text{for } m = 0\ldots(3N+6)(K-k)-3N-6, \tag{7.83$_2$}$$
$$\|\nabla^{(m)} E^i_{u_{k+1}}\|_0 \leq C\delta_k^{1/2}\mu_{k+1}^{m} \quad \text{for } m = 1\ldots(3N+6)(K-k)-3N-5,\ i=1,2. \tag{7.83$_3$}$$

At this point, the estimate $(7.11)_4$ at the counter $k+1$ already follows, directly from the above:

$$\begin{aligned}&\|\nabla^{(m)}\big((\nabla u_{k+1})^T\nabla E^1_{u_{k+1}}\big)\|_0 \leq C\sum_{p+q=m}\mu_{k+1}^p\delta_k^{1/2}\mu_{k+1}^{q+1} \leq C\delta_k^{1/2}\mu_{k+1}^{m+1}\\ &\text{for all } m = 0\ldots(3N+6)(K-k)-3N-6.\end{aligned}$$

To demonstrate the finer bounds $(7.11)_1$-$(7.11)_3$, we decompose, for any $i,j,s=1\ldots2$:

$$\langle\partial_{ij}u_{k+1}, E^s_{u_{k+1}}\rangle = \langle\partial_{ij}\bar{U}, E^s_{\bar{U}}\rangle + \langle\partial_{ij}(u_{k+1}-\bar{U}), E^s_{\bar{U}}\rangle + \langle\partial_{ij}u_{k+1}, (E^s_{u_{k+1}} - E^s_{\bar{U}})\rangle. \tag{7.84}$$

For the third term above, we recall (7.80) and $(7.83)_2$ and get:

$$\begin{aligned}&\|\nabla^{(m)}\langle\partial_{ij}u_{k+1}, (E^s_{u_{k+1}} - E^s_{\bar{U}})\rangle\|_0 \leq C\delta_k\mu_{k+1}^{m+1}\\ &\text{for } m = 0\ldots(3N+6)(K-k)-3N-6,\ i,j,s=1\ldots2.\end{aligned} \tag{7.85}$$

For the second term in the right hand side of (7.84) and in the specific case of $s=1$, we write:

$$\begin{aligned}\langle\partial_{ij}(u_{k+1}-\bar{U}), E^1_{\bar{U}}\rangle = {}& \partial_i\Big(\frac{\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}b\Big)\langle\partial_j E^2_{\bar{U}}, E^1_{\bar{U}}\rangle + \partial_j\Big(\frac{\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}b\Big)\langle\partial_i E^2_{\bar{U}}, E^1_{\bar{U}}\rangle\\ &+ \frac{\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}b\langle\partial_{ij}E^2_{\bar{U}}, E^1_{\bar{U}}\rangle + \Big\langle\partial_{ij}\Big(T_{\bar{U}}\Big(\frac{\bar{\Gamma}(\mu_{k+1}x_2)}{\mu_{k+1}}b^2e_2 + \bar{\bar{W}}\Big)\Big), E^1_{\bar{U}}\Big\rangle,\end{aligned}$$

using that $\langle E^1_{\bar{U}}, E^2_{\bar{U}}\rangle = 0$. Consequently, by (7.13), $(7.55)_3$, $(7.55)_4$, (7.68):

$$\begin{aligned}&\|\nabla^{(m)}\langle\partial_{ij}(u_{k+1}-\bar{U}), E^1_{\bar{U}}\rangle\|_0\\ &\leq C\sum_{p+q+t=m}\Big\|\nabla^{(p+1)}\Big(\frac{\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}b\Big)\Big\|_0\|\nabla^{(q+1)}E^2_{\bar{U}}\|_0\|\nabla^{(t)}E^1_{\bar{U}}\|_0\\ &\quad+\sum_{p+q+t=m}\Big\|\nabla^{(p)}\Big(\frac{\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}b\Big)\Big\|_0\|\nabla^{(q+2)}E^2_{\bar{U}}\|_0\|\nabla^{(t)}E^1_{\bar{U}}\|_0\\ &\quad+C\sum_{p+q=m}\ \sum_{t+s=p+2}\|\nabla^{(t)}T_{\bar{U}}\|_0\Big\|\nabla^{(s)}\Big(\frac{\bar{\Gamma}(\mu_{k+1}x_2)}{\mu_{k+1}}b^2e_2+\bar{\bar{W}}\Big)\Big\|_0\|\nabla^{(q)}E^1_{\bar{U}}\|_0\\ &\leq C\delta_k\mu_{k+1}^{m+1} \quad \text{for all } m=0\ldots(3N+6)(K-k)-3N-6,\ i,j=1\ldots2.\end{aligned} \tag{7.86}$$

In conclusion, (7.84), (7.85) and $(7.57)_1$ imply:

$$\begin{aligned}\|\nabla^{(m)}\langle\partial_{ij}u_{k+1}, E^1_{u_{k+1}}\rangle\|_0 &\leq \|\nabla^{(m)}\langle\partial_{ij}\bar{U}, E^1_{\bar{U}}\rangle\|_0 + C\delta_k\mu_{k+1}^{m+1}\\ &\leq \|\nabla^{(m)}\langle\partial_{ij}u_k, E^1_{u_k}\rangle\|_0 + C\delta_k^{1/2}\kappa^{m+1} + C\delta_k\mu_{k+1}^{m+1}\\ &\leq \|\nabla^{(m)}\langle\partial_{ij}u_k, E^1_{u_k}\rangle\|_0 + C\frac{\delta_k^{1/2}}{\sigma^{N/2}}\mu_{k+1}^{m+1}.\end{aligned} \tag{7.87}$$

From $(7.11)_1$ we recall that:

$$\begin{aligned}&\|\nabla^{(m)}\langle\partial_{ij}u_k, E^1_{u_k}\rangle\|_0 \leq C\frac{\delta_{k-1}^{1/2}}{\sigma^{N/2}}\mu_k^{m+1} \leq C\frac{\delta_k^{1/2}}{\sigma^{N/2}}\mu_{k+1}^{m+1}\\ &\text{for all } m = 0\ldots(3N+6)(K-k)-3N-6,\ i,j = 1\ldots 2,\ k = 1\ldots K-1,\end{aligned}$$

whereas at $k=0$ we likewise get by (7.6):

$$\begin{aligned}&\|\nabla^{(m)}\langle\partial_{ij}u_k, E^s_{u_k}\rangle\|_0 \leq C\delta_0^{1/2}\mu_0^{m+1} \leq C\frac{\delta_k^{1/2}}{\sigma^N}\mu_{k+1}^{m+1}\\ &\text{for all } m = 0\ldots(3N+6)K,\ i,j,s = 1\ldots 2,\ k = 0.\end{aligned} \tag{7.88}$$

Combining with (7.87), in either case we obtain the validity of $(7.11)_1$ at the $k+1$ counter.

14. (Closing the inductive bounds $(7.11)_2$, $(7.11)_3$) Finally, for the second term in the right hand side of (7.84) and in the specific case of $s=2$, we write:

$$\begin{aligned}\langle\partial_{ij}(u_{k+1}-\bar U), E^2_{\bar U}\rangle = {}& \partial_{ij}\Big(\frac{\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}b\Big) - \frac{\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}b\langle\partial_j E^2_{\bar U}, \partial_j E^1_{\bar U}\rangle\\ & + \Big\langle\partial_{ij}\Big(T_{\bar U}\Big(\frac{\bar\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}b^2e_2 + \bar{\bar W}\Big)\Big), E^2_{\bar U}\Big\rangle,\end{aligned} \tag{7.89}$$

where we used that $\langle E^2_{\bar U}, E^2_{\bar U}\rangle = 1$. Derivatives of the last two terms in the right hand side of (7.89) may be bounded as in (7.86), by (7.50), $(7.55)_3$, $(7.55)_4$, (7.68):

$$\begin{aligned}&\Big\|\nabla^{(m)}\Big(\frac{\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}b\langle\partial_j E^2_{\bar U}, \partial_j E^1_{\bar U}\rangle\Big)\Big\|_0\\ &+ \Big\|\nabla^{(m)}\Big(\Big\langle\partial_{ij}\Big(T_{\bar U}\Big(\frac{\bar\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}b^2e_2 + \bar{\bar W}\Big)\Big), E^2_{\bar U}\Big\rangle\Big)\Big\|_0 \leq C\delta_k\mu_{k+1}^{m+1}.\end{aligned} \tag{7.90}$$

We analyze the first term in the right hand side of (7.89):

$$\begin{aligned}\Big\|\nabla^{(m)}\partial_{11}\Big(\frac{\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}b\Big)\Big\|_0 &\leq C\sum_{p+q=m}\mu_{k+1}^{p-1}\|\nabla^{(q+2)}b\|_0\\ &\leq C\frac{\delta_k^{1/2}}{(\mu_{k+1}/\kappa)^2}\mu_{k+1}^{m+1} \leq C\frac{\delta_k^{1/2}}{\sigma^N}\mu_{k+1}^{m+1},\\ \Big\|\nabla^{(m)}\partial_{12}\Big(\frac{\Gamma(\mu_{k+1}x_2)}{\mu_{k+1}}b\Big)\Big\|_0 &\leq C\sum_{p+q=m}\mu_{k+1}^{p-1}\|\nabla^{(q+2)}b\|_0 + C\sum_{p+q=m}\mu_{k+1}^{p}\|\nabla^{(q+1)}b\|_0\\ &\leq C\frac{\delta_k^{1/2}}{(\mu_{k+1}/\kappa)}\mu_{k+1}^{m+1} \leq C\frac{\delta_k^{1/2}}{\sigma^{N/2}}\mu_{k+1}^{m+1}.\end{aligned} \tag{7.91}$$

In conclusion, (7.84), (7.85), (7.89), (7.90), (7.91) and having $\delta_k^{1/2}\sigma^N \leq 1$, imply:

$$\begin{aligned}&\|\nabla^{(m)}\langle\partial_{11}u_{k+1}, E^2_{u_{k+1}}\rangle\|_0 \leq \|\nabla^{(m)}\langle\partial_{11}u_k, E^2_{u_k}\rangle\|_0 + C\frac{\delta_k^{1/2}}{\sigma^N}\mu_{k+1}^{m+1},\\ &\|\nabla^{(m)}\langle\partial_{12}u_{k+1}, E^2_{u_{k+1}}\rangle\|_0 \leq \|\nabla^{(m)}\langle\partial_{12}u_k, E^2_{u_k}\rangle\|_0 + C\frac{\delta_k^{1/2}}{\sigma^{N/2}}\mu_{k+1}^{m+1}\\ &\text{for all } m = 0\ldots(3N+6)(K-k)-3N-6.\end{aligned} \tag{7.92}$$

We now claim that both first terms in the right hand sides above are bounded by the respective second terms. Indeed, for $k \geq 1$ this follows from $(7.11)_2$, $(7.11)_3$, namely:

$$\begin{aligned}
&\|\nabla^{(m)}\langle\partial_{11}u_k, E^2_{u_k}\rangle\|_0 \leq C\frac{\delta_{k-1}^{1/2}}{\sigma^N}\mu_k^{m+1} \leq C\frac{\delta_k^{1/2}}{\sigma^N}\mu_{k+1}^{m+1},\\
&\|\nabla^{(m)}\langle\partial_{12}u_k, E^2_{u_k}\rangle\|_0 \leq C\frac{\delta_{k-1}^{1/2}}{\sigma^{N/2}}\mu_k^{m+1} \leq C\frac{\delta_k^{1/2}}{\sigma^{N/2}}\mu_{k+1}^{m+1},\\
&\text{for all } m = 0\dots(3N+6)(K-k),\ k = 1\dots K-1,
\end{aligned}$$

whereas at $k = 0$ the same bounds follow by (7.88). Therefore, (7.92) become:

$$\begin{aligned}
&\|\nabla^{(m)}\langle\partial_{11}u_{k+1}, E^2_{u_{k+1}}\rangle\|_0 \leq C\frac{\delta_k^{1/2}}{\sigma^N}\mu_{k+1}^{m+1},\\
&\|\nabla^{(m)}\langle\partial_{12}u_{k+1}, E^2_{u_{k+1}}\rangle\|_0 \leq C\frac{\delta_k^{1/2}}{\sigma^{N/2}}\mu_{k+1}^{m+1} \qquad \text{for all } m = 0\dots(3N+6)(K-k)-3N-6,
\end{aligned}$$

which is exactly $(7.11)_2$ and $(7.11)_3$ at the counter $k+1$. The proof of Theorem 1.2 is complete. ∎

8. Nash-Kuiper's scheme: the proof of Theorem 1.3

In this section we exhibit the details of the Nash-Kuiper iteration scheme. The proof uses the double exponential ansatz on the progression of frequencies and defect measures that appeared in [17], and was similarly applied in [10, proof of Theorem 1.1].

Proof of Theorem 1.3

1. Fix α as in (1.13) and $\epsilon \in (0,1)$. We will obtain $\bar{u}$ as the limit of $\{u_n \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^4)\}_{n=1}^\infty$ converging in $\mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^4)$. These immersions will be constructed iteratively, starting from u_0 obtained by Theorem 4.1, and then by successive application of (1.12) with:

$$u = u_n, \qquad \delta = \delta_n, \qquad \mu = \mu_n, \qquad \sigma = \sigma_{n+1},$$

producing $\tilde{u} = u_{n+1}$. The progression of the above parameters is as follows. Fix θ and τ in:

$$\alpha < \theta < \min\Big\{\frac{r+\beta}{2}, \frac{1}{1+2J/S}\Big\}, \qquad \tau = \underline{\tau} + 1, \tag{8.1}$$

where $\underline{\tau}$ is as in Theorem 4.1. In particular, there holds:

$$\frac{J}{S} < \frac{1}{2\theta} - \frac{1}{2} \quad \text{ and } \quad \tau > 2 + \frac{1}{r+\beta}. \tag{8.2}$$

We define for all $n \geq 0$:

$$\delta_n = \frac{1}{a^{b^n}}, \quad \mu_n = a^{\tau + \frac{b^n - 1}{2\theta}}, \quad \sigma_{n+1} = \Big(\frac{\delta_n}{\delta_{n+1}}\Big)^{1/S} \text{ so that } \delta_{n+1} = \frac{\delta_n}{\sigma_{n+1}^S}, \tag{8.3}$$

for some $a, b > 1$, whose magnitudes are specified by the requirements in the course of the proof. In general, $b - 1 > 0$ will be sufficiently small and a sufficiently large (in that order).

2. From (8.3) we read the initial parameters:

$$\delta_0 = \frac{1}{a}, \qquad \mu_0 = a^\tau.$$

For a large so that $\delta_0 < \underline{\delta}$, we apply Theorem 4.1 obtaining an immersion $u_0 \in \mathcal{C}^2(\bar{\omega}, \mathbb{R}^4)$ in:

$$\begin{aligned} & \frac{1}{\underline{\gamma}}\mathrm{Id}_2 \leq (\nabla u_0)^T \nabla u_0 \leq \underline{\gamma}\mathrm{Id}_2 \quad \text{ in } \bar{\omega}, \\ & \|\mathcal{D}(g - \delta_0 H_0, u_0)\|_0 \leq \frac{r_0}{4}\delta_0, \\ & \|u_0 - \underline{u}\|_0 \leq \frac{r_0}{4}\delta_0 \leq \frac{\epsilon}{2} \\ & \|u_0\|_2 \leq \|\underline{u}\|_0 + \Big(C\delta_0 + C + \frac{C}{\delta_0^{\underline{\tau}}}\Big) \leq \frac{C}{\delta_0^{\underline{\tau}}} \leq \delta_0^{1/2}\mu_0, \end{aligned} \tag{8.4}$$

where the third bound again follows for a large enough (in function of ϵ) and the fourth bound in view of (8.1) because:

$$\delta_0^{1/2+\underline{\tau}}\mu_0 = a^{\tau-1/2-\underline{\tau}} = a^{1/2} \geq C,$$

likewise for a sufficiently large (in function of the constant C above that depends only on $\omega, \underline{u}, g$). As a byproduct, we get: $\delta_0^{1/2}\mu_0 \geq 1$.

3. We will show that if $b - 1 > 0$ is sufficiently small and a is sufficiently large (in that order) then one can proceed with the induction on n. We first show that for all $n \geq 0$:

$$\sigma_{n+1} \geq \underline{\sigma}, \qquad \sigma_{n+1}^p \delta_n^{1/2} \leq 1, \qquad \delta_{n+1} \leq \underline{\delta}, \qquad \mu_{n+1}\delta_{n+1}^{1/2} \geq 1, \tag{8.5$_1$}$$

$$\frac{r_0}{5}\frac{\delta_n}{\sigma_{n+1}^S} + \frac{\|g\|_{r,\beta}}{\mu_n^{r+\beta}} \leq \frac{r_0}{4}\delta_{n+1}, \tag{8.5$_2$}$$

$$C\mu_n\delta_n^{1/2}\sigma_{n+1}^J \leq \delta_{n+1}^{1/2}\mu_{n+1}, \tag{8.5$_3$}$$

where C is the constant appearing in (1.12). In order to show the four bounds in $(8.5)_1$, we use the definition (8.3). The first of the claimed bounds follows when a is sufficiently large (after possibly choosing $b - 1$ sufficiently small):

$$\sigma_{n+1} = a^{(b^{n+1}-b^n)/S} = a^{b^n(b-1)/S} \geq a^{(b-1)/S} \geq \underline{\sigma}.$$

For the second bound, we have:

$$\sigma_{n+1}^p\delta_n^{1/2} = a^{b^n((b-1)p/S-1/2)} \leq 1,$$

if only $(b-1)p/S \leq 1/2$, which is assured by taking $b-1$ small. The third bound is clear since $\delta_{n+1} \leq \delta_0 \leq \underline{\sigma}$, while the fourth bound is due to:

$$\mu_{n+1}\delta_{n+1}^{1/2} = a^{\tau+b^{n+1}((\frac{1}{2\theta}-\frac{1}{2})-\frac{1}{2\theta}} \geq a^{\tau+b((\frac{1}{2\theta}-\frac{1}{2})-\frac{1}{2\theta}} = a^{(\tau-\frac{1}{2})+(b-1)(\frac{1}{2\theta}-\frac{1}{2})} \geq 1$$

since $\frac{1}{2\theta} - \frac{1}{2} \geq 0$, so that, for $b-1$ sufficiently small, the positive term $\tau - \frac{1}{2}$ in last exponent above prevails. This ends the proof of $(8.5)_1$. To show $(8.5)_2$, we observe that, by definition:

$$\frac{r_0}{5}\frac{\delta_n}{\sigma_{n+1}^S} = \frac{r_0}{5}\delta_{n+1},$$

and thus it suffices to justify the smallness of $\frac{\|g\|_{r,\beta}}{\mu_n^{r+\beta}}\frac{1}{\delta_{n+1}}$, equivalent to the largeness of $\delta_{n+1}\mu_n^{r+\beta}$ (in function of $\omega, \underline{u}, g, \theta$). Observe that $\frac{r+\beta}{2\theta} - 1 > 0$ by the upper bound on θ in (8.1). Therefore:

$$\begin{aligned} \delta_{n+1}\mu_n^{r+\beta} &= a^{(\tau+\frac{b^n-1}{2\theta})(r+\beta)-b^{n+1}} = a^{b^n(\frac{r+\beta}{2\theta}-b)+(\tau-\frac{1}{2\theta}(r+\beta)} \\ &\geq a^{(\frac{r+\beta}{2\theta}-b)+(\tau-\frac{1}{2\theta}(r+\beta)} = a^{\tau(r+\beta)-b} = a^{\tau(r+\beta)-1-(b-1)}, \end{aligned}$$

for $b-1$ small. The above right hand side is as large as one wants, again by taking $b-1$ small and a large (in that order), because $\tau(r+\beta)-1>0$ by the second inequality in (8.2). This ends the proof of $(8.5)_2$. Finally, $(8.5)_3$ is equivalent to the largeness of:

$$\frac{\mu_{n+1}}{\mu_n}\big(\frac{\delta_{n+1}}{\delta_n}\big)^{1/2}\frac{1}{\sigma_{n+1}^J} = a^{b^n\frac{b-1}{2\theta}-b^n(b-1)/2-b^n(b-1)J/S} = a^{b^n(b-1)(\frac{1}{2\theta}-\frac{1}{2}-\frac{J}{S})} \geq a^{(b-1)(\frac{1}{2\theta}-\frac{1}{2}-\frac{J}{S})},$$

where we used the first inequality in (8.2). Clearly, for a large (after $b-1>0$ has been priorly fixed), the right hand side above is as large as desired. This ends the proof of $(8.5)_3$.

4. We thus see that as long as:

$$\frac{1}{2\underline{\gamma}} \leq (\nabla u_n)^T\nabla u_n \leq 2\underline{\gamma}\mathrm{Id}_2 \quad \text{ in } \bar{\omega}, \tag{8.6}$$

one can define u_{n+1} by the indicated application of (1.12). Noting the bound $b^i\geq 1+(\ln b)i$ valid for all $i\geq 0$ and $b\geq 1$, we then get:

$$\begin{aligned}\|u_{n+1}-u_0\|_1 &\leq C\sum_{i=1}^{n+1}\delta_i^{1/2} = C\sum_{i=1}^{n+1}\frac{1}{a^{b^i/2}} \leq C\sum_{i=1}^{n+1}\Big(\frac{1}{a}\Big)^{1/2+(\ln b)i/2}\\ &\leq \frac{C}{a^{1/2}}\sum_{i=1}^{\infty}\Big(\frac{1}{a}\Big)^{(\ln b)i/2} = \frac{C}{a^{1/2}(a^{(\ln b)/2}-1)} \leq \frac{\epsilon}{2},\end{aligned} \tag{8.7}$$

where the last inequality is valid for a sufficiently large. The above in particular implies:

$$\begin{aligned}&\|(\nabla u_n)^T\nabla u_n - (\nabla u_0)^T\nabla u_0\|_0 \leq \|\nabla(u_n-u_0)\|_0\big(\|\nabla u_n\|_0+\|\nabla u_0\|\big)\\ &\leq \|\nabla(u_n-u_0)\|_0\big(2\|\nabla u_0\|_0+\|\nabla(u_n-u_0)\|\big) \leq \|u_{n+1}-u_n\|_1(C+1)\leq C\epsilon,\end{aligned}$$

which means that for ϵ small enough (or, more generally, for a sufficiently large a), the immersion estimate (8.6) is valid for u_{n+1} in virtue of the first condition in (8.4). Consequently, the infinite sequence $\{u_n\in\mathcal{C}^2(\bar{\omega},\mathbb{R}^4)\}_{n=0}^{\infty}$ is well defined and it satisfies, for all $n\geq 0$:

$$\begin{aligned}&\|u_{n+1}-u_n\|_1\leq C\delta_n^{1/2}, \qquad \|u_n\|_2\leq \delta_n^{1/2}\mu_n,\\ &\|\mathcal{D}(g-\delta_nH_0,u_n)\|_0\leq \frac{r_0}{4}\delta_n.\end{aligned} \tag{8.8}$$

5. We are now ready to conclude the proof. Firstly, we show that $\{u_n\}_{n=0}^{\infty}$ is Cauchy, hence converges in $\mathcal{C}^{1,\alpha}(\bar{\omega},\mathbb{R}^4)$. To this end, we use the interpolation inequality in Hölder spaces:

$$\|u_{n+1}-u_n\|_{1,\alpha}\leq C\|u_{n+1}-u_n\|_2^{\alpha}\|u_{n+1}-u_n\|_1^{1-\alpha}.$$

Since $\{\delta_n^{1/2}\mu_n\}_{n=1}^{\infty}$ is an increasing sequence, the above and (8.8) imply for all $n\geq 0$:

$$\begin{aligned}\|u_{n+1}-u_n\|_{1,\alpha} &\leq C(\delta_{n+1}^{1/2}\mu_{n+1})^{\alpha}\delta_n^{(1-\alpha)/2}\\ &\leq C\big(a^{\tau+\frac{b^{n+1}-1}{2\theta}-b^{n+1}/2}\big)^{\alpha}a^{-b^n(1-\alpha)/2}\leq Ca^{(\tau-\frac{1}{2\theta})\alpha}a^{-b^nq},\end{aligned} \tag{8.9}$$

where $q>0$ is a positive constant, independent of n, given in:

$$q=\frac{1-\alpha}{2}-b\alpha\big(\frac{1}{2\theta}-\frac{1}{2}\big)=\frac{1}{2}\big(1-\frac{\alpha}{\theta}\big)-(b-1)\alpha\big(\frac{1}{2\theta}-\frac{1}{2}\big)>0,$$

if only $b-1$ is sufficiently small, since $1-\alpha/\theta>0$ in virtue of the lower bound in (8.1). Consequently, (8.9) implies the aforementioned Cauchy property in:

$$\|u_{n+1}-u_n\|_{1,\alpha}\leq Ca^{(\tau-\frac{1}{2\theta})\alpha}a^{-q(1+(\ln b)n)} = Ca^{(\tau-\frac{1}{2\theta})\alpha-q}\big(a^{\ln b}\big)^{-n},$$

where the terms in the right hand side above constitutes a converging power series. In conclusion, $\bar{u} \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^4)$ may be defined as the limit:

$$u_n \to \bar{u} \qquad \text{as } \ n \to \infty \quad \text{in } \ \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^4).$$

By (8.9) combined with the third statement in (8.4) there holds:

$$\|\bar{u} - \underline{u}\|_0 \leq \|\bar{u} - u_0\|_0 + \|u_0 - \underline{u}\|_0 \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

while by the last statement in (8.8) we obtain:

$$\|\mathcal{D}(g, u_n)\|_0 \leq \|\mathcal{D}(g - \delta_n H_0, u_n)\|_0 + \delta_n |H_0| \leq \Big(\frac{r_0}{4} + |H_0|\Big)\delta_n \to 0 \quad \text{as } \ n \to \infty.$$

Since the left hand side above clearly converges to $\|\mathcal{D}(g, \bar{u})\|_0$ as $n \to \infty$, there follows:

$$\mathcal{D}(g, \bar{u}) = 0 \quad \text{ in } \ \bar{\omega}.$$

This ends the proof of Theorem 1.3. ■

9. Appendix: Poznyak's theorem

For completeness, we present Poznyak's theorem and sketch its proof.

Theorem 9.1. [41, Theorem 4] *Let $\omega \subset \mathbb{R}^2$ be an open, bounded and simply connected set, and let $g \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{2\times 2}_{\mathrm{sym},>})$. Then, there exists $u \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^4)$ which solves (1.1) on $\bar{\omega}$.*

In the proof, the smooth immersion u satisfying (1.1) is sought to be of the following form, for some $\epsilon > 0$ and $w \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R})$, $v \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^2)$:

$$u(x) = \epsilon e^{w(x)} \Big(\cos \frac{v^1(x)}{\epsilon}, \sin \frac{v^1(x)}{\epsilon}, \cos \frac{v^2(x)}{\epsilon}, \sin \frac{v^2(x)}{\epsilon} \Big).$$

By a direct calculation one easily obtains:

$$(\nabla u)^T \nabla u = e^{2w} (\nabla v)^T \nabla v + 2\epsilon^2 e^{2w} \nabla w \otimes \nabla w.$$

Consequently, (1.1) is equivalent to:

$$(\nabla v)^T \nabla v = \tilde{g} \qquad \text{where } \ \tilde{g} \doteq e^{-2w} g - 2\epsilon^2 \nabla w \otimes \nabla w.$$

It now suffices to check that there exists $\epsilon > 0$ and $w \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R})$, such that the derived matrix field $\tilde{g} \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ is itself a Riemannian metric with zero Gaussian curvature:

$$\tilde{g} > 0 \quad \text{ and } \quad \kappa(\tilde{g}) = 0 \qquad \text{in } \ \bar{\omega}. \tag{9.1}$$

For $\epsilon = 0$, existence of a smooth w_0 such that $e^{-2w_0} g$ satisfies the second condition (since the first one holds trivially) in (9.1), is a consequence of the conformal equivalence of g with the Euclidean metric. In particular, one can take w_0 to be the solution of the Dirichlet problem:

$$\Delta_g w_0 = -\kappa(g) \quad \text{ in } \ \omega, \qquad\qquad w_0 = 0 \quad \text{ on } \ \partial\omega.$$

By applying the implicit function theorem, and in view of the formula for $\kappa(\tilde{g})$ in terms of $\kappa(g), \epsilon, w$, there follows in fact the existence of a one-parameter family $\epsilon \mapsto w(\epsilon)$ resulting in the validity of (9.1) for all $\epsilon > 0$ sufficiently small. This achieves the result in Theorem 9.1. ■

It would be interesting to see whether the main result of this paper could be recovered by applying convex integration directly to find w that solves $\kappa(\tilde{g}) = 0$.

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M.L.: University of Pittsburgh, Department of Mathematics, 139 University Place, Pittsburgh, PA 15260
E-mail address: lewicka@pitt.edu