THE ROBIN MEAN VALUE EQUATION II: ASYMPTOTIC HÖLDER REGULARITY

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Abstract. We show that solutions to the Robin mean value equations (RMV), introduced in [19], converge uniformly in the limit of the vanishing radius of averaging, to the unique solution of the Robin-Laplace boundary value problem (RL), posed on any $\mathcal{C}^{1,1}$-regular domain and with any bounded Borel right hand side. When compared with the case of continuous right hand side, analyzed in [19], the present more general setting presents significant technical challenges.

Along the way, we prove the asymptotic Hölder equicontinuity of solutions to (RMV): Lipschitz in the interior and $\mathcal{C}^{0,\alpha}$ up to the boundary, for any $\alpha \in (0,1)$. Our proofs employ martingale techniques, where (RMV) is interpreted as the dynamic programming principle for a discrete stochastic process, interpolating between the reflecting and the stopped-at-exit Brownian walks.

1. Introduction

In this paper, we continue the analysis of the *Robin mean value equation*, introduced in [19]. More precisely, we study the following family of integral problems, parametrised by $\epsilon \to 0+$:

$$(\mathrm{RMV})_\epsilon \qquad u_\epsilon(x) = \big(1 - \gamma s_\epsilon(x)\big) \fint_{B_\epsilon(x)\cap\mathcal{D}} u_\epsilon(y)\;\mathrm{d}y + \frac{\epsilon^2}{2(N+2)} f(x) \qquad \text{for all } x \in \bar{\mathcal{D}},$$

which are posed on a bounded domain $\mathcal{D} \subset \mathbb{R}^N$, with a given bounded, Borel function f, a positive constant γ, and s_ϵ appropriately given in (2.1). The weights s_ϵ interpolate between 0 in the interior points of $\mathcal{D}$ that are distanced from $\partial\mathcal{D}$ at least by ϵ, and a quantity of order ϵ on $\partial\mathcal{D}$. It has been shown in [19] that for f continuous, the unique solutions to $(\mathrm{RMV})_\epsilon$, in the limit of vanishing averaging radii $\epsilon \to 0$, approximate the solutions to the *Robin-Laplace problem*:

$$(\mathrm{RL}) \qquad -\Delta u = f \quad \text{in } \mathcal{D}, \qquad \frac{\partial u}{\partial \vec{n}} + \gamma u = 0 \quad \text{on } \partial\mathcal{D}.$$

The main purpose of this paper is to extend the approximation result $u_\epsilon \rightrightarrows u$ in $\bar{\mathcal{D}}$ to all bounded, Borel f-s. As a parallel statement, we also prove the asymptotic Hölder equicontinuity of the family $\{u_\epsilon\}_{\epsilon\to 0}$: Lipschitz in the interior and $\mathcal{C}^{0,\alpha}$ up to the boundary of $\mathcal{D}$, for any $\alpha \in (0,1)$.

We work under the following basic hypotheses:

$$(\mathrm{BH}) \qquad \left[\begin{array}{l}\text{The nonempty set } \mathcal{D} \subset \mathbb{R}^N \text{ is open, bounded, connected and of regularity } \mathcal{C}^{1,1}. \text{ The}\\ \text{function } f : \bar{\mathcal{D}} \to \mathbb{R} \text{ is bounded and Borel. The coefficient } \gamma > 0 \text{ is a positive constant.}\end{array}\right.$$

Recall that $\mathcal{D}$ being $\mathcal{C}^{1,1}$ regular signifies that $\partial\mathcal{D}$ is locally a graph of a $\mathcal{C}^{1,1}$ function, which is equivalent to the uniform (two-sided) supporting sphere condition; see (2.2) and [18] for details.

The following is the main result of this paper:

Theorem 1.1. *Assume (BH). Then $\{u_\epsilon\}_{\epsilon\to 0}$ converges uniformly on $\bar{\mathcal{D}}$ to $u \in \mathcal{C}(\bar{\mathcal{D}})$ that is the unique $W^{2,p}(\mathcal{D})$ solution to (RL).*

Key words and phrases. Robin problem, third boundary value problem, oblique boundary conditions, dynamic programming principle, random walks, finite difference approximations, viscosity solutions.

We point out that [20, Theorem 6.30] if $\mathcal{D}$ has regularity $\mathcal{C}^{1,\alpha}$ and $f\in L^p(\mathcal{D})$ with $p\in (1,\frac{1}{1-\alpha})$ then the unique solution to (RL) has regularity $u\in W^{2,p}(\mathcal{D})$. Consequently, under (BH) there holds: $u\in W^{2,p}(\mathcal{D})$ for any $p\in(1,\infty)$ and thus $u\in\mathcal{C}^{1,\alpha}(\bar{\mathcal{D}})$ for any $\alpha\in(0,1)$. The L^1 convergence of $\{u_\epsilon\}_{\epsilon\to 0}$ in Theorem 1.1 can be deduced by improving the direct estimates in [19], whereas the uniform convergence of these, in general, discontinuous approximants follows from:

Theorem 1.2. *Assume (BH). There exists $\delta_0 \ll 1$ such that for every $\delta\in(0,\delta_0)$ there is $\bar\epsilon>0$ with the following property. For all $\epsilon<\bar\epsilon$ and all $x_0,y_0\in\bar{\mathcal{D}}$ satisfying $|x_0-y_0|\leq\delta$, there holds:*

$$\big|u_\epsilon(x_0)-u_\epsilon(y_0)\big|\leq\frac{C\delta}{\mathrm{dist}(x_0,\mathcal{D})}\quad \text{and}\quad \big|u_\epsilon(x_0)-u_\epsilon(y_0)\big|\leq C\delta\Big(\log\frac{1}{\delta}\Big).$$

The constant C above may depend on $\mathcal{D}$, $\|f\|_{L^\infty(\mathcal{D})}$ and γ, but not on ϵ, x_0, y_0 or δ. [1]

Our proof of Theorem 1.2 is quite delicate, in particular we use martingale techniques involving various couplings of random walks and yielding estimates on the probabilistic representations of u_ϵ. Indeed, $(\mathrm{RMV})_\epsilon$ can be naturally interpreted as the dynamic programming principle along a discrete stochastic process $\{X_n^\epsilon\}_{n=0}^\infty$, which samples uniformly on the truncated balls $B_\epsilon(X_n^\epsilon)\cap\mathcal{D}$, and stops with probability $\gamma s_\epsilon(X_n^\epsilon)$ at each X_n^ϵ; the process accumulates values of f until its stopping time τ^ϵ. Alternatively, one can consider an infinite horizon process, where the accumulation procedure never stops, but the consecutive evaluations of f are instead weighted by the probability of not having had the opportunity to stop. As shown in [19], each u_ϵ has thus two representations:

$$(\mathrm{DPP})_\epsilon\qquad \begin{aligned} u_\epsilon(x) &= \frac{\epsilon^2}{2(N+2)}\mathbb{E}\Big[\sum_{n=0}^{\tau^{\epsilon,x}-1}\big(f\circ X_n^{\epsilon,x}\big)\Big]\\ &= \frac{\epsilon^2}{2(N+2)}\mathbb{E}\Big[\sum_{n=0}^{\infty}\big(f\circ X_n^{\epsilon,x}\big)\prod_{j=1}^{n}\big(1-\gamma s_\epsilon(X_j^{\epsilon,x})\big)\Big].\end{aligned}$$

Our approach suggests how to view more general, nonlinear operators (like Δ_p, Δ_∞) subject to the oblique-type boundary conditions, through their local averaged approximations. This approach has been previously successfully employed in the Dirichlet and Neumann cases [29, 28, 24, 1, 9, 17]. We also mention the recent papers [16, 23, 2] where the internal Hölder and Lipschitz estimates on solutions to the mean value equation related to the Dirichlet problem for Δ_p are derived.

1.1. **The structure of this paper.** In section 2 we recall the origin of the coefficient $s_\epsilon(x)$ in $(\mathrm{RMV})_\epsilon$ and quote the results obtained in [19]. In section 3 we recall the definition of the process $\{X_n^{\epsilon,x_0}\}_{n=0}^\infty$ in (3.2) and construct its stationary measure in Lemma 3.1, which is equivalent to the Lebesgue measure but more natural for the L^p estimates. In section 4 we repeat the probabilistic representations $(\mathrm{RMV})_\epsilon$. Section 5 we begin the discussion of convergence of $\{u_\epsilon\}_{\epsilon\to 0}^\infty$. We first show, in Theorem 5.1, that the associations $f\mapsto u_\epsilon$ are uniformly bounded as linear maps from L^p to L^1, with a bound that only depends on a given $p\in(1,\infty)$. By density of continuous functions f in L^∞ with respect to L^p norm, it immediately follows that $\{u_\epsilon\}_{\epsilon\to 0}^\infty$ converges in L^1 to some limit u. This unique limit is identified as the $W^{2,p}$ solution of (RL) in Theorem 5.2, where we use the estimate on the reminder term i the Taylor expansion of u, from [19].

Sections 6 - 10 are devoted to proving Theorem 1.2. In section 6 we treat the interior case and show the asymptotic Lipschitz continuity of $\{u_\epsilon\}_{\epsilon\to 0}$ away from $\partial\mathcal{D}$. To this end, we use reflection coupling of random walks in Lemma 6.4, following [21], and employ an iterative argument in the proof of Theorem 6.1. In section 8 we show the asymptotic Hölder equicontinuity of $\{u_\epsilon\}_{\epsilon\to 0}$ in the

[1] The dependence of the constant C in Theorems 1.2, 6.1, 8.1 and 10.1 on $\mathcal{D}$ involves only the maximal radius r in the two-sided supporting sphere condition and the diameter of $\mathcal{D}$.

region close to $\partial\mathcal{D}$, with the exponent $1/2$. This more delicate property is proved via a different coupling of the Robin processes. Namely, the internal supporting sphere property of $\mathcal{D}$ implies that the processes are unlikely to visit the ϵ neighborhood of $\partial\mathcal{D}$ too often, before reaching far enough into the interior, where Theorem 6.1 can be invoked. The expected distance increase in each such visit is bounded using a geometrical estimate in Lemma 7.2, while coupled rejection sampling bounds the chance that one of the processes is absorbed at the boundary whereas the other continues. In section 7 we derive the mentioned almost Lipschitz estimate (7.1) on the quantity $\fint_{B_\epsilon(x)\cap\mathcal{D}} y - x \, \mathrm{d}y$, which is related to the Prèkopa theorem on projections of log-concave densities and to the properties of the center of mass in truncations of a convex set as it passes a hyperplane. At this point, asymptotic equicontinuity of of $\{u_\epsilon\}_{\epsilon\to 0}$ is already established, yielding the result in Theorem 1.1. Sections 9 and 10 improve the Hölder exponent $\alpha = 1/2$ to any $\alpha \in (0,1)$, via a logarithmic bound claimed in the second estimate of Theorem 1.2.

1.2. **Notation.** Given a $\mathcal{C}^1$ domain $\mathcal{D}\subset\mathbb{R}^N$, we denote by $\vec{n}(x)$ the outward unit normal vector at $x\in\partial\mathcal{D}$, and by $\pi_{\partial\mathcal{D}}x$ the projection onto $\partial\mathcal{D}$ along the normal $\vec{n}(\pi_{\partial\mathcal{D}}x)$, defined for each $x\in\bar{\mathcal{D}}$ with sufficiently small distance from $\partial\mathcal{D}$. Unless specified otherwise, by C we denote any universal positive constant that may depend on $\mathcal{D}$, γ and f, but not on ϵ, x or other parameter quantities. The Landau symbols $\mathcal{O}$ and o likewise have the same uniformity properties. By $\epsilon \ll 1$ and $C \gg 1$ we mean a "sufficiently small" and a "sufficiently large" positive number.

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2. The Robin mean value equation: basic existence and convergence results in [19]

In this section we gather the definitions, the setup and some general preliminary results obtained in [19]. Recall that we work under the basic hypotheses (BH) and that we are concerned with the following family of integral equations, parametrised by $\epsilon > 0$:

$$(\mathrm{RMV})_\epsilon \qquad u_\epsilon(x) = \big(1-\gamma s_\epsilon(x)\big)\fint_{B_\epsilon(x)\cap\mathcal{D}} u_\epsilon(y)\,\mathrm{d}y + \frac{\epsilon^2}{2(N+2)}f(x) \qquad \text{for all } x\in\bar{\mathcal{D}}.$$

To define the coefficient $s_\epsilon(x)$ above, we introduce the notation (see Figure 2.1):

$$d_\epsilon(x) = \min\big\{1, \frac{1}{\epsilon}\mathrm{dist}(x,\partial\mathcal{D})\big\}\in[0,1] \qquad \text{for all } x\in\mathcal{D},\ \epsilon>0,$$

$$B_1^k = B_1(0)\subset\mathbb{R}^k, \qquad B_{1,d}^k = B_1^k\cap\{y_k<d\} \qquad \text{for all } \ d\in[0,1],$$

$$s_\epsilon(x) = \frac{|B_1^{N-1}|}{(N+1)|B^N_{1,d_\epsilon(x)}|}\cdot\epsilon\big(1-d_\epsilon(x)^2\big)^{\frac{N+1}{2}} \qquad \text{for all } x\in\bar{\mathcal{D}},\ \epsilon>0. \tag{2.1}$$

Recall also that $\mathcal{D}$ is said to be of class $\mathcal{C}^{1,1}$ provided that $\partial\mathcal{D}$ is locally a graph of a $\mathcal{C}^{1,1}$ function. Equivalently, such $\mathcal{D}$ satisfies the uniform (two-sided) supporting sphere condition, stated below. This result has been first shown in [22, Section 2] and then in the article [18], to which we refer for a self-contained discussion and an elementary proof.

Lemma. *An open, bounded set $\mathcal{D}\subset\mathbb{R}^N$ is of class $\mathcal{C}^{1,1}$ if and only if there exists a radius $r>0$ such that for every $x\in\partial\mathcal{D}$ there exist $a,b\in\mathbb{R}^N$ satisfying:*

$$B_r(a)\subset\mathcal{D}, \qquad B_r(b)\subset\mathbb{R}^N\setminus\bar{\mathcal{D}} \quad \textit{and} \quad |x-a| = |x-b| = r.$$

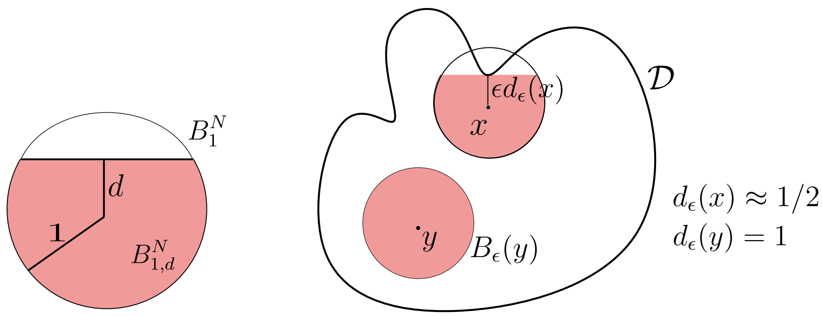

FIGURE 2.1. The referential truncated ball and the scaled distances from $\partial\mathcal{D}$.

Moreover, the global Lipschitz constant of $\vec{n}$ can be taken as the inverse of the supporting radius:

$$|\vec{n}(x) - \vec{n}(y)| \leq \frac{1}{r}|x - y| \qquad \text{for all } x, y \in \partial\mathcal{D}. \tag{2.2}$$

The following important properties of the coefficient s_ϵ were proved in [19]:

Lemma. *Let $\mathcal{D}$ be as in (BH). Then, for all $\epsilon \ll 1$ and all $x \in \bar{\mathcal{D}}$ we have:*

$$\frac{|B_\epsilon(x) \setminus \mathcal{D}|}{|B_\epsilon(0)|} \leq C\frac{s_\epsilon(x)}{\epsilon}, \tag{2.3}$$

$$\fint_{B_\epsilon(x)\cap\mathcal{D}} y - x \, \mathrm{d}y = -s_\epsilon(x)\vec{n}\big(\pi_{\partial\mathcal{D}}x\big) + \mathcal{O}\big(\epsilon s_\epsilon(x)\big), \tag{2.4}$$

$$\fint_{B_\epsilon(x)\cap\mathcal{D}} (y - x)^{\otimes 2} \, \mathrm{d}y = \frac{\epsilon^2}{N+2} Id_N + \mathcal{O}\big(\epsilon s_\epsilon(x)\big). \tag{2.5}$$

We now recall the main existence, uniqueness, comparison, and convergence result from [19]. It is the purpose of the present work to extend statement (ii) below to the general case of (BH).

Theorem. *Assume (BH) and let $\epsilon \ll 1$.*

(i) *Each problem $(\mathrm{RMV})_\epsilon$ has a unique solution u_ϵ that is Borel, bounded with a bound independent of ϵ, and obeys the comparison principle. When f is continuous / Hölder continuous / Lipschitz, then u_ϵ inherits the same regularity properties.*

(ii) *When $f \in \mathcal{C}(\bar{\mathcal{D}})$, then $\{u_\epsilon\}_{\epsilon\to 0}$ converges uniformly on $\bar{\mathcal{D}}$ to $u \in \mathcal{C}(\bar{\mathcal{D}})$ that is the unique viscosity solution to (RL). In fact, u coincides with the unique $W^{2,p}(\mathcal{D})$ solution to (RL).*

We conclude this section by a brief account of the classical regularity theory in the context of (RL). The Robin problem, called also the third boundary value problem / impedance boundary problem / convective boundary problem, has received large attention due to its many applications in science and engineering. Using Schauder estimates, it follows [13, Chapter 6.7] that on a bounded $\mathcal{C}^{2,\alpha}$-regular domain $\mathcal{D}$, the general strictly elliptic problem $Lu = f$ with $\mathcal{C}^\alpha(\bar{\mathcal{D}})$-regular coefficients and $f \in \mathcal{C}^\alpha(\bar{\mathcal{D}})$, subject to the oblique boundary conditions: $\langle\beta(x), \nabla u(x)\rangle + \gamma(x)u(x) = \phi(x)$ posed with $\gamma, \beta, \phi \in \mathcal{C}^{1,\alpha}(\partial D)$ where $\gamma\langle\beta, \vec{n}\rangle > 0$, has a unique solution $u \in \mathcal{C}^{2,\alpha}(\bar{\mathcal{D}})$ that satisfies the usual a-priori bounds. The modern theory for nonlinear boundary value problems modeled on (RL), is contained in the monograph [20]. It is shown in Theorem 1.26 there, that solutions to linear oblique problems in Lipschitz domains are Hölder continuous. Further, in Theorem 4.40 and Corollary 4.41 it is shown that regularity $\mathcal{C}^{1,\alpha}$ of $\mathcal{D}$ suffices for the solution regularity $u \in \mathcal{C}^{2,\alpha}(\bar{\mathcal{D}})$, provided that $f \in \mathcal{C}^\alpha(\bar{\mathcal{D}})$ and $\beta \in \mathcal{C}^{1,\alpha}(\partial\mathcal{D})$. We observe that for (RL), the obliqueness vector $\beta = \vec{n}$ is only Lipschitz and thus one cannot, in general, expect that $u \in \mathcal{C}^{2,\alpha}(\bar{\mathcal{D}})$. In [20, Theorem 6.30] it is shown that if $\mathcal{D}$ has regularity $\mathcal{C}^{1,\alpha}$ and $f \in L^p(\mathcal{D})$ with $p \in (1, \frac{1}{1-\alpha})$ then the unique solution to (RL) has regularity $u \in W^{2,p}(\mathcal{D})$. Consequently, under (BH) there holds: $u \in W^{2,p}(\mathcal{D})$ for any $p \in (1,\infty)$ and thus $u \in \mathcal{C}^{1,\alpha}(\bar{\mathcal{D}})$ for any $\alpha \in (0,1)$. Analysis of (RL) in non-smooth domains, including sets with a rectifiable topological boundary having finite $(N-1)$-dimensional Hausdorff measure, can be found in [26, 12, 7].

3. The Robin process and its stationary measure

We first recall the basic probability setting related to the equation $(\text{RMV})_\epsilon$, given in [19].

1. Consider the probability space $(B_1^N, \mathcal{B}, \frac{1}{|B_1^N|}\mathcal{L}_N)$ equipped with the standard Borel σ-algebra and the normalised Lebesgue measure, and define $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ as the countable product of B_1^N augmented by the unit interval (likewise equipped with Borel σ-algebra and Lebesgue measure):

$$\Omega_1 = (B_1^N)^{\mathbb{N}} \times (0,1) = \big\{(w,b);\ w = \{w^j\}_{j=1}^\infty,\ w^j \in B_1^N \quad \text{for all } j \in \mathbb{N} \quad \text{and } b \in (0,1)\big\}.$$

Further, the countable product of $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ is denoted by $(\Omega, \mathcal{F}, \mathbb{P})$, where:

$$\Omega = (\Omega_1)^{\mathbb{N}} = \big\{\omega = \{(w_i, b_i)\}_{i=1}^\infty;\ w_i = \{w_i^j\}_{j=1}^\infty,\ w_i^j \in B_1^N,\ b_i \in (0,1) \quad \text{for all } i,j \in \mathbb{N}\big\}.$$

For each $n \in \mathbb{N}$, the probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ is the product of n copies of $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and the σ-algebra $\mathcal{F}_n$ is identified with the sub-σ-algebra of $\mathcal{F}$, consisting of sets $A \times \prod_{i=n+1}^\infty \Omega_1$ for all $A \in \mathcal{F}_n$. Then $\{\mathcal{F}_n\}_{n=0}^\infty$ where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, is a filtration of $\mathcal{F}$.

2. For every $\epsilon \ll 1$, define the sequence of measurable functions $\big\{k_i^\epsilon : \Omega \times \bar{\mathcal{D}} \to \mathbb{N} \cup \{+\infty\}\big\}_{i=1}^\infty$:

$$k_i^\epsilon(\omega, x) = \min\big\{k \geq 1;\ x + \epsilon w_i^k \in B_\epsilon(x) \cap \mathcal{D}\big\} \qquad \text{for all } \ \omega \in \Omega,\ x \in \bar{\mathcal{D}}.$$

Since each k_i^ϵ is $\mathbb{P}$-a.s. finite, we further construct the sequence of vector-valued random variables $\{w_i^{\epsilon,x} : \Omega \to B_1^N\}_{i=1}^\infty$, corresponding to $\epsilon \ll 1$ and $x \in \bar{\mathcal{D}}$, by:

$$w_i^{\epsilon,x}(\omega) = w_i^{k_i^\epsilon(\omega,x)} \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

The procedure of generating $w_i^{k_i}$ is well known under the name of rejection sampling and it has the following measure preservation property: for every ϵ, x as above, for every Borel set $F \subset B_\epsilon(x) \cap \mathcal{D}$:

$$\begin{aligned} \mathbb{P}\big(x + \epsilon w_i^{\epsilon,x} \in F\big) &= \sum_{k=1}^\infty \mathbb{P}_i\big(\{x + \epsilon w_i^k \in F\} \cap \{k = k_i^\epsilon\}\big) \\ &= \frac{|F|}{|B_\epsilon(x)|} \cdot \sum_{k=1}^\infty \Big(1 - \frac{|B_\epsilon(x) \cap \mathcal{D}|}{|B_\epsilon(x)|}\Big)^{k-1} = \frac{|F|}{|B_\epsilon(x) \cap \mathcal{D}|}. \end{aligned} \tag{3.1}$$

Given $\epsilon \ll 1$, $x_0 \in \bar{\mathcal{D}}$, we recursively define the sequence of random variables $\big\{X_n^{\epsilon,x_0} : \Omega \to \bar{\mathcal{D}}\big\}_{n=0}^\infty$:

$$X_0^{\epsilon,x_0} \equiv x_0, \qquad X_n^{\epsilon,x_0}\big(w_1,\ldots,w_n\big) = X_{n-1}^{\epsilon,x_0}\big(w_1,\ldots,w_{n-1}\big) + \epsilon w_n^{\epsilon, X_{n-1}^{\epsilon,x_0}(w_1,\ldots,w_{n-1})}. \tag{3.2}$$

Each X_n^{ϵ,x_0} is $\mathcal{F}_n$-measurable and takes values in $\mathcal{D}$, for $n \geq 1$. We also observe that $X_n^{\epsilon,x_0}(\omega)$ is jointly measurable in ω and x_0, by the same property of k_i^ϵ. When no ambiguity arises, we will write $X_n^{x_0}$ or X_n to simplify notation.

3. For each $\epsilon \ll 1$, define the following probability measure on Borel subsets $F \subset \bar{\mathcal{D}}$:

$$\mu_\epsilon(F) = \frac{1}{\int_{\mathcal{D}} \frac{|B_\epsilon(x)\cap\mathcal{D}|}{|B_\epsilon(x)|}\,\mathrm{d}x} \cdot \int_F \frac{|B_\epsilon(x) \cap \mathcal{D}|}{|B_\epsilon(x)|}\,\mathrm{d}x = \frac{1}{\int_{\mathcal{D}} |B_\epsilon(x) \cap \mathcal{D}|\,\mathrm{d}x} \cdot \int_F |B_\epsilon(x) \cap \mathcal{D}|\,\mathrm{d}x.$$

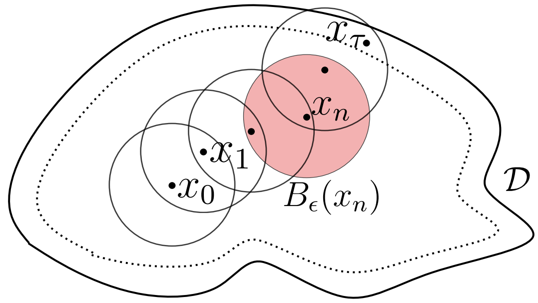

FIGURE 3.1. Positions of the process defined in (3.2).

It is clear that μ_ϵ is absolutely continuous with respect to the Lebesgue measure, and that its density function $x \mapsto |B_\epsilon(x) \cap \mathcal{D}|$ is Lipschitz and satisfies:

$$\frac{1}{3} \leq \frac{|B_\epsilon(x) \cap \mathcal{D}|}{|B_\epsilon(x)|} \leq 1 \quad \text{ for all } x \in \bar{\mathcal{D}} \text{ and } \epsilon \ll 1.$$

Consequently, μ_ϵ is equivalent to the Lebesgue measure:

$$\frac{|F|}{3|\mathcal{D}|} \leq \mu_\epsilon(F) \leq \frac{3|F|}{|\mathcal{D}|}.$$

We define the product probability measure $\mathbb{P}\times\mu_\epsilon$ on the space $\Omega\times\bar{\mathcal{D}}$ equipped with the product σ-algebra of $\mathcal{F}$ with the Borel σ-algebra on $\bar{\mathcal{D}}$. By $L^p(\mu_\epsilon)$ we denote the Banach space of p-integrable – with respect to μ_ϵ – Borel functions on $\bar{\mathcal{D}}$, for any power $p \in (1,\infty)$. Likewise, $L^p(\mathbb{P}\times\mu_\epsilon)$ is the space of p-integrable, with respect to the measure $\mathbb{P}\times\mu_\epsilon$, functions on $\Omega\times\bar{\mathcal{D}}$.

Lemma 3.1. *Assume (BH) and let $\epsilon \ll 1$. Then the measure μ_ϵ is stationary for the Robin process $\{X_n^{\epsilon,x_0}\}_{n=0}^\infty$. More precisely, for all $n \geq 0$ there holds:*

(a) $\mu_\epsilon(F) = \big(\mathbb{P}\times\mu_\epsilon\big)\big(X_n^{\epsilon,x_0} \in F\big)$, *for every Borel subset $F \subset \bar{\mathcal{D}}$,*

(b) $\displaystyle\int_{\mathcal{D}} g(x_0)\;\mathrm{d}\mu_\epsilon(x_0) = \int_{\Omega\times\bar{\mathcal{D}}} \big(g\circ X_n^{\epsilon,x_0}\big)(\omega)\;\mathrm{d}(\mathbb{P}\times\mu_\epsilon)(\omega,x_0)$, *for every Borel function $g : \bar{\mathcal{D}} \to \mathbb{R}$ that is either nonnegative or Lebesgue integrable.*

Proof. Assertion (a) for $n = 1$ follows by Fubini's theorem and (3.1):

$$\begin{aligned} \big(\mathbb{P}\times\mu_\epsilon\big)\big(X_1^{\epsilon,x_0} \in F\big) &= \int_{\mathcal{D}} \mathbb{P}\big(x_0 + \epsilon w_1^{\epsilon,x_0} \in F\big)\;\mathrm{d}\mu_\epsilon(x_0) \\ &= \frac{1}{\int_{\mathcal{D}} |B_\epsilon(x_0)\cap\mathcal{D}|\;\mathrm{d}x_0} \cdot \int_{\mathcal{D}} |F \cap B_\epsilon(x_0)|\;\mathrm{d}x_0 = \frac{\int_F |B_\epsilon(x_0)\cap\mathcal{D}|\;\mathrm{d}x_0}{\int_{\mathcal{D}} |B_\epsilon(x_0)\cap\mathcal{D}|\;\mathrm{d}x_0} = \mu_\epsilon(F). \end{aligned} \tag{3.3}$$

We thus see that μ_ϵ coincides with the push-forward measure $X_1^\epsilon\#(\mathbb{P}\times\mu_\epsilon)$ of $\mathbb{P}\times\mu_\epsilon$, via the measurable map $\Omega\times\bar{\mathcal{D}} \ni (\omega, x_0) \mapsto X_1^{\epsilon,x_0} \in \bar{\mathcal{D}}$. This yields (b) for $n = 1$, by means of the change of variable formula. The general case $n > 1$ is obtained by induction, applying (b) at $n = 1$ to the nonnegative function $g(y) = \mathbb{P}(X_1^{\epsilon,y} \in F)$ twice, in:

$$\big(\mathbb{P}\times\mu_\epsilon\big)\big(X_{n+1}^{\epsilon,x_0} \in F\big) = \int_{\Omega_n\times\mathcal{D}} \mathbb{P}\big(X_1^{\epsilon,X_n^{\epsilon,x_0}} \in F\big)\;\mathrm{d}\big(\mathbb{P}\times\mu_\epsilon\big) = \int_{\mathcal{D}} \mathbb{P}(X_1^{\epsilon,y} \in F)\;\mathrm{d}\mu_\epsilon(y) = \mu_\epsilon(F).$$

As before, one concludes that $\mu_\epsilon = X_{n+1}^\epsilon\#(\mathbb{P}\times\mu_\epsilon)$, which yields (b). ∎

4. Two probabilistic representations of u_ϵ.

For each $\epsilon \ll 1$, $x_0 \in \bar{\mathcal{D}}$ recall the definition of the Robin process $\{X_n^{\epsilon,x_0}\}_{n=0}^\infty$ in (3.2) and define the $\mathcal{F}$-measurable $\tau^{\epsilon,x_0} : \Omega \to \mathbb{N} \cup \{+\infty\}$ by:

$$\tau^{\epsilon,x_0}(\omega) = \min\big\{n \geq 1;\ b_n < \gamma s_\epsilon(X_{n-1}^{\epsilon,x_0})\big\}.$$

We further define the random variables $\big\{\Lambda_n^{\epsilon,x_0} : \Omega \to \mathbb{R}\big\}_{n=0}^\infty$ in:

$$\Lambda_n^{\epsilon,x_0}(\omega) = \prod_{j=1}^n \big(1 - \gamma \cdot (s_\epsilon \circ X_{j-1}^{\epsilon,x_0})(\omega)\big). \tag{4.1}$$

For $n = 0$ we adopt the convention that $\Lambda_0 = 1$. When no ambiguity arises we write $\Lambda_n^{x_0}$ and τ^{x_0}, or Λ_n and τ, to alleviate the notation. Note that each $\Lambda_n^{x_0}(\omega)$ as well as τ^{x_0}, is jointly measurable in ω and x_0, by the same property of X_n^{ϵ,x_0}. We now set:

$$\begin{aligned} u^\epsilon(x_0) &= \int_\Omega \sum_{i=0}^{\tau^{\epsilon,x_0}-1} \frac{\epsilon^2}{2(N+2)} \big(f \circ X_i^{\epsilon,x_0}\big)(\omega)\ \mathrm{d}\mathbb{P}(\omega), \\ \bar{u}^\epsilon(x_0) &= \frac{\epsilon^2}{2(N+2)} \mathbb{E}\Big[\sum_{i=0}^\infty (f \circ X_i^{\epsilon,x_0})\Lambda_i^{\epsilon,x_0}\Big] \\ &= \int_\Omega \sum_{i=0}^\infty \frac{\epsilon^2}{2(N+2)} \big(f \circ X_i^{\epsilon,x_0}\big)(\omega) \cdot \prod_{j=1}^i \big(1 - \gamma(s_\epsilon \circ X_{j-1}^{\epsilon,x_0})(\omega)\big)\ \mathrm{d}\mathbb{P}(\omega). \end{aligned} \tag{4.2}$$

As shown in [19], the following representation result:

Lemma. *Assume (BH). For each $\epsilon \ll 1$, the functions $u^\epsilon, \bar{u}^\epsilon$ are well defined a.e. in $\bar{\mathcal{D}}$. After adjusting on a negligible set, they coincide with the unique bounded Borel solution u_ϵ to $(\mathrm{RMV})_\epsilon$:*

$$u^\epsilon = \bar{u}^\epsilon = u_\epsilon.$$

Equivalence of the two representations in (4.2) can be seen directly. Given $x_0 \in \bar{\mathcal{D}}$ and $\epsilon \ll 1$, consider the integrable random variable:

$$F^{\epsilon,x_0} = \sum_{i=0}^{\tau^{\epsilon,x_0}-1} f \circ X_i^{\epsilon,x_0} = \sum_{i=0}^\infty \big(f \circ X_i^{\epsilon,x_0}\big) \cdot \mathbb{1}_{\{i<\tau^{\epsilon,x_0}\}}.$$

Let $\mathcal{G}$ be the sub-σ-algebra of $\mathcal{F}$, where we suppress the dependence on the auxiliary variables b_n. Namely, we set $\mathcal{G} = (\mathcal{G}_1)^{\mathbb{N}}$ where $\mathcal{G}_1 \subset \mathcal{F}_1$ consists of all the Cartesian products of: measurable subsets of $(B_1^N)^{\mathbb{N}}$, and the entire interval $(0,1)$. Then:

$$\mathbb{E}\big(\mathbb{1}_{\{i<\tau^{\epsilon,x_0}\}} \mid \mathcal{G}\big) = \mathbb{E}\Big(\prod_{j=1}^i \mathbb{1}_{\{b_j \geq \gamma s_\epsilon(X_{j-1}^{\epsilon,x_0})\}} \mid \mathcal{G}\Big) = \prod_{j=1}^i \big(1 - \gamma s_\epsilon(X_{j-1}^{\epsilon,x_0})\big) = \Lambda_i^{\epsilon,x_0} \quad \mathbb{P} - \text{a.s. in } \Omega,$$

which implies:

$$\begin{aligned} u^\epsilon(x_0) &= \frac{\epsilon^2}{2(N+2)} \mathbb{E}\big[F^{\epsilon,x_0}\big] = \frac{\epsilon^2}{2(N+2)} \mathbb{E}\big[\mathbb{E}\big(F^{\epsilon,x_0} \mid \mathcal{G}\big)\big] \\ &= \frac{\epsilon^2}{2(N+2)} \mathbb{E}\Big[\sum_{i=0}^\infty \big(f \circ X_i^{\epsilon,x_0}\big) \cdot \mathbb{E}\big(\mathbb{1}_{\{i<\tau^{\epsilon,x_0}\}} \mid \mathcal{G}\big)\Big] \\ &= \frac{\epsilon^2}{2(N+2)} \mathbb{E}\Big[\sum_{i=0}^\infty (f \circ X_i^{\epsilon,x_0})\Lambda_i^{\epsilon,x_0}\Big] = \bar{u}^\epsilon(x_0). \end{aligned}$$

5. The L^1 convergence.

We will use the second probabilistic representation of u_ϵ, namely $\bar{u}^\epsilon$ in (4.2). The first main result of this section is as follows:

Theorem 5.1. *Assume (BH). Fix $\epsilon \ll 1$ and $p \in (1,\infty)$. The map $L^p(\mathcal{D}) \ni f \mapsto u_\epsilon \in L^1(\mathcal{D})$ given in (4.2) is well defined, linear, and it satisfies:*

$$\|u_\epsilon\|_{L^1(\mu_\epsilon)} \leq \frac{C}{1-2^{1-1/p}}\|f\|_{L^p(\mu_\epsilon)}, \tag{5.1}$$

where C is a constant that is independent of p, ϵ and f, but it may depend on γ and $\mathcal{D}$. Recall that by $L^p(\mu_\epsilon)$ we denote the Banach space of μ_ϵ-integrable Borel functions on $\bar{\mathcal{D}}$.

Proof. **1.** Denote by v_ϵ the solution to $(\mathrm{RMV})_\epsilon$ with $f \equiv 1$. We first observe that the following sequence of random variables $\{M_n\}_{n=0}^\infty$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$:

$$M_n = (v_\epsilon \circ X_n)\Lambda_n + \frac{\epsilon^2}{2(N+2)}\sum_{i=0}^{n-1}\Lambda_i,$$

where we adopt the convention that $M_0 = v_\epsilon(x_0)$. Indeed, $(\mathrm{RMV})_\epsilon$ yields:

$$\begin{aligned}\mathbb{E}\big(M_{n+1} - M_n \mid \mathcal{F}_n\big) &= \Big(\fint_{B_\epsilon(X_n)\cap\mathcal{D}} v_\epsilon(y)\,\mathrm{d}y\Big)\cdot\Lambda_{n+1} - v_\epsilon(X_n)\cdot\Lambda_n + \frac{\epsilon^2}{2(N+2)}\Lambda_n \\ &= \Big(\big(1-\gamma s_\epsilon(X_n)\big)\fint_{B_\epsilon(X_n)\cap\mathcal{D}} v_\epsilon(y)\,\mathrm{d}y - v_\epsilon(X_n) + \frac{\epsilon^2}{2(N+2)}\Big)\Lambda_n \\ &= 0 \qquad \mathbb{P}-\text{a.s. in } \Omega.\end{aligned}$$

Using equiboundedness of $\{v_\epsilon\}_{\epsilon\to 0}$ in [19, Theorem 1.1 (a)] we get:

$$\frac{\epsilon^2}{2(N+2)}\mathbb{E}\Big[\sum_{i=0}^n \Lambda_i\Big] \leq \mathbb{E}\big[M_{n+1}\big] + C = \mathbb{E}\big[M_0\big] + C = v_\epsilon(x_0) + C \leq C,$$

where C is a constant depending only on γ and $\mathcal{D}$. Since $\{\Lambda_i\}_{i=0}^\infty$ is a decreasing sequence, if follows that for $n_\epsilon = \lceil\frac{2C}{\epsilon^2}\rceil$ we thus get: $\mathbb{E}\big[\Lambda_{n_\epsilon}^{\epsilon,x_0}\big] \leq \frac{C}{\epsilon^2 n_\epsilon} \leq \frac{1}{2}$ for all $x_0 \in \bar{\mathcal{D}}$. Consequently:

$$\mathbb{E}\big(\Lambda_{(k+1)n_\epsilon}^{\epsilon,x_0} \mid \mathcal{F}_{kn_\epsilon}\big) = \Lambda_{kn_\epsilon}^{\epsilon,x_0}\cdot\mathbb{E}\big(\Lambda_{n_\epsilon}^{\epsilon,X_{kn_\epsilon}^{\epsilon,x_0}} \mid \mathcal{F}_{kn_\epsilon}\big) \leq \frac{1}{2}\Lambda_{kn_\epsilon}^{\epsilon,x_0} \qquad \mathbb{P}-\text{a.s. in } \Omega,$$

and so we get by induction:

$$\mathbb{E}\big[\Lambda_{kn_\epsilon}^{\epsilon,x_0}\big] \leq \frac{1}{2^k} \quad \text{for all } \ x_0 \in \bar{\mathcal{D}} \ \text{ and all } \ k \geq 0$$

Further, denoting $q = \frac{p}{p-1}$ the Sobolev exponent dual to p, we get:

$$\Big(\int_{\Omega\times\mathcal{D}}\big|\Lambda_n^{\epsilon,x_0}(\omega)\big|^q\,\mathrm{d}(\mathbb{P}\times\mu_\epsilon)\Big)^{1/q} \leq \Big(\int_D \mathbb{E}\big[\Lambda_n^{\epsilon,x_0}\big]\,\mathrm{d}\mu_\epsilon(x_0)\Big)^{1/q} \leq \frac{1}{2^{k/q}} \quad \text{for all } \ n \geq kn_\epsilon \tag{5.2}$$

2. Towards estimating the left hand side of (5.1), we apply Young's inequality, followed by Lemma 3.1 (b) used to $g = |f|^p \in L^1(\mu_\epsilon)$ in:

$$\begin{aligned}\sum_{i=kn_\epsilon}^{(k+1)n_\epsilon}\big\|(f\circ X_i^{\epsilon,x_0})\Lambda_i^{\epsilon,x_0}\big\|_{L^1(\mathbb{P}\times\mu_\epsilon)} &\leq \sum_{i=kn_\epsilon}^{(k+1)n_\epsilon}\big\|f\circ X_i^{\epsilon,x_0}\big\|_{L^p(\mathbb{P}\times\mu_\epsilon)}\cdot\big\|\Lambda_i^{\epsilon,x_0}\big\|_{L^q(\mathbb{P}\times\mu_\epsilon)} \\ &\leq \frac{n_\epsilon}{2^{k/q}}\|f\|_{L^p(\mu_\epsilon)} \quad \text{for all } \ k \geq 0.\end{aligned} \tag{5.3}$$

In conclusion, $\bar{u}^\epsilon$ is well defined and:

$$\begin{aligned}\|\bar{u}^\epsilon\|_{L^1(\mu_\epsilon)} &\leq \frac{\epsilon^2}{2(N+2)}\sum_{k=0}^{\infty}\sum_{i=kn_\epsilon}^{(k+1)n_\epsilon}\left\|(f\circ X_i^{\epsilon,x_0})\Lambda_i^{\epsilon,x_0}\right\|_{L^1(\mathbb{P}\times\mu_\epsilon)} \\ &\leq \frac{\epsilon^2}{2(N+2)}\frac{2C+1}{\epsilon^2}\|f\|_{L^p(\mu_\epsilon)}\cdot\sum_{k=0}^{\infty}\frac{1}{2^{k/q}}\end{aligned}$$

proving the claimed bound. The linearity of the map $f\mapsto\bar{u}^\epsilon$ is obvious. ■

At this point, we immediately see that under assumption (BH), the family $\{\bar{u}^\epsilon\}_{\epsilon\to 0}$ converges in $L^1(\mathcal{D})$ to some Borel, bounded $u:\bar{\mathcal{D}}\to\mathbb{R}$. Indeed, fix any $p\in(1,\infty)$ and approximate the given f in $L^p(\mathcal{D})$ by a sequence $\{f_n\in\mathcal{C}(\bar{\mathcal{D}})\}_{n=1}^{\infty}$. Since $\{u_\epsilon^{f_n}\}_{\epsilon\to 0}$ converge uniformly on $\bar{\mathcal{D}}$ in virtue of [19, Theorem 1.1], it follows by Theorem 5.1 that the family $\{\bar{u}_\epsilon\}_{\epsilon\to 0}$ associated with f, is Cauchy in $L^1(D)$. In fact, the unique limit of $\{\bar{u}_\epsilon\}_{\epsilon\to 0}$ can be identified as the $W^{2,p}$ solution of (RL). Towards the proof, we need to recall the following statement from [19, Theorem 7.1]:

Lemma. *Assume (BH) and let $u\in\mathcal{C}^1(\bar{\mathcal{D}})$ be the unique $W^{2,p}$ solution to (RL). Define the following uniformly bounded sequence of Borel "reminder" functions $\{R_\epsilon\}_{\epsilon\to 0}$:*

$$R_\epsilon(x) = u(x) = \big(1-\gamma s_\epsilon(x)\big)\fint_{B_\epsilon(x)\cap\mathcal{D}} u(y)\,\mathrm{d}y - \frac{\epsilon^2}{2(N+2)}f(x).$$

Then, for every $\epsilon\ll 1$ and $x_0\in\bar{\mathcal{D}}$ we have:

$$u(x_0)-u_\epsilon(x_0) = \lim_{n\to\infty}\mathbb{E}\Big[\sum_{i=0}^{n}(R_\epsilon\circ X_i^{\epsilon,x_0})\mathbb{1}_{\{d_\epsilon(X_i^{x_0})\geq 1\}}\Lambda_i^{\epsilon,x_0}\Big]. \tag{5.4}$$

Moreover, there exists a family of positive Borel functions $\{h_\epsilon : B_\epsilon(0)\to\mathbb{R}\}_{\epsilon\to 0}$ that are probability densities: $\int_{B_\epsilon(0)}h_\epsilon(y)\,\mathrm{d}y=1$, and such that whenever $dist(x_0,\partial\mathcal{D})\geq\epsilon$, there holds:

$$R_\epsilon(x_0) = \frac{\epsilon^2}{2(N+2)}\Big(\int_{B_\epsilon(x_0)}h_\epsilon(x_0-y)f(y)\,\mathrm{d}y - f(x_0)\Big). \tag{5.5}$$

Here is the second main result of this section. In the next sections, we will show that the below convergence is actually uniform.

Theorem 5.2. *Assume (BH) and let $u\in\mathcal{C}^1(\bar{\mathcal{D}})$ be the unique $W^{2,p}$ solution to (RL). Then $\{u_\epsilon\}_{\epsilon\to 0}$ converge to u in $L^1(\mathcal{D})$.*

Proof. By (5.4) we obtain:

$$\begin{aligned}\int_{\mathcal{D}}|u-\bar{u}^\epsilon|\,\mathrm{d}\mu_\epsilon &\leq \sum_{i=0}^{\infty}\int_{\Omega\times\mathcal{D}}\big(|R_\epsilon|\circ X_i^{\epsilon,x_0}\big)\mathbb{1}_{\{d_\epsilon(X_i^{\epsilon,x_0})\geq 1\}}\cdot\Lambda_i^{\epsilon,x_0}\,\mathrm{d}(\mathbb{P}\times\mu_\epsilon)(\omega,x_0) \\ &\leq \sum_{i=1}^{\infty}\left\|\big(R_\epsilon\circ X_i^{\epsilon,x_0}\big)\mathbb{1}_{\{d_\epsilon(X_i)\geq 1\}}\right\|_{L^p(\mathbb{P}\times\mu_\epsilon)}\cdot\left\|\Lambda_i^{\epsilon,x_0}\right\|_{L^q(\mathbb{P}\times\mu_\epsilon)} \\ &= \left\|R_\epsilon\cdot\mathbb{1}_{\{d_\epsilon\geq 1\}}\right\|_{L^p(\mu_\epsilon)}\cdot\sum_{i=1}^{\infty}\left\|\Lambda_i^{\epsilon,x_0}\right\|_{L^q(\mathbb{P}\times\mu_\epsilon)},\end{aligned} \tag{5.6}$$

where in the last equality we used the stationarity property in Lemma 3.1 (b) to the nonnegative Borel function $g = |R_\epsilon|^p \cdot \mathbb{1}_{\{d_\epsilon \geq 1\}}$. Now, similarly as in (5.3), using (5.2) we get:

$$\sum_{i=1}^{\infty} \big\|\Lambda_i^{\epsilon,x_0}\big\|_{L^q(\mathbb{P}\times\mu_\epsilon)} = \sum_{k=0}^{\infty} \sum_{i=kn_\epsilon}^{(k+1)n_\epsilon} \big\|\Lambda_i^{\epsilon,x_0}\big\|_{L^q(\mathbb{P}\times\mu_\epsilon)} \leq \sum_{k=0}^{\infty} \frac{n_\epsilon}{2^{k/q}} \leq \frac{C}{\epsilon^2}\cdot\frac{1}{2^{1-1/p}}.$$

On the other hand, (5.5) gives:

$$\big\|R_\epsilon \cdot \mathbb{1}_{\{d_\epsilon\geq 1\}}\big\|_{L^p(\mu_\epsilon)} = \frac{\epsilon^2}{2(N+2)}\big\|h_\epsilon * f - f\big\|_{L^p(\mu_\epsilon)} \leq C\epsilon^2 \big\|h_\epsilon * f - f\big\|_{L^p(\mathcal{D})} = o(\epsilon^2).$$

Hence (5.6) yields that:

$$\lim_{\epsilon\to 0} \|u - \bar u^\epsilon\|_{L^1(\mu_\epsilon)} = 0$$

and ends the proof. ■

6. A proof of asymptotic equicontinuity - Lipschitz in the interior

In this section, we deduce the asymptotic equicontinuity bound in Theorem 1.2 for the interior case, namely that of x_0 located away from $\partial\mathcal{D}$. Our main result is:

Theorem 6.1. *Assume (BH). There exists $\delta_0 \ll 1$ such that for every $\delta \in (0, \delta_0)$ there is $\bar\epsilon > 0$ with the following property. For all $\epsilon < \bar\epsilon$ and all $x_0, y_0 \in \mathcal{D}$ satisfying $|x_0 - y_0| \leq \delta$ we have:*

$$|u_\epsilon(x_0) - u_\epsilon(y_0)| \leq C \cdot \frac{\delta}{\operatorname{dist}(x_0, \partial\mathcal{D})}. \tag{6.1}$$

The constant C may depend on $\mathcal{D}$, γ and f, but not on ϵ, x_0, y_0 or δ.

We begin by deriving an auxiliary estimate for the probability of exiting a ball before exiting a half-space. This estimate has been noted in [16]; we present it here for convenience of the reader.

Lemma 6.2. *Let $\{Y_n^\epsilon : \Omega \to \mathbb{R}^N\}_{n=1}^\infty$ be the simple ϵ-random walk in $\mathbb{R}^N$, started at $Y_0^\epsilon \equiv 0$ and given by: $Y_{n+1}^\epsilon(\omega) = Y_n^\epsilon + \epsilon w_{n+1}$, where $w_{n+1} \in B_1^N$. For any $d \in [0,1]$ define the stopping time:*

$$\tau_d^\epsilon(\omega) = \min\big\{n \geq 0;\ Y_n^\epsilon(\omega) \notin B_{1,d}^N\big\},$$

Then, for all $d \in [0,1]$ and $\epsilon \ll 1$, there holds:

$$\mathbb{E}\big[\tau_d^\epsilon\big] \leq \frac{N+2}{\epsilon^2}(d+\epsilon)(1+\epsilon), \tag{6.2}$$

$$\mathbb{P}\big(|Y_{\tau_d^\epsilon}| \geq 1\big) \leq N(d+\epsilon)(1+\epsilon). \tag{6.3}$$

Proof. To alleviate the notation, we suppress the fixed step parameter $\epsilon > 0$. Consider the sequence of random variables $\{M_n\}_{n=0}^\infty$:

$$M_n = \big(d + \epsilon - \langle Y_n, e_N\rangle\big)\cdot\big(1 + \epsilon + \langle Y_n, e_N\rangle\big) + \frac{n\epsilon^2}{N+2},$$

which is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$, because:

$$\begin{aligned}
&\mathbb{E}\big(M_{n+1} - M_n \mid \mathcal{F}_n\big) \\
&\quad = (d-1)\Big(\fint_{B_\epsilon(Y_n)} y^N \,\mathrm{d}y - \langle Y_n, e_N\rangle\Big) - \Big(\fint_{B_\epsilon(Y_n)} |y^N|^2 \,\mathrm{d}y - \langle Y_n, e_N\rangle^2\Big) + \frac{\epsilon^2}{N+2} \\
&\quad = -\fint_{B_\epsilon(0)} |y^N|^2 \,\mathrm{d}y + \frac{\epsilon^2}{N+2} = 0 \quad \mathbb{P} - \text{a.s. in } \Omega.
\end{aligned}$$

Apply now Doob's Optional Stopping to the martingale $\{M_{n\wedge m}\}_{n=0}^\infty$ with the stopping time τ_d:

$$(d+\epsilon)(1+\epsilon)=\mathbb{E}\big[M_0\big]=\mathbb{E}\big[M_{\tau_d\wedge m}\big]\geq\frac{\epsilon^2}{N+2}\mathbb{E}\big[\tau_d\wedge m\big].$$

Passing to the limit with $m\to\infty$ proves (6.2). To show (6.3), we recall that $|Y_n|^2-\frac{N}{N+2}n\epsilon^2$ is a martingale, so that Doob's Optional Stopping and (6.2) directly yield:

$$\mathbb{E}\big[|Y_{\tau_d}|^2\big]=\frac{N}{N+2}\epsilon^2\mathbb{E}\big[\tau_d\big]\leq N(d+\epsilon)(1+\epsilon).$$

By Chebyshev's inequality we easily obtain (6.3), which ends the proof of the Lemma. ∎

By a scaling argument, we arrive at:

Corollary 6.3. *Let $\{Y_n^\epsilon:\Omega\to\mathbb{R}^N\}_{n=1}^\infty$ be as in Lemma 6.2. For $R>\rho\geq 0$ define:*

$$\tau_R^\epsilon=\min\big\{n\geq 0;\ |Y_n^\epsilon|\geq R\big\},\qquad \tau_\rho^\epsilon=\min\big\{n\geq 0;\ \langle Y_n^\epsilon,e_N\rangle\geq r\big\}.$$

Then, for all $\epsilon\ll R$ there holds:

$$\mathbb{E}\big[\tau_R^\epsilon\wedge\tau_\rho^\epsilon\big]\leq\frac{N+2}{\epsilon^2}(\rho+\epsilon)(R+\epsilon), \tag{6.4}$$

$$\mathbb{P}\big(|Y_{\tau_R^\epsilon\wedge\tau_\rho^\epsilon}|\geq R\big)\leq\frac{N}{R^2}(\rho+\epsilon)(R+\epsilon). \tag{6.5}$$

The following is the main technical result of this section:

Lemma 6.4. *Assume (BH). There exists $\delta_0\ll 1$ and $\eta\in(0,1)$ such that the following holds. For every $\delta\in(0,\delta_0)$ and $R>\delta$ there is $\bar\epsilon>0$ satisfying:*

$$\big|u_\epsilon(x_0)-u_\epsilon(y_0)\big|\leq\eta\cdot\sup_{(x,y)\in A_1^{R,\epsilon,x_0}}\big|u_\epsilon(x)-u_\epsilon(y)\big|+2(N+1)\frac{\delta}{R}\|u_\epsilon\|_{L^\infty(\mathcal{D})}+2R\delta\|f\|_{L^\infty(\mathcal{D})}$$

$$\textit{for all}\ \ \epsilon<\bar\epsilon\quad\textit{and}\quad x_0,y_0\in\mathcal{D}\ \ \textit{such that}\ \ |x_0-y_0|\leq\delta\ \ \textit{and}\ \ \mathrm{dist}(x_0,\partial\mathcal{D})>2R,$$

where we define:

$$A_1^{R,\epsilon,x_0}=\big\{(x,y)\in\mathcal{D}\times\mathcal{D};\ |x-x_0|<2R\ \ \textit{and}\ \ |x-y|<3\epsilon\big\}.$$

Proof. **1.** We fix $\delta\ll 1$ and $R>\delta$. Let x_0,y_0 be as in the statement of the Lemma and define H to be the hyperplane bisecting the segment $[x_0,y_0]$. For each $\epsilon\ll\delta$, consider the following stopping time relative to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$, given through the process $\{X_n^{\epsilon,x_0}\}_{n=0}^\infty$ in (3.2):

$$\tau_H=\min\Big\{n\geq 0;\ |X_n^{\epsilon,x_0}-x_0|\geq R\ \text{ or }\ \mathrm{dist}(X_n^{\epsilon,x_0},H)\leq\frac{\epsilon}{2}\Big\}.$$

To alleviate the notation, we suppress the fixed step parameter $\epsilon>0$ and the superscript x_0 in X_n. By (6.4) it follows that:

$$\mathbb{E}\big[\tau_H\big]\leq\frac{N+2}{\epsilon^2}\cdot\Big(\frac{\delta}{2}+\frac{\epsilon}{2}\Big)(R+\epsilon),$$

and for $\epsilon\ll\delta$ we further deduce:

$$\mathbb{E}\big[\tau_H+1\big]\leq(N+3)\cdot\frac{R\delta}{\epsilon^2}. \tag{6.6}$$

Define the sequence of vector-valued random variables $\{Y_n:\Omega\to\mathcal{D}\}_{n=0}^{\tau_H+1}$, setting $Y_0\equiv y_0$ and:

$$Y_n=\left\{\begin{array}{lll} X_{\tau_H+1} & \text{if }\ n=\tau_H+1\quad\text{and}\quad \mathrm{dist}(X_{\tau_H},H)\leq\frac{\epsilon}{2}\ \text{ with }\ X_{\tau_H+1}\in B_\epsilon(Y_{\tau_H})\\ \mathrm{refl}_H(X_n) & \text{otherwise,}\end{array}\right.$$

where by refl_H we denote the reflection across the hyperplane H. Consider a further sequence $\{M_{n\wedge(\tau_H+1)}\}_{n=0}^\infty$, defined through:

$$M_n = \big|u_\epsilon(X_n) - u_\epsilon(Y_n)\big| + \frac{\epsilon^2 n}{N+2}\|f\|_{L^\infty(\mathcal{D})}.$$

2. In this step, we show that $\{M_{n\wedge(\tau_H+1)}\}_{n=0}^\infty$ is a submartingale relative to $\{\mathcal{F}_n\}_{n=0}^\infty$. Call $\tilde H$ the hyperplane parallel to H, passing through 0. When $n<\tau_H$ or when $n=\tau_H$ and $\mathrm{dist}(X_n,H)>\frac{\epsilon}{2}$:

$$\begin{aligned}
\mathbb{E}\big(M_{n+1}\mid \mathcal{F}_n\big) &= \mathbb{E}\big(\big|u_\epsilon(X_{n+1}) - u_\epsilon(Y_{n+1})\big| \mid \mathcal{F}_n\big) + \frac{\epsilon^2(n+1)}{N+2}\|f\|_{L^\infty(\mathcal{D})}\\
&= \int_{\Omega_1}\big|u_\epsilon(X_n+\epsilon w_{n+1}^1) - u_\epsilon(Y_n + \epsilon\, \mathrm{refl}_{\tilde H}(w_{n+1}^1))\big|\;\mathrm{d}\mathbb{P}_1(w_{n+1}) + \frac{\epsilon^2(n+1)}{N+2}\|f\|_{L^\infty(\mathcal{D})}\\
&\geq \Big|\fint_{B_\epsilon(X_n)} u_\epsilon(y)\;\mathrm{d}y - \fint_{B_\epsilon(Y_n)} u_\epsilon(y)\;\mathrm{d}y\Big| + \frac{\epsilon^2}{2(N+2)}|f(X_n)-f(Y_n)| + \frac{\epsilon^2 n}{N+2}\|f\|_{L^\infty(\mathcal{D})}\\
&\geq \Big|\Big(\fint_{B_\epsilon(X_n)} u_\epsilon(y)\;\mathrm{d}y + \frac{\epsilon^2}{2(N+2)}f(X_n)\Big) - \Big(\fint_{B_\epsilon(Y_n)} u_\epsilon(y)\;\mathrm{d}y + \frac{\epsilon^2}{2(N+2)}f(Y_n)\Big)\Big|\\
&\qquad\qquad + \frac{\epsilon^2 n}{N+2}\|f\|_{L^\infty(\mathcal{D})}\\
&= \big|u_\epsilon(X_n) - u_\epsilon(Y_n)\big| + \frac{\epsilon^2 n}{N+2}\|f\|_{L^\infty(\mathcal{D})} = M_n \quad \mathbb{P}-\text{a.s. in }\; \Omega,
\end{aligned}$$

where we used (RMV)_ϵ in the penultimate equality. On the other hand, for the case when $n=\tau_H$ and $\mathrm{dist}(X_n,H)\leq\frac{\epsilon}{2}$, there holds, $\mathbb{P}$-a.s. in Ω:

$$\begin{aligned}
&\mathbb{E}\big(\big|u_\epsilon(X_{n+1}) - u_\epsilon(Y_{n+1})\big| \mid \mathcal{F}_n\big)\\
&\quad= \int_{\{X_n+\epsilon w_{n+1}^1\not\in B_\epsilon(Y_n)\}}\big|u_\epsilon(X_n+\epsilon w_{n+1}^1) - u_\epsilon(Y_n + \;\mathrm{refl}_{\tilde H}(w_{n+1}^1))\big|\;\mathrm{d}\mathbb{P}_1(w_{n+1})\\
&\quad\geq \Big|\int_{\Omega_1} u_\epsilon\big(X_n+\epsilon w_{n+1}^1\big)\;\mathrm{d}\mathbb{P}_1(w_{n+1})\\
&\qquad - \int_{\{X_n+\epsilon w_{n+1}^1\in B_\epsilon(Y_n)\}} u_\epsilon\big(X_n+\epsilon w_{n+1}^1\big)\;\mathrm{d}\mathbb{P}_1(w_{n+1})\\
&\qquad - \int_{\{X_n+\epsilon w_{n+1}^1\not\in B_\epsilon(Y_n)\}} u_\epsilon\big(Y_n+\;\mathrm{refl}_{\tilde H}(w_{n+1}^1)\big)\;\mathrm{d}\mathbb{P}_1(w_{n+1})\Big|\\
&\quad= \Big|\int_{\Omega_1} u_\epsilon\big(X_n+\epsilon w_{n+1}^1\big)\;\mathrm{d}\mathbb{P}_1(w_{n+1}) - \int_{\Omega_1} u_\epsilon\big(Y_n+\epsilon w_{n+1}^1\big)\;\mathrm{d}\mathbb{P}_1(w_{n+1})\Big|.
\end{aligned}$$

Applying the dynamic programming principle (RMV)_ϵ, we conclude that:

$$\begin{aligned}
\mathbb{E}\big(\big|u_\epsilon(X_{n+1}) - u_\epsilon(Y_{n+1})\big| \mid \mathcal{F}_n\big) &\geq \Big|\fint_{B_\epsilon(X_n)} u_\epsilon(y)\;\mathrm{d}y - \fint_{B_\epsilon(Y_n)} u_\epsilon(y)\;\mathrm{d}y\Big|\\
&= \Big|\Big(u_\epsilon(X_n) - \frac{\epsilon^2}{2(N+2)}f(X_n)\Big) - \Big(u_\epsilon(Y_n) - \frac{\epsilon^2}{2(N+2)}f(Y_n)\Big)\Big|\\
&\geq \big|u_\epsilon(X_n) - u_\epsilon(Y_n)\big| - \frac{\epsilon^2}{N+2}\|f\|_{L^\infty(\mathcal{D})} \quad \mathbb{P}-\text{a.s. in }\; \Omega.
\end{aligned}$$

Consequently, there follows the claimed submartingale property:

$$\mathbb{E}\big(M_{n+1} \mid \mathcal{F}_n\big) = \mathbb{E}\big(\big|u_\epsilon(X_{n+1}) - u_\epsilon(Y_{n+1})\big| \mid \mathcal{F}_n\big) + \frac{\epsilon^2(n+1)}{N+2}\|f\|_{L^\infty(\mathcal{D})}$$
$$\geq \big|u_\epsilon(X_n) - u_\epsilon(Y_n)\big| + \frac{\epsilon^2 n}{N+2}\|f\|_{L^\infty(\mathcal{D})} = M_n \quad \mathbb{P}-\text{a.s. in } \Omega.$$

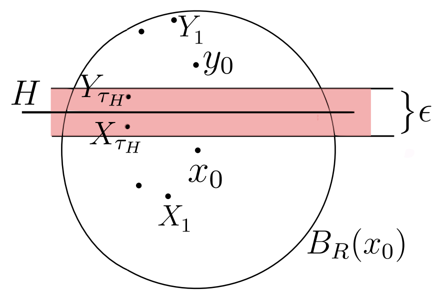

FIGURE 6.1. The hyperplane H and the symmetric processes $\{X_n, Y_n\}_{n=0}^{\tau_H}$ in the proof of Lemma 6.4.

3. Applying Doob's Optional Stopping with the stopping time τ_H+1, we get in virtue of (6.6):

$$\begin{aligned}\big|u_\epsilon(x_0) - u_\epsilon(y_0)\big| &= \mathbb{E}\big[M_0\big] \leq \mathbb{E}\big[M_{\tau_H+1}\big] \\ &= \mathbb{E}\big[|u_\epsilon(X_{\tau_H+1}) - u_\epsilon(Y_{\tau_H+1})|\big] + \frac{\epsilon^2\|f\|_{L^\infty(\mathcal{D})}}{N+2}\cdot\mathbb{E}\big[\tau_H+1\big] \\ &\leq \mathbb{E}\big[|u_\epsilon(X_{\tau_H+1}) - u_\epsilon(Y_{\tau_H+1})|\big] + \frac{N+3}{N+2}R\delta\cdot\|f\|_{L^\infty(\mathcal{D})}.\end{aligned} \tag{6.7}$$

To estimate the first term of (6.7), we split the effective domain of integration $\{X_{\tau_H+1} \neq Y_{\tau_H+1}\}$ according to the definition of Y_n, into two parts so that:

$$\begin{aligned}\mathbb{E}\big[|u_\epsilon(X_{\tau_H+1}) - u_\epsilon(Y_{\tau_H+1})|\big] = &\int_{\{\mathrm{dist}(X_{\tau_H},H)>\frac{\epsilon}{2}\}} |u_\epsilon(X_{\tau_H+1}) - u_\epsilon(Y_{\tau_H+1})|\,\mathrm{d}\mathbb{P} \\ &+ \int_{\{\mathrm{dist}(X_{\tau_H},H)\leq\frac{\epsilon}{2}\}\cap\{X_{\tau_H+1}\notin B_\epsilon(Y_{\tau_H})\}} |u_\epsilon(X_{\tau_H+1}) - u_\epsilon(Y_{\tau_H+1})|\,\mathrm{d}\mathbb{P}.\end{aligned} \tag{6.8}$$

In the first integral above, we apply Corollary 6.3 with R and $r = \frac{|x_0-y_0|}{2} - \frac{\epsilon}{2} \leq \frac{\delta}{2} - \frac{\epsilon}{2}$, to the effect that, when $\epsilon \ll \delta$:

$$\mathbb{P}\Big(\mathrm{dist}(X_{\tau_H}, H) > \frac{\epsilon}{2}\Big) \leq \frac{N}{R^2}\Big(\frac{\delta}{2} + \frac{\epsilon}{2}\Big)\big(R+\epsilon\big) \leq (N+1)\frac{\delta}{R}.$$

This yields:

$$\int_{\{\mathrm{dist}(X_{\tau_H},H)>\frac{\epsilon}{2}\}} |u_\epsilon(X_{\tau_H+1}) - u_\epsilon(Y_{\tau_H+1})|\,\mathrm{d}\mathbb{P} \leq 2N\frac{\delta}{R}\|u_\epsilon\|_{L^\infty(\mathcal{D})}. \tag{6.9}$$

For the second integral in (6.8), we observe that:

$$\mathbb{P}\Big(\big\{\mathrm{dist}(X_{\tau_H}, H) \leq \frac{\epsilon}{2}\big\} \cap \big\{X_{\tau_H+1} \notin B_\epsilon(Y_{\tau_H})\big\}\Big) \leq \frac{\big|B_\epsilon(0) \setminus B_\epsilon(\epsilon e_N)\big|}{|B_\epsilon(0)|} = \eta \in (0,1).$$

Since on the displayed event there holds: $|X_{\tau_H+1} - x_0| \leq R + 2\epsilon < 2R$ and $|X_{\tau_H+1} - Y_{\tau_H+1}| < 3\epsilon$, we conclude that:

$$\int_{\{\mathrm{dist}(X_{\tau_H},H)\leq\frac{\epsilon}{2}\}\cap\{X_{\tau_H+1}\not\in B_\epsilon(Y_{\tau_H})\}} |u_\epsilon(X_{\tau_H+1}) - u_\epsilon(Y_{\tau_H+1})|\,\mathrm{d}\mathbb{P} \leq \eta\cdot \sup_{(x,y)\in A_1} |u_\epsilon(x) - u_\epsilon(y)|. \tag{6.10}$$

The bounds (6.7), (6.9) and (6.10) imply now the main estimate of Lemma 6.4. ■

Proof of Theorem 6.1.

1. Fix $\delta \ll 1$ and let $|x_0 - y_0| \leq \delta$. In view of the equiboundedness of solutions $\{u_\epsilon\}_{\epsilon\to 0}$ to $(\mathrm{RMV})_\epsilon$, it suffices to prove (6.1) under the extra assumption $\mathrm{dist}(x_0, \partial\mathcal{D}) > 3\delta$. We first use Lemma 6.4 with $R = \frac{1}{3}\mathrm{dist}(x_0, \partial\mathcal{D})$ and obtain that for all $\epsilon < \bar\epsilon \ll \delta$ there holds:

$$\begin{aligned}\big|u_\epsilon(x_0) - u_\epsilon(y_0)\big| \leq \eta\cdot\sup\Big\{\big|u_\epsilon(x) - u_\epsilon(y)\big|;\ |x - x_0| < \frac{2}{3}\mathrm{dist}(x_0, \partial\mathcal{D}) \ \text{ and }\ |x-y| < 3\epsilon\Big\}\\ + C\cdot\frac{\delta}{\mathrm{dist}(x_0,\partial\mathcal{D})},\end{aligned} \tag{6.11}$$

again, in view of the equiboundedness of $\{u_\epsilon\}_{\epsilon\to 0}$.

2. To conclude the proof, we will estimate $|u_\epsilon(\bar x_0) - u_\epsilon(\bar y_0)|$ for all couples $(\bar x_0, \bar y_0) \in \mathcal{D}\times\mathcal{D}$ satisfying $\mathrm{dist}(\bar x_0, \partial\mathcal{D}) > \delta$ and $|\bar x_0 - \bar y_0| < 3\bar\epsilon$. We apply Lemma 6.4 with $R = \delta^2$ and get that:

$$\big|u_\epsilon(\bar x_0) - u_\epsilon(\bar y_0)\big| \leq \eta\cdot \sup_{(x,y)\in A_1^{\delta^2,\epsilon,\bar x_0}} \big|u_\epsilon(x) - u_\epsilon(y)\big| + C\delta. \tag{6.12}$$

For every integer $k \geq 1$ define the iterated domain:

$$A_k = \big\{(x,y)\in\mathcal{D}\times\mathcal{D};\ |x - \bar x_0| < 2k\delta^2 \ \text{ and }\ |x-y| < 3\epsilon\big\}.$$

Applying k times the bound (6.12), we obtain the following estimate:

$$\begin{aligned}\big|u_\epsilon(\bar x_0) - u_\epsilon(\bar y_0)\big| &\leq \eta^k\cdot\sup_{(x,y)\in A_k}\big|u_\epsilon(x) - u_\epsilon(y)\big| + C\delta\sum_{i=0}^{k-1}\eta^i\\ &\leq 2\eta^k\|u_\epsilon\|_{L^\infty(\mathcal{D})} + \frac{C\delta}{1-\eta},\end{aligned} \tag{6.13}$$

valid for all sufficiently small ϵ, provided that:

$$\delta - 2k\delta^2 > 2R = 2\delta^2. \tag{6.14}$$

Define k in a manner to ensure that $\eta^k \leq \delta$, namely:

$$k = \Big\lceil\frac{\log\delta}{\log\eta}\Big\rceil.$$

It is easy to verify that the iteration validity condition (6.14) is satisfied for all $\delta \leq \delta_0 \ll 1$. Thus, (6.13) and (6.11) imply (6.1), again by the invoked equiboundedness. ■

7. Boundary estimates I

In this and the following section, we complete the proof of Theorem 1.2, for the case when x_0 is close to $\partial\mathcal{D}$. We begin by some geometrical observations that are of independent interest:

Lemma 7.1. *Assume (BH). Then there exists a constant $C>0$, depending only on $\mathcal{D}$ such that for all $x_0, y_0 \in \mathcal{D}$ there holds:*

$$\Big|\fint_{B_\epsilon(x_0)\cap\mathcal{D}} y\;\mathrm{d}y - \fint_{B_\epsilon(y_0)\cap\mathcal{D}} y\;\mathrm{d}y\Big| \leq |x_0-y_0| + C\big(s_\epsilon(x_0)+s_\epsilon(y_0)\big)\cdot\big(|x_0-y_0|+\epsilon\big). \tag{7.1}$$

Proof. **1.** We first observe that (7.1) holds for $x_0, y_0 \notin \partial\mathcal{D} + B_\epsilon(0)$, because then the left hand side of (7.1) equals $|x_0-y_0|$. Likewise, for $|x_0-y_0|\geq\delta$, where $\delta>0$ is some fixed parameter:

$$\begin{aligned}\Big|\fint_{B_\epsilon(x_0)\cap\mathcal{D}} y\;\mathrm{d}y - \fint_{B_\epsilon(y_0)\cap\mathcal{D}} y\;\mathrm{d}y\Big| &\leq |x_0-y_0| + C\big(s_\epsilon(x_0)+s_\epsilon(y_0)\big)\\ &\leq |x_0-y_0| + \frac{C}{\delta}\big(s_\epsilon(x_0)+s_\epsilon(y_0)\big)|x_0-y_0|,\end{aligned}$$

by using (2.4). It thus suffices to treat the case:

$$x_0, y_0\in\partial\mathcal{D}+B_\delta(0) \quad\text{ and }\quad |x_0-y_0|<\delta\ll\min\big\{1,\frac{r}{2}\big\},$$

where r is the radius in the uniform supporting sphere condition (BH) and δ is small enough, in particular, for the projections $\pi_{\partial\mathcal{D}}x_0, \pi_{\partial\mathcal{D}}y_0$ to be well defined. We further observe that:

$$\big|\vec{n}(\pi_{\partial\mathcal{D}}x_0)-\vec{n}(\pi_{\partial\mathcal{D}}y_0)\big| \leq \frac{4}{r}|x_0-y_0|, \tag{7.2}$$

because by (2.2) it follows that:

$$\begin{aligned}\big|\vec{n}(\pi_{\partial\mathcal{D}}x_0)-\vec{n}(\pi_{\partial\mathcal{D}}y_0)\big| &\leq \frac{1}{r}\big|\pi_{\partial\mathcal{D}}x_0)-\pi_{\partial\mathcal{D}}y_0)\big|\\ &= \frac{1}{r}\Big|(x_0-y_0)+\mathrm{dist}(x_0,\partial\mathcal{D})\vec{n}(\pi_{\partial\mathcal{D}}x_0)-\mathrm{dist}(y_0,\partial\mathcal{D})\vec{n}(\pi_{\partial\mathcal{D}}y_0)\Big|\\ &\leq \frac{1}{r}\Big(|x_0-y_0|+\big|\mathrm{dist}(x_0,\partial\mathcal{D})-\mathrm{dist}(y_0,\partial\mathcal{D})\big|+\delta\cdot\big|\vec{n}(\pi_{\partial\mathcal{D}}x_0)-\vec{n}(\pi_{\partial\mathcal{D}}y_0)\big|\Big)\\ &\leq \frac{2}{r}|x_0-y_0|+\frac{1}{2}\big|\vec{n}(\pi_{\partial\mathcal{D}}x_0)-\vec{n}(\pi_{\partial\mathcal{D}}y_0)\big|.\end{aligned} \tag{7.3}$$

Subtracting the second term in the right hand side from the left hand side, we obtain (7.2). Our second preliminary estimate is:

$$\big|\langle\pi_{\partial\mathcal{D}}x_0-\pi_{\partial\mathcal{D}}y_0,\vec{n}(\pi_{\partial\mathcal{D}}x_0)\rangle\big|,\ \ \big|\langle\pi_{\partial\mathcal{D}}x_0-\pi_{\partial\mathcal{D}}y_0,\vec{n}(\pi_{\partial\mathcal{D}}y_0)\rangle\big|\leq C|x_0-y_0|^2. \tag{7.4}$$

To prove (7.4), observe that the boundary projections of x_0, y_0 are not in the supporting balls corresponding to y_0, x_0 respectively:

$$\pi_{\partial\mathcal{D}}y_0\notin B_r\big(\pi_{\partial\mathcal{D}}x_0-r\vec{n}(\pi_{\partial\mathcal{D}}x_0)\big)\cup B_r\big(\pi_{\partial\mathcal{D}}x_0+r\vec{n}(\pi_{\partial\mathcal{D}}x_0)\big).$$

It follows that:

$$r^2\leq\big|\pi_{\partial\mathcal{D}}x_0-\pi_{\partial\mathcal{D}}y_0\big|^2\pm 2r\langle\pi_{\partial\mathcal{D}}x_0-\pi_{\partial\mathcal{D}}y_0,\vec{n}(\pi_{\partial\mathcal{D}}x_0)\rangle+r^2,$$

which yields, in view of (7.3) and (7.2):

$$\pm\big\langle\pi_{\partial\mathcal{D}}x_0-\pi_{\partial\mathcal{D}}y_0,\vec{n}(\pi_{\partial\mathcal{D}}x_0)\big\rangle\leq\frac{1}{2r}\big|\pi_{\partial\mathcal{D}}x_0-\pi_{\partial\mathcal{D}}y_0\big|^2\leq\frac{4}{r^2}|x_0-y_0|^2,$$

implying the first estimate in (7.4). The second estimate follows by a symmetric argument.

2. Define:

$$w_1 = \frac{1}{2}\big(\vec n(\pi_{\partial\mathcal D}x_0) + \vec n(\pi_{\partial\mathcal D}y_0)\big), \qquad w_2 = \frac{1}{2}\big(\vec n(\pi_{\partial\mathcal D}x_0) - \vec n(\pi_{\partial\mathcal D}y_0)\big),$$

so that: $\vec n(\pi_{\partial\mathcal D}(x_0)) = \vec w_1 + \vec w_2$ and $\vec n(\pi_{\partial\mathcal D}y_0) = \vec w_1 - \vec w_2$. By (7.2) and (7.4) there holds:

$$\Big|\big\langle \pi_{\partial\mathcal D}x_0 - \pi_{\partial\mathcal D}y_0, \vec w_1\big\rangle\Big| \leq C|x_0-y_0|^2 \quad \text{and} \quad |\vec w_2| \leq C|x_0-y_0|. \tag{7.5}$$

Now, using the expansion (2.4) and (7.5) we get:

$$\begin{aligned}
\Big|\fint_{B_\epsilon(x_0)\cap\mathcal D} y\;\mathrm{d}y - \fint_{B_\epsilon(y_0)\cap\mathcal D} y\;\mathrm{d}y\Big| \\
&\leq \Big|\big(x_0 - s_\epsilon(x_0)\vec n(\pi_{\partial\mathcal D}x_0)\big) - \big(y_0 - s_\epsilon(y_0)\vec n(\pi_{\partial\mathcal D}y_0)\big)\Big| + C\epsilon\big(s_\epsilon(x_0)+s_\epsilon(y_0)\big) \\
&= \Big|\Big(\pi_{\partial\mathcal D}x_0 - (s_\epsilon(x_0) + \mathrm{dist}(x_0,\partial\mathcal D))\vec n(\pi_{\partial\mathcal D}x_0)\Big) \\
&\qquad - \Big(\pi_{\partial\mathcal D}y_0 - (s_\epsilon(y_0) + \mathrm{dist}(y_0,\partial\mathcal D))\vec n(\pi_{\partial\mathcal D}y_0)\Big)\Big| \\
&= \Big|V - \big(s_\epsilon(x_0)+s_\epsilon(y_0)\big)\vec w_2\Big| + C\epsilon\big(s_\epsilon(x_0)+s_\epsilon(y_0)\big) \\
&\leq \big|V\big| + C\big(s_\epsilon(x_0)+s_\epsilon(y_0)\big)\cdot\big(|x_0-y_0|+\epsilon\big),
\end{aligned} \tag{7.6}$$

where:

$$\begin{aligned}
V = \Big(\pi_{\partial\mathcal D}x_0 - \pi_{\partial\mathcal D}y_0\Big) - \Big(\big(s_\epsilon(x_0) + \mathrm{dist}(x_0,\partial\mathcal D)\big) - \big(s_\epsilon(y_0) + \mathrm{dist}(y_0,\partial\mathcal D)\big)\Big)\vec w_1 \\
- \Big(\mathrm{dist}(x_0,\partial\mathcal D)) + \mathrm{dist}(y_0,\partial\mathcal D)\Big)\vec w_2.
\end{aligned}$$

Since $\langle \vec w_1, \vec w_2\rangle = 0$ and observing the easy decomposition:

$$\begin{aligned}
x_0 - y_0 = \Big(\pi_{\partial\mathcal D}x_0 - \pi_{\partial\mathcal D}y_0\Big) - \Big(\mathrm{dist}(x_0,\partial\mathcal D) - \mathrm{dist}(y_0,\partial\mathcal D)\Big)\vec w_1 \\
- \Big(\mathrm{dist}(x_0,\partial\mathcal D) + \mathrm{dist}(y_0,\partial\mathcal D)\Big)\vec w_2,
\end{aligned} \tag{7.7}$$

we directly compute:

$$\begin{aligned}
&|V|^2 - |x_0-y_0|^2 \\
&\quad = \Big(\Big|\big(s_\epsilon(x_0) + \mathrm{dist}(x_0,\partial\mathcal D)\big) - \big(s_\epsilon(y_0) + \mathrm{dist}(y_0,\partial\mathcal D)\big)\Big|^2 - \Big|\mathrm{dist}(x_0,\partial\mathcal D) - \mathrm{dist}(y_0,\partial\mathcal D)\Big|^2\Big)|\vec w_1|^2 \\
&\qquad - 2\big\langle \pi_{\partial\mathcal D}(x_0) - \pi_{\partial\mathcal D}(y_0), \big(s_\epsilon(x_0) - s_\epsilon(y_0)\big)\vec w_1\big\rangle.
\end{aligned}$$

The key ingredient for the further arguments is that the first term above is nonpositive. This statement follows from the implication (7.8) that we will prove in the independent Lemma 7.2. We now readily conclude that:

$$\begin{aligned}
|V|^2 - |x_0-y_0|^2 &\leq -2\big\langle \pi_{\partial\mathcal D}x_0 - \pi_{\partial\mathcal D}y_0, \big(s_\epsilon(x_0)-s_\epsilon(y_0)\big)\vec w_1\big\rangle \\
&\leq C\big(s_\epsilon(x_0)+s_\epsilon(y_0)\big)\cdot|x_0-y_0|^2,
\end{aligned}$$

in virtue of (7.5). Consequently, it follows that:

$$|V| - |x_0-y_0| = \frac{|V|^2 - |x_0-y_0|^2}{|V| + |x_0-y_0|} \leq C\big(s_\epsilon(x_0)+s_\epsilon(y_0)\big)\cdot|x_0-y_0|,$$

yielding (7.1) in view of (10.7). ■

The proof of Lemma 7.1 relied on the following observation, which amounts to the monotonicity and the 1-Lipschitz continuity of the center of mass of a truncated unit ball in $\mathbb{R}^N$, as it passes a hyperplane:

Lemma 7.2. *The function:*

$$\Psi(d) = \frac{(1-d^2)^{\frac{N+1}{2}}}{N+1} \cdot \frac{1}{\int_{-1}^{d}(1-s^2)^{\frac{N-1}{2}}}\ \mathrm{d}s \qquad \textit{for all } \ d\in[0,1].$$

is decreasing and has Lipschitz constant 1*. Consequently, for all* $x_0, y_0 \in \bar{\mathcal{D}}$ *there holds:*

$$\mathrm{dist}(x_0,\partial\mathcal{D}) \leq \mathrm{dist}(y_0,\partial\mathcal{D}) \quad \Rightarrow \quad 0 \leq s_\epsilon(x_0) - s_\epsilon(y_0) \leq \mathrm{dist}(y_0,\partial\mathcal{D}) - \mathrm{dist}(x_0,\partial\mathcal{D}). \tag{7.8}$$

The implication (7.8) is indeed a direct consequence of the claimed properties of Ψ, because $s_\epsilon(x) = \epsilon\Psi(d_\epsilon(x))$. We will now present two proofs of Lemma 7.2. The first proof is self-contained, whereas the second proof uses the Prèkopa theorem [30] on the log-concave functions.

The first proof of Lemma 7.2.

1. We need to show that $0 < -\Psi' \leq 1$ on $(0,1)$. The first inequality is clear, because:

$$\Psi'(d) = \frac{-(N+1)d(1-d^2)^{\frac{N-1}{2}}\int_{-1}^{d}(1-s^2)^{\frac{N-1}{2}}\ \mathrm{d}s - (1-d^2)^N}{(N+1)\cdot\Big(\int_{-1}^{d}(1-s^2)^{\frac{N-1}{2}}\ \mathrm{d}s\Big)^2} < 0 \qquad \text{for all } \ d\in(0,1).$$

In particular, for $N=1$ we have $\Psi'(d) = -\frac{1}{2}$ and hence it remains to deduce the second claimed inequality: $-\Psi'(d)\leq 1$ for $N\geq 2$. Equivalently, we will check that:

$$(N+1)d(1-d^2)^{\frac{N-1}{2}}\int_{-1}^{d}(1-s^2)^{\frac{N-1}{2}}\ \mathrm{d}s + (1-d^2)^N \leq (N+1)\cdot\Big(\int_{-1}^{d}(1-s^2)^{\frac{N-1}{2}}\ \mathrm{d}s\Big)^2, \tag{7.9}$$

for all $d\in[0,1]$. To this end, we compute the derivative of the left hand side in (7.9):

$$\begin{aligned}(N+1)(1-d^2)^{\frac{N-3}{2}}\Big(\int_{-1}^{d}(1-s^2)^{\frac{N-1}{2}}\ \mathrm{d}s\Big)\cdot(1-d^2N) - (N-1)d(1-d^2)^{N-1}\\ \leq (N+1)(1-d^2)^{\frac{N-1}{2}}\int_{-1}^{d}(1-s^2)^{\frac{N-1}{2}}\ \mathrm{d}s\end{aligned}$$

and observe that it is less than or equal to the derivative of the right hand side in (7.9), which is:

$$2(N+1)\Big(\int_{-1}^{d}(1-s^2)^{\frac{N-1}{2}}\ \mathrm{d}s\Big)\cdot\big(1-d^2\big)^{\frac{N-1}{2}}.$$

To conclude (7.9) it thus suffices to check its validity at $d=0$, namely: $\int_0^1(1-s^2)^{\frac{N-1}{2}}\ \mathrm{d}s \geq \frac{1}{\sqrt{N+1}}$. This will be implied by the following bound, proved in the next step:

$$\int_0^1(1-s^2)^{\frac{N-1}{2}}\ \mathrm{d}s \geq \frac{1}{\sqrt{N}}. \tag{7.10}$$

2. We show (7.10) by induction on N. For $N=1$, both sides are equal 1. For $N=2$, we have:

$$\int_0^1(1-s^2)^{1/2}\ \mathrm{d}s = \int_0^{\pi/2}\cos^2 t\ \mathrm{d}t = \frac{\pi}{4} \geq \frac{1}{\sqrt{2}}.$$

For $N\geq 3$ we change variable in: $\int_0^1(1-s^2)^{\frac{N-1}{2}}~\mathrm{d}s=\int_0^{\pi/2}\cos^N t~\mathrm{d}t$ and integrate by parts:

$$\int_0^{\pi/2}\cos^N t~\mathrm{d}t=(N-1)\int_0^{\pi/2}\cos^{N-2}t\cdot\sin^2 t~\mathrm{d}t=(N-1)\int_0^{\pi/2}\cos^{N-2}t~\mathrm{d}t-(N-1)\int_0^{\pi/2}\cos^N t~\mathrm{d}t.$$

The above is a well-known calculation, that yields by the induction assumption:

$$\begin{split}\int_0^1(1-s^2)^{\frac{N-1}{2}}~\mathrm{d}s=\frac{N-1}{N}\int_0^{\pi/2}\cos^{N-2}t~\mathrm{d}t=\frac{N-1}{N}\int_0^1(1-s^2)^{\frac{N-3}{2}}~\mathrm{d}s\\ \geq\frac{N-1}{N}\cdot\frac{1}{\sqrt{N-2}}\geq\frac{1}{\sqrt{N}}.\end{split}$$

Consequently (7.10) follows, completing the direct proof of Lemma 7.2. ■

The second proof of Lemma 7.2.

1. Recall that $\Psi(d)=-\fint_{B_{1,d}^N}y_N~\mathrm{d}y$ by (2.4). We will show the claimed properties of Ψ in a more general set-up, namely when B_1^N is replaced by an arbitrary open, bounded, convex set $A\subset\mathbb{R}^N$. Without loss of generality, we assume that the projection of A on the e_N axis equals the interval $(0,h)$. Define the function $\Psi_A:(0,h)\to\mathbb{R}$:

$$\Psi_A(d)=-\fint_{A\cap\{y_N<d\}}y_N~\mathrm{d}y=-\frac{\int_0^d tg(t)~\mathrm{d}t}{\int_0^d g(t)~\mathrm{d}t}\qquad\text{for all }\ d\in(0,h),$$

where $g:\mathbb{R}\to[0,\infty)$ is given as the $(N-1)$-dimensional Lebesgue measure of the sections of A in: $g(t)=\big|\{y\in\mathbb{R}^{N-1};~(y_1\dots y_{N-1},t)\in A\}\big|$. By convexity of A it follows that g is continuous, and we also have: $g=0$ on $\mathbb{R}\setminus(0,h)$ and $g>0$ on $(0,h)$.

2. To show that Ψ_A is decreasing, we simply compute:

$$\Psi_A'(d)=\frac{-g(t)\int_0^d(d-t)g(t)~\mathrm{d}t}{\int_0^d g(t)~\mathrm{d}t}<0\qquad\text{for all }\ d\in(0,h).$$

To show that Ψ_A has Lipschitz constant 1, we are going to deduce that the function $(0,h)\ni d\mapsto -\Psi_A(d)-d$ is nonincreasing. Integrate by parts to get:

$$-\Psi_A(d)-d=-\frac{\int_0^d\int_0^t g(s)~\mathrm{d}s~\mathrm{d}t}{\int_0^d g(t)~\mathrm{d}t}$$

and observe that the claimed property is equivalent to the following function being nonincreasing:

$$(0,h)\ni d\mapsto\frac{\bar g'(d)}{\bar g(d)}\qquad\text{where:}\quad\bar g(d)=\int_0^d\int_0^t g(s)~\mathrm{d}s~\mathrm{d}t.$$

Equivalently, we will show that the above defined function $\bar g:\mathbb{R}\to[0,\infty)$ is log-concave, i.e.:

$$\bar g\big(\lambda d_1+(1-\lambda)d_2\big)\geq\bar g(d_1)^\lambda\bar g(d_2)^{1-\lambda}\qquad\text{for all }\ \lambda\in(0,1)\ \text{ and all }\ d_1,d_2\in\mathbb{R}.\tag{7.11}$$

Recall now the celebrated Prékopa theorem (the formulation we use is that of [30, Theorem 6]), which states that if a given function $h:\mathbb{R}^{n+m}\to[0,\infty)$ is log-concave, then the marginal function $\mathbb{R}^m\ni d\mapsto\int_{\mathbb{R}^n}h(z,d)~\mathrm{d}z$ must be log-concave as well. We put $n=N+1$, $m=1$ and set h to be the characteristic function of the following set $\tilde A\subset\mathbb{R}^{N+2}$:

$$\tilde A=\big\{(y,t,d);~d\in(0,h),~t\in(0,d),~y\in A,~y_N<t\big\}.$$

Since $\tilde{A}$ is convex, h is log-concave. The Prékopa theorem yields thus log-concavity of:

$$\mathbb{R} \ni d \mapsto \int_{\mathbb{R}^{N+1}} h(y,t,d)\,\mathrm{d}(y,t) = \big|\{(y,t)\in\mathbb{R}^{N+1};\, t\in(0,d),\ y\in A,\ y_N<t\}\big| = \bar{g}(t),$$

as requested in (7.11). This ends the second proof of Lemma 7.2. ∎

We conclude this section by another geometric observation. Note that the exponent $(-N)$ below, may be replaced by any exponent less than $2-N$.

Lemma 7.3. *Assume (BH) and let $\bar{r}>0$ be any radius that is strictly smaller than the uniform inner supporting sphere radius r of $\mathcal{D}$, defined in (8.1). Then there exists a constant $C_l>0$, depending only on $\mathcal{D},\bar{r}$ such that for all $\epsilon \ll 1$ the following holds. Consider a supporting ball $B_{\bar{r}}(y_0)$ and let $x_0\in \bar{B}_{\bar{r}}(y_0)\setminus B_{\bar{r}/2}(y_0)$. Then:*

$$⨍_{B_\epsilon(x_0)\cap\mathcal{D}} |y-y_0|^{-N}\,\mathrm{d}y \geq |x_0-y_0|^{-N} + C_l\big(s_\epsilon(x_0)+\epsilon^2\big).$$

Proof. We Taylor expand the function $B_\epsilon(x_0)\cap\mathcal{D}\ni y\mapsto \phi(y)=|y-y_0|^{-N}$ at x_0:

$$|y-y_0|^{-N} = |x_0-y_0|^{-N} + \big\langle \nabla\phi(x_0), y-x_0\big\rangle + \frac{1}{2}\big\langle \nabla^2\phi(x_0) : (y-x_0)^{\otimes 2}\big\rangle + \mathcal{O}(\epsilon^3) \tag{7.12}$$

and calculate the relevant derivatives:

$$\begin{aligned}\nabla\phi(x_0) &= -N|x_0-y_0|^{-N-2}(x_0-y_0)\\ \nabla^2\phi(x_0) &= N|x_0-y_0|^{-N-4}\big((N+2)(x_0-y_0)^{\otimes 2} - |x_0-y_0|^2 Id_N\big).\end{aligned}$$

The claim now follows by integrating (7.12) on $B_\epsilon(x_0)\cap\mathcal{D}$ and observing that:

$$\begin{aligned}&-N|x_0-y_0|^{-N-2}\big\langle x_0-y_0, ⨍_{B_\epsilon(x_0)\cap\mathcal{D}} y-x_0\,\mathrm{d}y\big\rangle\\ &\qquad = N|x_0-y_0|^{-N-1}\Big\langle \frac{x_0-y_0}{|x_0-y_0|}, \vec{n}(\pi_{\partial\mathcal{D}}x_0)\Big\rangle\cdot s_\epsilon(x_0) + \mathcal{O}\big(\epsilon s_\epsilon(x_0)\big) \geq C_l s_\epsilon(x_0)\end{aligned}$$

by (2.4), and:

$$\begin{aligned}&N|x_0-y_0|^{-N-4}\big\langle (N+2)(x_0-y_0)^{\otimes 2} - |x_0-y_0|^2 Id_N : ⨍_{B_\epsilon(x_0)\cap\mathcal{D}} (y-x_0)^{\otimes 2}\,\mathrm{d}y\big\rangle\\ &\qquad = \frac{2N}{N+2}|x_0-y_0|^{-N-2}\cdot\epsilon^2 + \mathcal{O}\big(\epsilon s_\epsilon(x_0)\big) \geq C_l\epsilon^2\end{aligned}$$

by using (2.5). This ends the proof of the Lemma. ∎

8. A proof of asymptotic equicontinuity - Hölder 1/2 up to the boundary

In this section, we show a boundary counterpart of Theorem 6.1, which already implies the asymptotic equicontinuity of $\{u_\epsilon\}_{\epsilon\to 0}$, albeit in a weaker regularity regime than that claimed in Theorem 1.2. The main result of this section is:

Theorem 8.1. *Assume (BH). There exists $\delta_0 \ll 1$ such that for every $\delta\in(0,\delta_0)$ there is $\bar{\epsilon}>0$ with the following property. For all $\epsilon<\bar{\epsilon}$ and all $x_0,y_0\in\bar{\mathcal{D}}$ satisfying $|x_0-y_0|\leq\delta$ with $\mathrm{dist}(x_0,\partial\mathcal{D}),\ \mathrm{dist}(y_0,\partial\mathcal{D})\leq \delta^{1/2}$, we have:*

$$|u_\epsilon(x_0)-u_\epsilon(y_0)| \leq C\delta^{1/2}.$$

The constant C may depend on $\mathcal{D}$, γ and f, but not on ϵ, x_0, y_0 or δ.

We start with a definition of quantities that will be used throughout the remaining discussion.

Definition 8.2. *Let $\mathcal{D}$ be as in (BH) and fix a radius $\rho > 0$ that is strictly smaller than some uniform inner supporting sphere radius r of $\mathcal{D}$, given by the property:*

$$\text{for every } x \in \partial\mathcal{D} \text{ exists } B_r(a) \subset \mathcal{D} \quad \text{such that} \quad |x-a| = r. \tag{8.1}$$

We define a continuous function $Z^\rho : \bar{\mathcal{D}} \to \mathcal{D}$, satisfying:

$$x \in \bar{B}_\rho\big(Z^\rho(x)\big) \subset \bar{\mathcal{D}} \quad \text{for all } x \in \bar{\mathcal{D}},$$

as follows. When $x \in \bar{\mathcal{D}}$ and $\operatorname{dist}(x, \partial\mathcal{D}) \leq \rho$, there exists exactly one $Z^\rho(x) \in \mathcal{D}$ that is the center of the inner ρ-supporting sphere at $\pi_{\partial\mathcal{D}}x$. For $x \in \mathcal{D}$ with $\operatorname{dist}(x, \partial\mathcal{D}) > \rho$, we set $Z^\rho(x) = x$. Writing now $\{\mathcal{Z}_n^{\epsilon,\rho,x_0} = Z^\rho \circ X_n^{\epsilon,x_0}\}_{n=0}^\infty$, it is straightforward that there holds:

$$\big|\mathcal{Z}_{n+1}^{\epsilon,\rho,x_0} - X_{n+1}^{\epsilon,x_0}\big| \leq \big|\mathcal{Z}_n^{\epsilon,\rho,x_0} - X_{n+1}^{\epsilon,x_0}\big|. \tag{8.2}$$

For every $\epsilon \ll 1, x_0 \in \bar{\mathcal{D}}$ and $h \in (\epsilon, \frac{\rho}{2} - \epsilon)$, we further define the stopping time $\bar{\tau}^{\epsilon,h,x_0}$ and two sequences of random variables $\{\Theta_n^{\epsilon,x_0}, S_n^{\epsilon,x_0}\}_{n=0}^\infty$ by:

$$\begin{split} & \bar{\tau}^{\epsilon,h,x_0} = \min\big\{n \geq 0;\ \big|X_n^{\epsilon,x_0} - \mathcal{Z}_n^{\epsilon,\rho,x_0}\big| < \rho - h\big\} = \min\big\{n \geq 0;\ \operatorname{dist}\big(X_n^{\epsilon,x_0}, \partial\mathcal{D}\big) > h\big\}, \\ & \Theta_n^{\epsilon,x_0} = \epsilon\sum_{j=0}^{n-1} \mathbb{1}_{\{d_\epsilon(X_j)<1\}}, \qquad S_n^{\epsilon,x_0} = \sum_{j=0}^{n-1} s_\epsilon(X_j^{\epsilon,x_0}). \end{split} \tag{8.3}$$

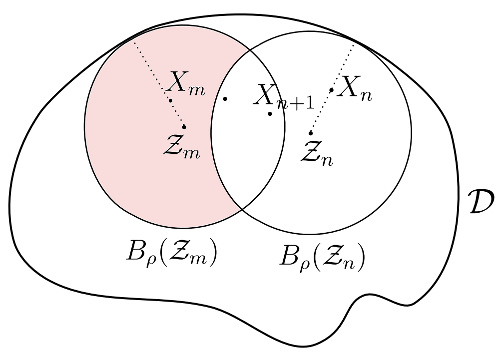

FIGURE 8.1. The auxiliary balls $B_\rho(\mathcal{Z}_n)$ and the process $\{X_n\}_{n=0}^\infty$.

Lemma 8.3. *Let $\mathcal{D}$ be as in (BH) and let $\bar{r} < r$ with r as in (8.1). Then, there exists a constant $C > 0$ depending only on $\mathcal{D}$ and $\bar{r}$, such that for every $\epsilon \ll 1$, $x_0 \in \bar{\mathcal{D}}$ and $h \in \big(\epsilon, \frac{\bar{r}}{2} - \epsilon\big)$:*

$$\mathbb{E}\big[\epsilon^2\bar{\tau}^{\epsilon,h,x_0} + S_{\bar{\tau}^{\epsilon,h,x_0}}^{\epsilon,x_0}\big] \leq Ch. \tag{8.4}$$

Proof. As usual, we drop the superscripts $\epsilon, \bar{r}, h$ and x_0 to alleviate the notation. It suffices to consider the case $|x_0 - \mathcal{Z}_0^{\epsilon,\bar{r},x_0}| \geq \bar{r} - h$. Define now the random variables:

$$M_n = \big|X_n - \mathcal{Z}_n\big|^{-N} - C_l(n\epsilon^2 + S_n),$$

where C_l is the constant from Lemma 7.3. We deduce that $\{M_{\bar{\tau}\wedge n}\}_{n=0}^\infty$ is a submartingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$, because whenever $n < \bar{\tau}$ then $|X_n - \mathcal{Z}_n| \in [\frac{\bar{r}}{2}, \bar{r}]$ implies, in virtue of Lemma 7.3, that:

$$\begin{split} \mathbb{E}\big(M_{n+1} - M_n \mid \mathcal{F}_n\big) & = \mathbb{E}\big(\big|X_{n+1} - \mathcal{Z}_{n+1}\big|^{-N} \mid \mathcal{F}_n\big) - \big|X_n - \mathcal{Z}_n\big|^{-N} - C_l\big(\epsilon^2 + s_\epsilon(X_n)\big) \\ & \geq \fint_{B_\epsilon(X_n)\cap\mathcal{D}} |y - \mathcal{Z}_n|^{-N}\,\mathrm{d}y - \big|X_n - \mathcal{Z}_n\big|^{-N} - C_l\big(\epsilon^2 + s_\epsilon(X_n)\big) \geq 0. \end{split}$$

Above, we have used (8.2) to replace $\mathbb{E}\big(\big|X_{n+1}-\mathcal{Z}_{n+1}\big|^{-N}\mid\mathcal{F}_n\big)$ by $\mathbb{E}\big(\big|X_{n+1}-\mathcal{Z}_{n}\big|^{-N}\mid\mathcal{F}_n\big)$ and (3.1) to further replace it by $\fint_{B_\epsilon(X_n)\cap\mathcal{D}}|y-\mathcal{Z}_n|^{-N}\,\mathrm{d}y$. For each fixed $i\geq 1$, we may apply Doob's Optional Stopping to the finite stopping time $\bar\tau\wedge i$, to the effect that:

$$\begin{aligned}\bar r^{-N}\leq|x_0-\mathcal{Z}_0|^{-N}=\mathbb{E}\big[M_0\big]\leq\mathbb{E}\big[M_{\bar\tau\wedge i}\big]=\mathbb{E}\big[\big|X_{\bar\tau\wedge i}-\mathcal{Z}_{\bar\tau\wedge i}\big|^{-N}\big]-C_l\cdot\mathbb{E}\big[\epsilon^2(\bar\tau\wedge i)+S_{\bar\tau\wedge i}\big]\\ \leq\big(\bar r-h-\epsilon\big)^{-N}-C_l\cdot\mathbb{E}\big[\epsilon^2(\bar\tau\wedge i)+S_{\bar\tau\wedge i}\big].\end{aligned}$$

Passing to the limit with $i\to\infty$, we get:

$$\mathbb{E}\big[\epsilon^2\bar\tau+S_{\bar\tau}\big]\leq\frac{1}{C_l}\Big(\big(\bar r-h-\epsilon\big)^{-N}-\bar r^{-N}\Big)\leq\frac{N(h+\epsilon)}{C_l}\cdot\bar r^{-N-1},$$

which concludes the proof of (8.4) with $C=\frac{2N\bar r^{-N-1}}{C_l}$. ∎

Proof of Theorem 8.1.

1. Fix $\epsilon\ll 1$ and $\bar r<r$ with r as in (8.1). Given x_0,y_0 as in the statement of the Lemma, consider the sequences of random variables $\{\mathcal{Z}_n^{\epsilon,\bar r,x_0},\mathcal{Z}_n^{\epsilon,\bar r,y_0},S_n^{\epsilon,x_0},S_n^{\epsilon,y_0}\}_{n=0}^\infty$ as defined in Definition 8.2), together with the stopping times $\bar\tau^{\epsilon,h,x_0},\bar\tau^{\epsilon,h,y_0}$ where we set:

$$h=\delta^{1/2}.$$

To alleviate the notation, we drop the superscripts ϵ,h and denote:

$$X_n=X_n^{x_0},\qquad Y_n=Y_n^{y_0},\qquad\bar\tau=\bar\tau^{x_0}\wedge\bar\tau^{y_0}.$$

We now define the sequence of random variables $\{M_n\}_{n=0}^\infty$:

$$M_n=u_\epsilon(X_n)\cdot\prod_{k=1}^n\big(1-\gamma s_\epsilon(X_{k-1})\big)+\frac{\epsilon^2}{2(N+2)}\sum_{j=1}^{n-1}\Big(f(X_j)\cdot\prod_{k=1}^j\big(1-\gamma s_\epsilon(X_{k-1})\big)\Big)\tag{8.5}$$

and check that it is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$. Indeed, (RMV)$_\epsilon$ yields:

$$\begin{aligned}\mathbb{E}\big(M_{n+1}-M_n\mid\mathcal{F}_n\big)=\Big(\fint_{B_\epsilon(X_n)\cap\mathcal{D}}u_\epsilon(y)\,\mathrm{d}y\Big)\cdot\Lambda_{n+1}\\ -u_\epsilon(X_n)\cdot\Lambda_n+\frac{\epsilon^2}{2(N+2)}f(X_n)\cdot\Lambda_n=0\quad\mathbb{P}-\text{a.s. in }\ \Omega.\end{aligned}$$

Applying Doob's Optional Stopping to the martingale $\{M_{n\wedge\bar\tau}\}_{n=0}^\infty$ that is equibounded by the random variable $C+C\epsilon^2\bar\tau$ whose integrability follows from (8.4) because $\mathrm{dist}(x_0,\partial\mathcal{D})\leq\delta^{1/2}$ and $h=\delta^{1/2}$, we obtain:

$$u_\epsilon(x_0)=\mathbb{E}\big[M_0\big]=\mathbb{E}\big[M_{\bar\tau}\big]=\mathbb{E}\big[u_\epsilon(X_{\bar\tau})\cdot\Lambda_{\bar\tau}\big]+\frac{\epsilon^2}{2(N+2)}\sum_{j=1}^{\bar\tau-1}\mathbb{E}\big[f(X_j)\cdot\Lambda_j\big].$$

Consequently, again by (8.4) and using the inductively validated inequality: $\prod_{k=1}^n(1-a_i)\geq 1-\sum_{k=1}^n a_i$ that holds for any n-tuple $\{a_i\in[0,1]\}_{i=1}^n$, we arrive at:

$$\big|u_\epsilon(x_0)-\mathbb{E}\big[u_\epsilon(X_{\bar\tau})\big]\big|\leq C\mathbb{E}\Big[1-\prod_{k=1}^{\bar\tau}\big(1-\gamma s_\epsilon(X_{k-1})\big)\Big]+C\epsilon^2\mathbb{E}\big[\bar\tau\big]\leq C\mathbb{E}\big[\gamma S_{\bar\tau}^{x_0}\big]+C\delta^{1/2}\leq C\delta^{1/2}.$$

Clearly, the same bound holds for the $\{Y_n\}_{n=0}^\infty$ process:

$$\big|u_\epsilon(y_0)-\mathbb{E}\big[u_\epsilon(Y_{\bar\tau})\big]\big|\leq C\delta^{1/2}.$$

Summing the last two bounds, we get:

$$\big|u_\epsilon(x_0)-u_\epsilon(y_0)\big|\leq \mathbb{E}\big[\big|u_\epsilon(X_{\bar\tau})-u_\epsilon(Y_{\bar\tau})\big|\big]+C\delta^{1/2}. \tag{8.6}$$

2. In this and the next step, we proceed to estimating $\mathbb{E}\big[\big|u_\epsilon(X_{\bar\tau})-u_\epsilon(Y_{\bar\tau})\big|\big]$ in (8.6). This will be done with the help of Corollary 6.1, where we note:

$$\mathrm{dist}(X_{\bar\tau},\partial\mathcal{D})>h=\delta^{1/2}\quad \text{ or }\quad \mathrm{dist}(Y_{\bar\tau},\partial\mathcal{D})>\delta^{1/2}. \tag{8.7}$$

To this end, we first estimate the following conditional expectation:

$$\begin{split}\mathbb{E}\big(|X_{n+1}-Y_{n+1}|^2\mid\mathcal{F}_n\big) &= \mathbb{E}\big(\big|(X_{n+1}-Y_{n+1})-(X_n-Y_n)\big|^2\mid\mathcal{F}_n\big)-|X_n-Y_n|^2\\ &\qquad + 2\big\langle\mathbb{E}\big(X_{n+1}-Y_{n+1}\mid\mathcal{F}_n\big),X_n-Y_n\big\rangle\\ &\leq \mathbb{E}\big(|2\epsilon\cdot\mathbb{1}_{\{k^\epsilon_{n+1}(\omega,X_n)\neq k^\epsilon_{n+1}(\omega,Y_n)\}}|^2\mid\mathcal{F}_n\big)+|X_n-Y_n|^2+C\big(|X_n-Y_n|^2+\epsilon\big)\big(s_\epsilon(X_n)+s_\epsilon(Y_n)\big),\end{split}$$

$\mathbb{P}$-a.s. in Ω, using the fact that $k^\epsilon_{n+1}(\omega,X_n)=k^\epsilon_{n+1}(\omega,Y_n)$ implies $X_{n+1}-Y_{n+1}=X_n-Y_n$, and applying (7.1) to bound the product intermediate term above. Further, (2.3) results in:

$$\begin{split}\mathbb{E}\big(|2\epsilon\cdot\mathbb{1}_{\{k^\epsilon_{n+1}(\omega,X_n)\neq k^\epsilon_{n+1}(\omega,Y_n)\}}|^2\mid\mathcal{F}_n\big)&\leq 4\epsilon^2\Big(\mathbb{E}\big(\mathbb{1}_{\{k^\epsilon_{n+1}(\omega,X_n)\neq 1\}}\mid\mathcal{F}_n\big)+\mathbb{E}\big(\mathbb{1}_{\{k^\epsilon_{n+1}(\omega,Y_n)\neq 1\}}\mid\mathcal{F}_n\big)\\ &\leq C\epsilon\big(s_\epsilon(X_n)+s_\epsilon(Y_n)\big)\quad \mathbb{P}-\text{a.s. in }\ \Omega,\end{split}$$

so that:

$$\mathbb{E}\big(|X_{n+1}-Y_{n+1}|^2\mid\mathcal{F}_n\big)\leq |X_n-Y_n|^2\big(1+C(s_\epsilon(X_n)+s_\epsilon(Y_n))\big)+C\epsilon\big(s_\epsilon(X_n)+s_\epsilon(Y_n)\big). \tag{8.8}$$

Define the sequence of random variables $\{Q_n\}_{n=0}^\infty$ by:

$$Q_n=|X_n-Y_n|^2\cdot e^{-C(S_n^{x_0}+S_n^{y_0})}-C\epsilon\big(S_n^{x_0}+S_n^{y_0}\big).$$

We now check that $\{Q_n\}_{n=0}^\infty$ is a supermartingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$:

$$\begin{split}\mathbb{E}\big(Q_{n+1}-Q_n\mid\mathcal{F}_n\big)&=\mathbb{E}\big(|X_{n+1}-Y_{n+1}|^2\mid\mathcal{F}_n\big)\cdot e^{-C(S_{n+1}^{x_0}+S_{n+1}^{y_0})}\\ &\qquad -|X_n-Y_n|^2\cdot e^{-C(S_n^{x_0}+S_n^{y_0})}-C\epsilon\big(s_\epsilon(X_n)+s_\epsilon(Y_n)\big)\\ &\leq |X_n-Y_n|^2\cdot e^{-C(S_{n+1}^{x_0}+S_{n+1}^{y_0})}\Big(1+C\big(s_\epsilon(X_n)+s_\epsilon(Y_n)\big)-e^{C(s_\epsilon(X_n)+s_\epsilon(Y_n))}\Big)\leq 0,\end{split}$$

in view of the fact that $S_{n+1}^{x_0}$ and $S_{n+1}^{y_0}$ are $\mathcal{F}_n$-measurable, and by (8.8). Application of Doob's Optional Stopping to the supermartingale $\{Q_{n\wedge\bar\tau}\}_{n=0}^\infty$, where each $|Q_{n\wedge\bar\tau}|$ is bounded by the random variable $C+C\epsilon(S_{\bar\tau}^{x_0}+S_{\bar\tau}^{y_0})$ that is integrable in view of (8.4), yields:

$$\delta^2\geq |x_0-y_0|^2=\mathbb{E}\big[Q_0\big]\geq\mathbb{E}\big[Q_{\bar\tau}\big]=\mathbb{E}\big[|X_{\bar\tau}-Y_{\bar\tau}|^2\cdot e^{-C(S_{\bar\tau}^{x_0}+S_{\bar\tau}^{y_0})}\big]-C\epsilon\cdot\mathbb{E}\big[S_{\bar\tau}^{x_0}+S_{\bar\tau}^{y_0}\big]. \tag{8.9}$$

3. We now complete estimating $\mathbb{E}\big[\big|u_\epsilon(X_{\bar\tau})-u_\epsilon(Y_{\bar\tau})\big|\big]$ in (8.6). Firstly, by (8.4) and the Chebyshev inequality, it follows that:

$$\mathbb{P}\big(S_{\bar\tau}^{x_0}+S_{\bar\tau}^{y_0}>1\big)\leq C\delta^{1/2}. \tag{8.10}$$

Secondly, by (8.9) and if only $\epsilon\ll\delta$:

$$\begin{split}\mathbb{E}\Big[|X_{\bar\tau}-Y_{\bar\tau}|^2\cdot\mathbb{1}_{\{S_{\bar\tau}^{x_0}+S_{\bar\tau}^{y_0}\leq 1\}}\Big]&\leq e^C\cdot\mathbb{E}\Big[|X_{\bar\tau}-Y_{\bar\tau}|^2\cdot e^{-C(S_{\bar\tau}^{x_0}+S_{\bar\tau}^{y_0})}\Big]\\ &\leq C\epsilon\mathbb{E}\big[S_{\bar\tau}^{x_0}+S_{\bar\tau}^{y_0}\big]+C\delta^2\leq C\epsilon\cdot\delta^{1/2}+C\delta^2\leq C\delta^2,\end{split}$$

so that we obtain:

$$\mathbb{E}\Big[|X_{\bar\tau}-Y_{\bar\tau}|\cdot\mathbb{1}_{\{S_{\bar\tau}^{x_0}+S_{\bar\tau}^{y_0}\leq 1\}}\Big]\leq C\delta. \tag{8.11}$$

We now decompose:

$$
\begin{aligned}
\mathbb{E}\big[\big|u_\epsilon(X_{\bar\tau})-u_\epsilon(Y_{\bar\tau})\big|\big] = {} & \mathbb{E}\Big[\big|u_\epsilon(X_{\bar\tau})-u_\epsilon(Y_{\bar\tau})\big|\cdot \mathbb{1}_{\{S^{x_0}_{\bar\tau}+S^{y_0}_{\bar\tau}>1\}}\Big] \\
& + \mathbb{E}\Big[\big|u_\epsilon(X_{\bar\tau})-u_\epsilon(Y_{\bar\tau})\big|\cdot \mathbb{1}_{\{S^{x_0}_{\bar\tau}+S^{y_0}_{\bar\tau}\leq 1\}}\cdot \mathbb{1}_{\{|X_{\bar\tau}-Y_{\bar\tau}|\geq \frac{\delta_0}{2}\}}\Big] \\
& + \mathbb{E}\Big[\big|u_\epsilon(X_{\bar\tau})-u_\epsilon(Y_{\bar\tau})\big|\cdot \mathbb{1}_{\{S^{x_0}_{\bar\tau}+S^{y_0}_{\bar\tau}\leq 1\}}\cdot \mathbb{1}_{\{|X_{\bar\tau}-Y_{\bar\tau}|< \frac{\delta_0}{2}\}}\Big].
\end{aligned}
\tag{8.12}
$$

The first term in the right hand side above is bounded by $C\delta^{1/2}$ in view of (8.10). The same estimate holds for the second term, because Chebyshev's inequality applied to (8.11) yields:

$$
\mathbb{P}\Big(\big\{S^{x_0}_{\bar\tau}+S^{y_0}_{\bar\tau}\leq 1\big\}\cap\big\{|X_{\bar\tau}-Y_{\bar\tau}|\geq \frac{\delta_0}{2}\big\}\Big)\leq \frac{C\delta}{\delta_0}.
$$

To treat the last term in (8.12), recall that for each $\eta\in(0,\delta_0)$, Theorem 6.1 gives $\bar\epsilon=\bar\epsilon(\eta)$ so that $|u_\epsilon(x)-u_\epsilon(y)|\leq \frac{C\eta}{\operatorname{dist}(x_0,\partial\mathcal{D})}$ holds for all $\epsilon<\bar\epsilon$ and all x,y satisfying $|x-y|\leq\eta$. Define:

$$
\tilde\epsilon = \min\Big\{\bar\epsilon\big(\frac{1}{2^k}\big);\ \frac{1}{2^k}\in[\delta,\delta_0)\Big\},
$$

where we decrease the threshold for the admissible δ below $\frac{\delta_0}{2}$ so that the set in the definition of $\tilde\epsilon$ is nonempty. Assume further that $\epsilon<\tilde\epsilon$. For each $\omega\in\big\{S^{x_0}_{\bar\tau}+S^{y_0}_{\bar\tau}\leq 1\big\}\cap\big\{|X_{\bar\tau}-Y_{\bar\tau}|<\frac{\delta_0}{2}\big\}$, apply now the above recalled statement to $\eta=\min\big\{\frac{1}{2^k};\frac{1}{2^k}\geq\max\{\delta,|X_{\bar\tau}-Y_{\bar\tau}|(\omega)\big\}$ and $x=X_{\bar\tau}(\omega)$, $y=Y_{\bar\tau}(\omega)$. Since $\eta\in[\delta,\delta_0)$, it follows that:

$$
\begin{aligned}
\big|u_\epsilon(X_{\bar\tau})-u_\epsilon(Y_{\bar\tau})\big| & \leq \frac{C\eta}{\delta^{1/2}}\leq C\cdot\frac{2(\delta+|X_{\bar\tau}-Y_{\bar\tau}|)}{\delta^{1/2}}\leq C\delta^{1/2}+C\frac{|X_{\bar\tau}-Y_{\bar\tau}|}{\delta^{1/2}} \\
& \qquad\qquad \text{a.s in } \ \big\{S^{x_0}_{\bar\tau}+S^{y_0}_{\bar\tau}\leq 1\big\}\cap\big\{|X_{\bar\tau}-Y_{\bar\tau}|<\frac{\delta_0}{2}\big\},
\end{aligned}
$$

where we also used (8.7). Consequently and in virtue of (8.11), we get:

$$
\begin{aligned}
\mathbb{E}\Big[\big|u_\epsilon(X_{\bar\tau})-u_\epsilon(Y_{\bar\tau})\big|\cdot \mathbb{1}_{\{S^{x_0}_{\bar\tau}+S^{y_0}_{\bar\tau}\leq 1\}}\cdot \mathbb{1}_{\{|X_{\bar\tau}-Y_{\bar\tau}|< \frac{\delta_0}{2}\}}\Big] & \leq C\delta^{1/2}+\frac{C}{\delta^{1/2}}\cdot\mathbb{E}\Big[\big|X_{\bar\tau}-Y_{\bar\tau}\big|\cdot \mathbb{1}_{\{S^{x_0}_{\bar\tau}+S^{y_0}_{\bar\tau}\leq 1\}}\Big] \\
& \leq C\delta^{1/2},
\end{aligned}
$$

Thus, (8.12) implies that:

$$
\mathbb{E}\big[\big|u_\epsilon(X_{\bar\tau})-u_\epsilon(Y_{\bar\tau})\big|\big]\leq C\delta^{1/2}
$$

and the claim of the Theorem 8.1 finally follows by recalling (8.6). ■

Combining Theorems 8.1 and 6.1, we obtain the Hölder $\mathcal{C}^{0,1/2}$-asymptotic regularity of $\{u_\epsilon\}_{\epsilon\to 0}$ in $\bar{\mathcal{D}}$, which yields Theorem 1.1 in view of the L^1 convergence in Theorem 5.2.

Corollary 8.4. *Assume (BH). There exists $\delta_0\ll 1$ such that for every $\delta\in(0,\delta_0)$ there is $\bar\epsilon>0$ with the following property. For all $\epsilon<\bar\epsilon$ and all $x_0,y_0\in\bar{\mathcal{D}}$ satisfying $|x_0-y_0|\leq\delta$, there holds:*

$$
\big|u^\epsilon(x_0)-u^\epsilon(y_0)\big|\leq C\delta^{1/2}.
$$

The constant C above may depend on $\mathcal{D}$, $\|f\|_{L^\infty(\mathcal{D})}$ and γ, but not on ϵ, x_0, y_0 or δ.

9. Boundary estimates II

In this section we derive further geometrical observations towards an improvement of the bound in Theorem 8.1 and a complete proof of Theorem 1.2.

Lemma 9.1. *Assume (BH). Then there exists a constant $C>0$ depending only on $\mathcal{D}$, such that for all $x_0, y_0 \in \bar{\mathcal{D}}$ there holds:*

$$\fint_{B_\epsilon(y_0)\cap\mathcal{D}} |y-x_0|^2\,\mathrm{d}y \leq |x_0-y_0|^2\cdot e^{Cs_\epsilon(y_0)} + C\epsilon^2. \tag{9.1}$$

Proof. We decompose:

$$\begin{aligned}\fint_{B_\epsilon(y_0)\cap\mathcal{D}} |y-x_0|^2 &= |y_0-x_0|^2 + 2\Big\langle y_0-x_0, \fint_{B_\epsilon(y_0)\cap\mathcal{D}} y-y_0\,\mathrm{d}y\Big\rangle + \fint_{B_\epsilon(y_0)\cap\mathcal{D}} |y-y_0|^2\,\mathrm{d}y \\ &\leq |y_0-x_0|^2 + 2\Big\langle y_0-x_0, \fint_{B_\epsilon(y_0)\cap\mathcal{D}} y-y_0\,\mathrm{d}y\Big\rangle + \epsilon^2.\end{aligned} \tag{9.2}$$

To estimate the linear term above, observe first that it is null when $\mathrm{dist}(y_0,\partial\mathcal{D})>\epsilon$. In the opposite case, we will use the representation (7.7), recalling the definitions:

$$w_1 = \frac{1}{2}\big(\vec{n}(\pi_{\partial\mathcal{D}}x_0) + \vec{n}(\pi_{\partial\mathcal{D}}y_0)\big), \qquad w_2 = \frac{1}{2}\big(\vec{n}(\pi_{\partial\mathcal{D}}x_0) - \vec{n}(\pi_{\partial\mathcal{D}}y_0)\big),$$

as in the proof of Lemma 7.1. Since by (2.4) we have:

$$\fint_{B_\epsilon(y_0)\cap\mathcal{D}} y-y_0\,\mathrm{d}y = -s_\epsilon(y_0)\big(w_1-w_2\big) + O(\epsilon^2),$$

it follows that:

$$\begin{aligned}&\Big\langle \pi_{\partial\mathcal{D}}y_0 - \pi_{\partial\mathcal{D}}x_0, \fint_{B_\epsilon(y_0)\cap\mathcal{D}} y-y_0\,\mathrm{d}y\Big\rangle \leq Cs_\epsilon(y_0)\cdot|y_0-x_0|^2 + O(\epsilon^2),\\ &\big(\mathrm{dist}(x_0,\partial\mathcal{D}) - \mathrm{dist}(y_0,\partial\mathcal{D})\big)\Big\langle w_1, \fint_{B_\epsilon(y_0)\cap\mathcal{D}} y-y_0\,\mathrm{d}y\Big\rangle \leq s_\epsilon(y_0)\cdot\mathrm{dist}(y_0,\partial\mathcal{D}) + O(\epsilon^2) = O(\epsilon^2),\\ &\big(\mathrm{dist}(x_0,\partial\mathcal{D}) + \mathrm{dist}(y_0,\partial\mathcal{D})\big)\Big\langle w_2, \fint_{B_\epsilon(y_0)\cap\mathcal{D}} y-y_0\,\mathrm{d}y\Big\rangle \leq Cs_\epsilon(y_0)\cdot|y_0-x_0|^2 + O(\epsilon^2),\end{aligned}$$

where we used (7.5) in the second and third estimate, and the assumption $\mathrm{dist}(y_0,\partial\mathcal{D})\leq\epsilon$ in the second estimate. Summing the three above bounds, and recalling (9.2), we get:

$$\fint_{B_\epsilon(y_0)\cap\mathcal{D}} |y-x_0|^2\,\mathrm{d}y \leq |y_0-x_0|^2\big(1+\bar{C}_u s_\epsilon(y_0)\big) + \bar{C}_u\epsilon^2,$$

where the linear term in (9.2) is bounded by $Cs_\epsilon(y_0)|y_0-x_0|^2 + O(\epsilon^2)$. The proof is done. ∎

Corollary 9.2. *Assume (BH). Then for every $x_0\in\bar{\mathcal{D}}$ and $\epsilon\ll 1$, the following sequence of random variables is a supermartingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$:*

$$\Big\{|X_n^{\epsilon,x_0} - x_0|^2\cdot\exp\Big(-C\sum_{j=0}^{n-1} s_\epsilon(X_j^{\epsilon,x_0})\Big) - Cn\epsilon^2\Big\}_{n=0}^\infty,$$

with the same constant $C>0$ as in (9.1).

Proof. The proof is straightforward, because:

$$\begin{aligned}\mathbb{E}\Big(|X_{n+1}-x_0|^2\cdot\exp\big(-C\sum_{j=0}^{n}s_\epsilon(X_j)\big)-C(n+1)\epsilon^2\mid\mathcal{F}_n\Big)\\ &=\fint_{B_\epsilon(X_n)\cap\mathcal{D}}|x-x_0|^2\,\mathrm{d}x\cdot\exp\big(-C\sum_{j=0}^{n}s_\epsilon(X_j)\big)-C(n+1)\epsilon^2\\ &\leq|X_n-x_0|\cdot\exp\big(-C\sum_{j=0}^{n-1}s_\epsilon(X_j)\big)-Cn\epsilon^2,\end{aligned}$$

by applying (9.1) to x_0 and $y_0=X_n$. ■

Lemma 9.3. *Let $\mathcal{D}$ be as in (BH) and let $\bar r<r$ with r as in (8.1). Then, there exists a constant $C_u>1$ depending only on $\mathcal{D}$ and $\bar r$, such that for every $\epsilon\ll1$, $x_0\in\bar{\mathcal{D}}$ and $h\in\big(\epsilon,\frac{\bar r}{2}-\epsilon\big)$:*

$$\mathbb{E}\big[\Theta^{\epsilon,x_0}_{\bar\tau^{\epsilon,h,x_0}}\big]\leq C_uh. \tag{9.3}$$

Moreover, for all $k\geq1$ there holds:

$$\mathbb{P}\Big(\Theta^{\epsilon,x_0}_{\bar\tau^{\epsilon,h,x_0}}\geq k\big(\epsilon+2C_uh\big)\Big)\leq\frac{1}{2^k}. \tag{9.4}$$

Proof. **1.** Given $\bar r,h,\epsilon,x_0$ as in the statement and a constant $\lambda>0$, consider the sequence of random variables $\{M_n\}_{n=0}^\infty$ in:

$$M_n=\lambda\Theta_n-S_{n+1}=\lambda\epsilon\Big(\sum_{j=0}^{n-1}\mathbb{1}_{\{d_\epsilon(X_j^{\epsilon,x_0})<1\}}\Big)-\Big(\sum_{j=0}^{n}s_\epsilon(X_j^{\epsilon,x_0})\Big),$$

adopting the convention that $M_0=-s_\epsilon(x_0)$. We claim that $\{M_n\}_{n=0}^\infty$ is a supermartingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$ provided that $\lambda=\lambda(\mathcal{D})$ is chosen appropriately. Indeed:

$$\mathbb{E}\big(M_{n+1}-M_n\mid\mathcal{F}_n\big)=\lambda\epsilon\mathbb{1}_{\{d_\epsilon(X_n)<1\}}-\mathbb{E}\big(s_\epsilon(X_{n+1})\mid\mathcal{F}_n\big)\leq0\qquad\mathbb{P}-\text{a.s. in }\Omega,$$

follows by observing that on the event $\{\mathrm{dist}(X_n,\partial\mathcal{D})\geq\epsilon\}$ the quantity above is clearly nonpositive, whereas on the event $\{\mathrm{dist}(X_n,\partial\mathcal{D})<\epsilon\}$ it is still nonpositive upon choosing λ so small that:

$$\mathbb{E}\big(s_\epsilon(X_{n+1})\mid\mathcal{F}_n\big)=\fint_{B_\epsilon(X_n)\cap\mathcal{D}}s_\epsilon(y)\,\mathrm{d}y\geq c\epsilon\fint_{B_\epsilon(X_n)\cap\mathcal{D}}(1-d_\epsilon(y)^2)^{\frac{N+1}{2}}\,\mathrm{d}y\geq\lambda\epsilon.$$

The supermartingale property and Doob's theorem applied to the finite stopping times $\bar\tau\wedge i$ yield:

$$\mathbb{E}\big[\lambda\Theta_{\bar\tau\wedge i}\big]=\mathbb{E}\big[M_{\bar\tau\wedge i}\big]+\mathbb{E}\big[S_{(\bar\tau\wedge i)+1}\big]\leq\mathbb{E}\big[M_0\big]+\mathbb{E}\big[S_{\bar\tau+1}\big]\leq\mathbb{E}\big[S_{\bar\tau}\big]+\epsilon\leq Ch,$$

in virtue of (8.4). Consequently, (9.3) follows with $C_u=C/\lambda$ by passing to the limit $i\to\infty$.

2. To show (9.4), we first use Chebyshev's inequality in (8.4), to arrive at:

$$\mathbb{P}\Big(\Theta_{\bar\tau}\geq2C_uh\Big)\leq\frac12. \tag{9.5}$$

The general case $k>1$ follows by induction. To this end, for each $k\geq1$, define the stopping time:

$$T_k=\min\Big\{n\geq0;\ \Theta_n\geq k\big(\epsilon+2C_uh\big)\Big\}.$$

Then (9.4) is equivalent to: $\mathbb{P}\big(T_k\leq\bar\tau\big)\leq1/2^k$. Observe that:

$$\big\{T_{k+1}\leq\bar\tau\big\}=\big\{T_k\leq\bar\tau\big\}\cap\Big\{\Theta_{\bar\tau}^{X_{T_k}}\geq2C_uh\Big\}.$$

Taking conditional expectations we get:

$$\mathbb{P}\big(T_{k+1} \leq \bar\tau\big) = \mathbb{E}\big[\mathbb{1}_{\{T_{k+1}\leq\bar\tau\}}\big] = \mathbb{E}\Big[\mathbb{1}_{\{T_k\leq\bar\tau\}}\cdot\mathbb{E}\Big(\mathbb{1}_{\big\{\Theta_{\bar\tau}^{X_{T_k}}\geq 2C_u h\big\}} \mid \mathcal{F}_{T_k}\Big)\Big] \leq \frac{1}{2}\mathbb{E}\big[\mathbb{1}_{\{T_k\leq\bar\tau\}}\big] \leq \frac{1}{2^{k+1}},$$

by (9.5) applied with X_{T_k} in place of x_0, and by the induction assumption. The proof is done. ■

10. A proof of asymptotic equicontinuity - Hölder α up to the boundary, $\alpha\in(0,1)$

The main result of this section is:

Theorem 10.1. *Assume (BH). There exists $\delta_0 \ll 1$ such that for every $\delta\in(0,\delta_0)$ there is $\bar\epsilon>0$ with the following property. For all $\epsilon<\bar\epsilon$ and all $x_0,y_0\in\bar{\mathcal{D}}$ satisfying $|x_0-y_0|\leq\delta$ with $\mathrm{dist}(x_0,\partial\mathcal{D}),\ \mathrm{dist}(y_0,\partial\mathcal{D})\leq\frac{1}{9C_u\log\big(1/\delta\big)}$, we have:*

$$|u_\epsilon(x_0)-u_\epsilon(y_0)|\leq C\delta\cdot\log\big(\frac{1}{\delta}\big).$$

The threshold constant C_u is as in Lemma 9.3, whereas the constant $C>0$ may depend on $\mathcal{D}$, γ and f, but not on ϵ, x_0, y_0 or δ.

We start by defining yet another useful coupling.

Definition 10.2. *Given $x_0,y_0\in\bar{\mathcal{D}}$ and $\epsilon\ll1$, recall the set-up for the process $\{X_n^{\epsilon,x_0}\}_{n=0}^\infty$ in (3.2), and define the coupled process $\{Y_n^{\epsilon,y_0,x_0}\}_{n=0}^\infty$ inductively as follows. Set $Y_0=y_0$. At each n, denote by H_n the hyperplane in $\mathbb{R}^N$ that is orthogonal to and bisecting the segment $[X_n,Y_n]$. By refl_{H_n}, denote the reflection across H_n. Then, we set:*

$$\begin{aligned}&Y_{n+1}^{\epsilon,y_0,x_0} = \mathrm{refl}_{H_n}\big(X_n+\epsilon w_{n+1}^{\tilde k^\epsilon(\omega,X_n)}\big),\\ &\textit{where}\ \ \tilde k_{n+1}^\epsilon = \min\big\{\tilde k\geq 1;\ \mathrm{refl}_{H_n}\big(X_n+\epsilon w_{n+1}^{\tilde k}\big)\in\mathcal{D}\big\}.\end{aligned}$$

Lemma 10.3. *Let $\mathcal{D}$ be as in (BH) and let $\bar r<r$ with r as in (8.1). For a given $x_0,y_0\in\bar{\mathcal{D}}$, $\epsilon\ll1$ and $h\in(\epsilon,\min\{\frac{\bar r}{2}-\epsilon,1\}$, we define the sequence of random variables $\{Q_n^{\epsilon,h,x_0,y_0}\}_{n=0}^\infty$ in:*

$$Q_n^{\epsilon,h,x_0,y_0} = \big(4h-|X_n^{\epsilon,x_0}-Y_n^{\epsilon,y_0,x_0}|\big)\cdot|X_n^{\epsilon,x_0}-Y_n^{\epsilon,y_0,x_0}|,$$

where the coupled process Y_n^{ϵ,y_0,x_0} is as in Definition 10.2. Then there exists a constant $C>0$ depending only on $\mathcal{D}$ and $\bar r$, such that:

(i) *For every $\delta\in(0,h)$, conditions: $|X_n-Y_n|\in(\delta^2,3h)$ and $\epsilon<\frac{1}{2}\delta^4$ imply:*

$$\mathbb{E}\big(Q_{n+1}-Q_n\mid\mathcal{F}_n\big)\leq Ch\epsilon\big(\mathbb{1}_{\{d_\epsilon(X_n)<1\}}+\mathbb{1}_{\{d_\epsilon(Y_n)<1\}}\big)|X_n-Y_n|-\frac{4}{N+2}\epsilon^2\quad \mathbb{P}-a.s.\ in\ \ \Omega.$$

(ii) *In the setting of (i) there further holds:*

$$\mathbb{E}\big(Q_{n+1}\mid\mathcal{F}_n\big)\leq Q_n\cdot\exp\big(C\epsilon(\mathbb{1}_{\{d_\epsilon(X_n)<1\}}+\mathbb{1}_{\{d_\epsilon(Y_n)<1\}})\big)-\frac{4}{N+2}\epsilon^2\quad \mathbb{P}-a.s.\ in\ \ \Omega.$$

(iii) *Fix $\delta\in(0,h)$, and assume the conditions: $|x_0-y_0|<\delta$ and $\epsilon<\frac12\delta^4$. Define the sequence of random variables $\{\bar Q_n^{\epsilon,h,x_0,y_0}\}_{n=0}^\infty$ in:*

$$\bar Q_n^{\epsilon,h,x_0,y_0} = Q_n^{\epsilon,h,x_0,y_0}\cdot\exp\big(-C(\Theta_n^{\epsilon,x_0}+\Theta_n^{\epsilon,y_0,x_0})\big)+e^{-3C}n\epsilon^2,$$

where, consistently with (8.3), we set:

$$\Theta_n^{\epsilon,y_0,x_0} = \epsilon\sum_{j=0}^{n-1}\mathbb{1}_{\{d_\epsilon(Y_j^{\epsilon,y_0,x_0})<1\}}.$$

Define also the stopping time:

$$\begin{aligned}\sigma^{\epsilon,h,\delta,x_0,y_0} = \min\big\{n\geq 0;\ |X_n^{\epsilon,x_0}-x_0|\geq h \ \ \textit{or}\ \ |Y_n^{\epsilon,y_0,x_0}-y_0|\geq h \\ \textit{or}\ \ |X_n^{\epsilon,x_0}-Y_n^{\epsilon,y_0,x_0}|\leq\delta^2 \ \ \textit{or}\ \ \Theta_n^{\epsilon,x_0}+\Theta_n^{\epsilon,y_0,x_0}\geq 2\big\}.\end{aligned} \tag{10.1}$$

Then, $\big\{\bar Q^{\epsilon,h,x_0,y_0}_{n\wedge\sigma^{\epsilon,h,\delta,x_0,y_0}}\big\}_{n=0}^\infty$ *is a supermartingale with respect to the filtration* $\{\mathcal{F}_n\}_{n=0}^\infty$.

Proof. **1.** Fix x_0,y_0,h,ϵ as in the statement and let δ be as in (i). Assume that $|X_n-Y_n|\in(\delta^2,3h)$, where as usual we suppress the parameter superscripts in order to alleviate the notation. The first observation is that on the event $\{d_\epsilon(X_n)=d_\epsilon(Y_n)=1\}$, there holds $Y_{n+1}=\mathrm{refl}_{H_n}X_{n+1}$ and hence the vectors $X_{n+1}-Y_{n+1}$ and X_n-Y_n are parallel. A direct calculation now gives:

$$\begin{aligned}&|X_{n+1}-Y_{n+1}|-|X_n-Y_n| = 2\big\langle\epsilon w^1_{n+1},\frac{X_n-Y_n}{|X_n-Y_n|}\big\rangle,\\ &|X_{n+1}-Y_{n+1}|^2-|X_n-Y_n|^2 = -4\big\langle\epsilon w^1_{n+1},\frac{X_n-Y_n}{|X_n-Y_n|}\big\rangle^2+4\big\langle\epsilon w^1_{n+1},X_n-Y_n\big\rangle.\end{aligned}$$

Thus, we conclude (i) in this case:

$$\begin{aligned}\mathbb{E}\big(Q_{n+1}-Q_n\mid\mathcal{F}_n\big) &= 4\mathbb{E}\big(\big\langle\epsilon w^1_{n+1},2h\frac{X_n-Y_n}{|X_n-Y_n|}-(X_n-Y_n)\big\rangle\mid\mathcal{F}_n\big)\\ &\quad+4\mathbb{E}\big(\big\langle\epsilon w^1_{n+1},\frac{X_n-Y_n}{|X_n-Y_n|}\big\rangle^2\mid\mathcal{F}_n\big)\\ &= -4\epsilon^2\fint_{B_1^N}y_1^2\,\mathrm{d}y = -\frac{4\epsilon^2}{N+2}\quad\mathbb{P}-\text{a.s. in }\ \Omega.\end{aligned} \tag{10.2}$$

2. On the other hand, on the event $\{d_\epsilon(X_n)<1\}\cup\{d_\epsilon(Y_n)<1\}$, we write:

$$Q_{n+1}-Q_n = \phi(X_{n+1}-Y_{n+1})-\phi(X_n-Y_n),$$

where the function $\phi:\mathbb{R}^N\to\mathbb{R}$ is given by: $\phi(v)=(4h-|v|)|v|$. We will apply the Taylor expansion below to $v=X_n-Y_n$ and $w=(X_{n+1}-Y_{n+1})-(X_n-Y_n)$:

$$\phi(v+w)-\phi(v) = \big\langle 4h\frac{v}{|v|}-2v,w\big\rangle+\mathcal{O}\big(\frac{h}{|v|}+1\big)|w|^2.$$

This expansion is valid, with the uniform bound in $\mathcal{O}$, for all $v\neq 0$ and $|w|<\frac12|v|$. Therefore:

$$\begin{aligned}\mathbb{E}\big(Q_{n+1}-Q_n\mid\mathcal{F}_n\big) =&\,\mathbb{E}\big(\big(\frac{4h}{|X_n-Y_n|}-2\big)\big\langle X_n-Y_n,(X_{n+1}-Y_{n+1})-(X_n-Y_n)\big\rangle\mid\mathcal{F}_n\big)\\ &+\mathcal{O}\big(\frac{h}{|X_n-Y_n|}+1\big)\epsilon^2\quad\mathbb{P}-\text{a.s. in }\ \Omega.\end{aligned} \tag{10.3}$$

Also, the same calculation as in (3.1), combined with the estimate (7.1) yield:

$$\begin{aligned}\mathbb{E}\big(X_{n+1}-Y_{n+1}\mid\mathcal{F}_n\big) &= \int_{\Omega_1}X_n+\epsilon w_{n+1}^{k_{n+1}}\,\mathrm{d}\mathbb{P}_1(w_{n+1})+\int_{\Omega_1}\mathrm{refl}_{H_n}\big(X_n+\epsilon w_{n+1}^{\tilde k_{n+1}}\big)\,\mathrm{d}\mathbb{P}_1(w_{n+1})\\ &= \Big|\fint_{B_\epsilon(X_n)\cap\mathcal{D}}y\,\mathrm{d}y-\fint_{B_\epsilon(Y_n)\cap\mathcal{D}}y\,\mathrm{d}y\Big|\\ &\leq |X_n-Y_n|+C\big(s_\epsilon(X_n)+s_\epsilon(Y_n)\big)\big(|X_n-Y_n|+\epsilon\big)\quad\mathbb{P}-\text{a.s. in }\ \Omega.\end{aligned}$$

In conclusion and using the facts that $|\frac{4h}{|X_n-Y_n|}-2|\leq\frac{8h}{|X_n-Y_n|}$ and $2\epsilon<h|X_n-Y_n|$, following from the assumed relative magnitudes of ϵ,δ,h and $|X_n-Y_n|$, the expansion (10.3) gives:

$$\begin{aligned}\mathbb{E}\big(Q_{n+1}-Q_n\mid\mathcal{F}_n\big) & \leq 8h\cdot C\big(s_\epsilon(X_n)+s_\epsilon(Y_n)\big)\big(|X_n-Y_n|+\epsilon\big)+\mathcal{O}\big(\frac{h}{|X_n-Y_n|}+1\big)\epsilon^2\\ & \leq Ch\epsilon|X_n-Y_n|+\mathcal{O}\big(\frac{h\epsilon^2}{\delta^2}\big)\\ & \leq Ch|X_n-Y_n|\leq Ch\epsilon|X_n-Y_n|-\frac{4}{N+2}\epsilon^2\quad\mathbb{P}-\text{a.s. in }\ \Omega.\end{aligned}\tag{10.4}$$

The estimates (10.2) and (10.4) conclude the proof of (i). Finally, since $Q_n\geq h|X_n-Y_n|$, the bound (i) directly implies (ii), because:

$$\mathbb{E}\big(Q_{n+1}\mid\mathcal{F}_n\big)\leq Q_n+C\epsilon(\mathbb{1}_{\{d_\epsilon(X_n)<1\}}+\mathbb{1}_{\{d_\epsilon(Y_n)<1\}})Q_n-\frac{4}{N+2}\epsilon^2\quad\mathbb{P}-\text{a.s. in }\ \Omega.$$

3. To show (iii), fix $n<\sigma$. Then the assumptions listed in (ii) hold, so we have:

$$\begin{aligned}\mathbb{E}\big(\bar{Q}_{n+1}\mid\mathcal{F}_n\big) & =\mathbb{E}\big(Q_{n+1}\cdot\exp\big(-C(\Theta^{x_0}_{n+1}+\Theta^{y_0,x_0}_{n+1})\big)\mid\mathcal{F}_n\big)+e^{-3C(n+1)}\epsilon^2\\ & \leq Q_n\cdot\exp\big(C\epsilon(\mathbb{1}_{\{d_\epsilon(X_n)<1\}}+\mathbb{1}_{\{d_\epsilon(Y_n)<1\}})\big)\cdot\exp\big(-C(\Theta^{x_0}_{n+1}+\Theta^{y_0,x_0}_{n+1})\big)\\ & \quad-\frac{4\epsilon^2}{N+2}\exp\big(-C(\Theta^{x_0}_{n+1}+\Theta^{y_0,x_0}_{n+1})\big)+e^{-3C}(n+1)\epsilon^2\\ & \leq Q_n\cdot\exp\big(-C(\Theta^{x_0}_{n}+\Theta^{y_0,x_0}_{n})\big)-\frac{4e^{-2C}}{N+2}\epsilon^2-e^{-3C}(n+1)\epsilon^2\\ & \leq Q_n\cdot\exp\big(-C(\Theta^{x_0}_{n}+\Theta^{y_0,x_0}_{n})\big)-e^{-3C}(n+1)\epsilon^2=\bar{Q}_n\quad\mathbb{P}-\text{a.s. in }\ \Omega.\end{aligned}$$

This ends the proof of the Lemma. ■

Corollary 10.4. *Let $\mathcal{D}$ be as in (BH) and let $\bar{r}<r$ with r as in (8.1). Then there exists a constant $C>0$, depending on $\mathcal{D}$ and $\bar{r}$, such that for every set of parameters as below:*

$$0<\delta<h<\min\big\{\frac{\bar{r}}{4},1\big\},\quad\epsilon<\min\big\{\frac{\bar{r}}{4},\frac{1}{2}\delta^4\big\},\quad\textit{for every}\ \ x_0,y_0\in\bar{\mathcal{D}}\ \ \textit{such that}\ \ |x_0-y_0|<\delta,$$

the following holds:

(i) $\epsilon^2\mathbb{E}\big[\sigma^{\epsilon,h,\delta,x_0,y_0}\big]\leq Ch\delta$,

(ii) $\mathbb{P}\big(|X^{\epsilon,x_0}_{\sigma^{\epsilon,h,\delta,x_0,y_0}}-x_0|\geq h\big)+\mathbb{P}\big(|Y^{\epsilon,y_0,x_0}_{\sigma^{\epsilon,h,\delta,x_0,y_0}}-y_0|\geq h\big)\leq\frac{C\delta}{h}$,

(iii) $\mathbb{E}\big[|X^{\epsilon,x_0}_{\sigma^{\epsilon,h,\delta,x_0,y_0}}-x_0|\big]+\mathbb{E}\big[|Y^{\epsilon,y_0,x_0}_{\sigma^{\epsilon,h,\delta,x_0,y_0}}-y_0|\big]\leq C\delta\cdot\big(\log\frac{1}{\delta}\big)$,

(iv) *If additionally:* $\mathrm{dist}(x_0,\partial\mathcal{D}),\mathrm{dist}(y_0,\partial\mathcal{D})<\frac{\bar{r}}{2}-(h+\epsilon)$, *then:*

$$\mathbb{E}\big[S^{\epsilon,x_0}_{\sigma^{\epsilon,h,\delta,x_0,y_0}}+S^{\epsilon,y_0,x_0}_{\sigma^{\epsilon,h,\delta,x_0,y_0}}\big]\leq C\delta\cdot\big(\log\frac{1}{\delta}\big),$$

where consistently with (8.3) we have defined: $S^{\epsilon,y_0,x_0}_n=\sum_{j=0}^{n-1}s_\epsilon(Y^{\epsilon,y_0,x_0}_j)$.

Proof. **1.** By the supermartingale property in Lemma 10.3 (iii), Doob's Optional Stopping applied to the stopping time $\sigma=\sigma^{\epsilon,h,\delta,x_0,y_0}$ yields:

$$e^{-3C}\epsilon^2\mathbb{E}\big[\sigma\big]\leq\mathbb{E}\big[\bar{Q}^{\epsilon,h,x_0,y_0}_\sigma\big]\leq\mathbb{E}\big[\bar{Q}_0\big]=(4h-|x_0-y_0|)|x_0-y_0|\leq 4h\delta,$$

as claimed in (i). For (ii), we apply Corollary 9.2 and Doob's Optional Stopping again, in:

$$\mathbb{E}\Big[|X_\sigma - x_0|^2 \cdot \exp\big(-C\sum_{j=0}^{\sigma-1} s_\epsilon(X_j)\big)\Big] \leq C\epsilon^2\mathbb{E}\big[\sigma\big],$$

where the constant C is as in (9.1). Since $\sum_{j=0}^{\sigma-1} s_\epsilon(X_j) \leq \Theta_\sigma^{\epsilon,x_0} \leq 2+\epsilon$, the assertion in (i) gives:

$$\mathbb{E}\big[|X_\sigma - x_0|^2\big] \leq Ch\delta.$$

Consequently, by Chebyshev's inequality we conclude the following inequality in (ii):

$$\mathbb{P}\big(|X_\sigma - x_0| \geq h\big) = \mathbb{P}\big(|X_\sigma - x_0|^2 \geq h^2\big) \leq \frac{1}{h^2}\mathbb{E}\big[|X_\sigma - x_0|^2\big] \leq \frac{C\delta}{h}.$$

The parallel bound for the coupled process $\{Y_n^{\epsilon,y_0,x_0}\}$ is deduced by observing that the coupling in Definition 10.2 is symmetric in distribution. Namely, for every $n \geq 0$ and every Borel set $F \subset \bar{\mathcal{D}}$, the same calculation as in (3.1) yields:

$$\mathbb{P}\big(Y_n^{\epsilon,y_0,x_0} \in F\big) = \mathbb{P}\big(X_n^{\epsilon,y_0} \in F\big), \qquad \mathbb{P}\big(Y_n^{\epsilon,x_0,y_0} \in F\big) = \mathbb{P}\big(X_n^{\epsilon,y_0} \in F\big). \tag{10.5}$$

2. To prove (iii), we integrate the distribution function (effectively, the integration takes place on the interval $(0, h+\epsilon)$), and apply the bound (ii):

$$\begin{aligned}
\mathbb{E}\big[|X^{\epsilon,x_0}_{\sigma^{\epsilon,h,\delta,x_0,y_0}} - x_0|\big] &= \int_0^\infty \mathbb{P}\big|(X^{\epsilon,x_0}_{\sigma^{\epsilon,h,\delta,x_0,y_0}} - x_0| \geq t\big)\,\mathrm{d}t \\
&\leq \int_0^\infty \mathbb{P}\big(|X^{\epsilon,x_0}_{\sigma^{\epsilon,t,\delta,x_0,y_0}} - x_0| \geq t\big) + \mathbb{P}\big(|Y^{\epsilon,y_0,x_0}_{\sigma^{\epsilon,t,\delta,x_0,y_0}} - y_0| \geq t\big)\,\mathrm{d}t \\
&\leq \epsilon + \delta + \int_\delta^h \mathbb{P}\big(|X^{\epsilon,x_0}_{\sigma^{\epsilon,t,\delta,x_0,y_0}} - x_0| \geq t\big) + \mathbb{P}\big(|Y^{\epsilon,y_0,x_0}_{\sigma^{\epsilon,t,\delta,x_0,y_0}} - y_0| \geq t\big)\,\mathrm{d}t \\
&\leq 2\delta + \int_\delta^h \frac{C\delta}{t}\,\mathrm{d}t = 2\delta + C\delta\cdot\big(\log\frac{1}{\delta} + \log h\big) \leq C\delta\cdot\big(\log\frac{1}{\delta}\big).
\end{aligned}$$

The estimate on $\mathbb{E}\big[|Y^{\epsilon,y_0,x_0}_{\sigma^{\epsilon,h,\delta,x_0,y_0}} - y_0|\big]$ follows in the same fashion, in view of (10.5).

3. As in the proof of Lemma 8.3, we deduce that $\{M_{\sigma^{\epsilon,h,\delta,x_0,y_0}\wedge n}\}_{n=0}^\infty$ is a submartingale with respect to the filtration $\{F_n\}_{n=0}^\infty$, where we define:

$$M_n = |X_n^{\epsilon,x_0} - \mathcal{Z}_n^{\epsilon,\bar{r},x_0}|^{-N} - C_l S_n^{\epsilon,x_0},$$

with the constant $C_l > 0$ is as in Lemma 7.3. In the proof of this fact, we use that $\mathrm{dist}(X_n, \partial\mathcal{D}) < \frac{\bar{r}}{2}$ for $n < \sigma^{\epsilon,h,\delta,x_0,y_0}$ in view of the assumed bound on $\mathrm{dist}(x_0, \partial\mathcal{D})$. Applying now Doob's Optional Stopping to the stopping time $\sigma = \sigma^{\epsilon,h,\delta,x_0,y_0}$, and applying the mean value theorem to the function $v \mapsto |v|^{-N}$ while noting $|X_\sigma^{\epsilon,x_0} - \mathcal{Z}_\sigma^{\epsilon,\bar{r},x_0}| > \frac{\bar{r}}{2}$, we get:

$$\begin{aligned}
C_l\mathbb{E}\big[S_\sigma^{\epsilon,x_0}\big] &= \mathbb{E}\big[|X_\sigma^{\epsilon,x_0} - \mathcal{Z}_\sigma^{\epsilon,\bar{r},x_0}|^{-N}\big] - \mathbb{E}\big[M_\sigma\big] \leq \mathbb{E}\big[|X_\sigma^{\epsilon,x_0} - \mathcal{Z}_\sigma^{\epsilon,\bar{r},x_0}|^{-N} - |x_0 - Z^{\bar{r}}(x_0)|^{-N}\big] \\
&\leq \mathbb{E}\big[\big(|X_\sigma^{\epsilon,x_0} - \mathcal{Z}_\sigma^{\epsilon,\bar{r},x_0}|^{-N} - |x_0 - Z^{\bar{r}}(x_0)|^{-N}\big)_+\big] \\
&\leq \frac{1}{N(\bar{r}/2)^{N+1}}\cdot\mathbb{E}\big[\big(|x_0 - Z^{\bar{r}}(x_0)| - |X_\sigma^{\epsilon,x_0} - Z^{\bar{r}}(X_\sigma^{\epsilon,x_0})|\big)_+\big] \\
&\leq C\mathbb{E}\big[|x_0 - X_\sigma^{\epsilon,x_0}|\big] \leq C\delta\cdot\log\big(\frac{1}{\delta}\big).
\end{aligned}$$

In the last two inequalities above we also used the fact that $Z^{\bar{r}}$ is a projection, and the already established bound in (iii). The parallel estimate on $\mathbb{E}\big[S^{\epsilon,y_0,x_0}_{\sigma^{\epsilon,h,\delta,x_0,y_0}}\big]$ follows by recalling (10.5). ■

Proof of Theorem 10.1.

1. Let $\bar r < r$ with the radius r as in (8.1). Take $\delta_0 > 0$ so small that for every $\delta \in (0, \delta_0)$ the following threshold quantity satisfies $\delta < h(\delta) < \min\{\frac{\bar r}{4}, 1\}$:

$$h(\delta) = \frac{1}{9C_u \cdot \log(1/\delta)}.$$

We set $\bar\epsilon = \frac{1}{2}\delta^4$, let $\epsilon \in (0, \bar\epsilon)$ and take $x_0, y_0 \in \bar{\mathcal{D}}$ with the properties indicated in the statement of the Theorem. To alleviate the notation, we will be writing $h = h(\delta)$.

Applying (9.4) with an integer $k \in [2\log\frac{1}{\delta}, 3\log\frac{1}{\delta}]$, we obtain:

$$\mathbb{P}\big(\Theta^{\epsilon,x_0}_{\bar\tau^{\epsilon,h,x_0}} \geq 1\big) \leq \mathbb{P}\big(\Theta^{\epsilon,x_0}_{\bar\tau^{\epsilon,h,x_0}} \geq k(\epsilon + 2C_u h)\big) \leq \frac{1}{2^k} \leq \frac{1}{e^{k/2}} \leq \delta, \tag{10.6}$$

since $k \leq 1/(3C_u \log\frac{1}{h}) \leq 1/(\epsilon + 2C_u h)$.

2. Recall that by (RMV)_ϵ, the sequence of random variables $\{M_n\}_{n=0}^\infty$ defined in (8.5):

$$M_n = u_\epsilon(X_n^{\epsilon,x_0})\Lambda_n^{\epsilon,x_0} + \frac{\epsilon^2}{2(N+2)}\sum_{j=0}^{n-1} f(X_j^{\epsilon,x_0}) \cdot \Lambda_j^{\epsilon,x_0}$$

is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$. Applying Doob's Optional Stopping with the stopping time $\sigma = \sigma^{\epsilon,h,\delta,x_0,y_0}$, the same calculation as in the proof of Theorem 8.1 in section 8, leads to the estimate:

$$\big|u_\epsilon(x_0) - \mathbb{E}\big[u_\epsilon(X_\sigma^{\epsilon,x_0})\big] \leq C\big(\mathbb{E}\big[\gamma S_\sigma^{\epsilon,x_0}\big] + \epsilon^2\mathbb{E}\big[\sigma\big]\big) \leq C\delta \cdot \log\big(\frac{1}{\delta}\big),$$

in virtue of Corollary 10.4 (i) and (iv). Similarly, the coupling symmetry (10.5) result is:

$$\big|u_\epsilon(y_0) - \mathbb{E}\big[u_\epsilon(X_\sigma^{\epsilon,y_0})\big] = \big|u_\epsilon(y_0) - \mathbb{E}\big[u_\epsilon(Y_\sigma^{\epsilon,y_0,x_0})\big] \leq C\delta \cdot \log\big(\frac{1}{\delta}\big),$$

and we see that:

$$|u_\epsilon(x_0) - u_\epsilon(y_0)| \leq \mathbb{E}\big[\big|u_\epsilon(X_\sigma^{\epsilon,x_0}) - u_\epsilon(Y_\sigma^{\epsilon,y_0,x_0})\big|\big] + C\delta \cdot \log\big(\frac{1}{\delta}\big) \tag{10.7}$$

We now estimate the expectation in the right hand side above, separately on the four events indicated in the definition (10.1). Firstly, by Corollary 10.4 we get:

$$\mathbb{E}\big[\big|u_\epsilon(X_\sigma^{\epsilon,x_0}) - u_\epsilon(Y_\sigma^{\epsilon,y_0,x_0})\big| \cdot \mathbb{1}_{\{|X_\sigma - x_0| \geq h \text{ or } |Y_\sigma - y_0| \geq h\}}\big] \leq \frac{C\delta}{h} \leq C\delta \cdot \log\big(\frac{1}{\delta}\big). \tag{10.8}$$

Secondly, by (10.6) and since $\sigma \leq \bar\tau^{\epsilon,h,x_0}$, we have:

$$\begin{aligned} \mathbb{E}\big[\big|u_\epsilon(X_\sigma^{\epsilon,x_0}) - u_\epsilon(Y_\sigma^{\epsilon,y_0,x_0})\big| \cdot \mathbb{1}_{\{\Theta_\sigma^{\epsilon,x_0} + \Theta_\sigma^{\epsilon,y_0,x_0} \geq 2\}}\big] &\leq C\Big(\mathbb{P}\big(\Theta_\sigma^{\epsilon,x_0} \geq 1\big) + \mathbb{P}\big(\Theta_\sigma^{\epsilon,y_0,x_0} \geq 1\big)\Big) \\ &\leq C\Big(\mathbb{P}\big(\Theta^{\epsilon,x_0}_{\bar\tau^{\epsilon,h,x_0}} \geq 1\big) + \mathbb{P}\big(\Theta^{\epsilon,y_0,x_0}_{\bar\tau^{\epsilon,h,y_0}} \geq 1\big)\Big) \leq C\delta. \end{aligned} \tag{10.9}$$

Finally, the weak bound in Corollary 8.4 gives:

$$\mathbb{E}\big[\big|u_\epsilon(X_\sigma^{\epsilon,x_0}) - u_\epsilon(Y_\sigma^{\epsilon,y_0,x_0})\big| \cdot \mathbb{1}_{\{|X_\sigma^{\epsilon,x_0} - Y_\sigma^{\epsilon,y_0,x_0}| \leq \delta^2\}}\big] \leq C\delta. \tag{10.10}$$

Combining (10.7), (10.8), (10.9) and (10.10), concludes the proof. ∎

This ends the proof of Theorem 1.2, in view of (6.1). Recall that the family $\{u_\epsilon\}_{\epsilon\to 0}$ is equibounded. The asymptotic equi-Hölder estimate in Theorem 1.2 implies also the asymptotic equicontinuity, in the sense that for every $\eta > 0$ there exist $\delta, \epsilon_0 > 0$ satisfying:

$$|u_\epsilon(x_0) - u_\epsilon(y_0)| \leq \eta \qquad \text{for all } \epsilon < \epsilon_0 \quad \text{and} \quad x_0, y_0 \in \mathcal{D} \text{ such that } |x_0 - y_0| \leq \delta.$$

Thus, a version of the Ascoli-Arzelà theorem yields uniform convergence of $\{u_\epsilon\}_{\epsilon\to 0}$ in $\bar{\mathcal{D}}$, up to a subsequence. Combining this observation with Theorem 5.2 we conclude the result of Theorem 1.1. We also recover the Hölder continuity of the limiting function: $u \in \mathcal{C}^{0,\alpha}(\bar{\mathcal{D}})$ for any $\alpha \in (0,1)$, because $t\log(1/t) \leq t^\alpha$ as $t \to 0$, when $\alpha < 1$.

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