DIMENSION REDUCTION FOR THIN FILMS PRESTRAINED BY SHALLOW CURVATURE

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ABSTRACT. We are concerned with the dimension reduction analysis for thin three-dimensional elastic films, prestrained via Riemannian metrics with weak curvatures.

For the prestrain inducing the incompatible version of the Föppl-von Kármán equations, we find the Γ-limits of the rescaled energies, identify the optimal energy scaling laws, and display the equivalent conditions for optimality in terms of both the prestrain components and the curvatures of the related Riemannian metrics. When the stretching-inducing prestrain carries no in-plane modes, we discover similarities with the previously described shallow shell models.

In higher prestrain regimes, we prove new energy upper bounds by constructing deformations as the Kirchhoff-Love extensions of the highly perturbative, Hölder-regular solutions to the Monge-Ampere equation obtained by means of convex integration.

1. INTRODUCTION

This paper is concerned with the dimension reduction analysis for prestrained thin three-dimensional elastic films. We assume that the prestrain corresponds to a family of Riemannian metrics with weak curvatures i.e. metrics deviating from the Euclidean metric by the order of power of the film's thickness. In various regimes of these scaling powers (separately for the stretching and bending-inducing prestrain), we complete the dimension reduction in the full range of parameters, as well as recover previous results in a unified manner.

There are essentially three new contributions in this paper:

(i) For the prestrain of order at least as that inducing the incompatible version of the Föppl-von Kármán equations in [13], we: find the Γ-limits of the rescaled energies; identify the optimal energy scaling laws; and display the equivalent conditions for optimality in terms of both the prestrain components and the curvatures of the related Riemannian metrics. Similarly to the case of large prestrain [18], we observe that one such condition is the non-vanishing of the lowest order terms in the curvatures $R_{12,12}, R_{12,13}, R_{13,23}$ along the midplate. These results are valid when $\alpha \geq 4, \gamma \geq 2$ in (1.1).

(ii) In the larger prestrain regime we propose new energy upper bounds, based on the construction of a sequence of deformations via the Kirchhoff-Love extension of the highly perturbative, Hölder-regular solutions to the Monge-Ampere equation obtained by convex integration. These results are valid when $\alpha \in (0, 4)$, $\gamma > 0$ in (1.1).

(iii) When the stretching-inducing prestrain is of order lower than that allowed in (i), but carries no in-plane modes, we still perform the full analysis as in (i) and discover similarities with both the theories in (i) and the shallow shell models of [14]. This corresponds to the case $\alpha, \gamma \geq 2$ and $S_{2 \times 2} \equiv 0$ in (1.1).

The remaining part of the introductory section will be devoted to the description of our results, and their discussion including connection to previous literature.

1.1. **The set-up.** Thin prestrained films arise in science and technology in a variety of situations and from a range of causes: inhomogeneous growth, plastic deformation, swelling or shrinkage driven by solvent absorption. In all these situations, the resulting shape is a consequence of the local and heterogeneous incompatibility of strains that leads to local elastic stresses. One approach towards understanding the coupling between residual stress and the ultimate shape of the body relies on the model of "non-Euclidean elasticity".

The model postulates that an elastic three-dimensional film Ω seeks to realize a configuration with a prescribed Riemannian metric $\mathcal{G}$. Although $\mathcal{G}$ always has a Lipschitz isometric immersion, one can show that any such immersion $u : \Omega \to \mathbb{R}^3$ necessarily changes its orientation in any neighbourhood of a point where the Riemann curvature of $\mathcal{G}$ is not zero. Excluding such nonphysical deformations leads to the elastic energy $I(u)$ which measures how far a given deformation u is from being an orientation preserving realization of $\mathcal{G}$. Equivalently, $I(u)$ quantifies the total pointwise deviation of the deformation gradient ∇u from $\mathcal{G}^{1/2}$, modulo orientation-preserving rotations. The infimum of I in absence of any forces or boundary conditions is then indeed strictly positive for a non-Euclidean $\mathcal{G}$, pointing to existence of residual stress.

This approach borrows from the theory of plasticity, in as much as it uses a multiplicative decomposition of the deformation gradient, and requires the notion of a reference configuration Ω with respect to which all displacements are measured. We assume that the elastic response derives from hyperelasticity, while the inelastic deformation follows different laws depending on their origin and it is encoded in the given prestrain $A = \sqrt{\mathcal{G}}$. We point out that this description follows the one-way coupling of growth to shape and ignores the feedback from shape to growth. On the frontiers of the related experimental modeling [4,9,10,19], we mention the halftone gel lithography method for polymeric materials that can swell by imbibing fluids. By blocking the ability of portions of plate to swell or causing them to swell inhomogeneously, it is possible to have the plate assume a variety of deformed shapes. Even more sophisticated techniques of biomimetic 4d printing allow for engineering of the 3d shape-morphing systems that mimic nastic plant motions where organs such as tendrils, leaves and flowers respond to the environmental stimuli [6].

In this paper we consider a family of $(\Omega^h, u^h, \mathcal{G}^h, A^h, I^h)_{h>0}$ as described above, but given in function of the film's thickness parameter h. Following a large body of previous literature on dimension reduction in nonlinear and non-Euclidean elasticity (see e.g. [12] and references therein), we are interested in predicting the scaling of $\inf I^h$ as $h \to 0$ and analyzing the asymptotic behaviour of the minimizing deformations u^h, from the curvatures of the prestrain. In analogy to the case of "shallow shells" considered in [14], here we treat the "shallow prestrain", i.e. we assume that the family of imposed tensors A^h consists of perturbation of Id of the order that is a power of the thickness. We explicitly distinguish between the stretching-generating leading order prestrain S and the bending-related B, and define:

$$A^h(x', x_3) = Id_3 + h^{\alpha/2}S(x') + h^{\gamma/2}x_3B(x')$$

$$\text{for all } \ x = (x', x_3) \in \Omega^h = \omega \times \big(-\frac{h}{2}, \frac{h}{2} \big), \tag{1.1}$$

where $S, B : \bar{\omega} \to \mathbb{R}^{3\times3}_{sym}$ and $\alpha, \gamma > 0$. The open, bounded set $\omega \subset \mathbb{R}^2$ with Lipschitz boundary is viewed as the midplate of the thin film Ω^h, on which we pose the energy of elastic deformations:

$$I^h(u^h) = \frac{1}{h}\int_{\Omega^h} W\big((\nabla u^h)(A^h)^{-1}\big) \,\mathrm{d}x \qquad \text{for all } \ u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3). \tag{1.2}$$

Here, $W : \mathbb{R}^{3\times3} \to \mathbb{R}$ is a Borel measurable density function, assumed to be $\mathcal{C}^2$ in a neighborhood of $SO(3)$ and to satisfy, for every $F \in \mathbb{R}^{3\times3}$ and $R \in SO(3)$:

$$W(R) = 0, \qquad W(RF) = W(F), \qquad W(F) \geq c\,\mathrm{dist}^2(F, SO(3)). \qquad (1.3)$$

1.2. **Gamma-convergence: case $\alpha \geq 4, \gamma \geq 2$.** Our first result analyzes the case of bending and stretching components in A^h (i.e. terms $h^{\alpha/2}S$, $h^{\gamma/2}x_3B$) with scaling at least h^2.

Theorem 1.1. *Let $\alpha \geq 4$, $\gamma \geq 2$. We have the following energy scaling and Γ-limit results.*

(i) When $\gamma = 2$ then $\inf I^h \leq Ch^4$, and $h^{-4}I^h \xrightarrow{\Gamma} I$ where:

$$I(v,w) = \frac{1}{24}\int_\omega \mathcal{Q}_2(\nabla^2 v + B_{2\times2})\,\mathrm{d}x' + \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(\mathrm{sym}\,\nabla w + \frac{1}{2}(\nabla v)^{\otimes2} - \Big\{\begin{array}{ll} 0 & for\ \alpha > 4 \\ S_{2\times2} & for\ \alpha = 4 \end{array}\Big)\,\mathrm{d}x'.$$

(ii) When $\gamma \in (2, \alpha-2]$ then $\inf I^h \leq Ch^{2+\gamma}$, and $h^{-(2+\gamma)}I^h \xrightarrow{\Gamma} I$ where:

$$I(v,w) = \frac{1}{24}\int_\omega \mathcal{Q}_2(\nabla^2 v + B_{2\times2})\,\mathrm{d}x' + \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(\mathrm{sym}\,\nabla w - \Big\{\begin{array}{ll} 0 & for\ \gamma < \alpha-2 \\ S_{2\times2} & for\ \gamma = \alpha-2 \end{array}\Big)\,\mathrm{d}x'.$$

(iii) When $\gamma > \alpha-2$ then $\inf I^h \leq Ch^\alpha$, and $h^{-\alpha}I^h \xrightarrow{\Gamma} I$ where:

$$I(v,w) = \frac{1}{24}\int_\omega \mathcal{Q}_2(\nabla^2 v)\,\mathrm{d}x' + \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(\mathrm{sym}\,\nabla w + \Big\{\begin{array}{ll} 0 & for\ \alpha > 4 \\ \frac{1}{2}(\nabla v)^{\otimes2} & for\ \alpha = 4 \end{array}\ \mathrm{d}x' - S_{2\times2}\Big)\,\mathrm{d}x'.$$

All Γ-limit functionals $I(v,w)$ above are defined on the scalar out-of-plane displacements $v \in W^{2,2}(\omega, \mathbb{R})$ and the in-plane displacements $w \in W^{1,2}(\omega, \mathbb{R}^2)$. The Γ-convergence statements are with respect to the following compactness properties (valid in each corresponding scaling regime, with convergence up to a subsequence):

$$y^h(x', x_3) \doteq (\bar{R}^h)^T u^h(x', hx_3) - c^h \to x' \quad in\ W^{1,2}(\omega, \mathbb{R}^3)$$
$$for\ some\ \bar{R}^h \in SO(3), c^h \in \mathbb{R}^3,$$
$$V^h(x') \doteq h^{-\delta/2}\int_{-1/2}^{1/2} y^h(x', x_3) - x'\,\mathrm{d}x_3 \to (0,0,v) \quad in\ W^{1,2}(\omega, \mathbb{R}^3), \qquad (1.4)$$
$$h^{-1}(V_1^h, V_2^h) \rightharpoonup w \quad in\ W^{1,2}(\omega, \mathbb{R}^2),$$

where $\delta = 2$ in case (i), $\delta = \gamma$ in case (ii), and $\delta = \alpha-2$ in case (iii).

The above theorem encompasses several cases studied before. The case $\gamma = 2$, $\alpha = 4$ in (i) has been covered in [13], and the case $\alpha = 2\gamma$, $\gamma > 2$ in (ii) was analyzed in [7]. We also refer to the paper [1], where the authors considered prestrain of the type: $A^h(x_3) = \big(Id_3 + h^{a-1}b\big(\frac{x_3}{h}\big)\big)^{-1}$ with $a > 2$. In our notation and for smooth tensor b, this is equivalent to having: $\alpha = 2a-2 > 2$, $\gamma = \alpha-2$, and $S = -b(0)$, $B = -b'(0)$ constant. When $a = 3$ this leads to a subcase of (i), while when $a > 3$ to a subcase of (ii) in which $I(v,w)$ can be always minimized to 0. Thus, the optimal scaling of I^h in that case must have scaling exponent larger than $2a - 2 = \alpha$, studied in [1]. We remark that the main contribution of [1] was, however, allowing W to depend on $\frac{x_3}{h}$ and $b : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}^{3\times3}_{sym}$ to have regularity L^∞.

When $S = B \equiv 0$ in ω then all three cases in Theorem 1.1 reduce to the von Karman and linear theories in classical nonlinear elasticity, derived in [5]. On the other hand, the limiting functional in case (ii) when $\gamma = \alpha - 2$ seems to be new with respect to the previous literature. Of interest is also the Γ-limit in (iii). When $\alpha = 4$, its minimization amounts to finding

the displacement v whose combined magnitude of the total induced curvature $\nabla^2 v$ and the deviation of the Gaussian curvature $\det \nabla^2 v$ from the given $-\text{curl}^T\text{curl}\, S_{2\times2}$, is the smallest.

In this line, we further remark that the in-plane displacement w is always slaved to S, B and v, and as such can be omitted all together:

Corollary 1.2. *In the context of Theorem 1.1, assume additionally that ω is simply connected. Then each stretching term may be replaced by the following squared distance from the space $\{\text{sym}\nabla w;\ w \in W^{1,2}(\omega,\mathbb{R}^2)\}$, where $\delta = 2$ in case (i), $\delta = \gamma$ in case (ii), and $\delta = \alpha - 2$ in (iii):*

$$\frac{1}{2} \min_{w\in W^{1,2}(\omega,\mathbb{R}^2)} \int_\omega \mathcal{Q}_2\Big(\text{sym}\,\nabla w + \left\{ \begin{array}{ll} 0 & \text{for } \delta > 2 \\ \frac{1}{2}(\nabla v)^{\otimes 2} & \text{for } \delta = 2 \end{array}\right. - \left\{ \begin{array}{ll} 0 & \text{for } \alpha > 2+\delta \\ S_{2\times 2} & \text{for } \alpha = 2+\delta \end{array}\right. \Big)\, \mathrm{d}x'.$$

Consequently, we have the equivalences below, where the congruency symbol $a \simeq b$ means that $a \leq Cb$ and $b \leq Ca$ with a constant C depending only on ω:

(i) *If $\gamma = 2$, $\alpha > 4$ then $h^{-4}I^h \xrightarrow{\Gamma} \bar{I}$, where:* $\bar{I}(v) \simeq \|\nabla^2 v + B_{2\times 2}\|^2_{L^2(\omega)} + \|\det\nabla^2 v\|^2_{H^{-2}(\omega)}$.

 If $\gamma = 2$, $\alpha = 4$ then: $h^{-4}I^h \xrightarrow{\Gamma} \bar{I}$, *where*

$$\bar{I}(v) \simeq \|\nabla^2 v + B_{2\times 2}\|^2_{L^2(\omega)} + \|\det\nabla^2 v + \text{curl}^T\text{curl}S_{2\times 2}\|^2_{H^{-2}(\omega)}.$$

(ii) *If $\gamma \in (2, \alpha-2]$ then $h^{-(2+\gamma)}I^h \xrightarrow{\Gamma} \bar{I}$, with:* $\bar{I}(v) \simeq \|\nabla^2 v + B_{2\times 2}\|^2_{L^2(\omega)} + \|\text{curl}^T\text{curl}S_{2\times 2}\|^2_{H^{-2}(\omega)}$. *Alternatively, the same rescaled energies Γ-converge to the constant limit:*

$$\bar{\bar{I}} \equiv \min\bar{I} \simeq \|\text{curl}\, B_{2\times 2}\|^2_{H^{-1}(\omega)} + \|\text{curl}^T\text{curl}S_{2\times 2}\|^2_{H^{-2}(\omega)}.$$

(iii) *If $\alpha = 4$, $\gamma > 2$ then $h^{-4}I^h \xrightarrow{\Gamma} \bar{I}$, where:* $\bar{I}(v) \simeq \|\nabla^2 v\|^2_{L^2(\omega)} + \|\det\nabla^2 v + \text{curl}^T\text{curl}S_{2\times 2}\|^2_{H^{-2}(\omega)}$.

 If $\gamma > \alpha - 2 > 2$ then $h^{-\alpha}I^h \xrightarrow{\Gamma} \bar{\bar{I}}$, where: $\bar{\bar{I}} \equiv \min I \simeq \|\text{curl}^T\text{curl}S_{2\times 2}\|^2_{H^{-2}(\omega)}$.

All Γ-limit functionals $\bar{I}(v)$ are defined on the scalar out-of-plane displacements $v \in W^{2,2}(\omega,\mathbb{R})$.

Next, we observe the bound on the infimum of the following limiting functional corresponding to $\gamma = 2, \alpha = 4$, which also contains cases (i) and (iii). The bound is consistent with the optimality conditions in Theorem 1.7.

Proposition 1.3. *Let $\omega \subset \mathbb{R}^2$ be open, bounded, simply connected. Denote $\bar{B} = B_{2\times 2}$, $\bar{S} = S_{2\times 2}$, and for $v \in W^{2,2}(\omega,\mathbb{R})$ define:*

$$\bar{I}_0(v) = \|\nabla^2 v + \bar{B}\|^2_{L^2(\omega)} + \|\det\nabla^2 v + \text{curl}^T\text{curl}\,\bar{S}\|^2_{H^{-2}(\omega)}.$$

Then, with some constants $c, C > 0$ depending only on ω we have:

$$\inf\bar{I}_0 \leq C\,\|\text{curl}\,\bar{B}\|^2_{H^{-1}(\omega)}\Big(1 + \|\bar{B}\|^2_{L^\infty(\omega)} + \|\text{curl}\,\bar{B}\|^2_{H^{-1}(\omega)}\Big)$$
$$+ C\,\|\det\bar{B} + \text{curl}^T\text{curl}\,\bar{S}\|^2_{H^{-2}(\omega)},$$

$$\inf\bar{I}_0 \geq c\,\|\text{curl}\,\bar{B}\|^2_{H^{-1}} + \frac{c}{1\vee a}\|\det\bar{B} + \text{curl}^T\text{curl}\,\bar{S}\|^2_{H^{-2}(\omega)},$$

where $a = \|\bar{B}\|^2_{L^\infty} + \|\det\bar{B} + \text{curl}^T\text{curl}\,\bar{S}\|^2_{H^{-2}} + \|\text{curl}\,\bar{B}\|^2_{H^{-1}}(1 + \|\bar{B}\|^2_{L^\infty} + \|\text{curl}\,\bar{B}\|^2_{H^{-1}}).$

We finally remark that the following two cases, which are not included in Theorem 1.1: $\alpha = 2\gamma$, $\gamma \in (0,2)$, and $\alpha \in (2,4), \gamma = \alpha - 2$ with S and B constant, have been discussed in [16] and [1], respectively. They lead to the limiting theories with Monge-Ampere constraints. In the present general setting, this case corresponds to taking $\delta \leq \alpha/2$ and it will appear in [8].

1.3. **Energy scaling: case $0 < \alpha < 4$ and $\gamma > 0$.** Our second result provides the energy bound when the prestrain in A^h is of order higher than h^2. Namely, we have:

Theorem 1.4. *Assume that $\omega \subset \mathbb{R}^2$ is simply connected and has $\mathcal{C}^{1,1}$-regular boundary. Let $\alpha \in (0, 4)$ and $\gamma > 0$. Then the following holds:*

(i) If $\alpha \in [4/7, 4)$ and $5\alpha/6 + 2/3 > 2 + \gamma$, then $\inf I^h \leq Ch^{2+\gamma}$.
(ii) If $\alpha \in [4/7, 4)$ and $5\alpha/6 + 2/3 \leq 2 + \gamma$, then $\inf I^h \leq Ch^\delta$ for every $\delta \in (0, 5\alpha/6 + 2/3)$.
(iii) If $\alpha \in (0, 4/7)$, then $\inf I^h \leq Ch^{2\alpha}$.

When $B = 0$ then either $\inf I^h \leq Ch^\delta$ for every $\delta \in (0, 5\alpha/6+2/3)$ or $\inf I^h \leq Ch^{2\alpha}$, depending on whether $\alpha \in [4/7, 4]$ or $\alpha \in (0, 4/7)$.

Figure 1 shows a diagram depicting various cases and the corresponding scaling exponents. The indicated energy bounds are obtained by constructing deformations u^h through the Kirchhoff-Love extension corresponding to the out-of-plane and the in-plane displacements (v, w), with regularity $\mathcal{C}^{1,\alpha}$ and satisfying $\frac{1}{2}(\nabla v)^{\otimes 2} + \mathrm{sym}\,\nabla w = S_{2\times2}$. Existence of such displacements is guaranteed by techniques of convex integration [2], for all $\alpha < 1/5$. This threshold implies the particular energy scaling bounds in Theorem 1.4. If we had $v \in W^{2,2}$ and $w \in W^{1,2}$ satisfying the same equation, then $\inf I^h$ may be further decreased. Indeed, in [16] we showed that existence of $v \in W^{2,2}(\omega, \mathbb{R})$ with $\det \nabla^2 v = -\mathrm{curl\,curl}\, S_{2\times2}$ yields $\inf I^h \leq Ch^{\alpha/2+2}$, for any $\alpha \in (2, 4)$ and $\gamma = \alpha/4$. Naturally, this bound is superior in any of the cases (i)-(iii).
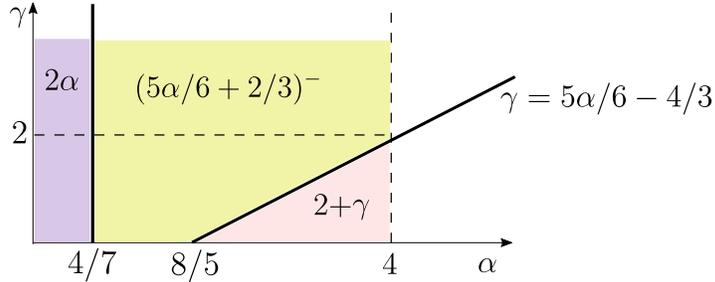

FIGURE 1. Bounding exponents of $\inf I^h$ in Theorem 1.4 (i), (ii), (iii).

1.4. **Gamma-convergence: case $S_{2\times2} \equiv 0$ and $\alpha, \gamma \geq 2$.** Our next result concerns scaling of the bending prestrain component $h^{\gamma/2}x_3 B$ as in Theorem 1.1 (i.e. at least h^2), but allowing for the stretching component $h^{\alpha/2}S$ in the out-of-plane directions to be of order h, provided that the leading order in-plane prestrain vanishes.

Theorem 1.5. *Let $S_{2\times2} \equiv 0$ in ω. For every $\alpha, \gamma \geq 2$ we have the following energy scaling and Γ-convergence results.*

(i) When $\alpha = 2$ then $\inf I^h \leq Ch^4$, and $h^{-4}I^h \xrightarrow{\Gamma} I$ where:

$$I(v, w) = \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(\nabla^2 v - 2\,\mathrm{sym}\nabla(S_{31}, S_{32}) + \left\{\begin{array}{ll} 0 & \text{for } \gamma > 2 \\ B_{2\times2} & \text{for } \gamma = 2 \end{array}\right\}\Big)\,\mathrm{d}x'$$
$$+ \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(\mathrm{sym}\,\nabla w + \frac{1}{2}(\nabla v)^{\otimes 2} - \frac{1}{2}(S_{31}, S_{32})^{\otimes 2}\Big)\,\mathrm{d}x'.$$

(ii) *When $\gamma = 2 < \alpha$ then $\inf I^h \leq Ch^4$, and $h^{-4}I^h \xrightarrow{\Gamma} I$ where:*

$$I(v,w) = \frac{1}{24}\int_\omega \mathcal{Q}_2\big(\nabla^2 v + B_{2\times 2}\big)\,\mathrm{d}x' + \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(\mathrm{sym}\,\nabla w + \frac{1}{2}(\nabla v)^{\otimes 2}\Big)\,\mathrm{d}x'.$$

(iii) *When $\alpha, \gamma > 2$ then $\inf I^h \leq Ch^{2+\alpha\wedge\gamma}$, and $h^{-(2+\alpha\wedge\gamma)}I^h \xrightarrow{\Gamma} I$ where:*

$$I(v,w) = \frac{1}{2}\int_\omega \mathcal{Q}_2\big(\mathrm{sym}\,\nabla w\big)\,\mathrm{d}x'$$
$$+ \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(\nabla^2 v - \begin{cases} 0 & \text{for } \gamma < \alpha \\ 2\,\mathrm{sym}\nabla(S_{31}, S_{32}) & \text{for } \alpha \leq \gamma \end{cases} + \begin{cases} 0 & \text{for } \alpha < \gamma \\ B_{2\times 2} & \text{for } \gamma \leq \alpha \end{cases}\Big)\,\mathrm{d}x'.$$

All Γ-limits $I(v,w)$ above are defined on the scalar out-of-plane displacements $v \in W^{2,2}(\omega, \mathbb{R})$ and the in-plane displacements $w \in W^{1,2}(\omega, \mathbb{R}^2)$. The Γ-convergences are with respect to the compactness statements in (1.4), which are valid in each corresponding scaling regime, and with $\delta = 2$ in case (i) and (ii), and $\delta = \alpha \wedge \gamma$ in case (iii).

We note that in cases (i) and (iii), the apparent stretching components S of A^h contribute to the bending term and are mixed with the original bending components B. In fact, the limiting functionals in those cases are of the same type as derived in [14, Theorem 5.2] where we considered the prestrained shallow shells. There, we assumed that the reference domains Ω^h were configured around the mid-surfaces $\{x' + hv_0(x');\ x' \in \omega\}$ rather than the flat mid-plate ω. Taking $(S_{31}, S_{32}) = \nabla v_0$ leads to the same energy $I(v,w)$, in which both bending and stretching are relative to the matching order tensors derived from v_0. We point out that the presence of similar tensors also occurred in [15] in the context of shells with varying thickness.

We also check that results of Theorem 1.5 are stronger than the general energy bounds in Theorem 1.4, which are however valid for any nonzero $S_{2\times 2}$. Since $\alpha, \gamma \geq 2$, it follows that $\inf I^h \leq Ch^{(5\alpha/6+2/3)-}$ whenever $\alpha < 4$. This leads to: $h^{7/3-}$ in case (i), which is a bound indeed inferior to h^4. Likewise, in case (iii) we get: $h^{(5\alpha/6+2/3)-} \gg h^{2+\alpha\wedge\gamma}$.

Finally, observe that the in-plane displacement w is slaved to S, B and v, and can be omitted by replacing the Γ-limits $I(v,w)$ by $\bar{I}(v)$:

Corollary 1.6. *In the context of Theorem 1.5, assume additionally that ω is simply connected. Then each stretching term may be replaced by the following squared distance from the space $\{\mathrm{sym}\nabla w;\ w \in W^{1,2}(\omega, \mathbb{R}^2)\}$, where $\delta = 2$ in cases (i), (ii) and $\delta = \alpha \wedge \gamma$ in case (iii):*

$$\frac{1}{2}\min_{w\in W^{1,2}(\omega,\mathbb{R}^2)}\int_\omega \mathcal{Q}_2\Big(\mathrm{sym}\,\nabla w + \begin{cases} 0 & \text{for } \delta > 2 \\ \frac{1}{2}(\nabla v)^{\otimes 2} & \text{for } \delta = 2 \end{cases} - \begin{cases} 0 & \text{for } \alpha > 2 \\ \frac{1}{2}(S_{31}, S_{32})^{\otimes 2} & \text{for } \alpha = 2 \end{cases}\Big)\,\mathrm{d}x'.$$

Consequently, we have the equivalences of Γ-limits $\bar{I}(v)$ defined on $v \in W^{2,2}(\omega, \mathbb{R})$:

(i) *If $\alpha = 2$, $\gamma > 2$ then $h^{-4}I^h \xrightarrow{\Gamma} \bar{I}$, where:*

$$\bar{I}(v) \simeq \|\nabla^2 v - 2\,\mathrm{sym}\nabla(S_{31}, S_{32})\|^2_{L^2(\omega)} + \|\det\nabla^2 v - \frac{1}{2}\mathrm{curl}^T\mathrm{curl}(S_{31}, S_{32})^{\otimes 2}\|^2_{H^{-2}(\omega)}.$$

If $\alpha = \gamma = 2$ then $h^{-4}I^h \xrightarrow{\Gamma} \bar{I}$, where:

$$\bar{I}(v) \simeq \|\nabla^2 v - 2\,\mathrm{sym}\nabla(S_{31}, S_{32}) + B_{2\times 2}\|^2_{L^2(\omega)} + \|\det\nabla^2 v - \frac{1}{2}\mathrm{curl}^T\mathrm{curl}(S_{31}, S_{32})^{\otimes 2}\|^2_{H^{-2}(\omega)}.$$

(ii) *If $\gamma = 2 < \alpha$ then $h^{-4}I^h \xrightarrow{\Gamma} \bar{I}$, where: $\bar{I}(v) \simeq \|\nabla^2 v + B_{2\times 2}\|^2_{L^2(\omega)} + \|\det\nabla^2 v\|^2_{H^{-2}(\omega)}.$*

(iii) If $\alpha, \gamma > 2$ then $h^{-(2+\alpha\wedge\gamma)}I^h \xrightarrow{\Gamma} \bar{\bar{I}}$, where:

$$\bar{\bar{I}} \equiv \min I \simeq \begin{cases} \|\text{curl}B_{2\times2}\|^2_{H^{-1}(\omega)} & \text{for } \gamma < \alpha \\ \|\nabla\text{curl}(S_{31}, S_{32})\|^2_{H^{-1}(\omega)} & \text{for } \gamma > \alpha \\ \|\text{curl}B_{2\times2} - \nabla\text{curl}(S_{31}, S_{32})\|^2_{H^{-1}(\omega)} & \text{for } \gamma = \alpha. \end{cases}$$

We remark that a bound on $\inf \bar{I}$ in cases (i) and (ii) above may be deduced from Proposition 1.3, again consistent with Theorem 1.8.

1.5. **Identification of the optimal scaling regimes.** Our final set of results concerns the optimality of the energy scalings implied by Theorems 1.1 and 1.5 and their connection to curvature of the prestrains A^h. Namely, in the setting of Theorem 1.1, we get:

Theorem 1.7. *Let $\alpha \geq 4$, $\gamma \geq 2$ and assume that ω is simply connected.*

(i) When $\gamma = 2$ then $ch^4 \leq \inf I^h \leq Ch^4$ with $c > 0$, if and only if:

$$\text{curl } B_{2\times2} \not\equiv 0, \quad or \quad \det B_{2\times2} + \begin{cases} 0 & \text{for } \alpha > 4 \\ \text{curl}^T\text{curl } S_{2\times2} & \text{for } \alpha = 4 \end{cases} \not\equiv 0 \quad in \ \omega.$$

(ii) When $\gamma \in (2, \alpha - 2]$ then $ch^{2+\gamma} \leq \inf I^h \leq Ch^{2+\gamma}$ with $c > 0$, if and only if:

$$\text{curl } B_{2\times2} \not\equiv 0, \quad or \quad \gamma = \alpha - 2 \ and \ \text{curl}^T\text{curl } S_{2\times2} \not\equiv 0 \quad in \ \omega.$$

(iii) When $\gamma > \alpha - 2$ then $ch^\alpha \leq \inf I^h \leq Ch^\alpha$ with $c > 0$, if and only if:

$$\text{curl}^T\text{curl } S_{2\times2} \not\equiv 0 \quad in \ \omega.$$

Consider now the following Riemann metrics on Ω^h:

$$\mathcal{G}^h = (A^h)^T A^h = Id_3 + 2h^{\alpha/2}S + h^\alpha S^2 + x_3\left(2h^{\gamma/2}B + 2h^{(\alpha+\gamma)/2}\text{sym}(SB)\right) + x_3^2 h^\gamma B^2. \quad (1.5)$$

We calculate the lowest order terms in the six Riemann curvatures of $\mathcal{G}^h$ on ω, at $x_3 = 0$:

$$\begin{aligned} R_{12,12} &\simeq -h^{\alpha/2}\text{curl curl } S_{2\times2} - h^\gamma \det B_{2\times2}, \\ R_{12,13} &\simeq -h^{\alpha/2}\partial_1\text{curl}(S_{13}, S_{23}) + h^{\gamma/2}(\text{curl}B_{2\times2})_1, \\ R_{12,23} &\simeq -h^{\alpha/2}\partial_2\text{curl}(S_{13}, S_{23}) + h^{\gamma/2}(\text{curl}B_{2\times2})_2, \\ R_{13,13} &\simeq -h^{\alpha/2}\partial^2_{11}S_{33} + 2h^{\gamma/2}\partial_1 B_{13}, \\ R_{13,23} &\simeq -h^{\alpha/2}\partial^2_{12}S_{33} + h^{\gamma/2}(\partial_1 B_{23} + \partial_2 B_{13}), \\ R_{23,23} &\simeq -h^{\alpha/2}\partial^2_{22}S_{33} + 2h^{\gamma/2}\partial_2 B_{23}. \end{aligned} \quad (1.6)$$

In all cases indicated in Theorem 1.7, both the optimality conditions and the energy scaling orders can be read from the first three curvatures above. These are: $R_{12,12}$ corresponding to stretching, and $R_{12,13}, R_{12,23}$ corresponding to bending. Indeed, when $\gamma = 2, \alpha = 4$ we have $R_{12,12} \simeq -h^2(\text{curl curl } S_{2\times2} + \det B_{2\times2})$ and $(R_{12,13}, R_{12,23}) \simeq h \text{ curl } B_{2\times2}$, which are the two tensors displayed in case (i). When $\gamma = 2, \alpha > 4$ the only difference is that $R_{12,12} \simeq -h^2 \det B_{2\times2}$, and in both cases $\inf I^h$ is of order equal to the square of the stretching order, i.e. h^4. When $\gamma > 2, \alpha = \gamma + 2$ we have $R_{12,12} \simeq -h^{\gamma/2+1}\text{curl curl } S_{2\times2}$ and $(R_{12,13}, R_{12,23}) \simeq h^{\gamma/2} \text{ curl } B_{2\times2}$, in agreement with case (ii). For $\gamma > 2, \alpha < \gamma+2$ the stretching-related curvature $R_{12,12}$ has order $h^{\gamma\wedge\frac{\alpha}{2}}$ which is strictly less than the bending-compatible order $h^{1+\gamma/2}$, so this term becomes irrelevant. The energy order in both cases equals the square of the compatible stretching order, i.e. $h^{\gamma+2}$. Finally, when $\gamma > \alpha-2$ we have: $R_{12,12} \simeq -h^{\alpha/2}\text{curl curl } S_{2\times2}$, while

$(R_{12,13}, R_{12,23})$ has order $h^{\frac{\gamma}{2} \wedge \frac{\alpha}{2}}$ which is strictly less than the stretching-compatible $h^{\alpha/2-1}$. Thus, the bending tensor is discarded on the basis of not contributing towards the energy, which is in agreement with case (iii). Similarly, the energy order equals the square of the stretching order i.e. h^α.

We note that the above observations are precisely the "small-curvature" regime counterparts of the findings in [18]. There, the authors considered a constant prestrain $A = A(x', x_3)$ independent of h. They proved that the optimal energy scaling, which is h^2, is superseded (i.e. $h^{-2} \inf I^h \to 0$ as $h \to 0$) if and only if $R_{12,12} = R_{12,13} = R_{12,23} \equiv 0$ on ω. The next viable scaling is then h^4, corresponding to the three remaining curvatures $R_{13,13}, R_{13,23}, R_{23,23}$. Further, the vanishing of these implies the energy scaling order h^6, and in general one has an infinite hierarchy of the limiting theories each valid in an appropriate regime of vanishing of components of $R_{..,..}$ together with their covariant derivatives [12].

We conjecture that the next viable energy scalings after those proved in Theorem 1.7 are: h^6 in case (i), $h^{(4+\gamma)\wedge 2\gamma}$ in case (ii) for $\gamma = \alpha - 2$, $h^{(4+\gamma)\wedge 2\gamma \wedge \alpha}$ in case (ii) for $\gamma < \alpha - 2$, and $h^{(2+\gamma)\wedge(2+\alpha)}$ in case (iii). By analogy, vanishing of the lowest order terms of $R_{13,13}, R_{13,23}, R_{23,23}$ given in (1.6) should then be responsible for even higher energy scalings.

We begin the discussion of optimality in the setting of Theorem 1.5 by stating:

Theorem 1.8. *Let* $S_{2\times 2} \equiv 0$ *in* ω *and let* $\alpha, \gamma \geq 2$. *Assume that* ω *is simply connected.*

(i) *When* $\alpha = 2$ *then* $ch^4 \leq \inf I^h \leq Ch^4$ *with* $c > 0$, *if and only if:*

$$
\begin{aligned}
&\text{when } \gamma > 2: \quad \nabla' \operatorname{curl}(S_{31}, S_{32}) \not\equiv 0 \quad \text{or} \quad \det \nabla'(S_{31}, S_{32}) \not\equiv 0 \quad \text{in } \omega, \\
&\text{when } \gamma = 2: \quad \operatorname{curl} B_{2\times 2} - \nabla' \operatorname{curl}(S_{31}, S_{32}) \not\equiv 0 \\
&\qquad\qquad \text{or } \det\big(B_{2\times 2} - 2\operatorname{sym}\nabla'(S_{31}, S_{32})\big) + \frac{1}{2}\operatorname{curl}^T \operatorname{curl}(S_{31}, S_{32})^{\otimes 2} \not\equiv 0.
\end{aligned}
$$

Equivalently, the last condition above may be rewritten as:

$$
3 \det \nabla'(S_{31}, S_{32}) - \big\langle \nabla' \operatorname{curl}(S_{31}, S_{32}), (S_{31}, S_{32})^\perp \big\rangle - 2\big\langle \operatorname{cof} B_{2\times 2} : \nabla'(S_{31}, S_{32}) \big\rangle + \det B_{2\times 2} \not\equiv 0.
$$

(ii) *When* $\gamma = 2 < \alpha$ *then* $ch^4 \leq \inf I^h \leq Ch^4$ *with* $c > 0$, *if and only if:*

$$
\operatorname{curl} B_{2\times 2} \not\equiv 0 \quad \text{or} \quad \det B_{2\times 2} \not\equiv 0 \quad \text{in } \omega.
$$

(iii) *When* $\alpha, \gamma > 2$ *then* $ch^{2+\alpha\wedge\gamma} \leq \inf I^h \leq Ch^{2+\alpha\wedge\gamma}$ *with* $c > 0$, *if and only if:*

$$
\begin{aligned}
&\operatorname{curl} B_{2\times 2} \not\equiv 0 \quad \text{and} \quad \gamma < \alpha, \quad \text{or} \quad \nabla' \operatorname{curl}(S_{31}, S_{32}) \not\equiv 0 \quad \text{and} \quad \gamma > \alpha, \\
&\qquad \text{or} \quad \operatorname{curl} B_{2\times 2} - \nabla' \operatorname{curl}(S_{31}, S_{32}) \not\equiv 0 \quad \text{and} \quad \gamma = \alpha.
\end{aligned}
$$

Firstly, observe that (ii) above coincides with the statement of Theorem 1.7 (i) when $S_{2\times 2} \equiv 0$, while (iii) gives more precise information than Theorem 1.7 (ii) and (iii). As before, we may

compute the lowest order terms in the Riemann curvatures of the metric $\mathcal{G}^h$ in (1.5), at $x_3 = 0$:

$$
\begin{aligned}
R_{12,12} &\simeq h^\alpha\Big(-3\det\nabla'(S_{31}, S_{32}) + S_{13}\partial_2\mathrm{curl}(S_{31}, S_{32}) - S_{32}\partial_1\mathrm{curl}(S_{31}, S_{32}) \Big) \\
&\quad - h^\gamma \det B_{2\times 2} + 2h^{(\alpha+\gamma)/2}\langle B_{2\times 2} : \mathrm{cof}\,\nabla'(S_{31}, S_{32})\rangle, \\
R_{12,13} &\simeq -h^{\alpha/2}\partial_1\mathrm{curl}(S_{31}, S_{32}) + h^{\gamma/2}\big(\mathrm{curl}\,B_{2\times 2}\big)_1, \\
R_{12,23} &\simeq -h^{\alpha/2}\partial_2\mathrm{curl}(S_{31}, S_{32}) + h^{\gamma/2}\big(\mathrm{curl}\,B_{2\times 2}\big)_2, \\
R_{13,13} &\simeq -h^{\alpha/2}\partial_{11}^2 S_{33} + 2h^{\gamma/2}\partial_1 B_{13}, \\
R_{13,23} &\simeq -h^{\alpha/2}\partial_{12}^2 S_{33} + h^{\gamma/2}\big(\partial_1 B_{23} + \partial_2 B_{13}\big), \\
R_{23,23} &\simeq -h^{\alpha/2}\partial_{22}^2 S_{33} + 2h^{\gamma/2}\partial_2 B_{23}.
\end{aligned}
\tag{1.7}
$$

Again, the optimality conditions and the energy scalings in all cases of Theorem 1.8, can be read from the first three curvatures above. When $\alpha = 2$, $\gamma > 2$ we have $R_{12,12} \simeq h^2\big(-3\det\nabla'(S_{31}, S_{32}) + \langle\nabla\mathrm{curl}(S_{31}, S_{32}), (S_{31}, S_{32})^\perp\rangle\big)$, $(R_{12,13}, R_{12,23}) \simeq -h\nabla'\mathrm{curl}(S_{31}, S_{32})$ and the fact that at least one of these expressions is nonzero is equivalent to the condition displayed in case (i). When $\alpha = \gamma = 2$, then get the full expressions: $R_{12,12} \simeq h^2\big(-3\det\nabla'(S_{31}, S_{32}) + \langle\nabla\mathrm{curl}(S_{31}, S_{32}), (S_{31}, S_{32})^\perp\rangle\big) - \det B_{2\times 2} + 2\langle B_{2\times 2} : \mathrm{cof}\,\nabla'(S_{31}, S_{32})\rangle\big)$ and $(R_{12,13}, R_{12,23}) \simeq h\big(\mathrm{curl}\,B_{2\times 2} - \nabla'\mathrm{curl}(S_{31}, S_{32})\big)$, which are again consistent with (i). In both cases $\inf I^h$ is of order equal to the square of the stretching order, i.e. h^4. For $\gamma = 2$, $\alpha > 2$ we get: $R_{12,12} \simeq -h^2\det B_{2\times 2}$ and $(R_{12,13}, R_{12,23}) \simeq h\,\mathrm{curl}\,B_{2\times 2}$, in agreement with case (ii). Finally, when $\alpha, \gamma > 2$ then $(R_{12,13}, R_{12,23})$ is of the order $h^{(\alpha\wedge\gamma)/2}$, with the corresponding coefficients indicated in (iii). The compatible stretching order in that case is $h^{1+(\alpha\wedge\gamma)/2}$ which is strictly larger than the order $h^{\alpha\wedge\gamma\wedge(\alpha+\gamma)/2}$ of $R_{12,12}$, as computed in (1.7). Thus, stretching-related curvature does not contribute towards the energy, and the exponent of the order of $\inf I^h$ equals twice the bending order plus 1, which yields $h^{2+\alpha\wedge\gamma}$.

We conjecture that the next viable energy scalings after those displayed in Theorem 1.8 are guided by the lowest order terms in $R_{13,13}, R_{13,23}, R_{23,23}$ given in (1.7).

1.6. **An outline.** We prove the convergence properties (1.4) and the lower bounds of both Theorems 1.1 and 1.5 in section 2. The proofs are written in a unified manner and are only specified to the two assumed prestrain scalings in Corollaries 2.5 and 2.6. Section 3 gathers all constructions of recovery sequence in this papers: in sections 3.2 and 3.3 we provide the upper bounds portion of the Γ-limit statements in Theorems 1.1 and 1.5, while section 3.1 is devoted to showing Theorem 1.4. Corollaries 1.2 and 1.6 rely on two decomposition results for symmetric matrix fields in section 4, where we also prove Proposition 1.3. The optimality conditions listed in Theorems 1.7 and 1.8 follow from the same decomposition results, while the identification in terms of Riemann curvatures in (1.6) and (1.7) is given in section 5.

2. Compactness and lower bounds

In this section, we investigate the asymptotic properties of minimizing sequences to I^h in the scaling sub-regimes of: $\lim_{h\to 0} h^{-2} \inf I^h = 0$. First, it follows from [13, Theorem 1.6] that:

Lemma 2.1. *Let $\delta > 0$ and assume that for a sequence $\{u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)\}_{h\to 0}$ there holds: $I^h(u^h) \le Ch^{2+\delta}$. Then there exists a sequence $\{R^h \in W^{1,2}(\omega, SO(3))\}_{h\to 0}$ such that:*

$$\frac{1}{h} \int_{\Omega^h} |\nabla u^h - R^h A^h|^2 \, dx \le Ch^{2+\delta\wedge\alpha\wedge\gamma}, \qquad \int_\omega |\nabla R^h|^2 \, dx' \le Ch^{\delta\wedge\alpha\wedge\gamma}. \tag{2.1}$$

Second, in the context of Lemma 2.1, the scaled integrands in $h^{-(2+\delta)}I^h(u^h)$, equal in virtue of frame invariance and by Taylor expanding W:

$$\frac{1}{h^{2+\delta}} W\big((R^h)^T \nabla u^h (A^h)^{-1}\big) = \frac{1}{2} D^2 W(Id_3)(G^h)^{\otimes 2} + o(|G^h|^2) \quad \text{as } h \to 0, \tag{2.2}$$

where we define the sequence of strains $\{G^h \in L^2(\Omega^1, \mathbb{R}^{3\times 3})\}_{h\to 0}$ in:

$$G^h(x', x_3) = \frac{1}{h^{1+\delta/2}} \Big(R^h(x')^T \nabla u^h(x', hx_3) A^h(x', hx_3)^{-1} - Id_3\Big) \quad \text{for all } x' \in \omega, \ x_3 \in \big(-\tfrac{1}{2}, \tfrac{1}{2}\big).$$

In view of the expansion (2.2), it is natural to expect that the limit of $h^{-(2+\delta)}I^h(u^h)$ quantifies the limit of $\{G^h\}_{h\to 0}$. By (2.1), a sufficient condition to get a subsequential convergence in:

$$G^h \rightharpoonup G \qquad \text{weakly in } L^2(\Omega^1, \mathbb{R}^{3\times 3}), \text{ as } h \to 0 \tag{2.3}$$

is thus: $1 + (\delta \wedge \alpha \wedge \gamma)/2 \ge 1 + \delta/2$, or equivalently:

$$\delta \le \alpha \wedge \gamma.$$

We have the following compactness result:

Lemma 2.2. *Let $\delta \in (0, \alpha \wedge \gamma]$ and assume that for a sequence $\{u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)\}_{h\to 0}$ there holds: $I^h(u^h) \le Ch^{2+\delta}$. Then there exist sequences $\{\bar{R}^h \in SO(3)\}_{h\to 0}$, $\{c^h \in \mathbb{R}^3\}_{h\to 0}$ such that there holds for the rescaled deformations $y^h(x', x_3) = (\bar{R}^h)^T u^h(x', hx_3) - c^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$:*

(i) $y^h \to x'$ *strongly in* $W^{1,2}(\Omega^1, \mathbb{R}^3)$, *as* $h \to 0$.

(ii) *Define:* $V^h(x') = h^{-\delta/2} \fint_{-1/2}^{1/2} y^h(x', x_3) - x' \, dx_3$. *Then, up to a subsequence:* $V^h \to V$ *as $h \to 0$ strongly in $W^{1,2}(\omega, \mathbb{R}^3)$. The limit $V \in W^{2,2}(\omega, \mathbb{R}^3)$ and it satisfies:*

$$\text{sym}(\nabla V)_{2\times 2} = \begin{cases} 0 & \text{for } \delta < \alpha \\ S_{2\times 2} & \text{for } \delta = \alpha. \end{cases}$$

In particular, for $\delta < \alpha$ or when $S_{2\times 2} = 0$, we get that $V = (0, 0, v)$ for an out-of-plane displacement $v \in W^{2,2}(\omega, \mathbb{R})$. The case $\delta = \alpha$ is viable only when $\text{curl}^T \text{curl} \, S_{2\times 2} \equiv 0$ in ω.

(iii) *The in-plane part of the limiting strain G in (2.3) satisfies:*

$$(\partial_3 G)_{2\times 2} = -\nabla^2 V_3 + \begin{cases} 0 & \text{for } \delta < \alpha \\ 2\big(\nabla(S_{31}, S_{32})\big)_{2\times 2} & \text{for } \delta = \alpha \end{cases} - \begin{cases} 0 & \text{for } \delta < \gamma \\ B_{2\times 2} & \text{for } \delta = \gamma. \end{cases}$$

Hence $G(x', 0)_{2\times 2}$ is well defined, and it equals: $\fint_{-1/2}^{1/2} G(x', x_3)_{2\times 2} \, dx_3 \in L^2(\omega, \mathbb{R}^{2\times 2})$.

Proof. **1.** As in the proof of [13, Theorem 1.2], we define:

$$\bar{R}^h = \tilde{R}^h \hat{R}^h \quad \text{where: } \tilde{R}^h = \mathbb{P}_{SO(3)} \fint_\omega R^h \, dx', \quad \hat{R}^h = \mathbb{P}_{SO(3)} \fint_{\Omega^h} (\tilde{R}^h)^T \nabla u^h \, dx,$$

where the uniqueness of the above projections follows in virtue of the bounds in (2.1). Further:

$$\|(\bar{R}^h)^T R^h - Id_3\|_{W^{1,2}(\omega)} \le Ch^{\delta/2}. \tag{2.4}$$

By choosing c^h so that $\fint_{\Omega^1} y^h \,\mathrm{d}x = \fint_\omega x' \,\mathrm{d}x'$, we obtain the following bounds that imply (i):

$$\fint_{\Omega^1} |(\nabla y^h - Id)_{3\times2}|^2 \,\mathrm{d}x \le Ch^\delta, \qquad \fint_{\Omega^1} |\partial_3 y^h|^2 \,\mathrm{d}x \le Ch^2.$$

2. Fix $s \ll 1$. From (2.4), (2.3) we observe the subsequential convergence as $h \to 0$, in:

$$\frac{1}{s}\big(G(x',x_3 + s) - G(x', x_3)\big)_{2\times2}$$

$$\leftarrow \frac{1}{s}\Big((\bar{R}^h)^T R^h \big(G^h(x', x_3 + s)A^h(x', hx_3 + hs) - G^h(x', x_s)A^h(x', hx_3)\big)\Big)_{2\times2} \tag{2.5}$$

$$= \frac{1}{s}\frac{1}{h^{1+\delta/2}}\Big(\nabla y^h(x', x_3 + s) - \nabla y^h(x', x_3)\Big)_{2\times2} - h^{\gamma/2-\delta/2}B(x')_{2\times2}.$$

To identify the limit of the first term in the right hand side above, we consider the asymptotics of: $h^{-(1+\delta/2)}\frac{1}{s}\big(y^h(x', x_3 + s) - y^h(x', x_3)\big) = h^{-(1+\delta/2)}\int_0^s \partial_3 y^h(x', x_3 + t)\,\mathrm{d}t$. Namely, we write:

$$\frac{1}{h^{1+\delta/2}}\big(\partial_3 y^h(x', x_3) - he_3\big) = \frac{1}{h^{\delta/2}}\Big((\bar{R}^h)^T \partial_3 u^h(x', hx_3) - e_3 \pm A^h(x', hx_3)e_3\Big)$$

$$= \frac{1}{h^{\delta/2}}(\bar{R}^h)^T\Big(\nabla u^h(x', hx_3) - R^h A^h(x', hx_3)\Big)e_3 + \frac{1}{h^{\delta/2}}\Big((\bar{R}^h)^T R^h(x') - Id_3\Big)A^h(x', hx_3)e_3$$

$$+ h^{\alpha/2-\delta/2}S(x')e_3 + h^{\gamma/2+1-\delta/2}x_3 B(x')e_3.$$

The first term in the right hand side above is bounded by Ch from (2.1), so it converges to 0. For the second term, we first denote the following subsequential limit:

$$\frac{1}{h^{\delta/2}}\Big((\bar{R}^h)^T R^h - Id_3\Big) \rightharpoonup P \quad \text{weakly in } W^{1,2}(\omega, \mathbb{R}^{3\times3}), \text{ as } h \to 0, \tag{2.6}$$

whose existence is implied by (2.4) and where the limiting field satisfies: $P \in W^{1,2}(\omega, so(3))$. Recalling (2.5) we hence obtain:

$$G(x', x_3)_{2\times2} - G(x', 0)_{2\times2} = x_3\big(\nabla(Pe_3)(x')\big)_{2\times2} + x_3 \begin{cases} 0 & \text{for } \delta < \alpha \\ \big(\nabla(Se_3)(x')\big)_{2\times2} & \text{for } \delta = \alpha \end{cases}$$

$$- x_3 \begin{cases} 0 & \text{for } \delta < \gamma \\ B(x')_{2\times2} & \text{for } \delta = \gamma. \end{cases}$$

3. We now identify the entries P_{31}, P_{32} of the limiting skew field in (2.6). We write:

$$\nabla V^h = \frac{1}{h^{\delta/2}}(\bar{R}^h)^T \fint_{-h/2}^{h/2} \big(\nabla u^h - R^h A^h\big)_{3\times2} \,\mathrm{d}x_3 + \frac{1}{h^{\delta/2}}\big((\bar{R}^h)^T R^h - Id_3\big)_{3\times2}$$

$$+ h^{\alpha/2-\delta/2}(\bar{R}^h)^T R^h S_{3\times2},$$

so by (2.6) we get, up to a subsequence:

$$\nabla V^h \to P_{3\times2} + \begin{cases} 0 & \text{for } \delta < \alpha \\ S_{3\times2} & \text{for } \delta = \alpha \end{cases} \quad \text{strongly in } L^2(\omega, \mathbb{R}^{3\times2}), \text{ as } h \to 0.$$

Since $\fint_\omega V^h \,\mathrm{d}x' = 0$, this proves the convergence statement in (ii), with ∇V given by the right hand side above. Equivalently, we record the useful formula:

$$P_{3\times2} = \nabla V - \begin{cases} 0 & \text{for } \delta < \alpha \\ S_{3\times2} & \text{for } \delta = \alpha. \end{cases} \tag{2.7}$$

Note now that the definitions of $\tilde{R}^h$ and $\hat{R}^h$ imply that: skew $\fint_\omega (\nabla V)_{2\times 2} \, \mathrm{d}x' = 0$, because:

$$\fint_\omega (\nabla V^h)_{2\times 2} \, \mathrm{d}x' = \frac{1}{h^{\delta/2}} \Big((\hat{R}^h)^T \fint_{\Omega^h} (\tilde{R}^h)^T \nabla u^h - \hat{R}^h \, \mathrm{d}x \Big)_{2\times 2} \in \mathbb{R}^{2\times 2}_{sym}. \tag{2.8}$$

There also holds: $\fint_\omega V \, \mathrm{d}x' = 0$. Consequently, in case $\delta < \alpha$ or $S_{2\times 2} = 0$ it follows that the tangential component of V must have gradient 0 and mean 0, and hence be equal to 0. This completes the proof of (ii), while from (2.7) we get:

$$((Pe_3)_1, (Pe_3)_2) = -(P_{31}, P_{32}) = -(\partial_1 V_3, \partial_2 V_3) + \begin{cases} 0 & \text{for } \delta < \alpha \\ (S_{31}, S_{32}) & \text{for } \delta = \alpha. \end{cases} \tag{2.9}$$

The formula in (iii) is now a consequence of step 2. ∎

Corollary 2.3. *In the context of Lemma 2.2, there holds:*

$$\liminf_{h\to 0} \frac{1}{h^{2+\delta}} I^h(u^h) \geq \frac{1}{2} \int_\omega \mathcal{Q}_2\big(\mathrm{sym}\, G(x',0)_{2\times 2}\big) \, \mathrm{d}x' + \frac{1}{24} \int_\omega \mathcal{Q}_2\big(\mathrm{sym}\, \partial_3 G_{2\times 2}\big) \, \mathrm{d}x',$$

where the quadratic form $\mathcal{Q}_2$ is defined in:

$$\mathcal{Q}_2(F_{2\times 2}) = \min\Big\{ \mathcal{Q}_3(\tilde{F}); \ \tilde{F} \in \mathbb{R}^{3\times 3} \ \text{with} \ \tilde{F}_{2\times 2} = F_{2\times 2} \Big\}, \quad \mathcal{Q}_3(F) = D^2 W(Id_3)(F, F).$$

Proof. Recalling (2.2) and the fact that $\{G^h\}_{h\to 0}$ is bounded in $L^2(\Omega^1, \mathbb{R}^3)$, we write:

$$\frac{1}{h^{2+\delta}} I^h(u^h) \geq \int_{\{h^{\delta/2}|G^h|\leq 1\}} \frac{1}{2} \mathcal{Q}_3(G^h) + o(1)|G^h|^2 \, \mathrm{d}x = \frac{1}{2} \int_{\Omega^1} \mathcal{Q}_3\big(\mathbb{1}_{\{h^{\delta/2}|G^h|\leq 1\}}|G^h|\big) \, \mathrm{d}x + o(1),$$

as $h \to 0$. Further, since the argument in $\mathcal{Q}_3$ in the right hand side above converges, weakly in $L^2(\Omega^1, \mathbb{R}^{3\times 3})$, to G by (2.3), the weak lowersemicontinuity argument implies that:

$$\liminf_{h\to 0} \frac{1}{h^{2+\delta}} I^h(u^h) \geq \frac{1}{2} \int_{\Omega^1} \mathcal{Q}_3(G) \, \mathrm{d}x \geq \frac{1}{2} \int_{\Omega^1} \mathcal{Q}_3(\mathrm{sym}\, G) \, \mathrm{d}x \geq \frac{1}{2} \int_{\Omega^1} \mathcal{Q}_2(\mathrm{sym}\, G_{2\times 2}) \, \mathrm{d}x.$$

Here, we also used that $\mathcal{Q}_3(G) = \mathcal{Q}_3(\mathrm{sym}\, G)$ and the definition of $\mathcal{Q}_2$. By Lemma 2.2 (iii) we get: $G(x', x_3)_{2\times 2} = G(x', 0)_{2\times 2} + x_3 \partial_3 G(x', \cdot)_{2\times 2}$, which concludes the proof. ∎

Lemma 2.4. *In the context of Lemma 2.2, we have the following subsequential convergence, weakly in $L^2(\omega, \mathbb{R}^{2\times 2})$ as $h \to 0$:*

$$\mathrm{sym}\, G(\cdot, 0)_{2\times 2} \leftharpoonup \frac{1}{h} \mathrm{sym}\big(\nabla V^h\big)_{2\times 2} - \frac{1}{h^{1+\delta/2}} \mathrm{sym}\big((\bar{R}^h)^T R^h - Id_3\big)_{2\times 2} \\ - h^{\alpha/2-1-\delta/2} \mathrm{sym}\big((\bar{R}^h)^T R^h S\big)_{2\times 2}. \tag{2.10}$$

Proof. We write:

$$\fint_{-1/2}^{1/2} G^h_{2\times 2} \, \mathrm{d}x_3 = \Big(\big((\bar{R}^h)^T R^h - Id_3\big)^T (\bar{R}^h)^T \cdot \frac{1}{h^{1+\delta/2}} \fint_{-h/2}^{h/2} (\nabla u^h - R^h A^h) \, \mathrm{d}x_3 \cdot (A^h)^{-1} \Big)_{2\times 2}$$

$$+ \Big((\bar{R}^h)^T \cdot \frac{1}{h^{1+\delta/2}} \fint_{-h/2}^{h/2} (\nabla u^h - R^h A^h) \, \mathrm{d}x_3 \Big)_{2\times 2}$$

$$+ \Big((\bar{R}^h)^T \cdot \frac{1}{h^{1+\delta/2}} \fint_{-h/2}^{h/2} (\nabla u^h - R^h A^h) \, \mathrm{d}x_3 \cdot \big((A^h)^{-1} - Id_3\big) \Big)_{2\times 2}.$$

The first and the third terms in the right hand side above converge to 0, in $L^2(\omega, \mathbb{R}^{2\times2})$ as $h \to 0$, because of (2.4) and (2.1). We rewrite the second term as:

$$\frac{1}{h}(\nabla V^h)_{2\times2} - \frac{1}{h^{1+\delta/2}}\big((\bar{R}^h)^T R^h - Id_3\big)_{2\times2} - h^{\alpha/2-1-\delta/2}\big((\bar{R}^h)^T R^h S\big)_{2\times2}.$$

This implies the result, because of the subsequential convergence weakly in $L^2(\omega, \mathbb{R}^{2\times2})$, of $\fint_{-1/2}^{1/2} G^h(\cdot, x_3)_{2\times2}\,\mathrm{d}x_3$ to $\fint_{-1/2}^{1/2} G(\cdot, x_3)_{2\times2}\,\mathrm{d}x_3 = G(\cdot, 0)_{2\times2}$. ∎

Corollary 2.5. *In the context of Lemma 2.2, assume further that $\delta \geq 2$ and $\alpha \geq 2+\delta$. Then, $h^{-1}(V_1^h, V_2^h)$ converges to some in-plane displacement w, up to a subsequence weakly in $W^{1,2}(\omega, \mathbb{R}^2)$ as $h \to 0$. Moreover, there holds:*

$$\mathrm{sym}\,G(\cdot, 0)_{2\times2} = \mathrm{sym}\nabla w + \begin{cases} 0 & \text{for } \delta > 2 \\ \frac{1}{2}(\nabla v)^{\otimes2} & \text{for } \delta = 2 \end{cases} - \begin{cases} 0 & \text{for } \alpha > 2+\delta \\ S_{2\times2} & \text{for } \alpha = 2+\delta. \end{cases}$$

Consequently, $\liminf_{h\to0} h^{-(2+\delta)} I^h(u^h)$ is bounded from below by:

$$\begin{aligned}
I(v, w) = \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(\nabla^2 v + \begin{cases} 0 & \text{for } \delta < \gamma \\ B_{2\times2} & \text{for } \delta = \gamma \end{cases}\Big)\,\mathrm{d}x' \\
+ \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(\mathrm{sym}\,\nabla w + \begin{cases} 0 & \text{for } \delta > 2 \\ \frac{1}{2}(\nabla v)^{\otimes2} & \text{for } \delta = 2 \end{cases} - \begin{cases} 0 & \text{for } \alpha > 2+\delta \\ S_{2\times2} & \text{for } \alpha = 2+\delta \end{cases}\Big)\,\mathrm{d}x',
\end{aligned} \tag{2.11}$$

which coincides with the functionals given in Theorem 1.1 in each of the indicated three cases.

Proof. The second term in the right hand side of (2.10) can be rewritten with the help of:

$$-\mathrm{sym}\big((\bar{R}^h)^T R^h - Id_3\big)_{2\times2} = \frac{1}{2}\mathrm{sym}\Big(\big((\bar{R}^h)^T R^h - Id_3\big)^T\big((\bar{R}^h)^T R^h - Id_3\big)\Big)_{2\times2}$$

By (2.6) and since $\delta \geq 1 + \delta/2$, we conclude the following subsequential convergence, strongly in $L^2(\omega, \mathbb{R}^{2\times2})$ as $h \to 0$:

$$-\frac{1}{h^{1+\delta/2}}\mathrm{sym}\big((\bar{R}^h)^T R^h - Id_3\big)_{2\times2} \rightharpoonup \begin{cases} 0 & \text{for } \delta > 2 \\ \frac{1}{2}(P^T P)_{2\times2} & \text{for } \delta = 2. \end{cases}$$

When $\alpha \geq 2+\delta$, then the third term in (2.10) converges, by (2.4), to 0 for $\alpha > 2+\delta$ and to $-S_{2\times2}$ for $\alpha = 2+\delta$. In this case, we also have $V = (0, 0, v)$ and (2.7) yields that $(P^T P)_{2\times2} = (\nabla v)^{\otimes2}$. This proves the formula for $G(\cdot, 0)_{2\times2}$, since the resulting weak subsequential convergence of the first term in the right hand side of (2.10): $\frac{1}{h}\mathrm{sym}\big(\nabla V^h\big)_{2\times2}$, is equivalent with the weak convergence of $h^{-1}(V_1^h, V_2^h)$ to some limiting $w \in W^{1,2}(\omega, \mathbb{R}^2)$.

Finally, the lower bound on $\liminf_{h\to0} h^{-(2+\delta)} I^h$ is derived from Corollary 2.3, upon recalling Lemma 2.2 (iii). The optimal values of the exponent δ given in Theorem 1.1 in function of $\alpha \geq 4$ and $\gamma \geq 2$, follow by a direct inspection. ∎

Corollary 2.6. *In the context of Lemma 2.2, assume further that $\delta \geq 2$ and $S_{2\times2} \equiv 0$ in ω. Then, $h^{-1}(V_1^h, V_2^h)$ converges to some in-plane displacement w, up to a subsequence weakly in $W^{1,2}(\omega, \mathbb{R}^2)$ as $h \to 0$. Moreover, there holds:*

$$\mathrm{sym}\,G(\cdot, 0)_{2\times2} = \mathrm{sym}\nabla w + \begin{cases} 0 & \text{for } \delta > 2 \\ \frac{1}{2}(\nabla v)^{\otimes2} & \text{for } \delta = 2 < \alpha \\ \frac{1}{2}(\nabla v)^{\otimes2} - \frac{1}{2}(S_{31}, S_{32})^{\otimes2} & \text{for } \delta = \alpha = 2. \end{cases} \tag{2.12}$$

Consequently, $\liminf_{h\to 0} h^{-(2+\delta)} I^h(u^h)$ *is bounded from below by:*

$$I(v,w) = \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(\nabla^2 v - \left\{\begin{array}{ll} 0 & for\ \delta < \alpha \\ 2\,\mathrm{sym}\big(\nabla(S_{31}, S_{32})\big)_{2\times 2} & for\ \delta = \alpha \end{array}\right. + \left\{\begin{array}{ll} 0 & for\ \delta < \gamma \\ B_{2\times 2} & for\ \delta = \gamma \end{array}\right.\Big)\,\mathrm{d}x'$$
$$+ \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(\mathrm{sym}\,\nabla w + \left\{\begin{array}{ll} 0 & for\ \delta > 2 \\ \frac{1}{2}(\nabla v)^{\otimes 2} & for\ \delta = 2 \end{array}\right. - \left\{\begin{array}{ll} 0 & for\ \alpha > 2 \\ \frac{1}{2}(S_{31}, S_{32})^{\otimes 2} & for\ \alpha = 2 \end{array}\right.\Big)\,\mathrm{d}x',$$
(2.13)

which coincides with the functionals given in Theorem 1.5 in each of the indicated cases.

Proof. Convergence of the second term in the right hand side of (2.10) follows as in the proof of Corollary 2.5. To show (2.12), observe that the third term in (2.10) can be now written as:

$$- h^{\alpha/2-1}\,\mathrm{sym}\Big(\frac{(\bar{R}^h)^T R^h - Id_3}{h^{\delta/2}}\cdot S\Big)_{2\times 2},$$
(2.14)

and we may identify its limit when $\alpha \geq 2$, in virtue of (2.6). Namely, when $\alpha > 2$ this limit is 0 and we recover the first two cases of (2.12). When $\alpha = 2$ then automatically $\delta = 2$ as well, and (2.14) converges subsequentially, weakly in $L^2(\omega, \mathbb{R}^{2\times 2})$ as $h \to 0$ to:

$$-\mathrm{sym}\big(PS\big)_{2\times 2} = \mathrm{sym}\Big(\big(\nabla v - (S_{31}, S_{32})\big)\otimes (S_{31}, S_{32})\Big) = \mathrm{sym}\big(\nabla v \otimes (S_{31}, S_{32})\big) - (S_{31}, S_{32})^{\otimes 2}$$

where we used (2.7). At the same time: $\frac{1}{2}(P^T P)_{2\times 2} = \frac{1}{2}\big(\nabla v - (S_{31}, S_{32})\big)^{\otimes 2}$, which concludes the proof of the last case in (2.12).

The lower bound in (2.13) follows from Corollary 2.3, Lemma 2.2 (iii) and (2.12). The optimal values of δ in function of $\alpha, \gamma \geq 2$, follow by a direct inspection. ∎

3. Recovery sequences

In this section, we construct sequences of deformations $\{u^h\}_{h\to 0}$ with the desired asymptotics of the energy $I^h(u^h)$. First, in subsection 3.1 we prove the upper bound stated in Theorem 1.4. Second, we complete the Γ-convergence results given in Theorems 1.1 and 1.5, by constructing recovery sequences for the general form of the limiting functional I. In subsection 3.2 we treat the case (2.11), and in subsection 3.3 the case (2.13).

3.1. **Case $0 < \alpha < 4$ and $\gamma > 0$.** Recall that by [2, Theorem 1.1], for every $a \in (0, 1/5)$ there exists $v \in \mathcal{C}^{1,a}(\bar{\omega}, \mathbb{R})$ and $w \in \mathcal{C}^{1,a}(\bar{\omega}, \mathbb{R}^2)$ satisfying:

$$\frac{1}{2}(\nabla v)^{\otimes 2} + \mathrm{sym}\,\nabla w = S_{2\times 2}.$$
(3.1)

We now regularize v, w to $v_\varepsilon \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R})$, $w_\varepsilon \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^2)$, by means of the family of standard convolution kernels $\{\phi_\varepsilon(x) = \varepsilon^{-2}\phi(x/\varepsilon)\}_{\varepsilon\to 0}$ where ε is a power of h to be chosen later:

$$v_\varepsilon = v * \phi_\varepsilon, \quad w_\varepsilon = w * \phi_\varepsilon \quad \text{and} \quad \varepsilon = h^t.$$

We will utilize the following bound, resulting from the commutator estimate [3, Lemma 1]:

$$
\begin{aligned}
\Big\| \frac{1}{2}(\nabla v_\varepsilon)^{\otimes 2} &+ \operatorname{sym} \nabla w_\varepsilon - S_{2\times 2} \Big\|_{\mathcal{C}^0(\omega)} \\
&\leq \Big\| \frac{1}{2}(\nabla v_\varepsilon)^{\otimes 2} + \operatorname{sym} \nabla w_\varepsilon - S_{2\times 2} * \phi_\varepsilon \Big\|_{\mathcal{C}^0(\omega)} + \| S_{2\times 2} * \phi_\varepsilon - S_{2\times 2} \|_{\mathcal{C}^0(\omega)} \\
&= \frac{1}{2} \big\| (\nabla v_\varepsilon)^{\otimes 2} - (\nabla v)^{\otimes 2} * \phi_\varepsilon \big\|_{\mathcal{C}^0(\omega)} + \| S_{2\times 2} * \phi_\varepsilon - S_{2\times 2} \|_{\mathcal{C}^0(\omega)} \\
&\leq C\varepsilon^{2a} + C\varepsilon^2 \leq C\varepsilon^{2a},
\end{aligned}
\tag{3.2}
$$

where the $C\varepsilon^2$ bound follows by Taylor expanding $S_{2\times 2}$ up to second order terms. Further, by [17, (4.5)], we get the uniform bounds:

$$
\|\nabla v_\varepsilon\|_{\mathcal{C}^0(\omega)} + \|\nabla w_\varepsilon\|_{\mathcal{C}^0(\omega)} \leq C, \qquad \|\nabla^2 v_\varepsilon\|_{\mathcal{C}^0(\omega)} + \|\nabla^2 w_\varepsilon\|_{\mathcal{C}^0(\omega)} \leq C\varepsilon^{a-1}.
\tag{3.3}
$$

- Let $\delta = \alpha/2$. We denote $s = (S_{31}, S_{32})^T$ and define $u^h \in \mathcal{C}^\infty(\bar\Omega^h, \mathbb{R}^3)$ by:

$$
u^h = id_3 + h^{\delta/2} \begin{bmatrix} 0 \\ v_\varepsilon \end{bmatrix} + h^\delta \begin{bmatrix} w_\varepsilon \\ 0 \end{bmatrix} + x_3 \Big(h^{\delta/2} \begin{bmatrix} -\nabla v_\varepsilon \\ 0 \end{bmatrix} + h^\delta \begin{bmatrix} 2s \\ S_{33} - \frac{1}{2}|\nabla v_\varepsilon|^2 \end{bmatrix} + h^{3\delta/2}\vec{b} \Big),
$$

with the higher order smooth correction vector field:

$$
\vec{b}(x') = \Big(-S_{33}\nabla v_\varepsilon + \frac{1}{2}|\nabla v_\varepsilon|^2 \nabla v_\varepsilon + (\nabla w_\varepsilon)^T \nabla v_\varepsilon, 2\langle \nabla v_\varepsilon, s\rangle \Big)^T.
$$

It follows that:

$$
\begin{aligned}
\nabla u^h(x', hx_3) = Id_3 &+ h^{\delta/2} \begin{bmatrix} 0 & -\nabla v_\varepsilon \\ \nabla v_\varepsilon & 0 \end{bmatrix} + h^\delta \begin{bmatrix} \nabla w_\varepsilon & 2s \\ 0 & S_{33} - \frac{1}{2}|\nabla v_\varepsilon|^2 \end{bmatrix} + h^{3\delta/2} \begin{bmatrix} 0 & 0 & \vec{b} \end{bmatrix} \\
&+ h^{1+\delta/2} x_3 \begin{bmatrix} -\nabla^2 v_\varepsilon & 0 \\ 0 & 0 \end{bmatrix} + \mathcal{O}\big(h^{1+\delta} + h^{1+3\delta/2}(\|\nabla^2 v_\varepsilon\| + \|\nabla^2 w_\varepsilon\|) \big).
\end{aligned}
$$

We now write $(A^h(x', hx_3))^{-1} = Id_3 - h^\delta S - h^{1+\gamma/2}x_3 B + \mathcal{O}(h^{2\delta} + h^{2+\gamma})$ and proceed with:

$$
\begin{aligned}
\nabla u^h(A^h)^{-1}(x', hx_3) = Id_3 &+ P^h + h^\delta \begin{bmatrix} \nabla w_\varepsilon - S_{2\times 2} & 0 \\ 0 & -\frac{1}{2}|\nabla v_\varepsilon|^2 \end{bmatrix} \\
&+ h^{3\delta/2} \begin{bmatrix} \nabla v_\varepsilon \otimes s & \frac{1}{2}|\nabla v_\varepsilon|^2 \nabla v_\varepsilon + (\nabla w_\varepsilon)^T \nabla v_\varepsilon \\ -S_{2\times 2}\nabla v_\varepsilon & \langle \nabla v_\varepsilon, s\rangle \end{bmatrix} + x_3 \Big(h^{1+\delta/2} \begin{bmatrix} -\nabla^2 v_\varepsilon & 0 \\ 0 & 0 \end{bmatrix} - h^{1+\gamma/2} B \Big) \\
&+ \mathcal{O}\big(h^{1+\delta} + h^{2\delta} + h^{1+\delta/2+\gamma/2} + h^{2+\gamma} \big) + \mathcal{O}\big(h^{1+3\delta/2} + h^{2+\delta/2+\gamma/2} \big)(\|\nabla^2 v_\varepsilon\| + \|\nabla^2 w_\varepsilon\|).
\end{aligned}
$$

Above, we used the following skew-symmetric matrix field:

$$
P^h = \begin{bmatrix} 0 & p^h \\ -p^h & 0 \end{bmatrix}, \qquad p^h = -h^{\delta/2}\nabla v_\varepsilon + h^\delta s.
$$

- Consider the rotation fields $q^h \in \mathcal{C}^\infty(\bar\omega, SO(3))$:

$$
q^h = \exp(-P^h) = Id_3 - P^h + \frac{1}{2}(P^h)^2 - \frac{1}{6}(P^h)^3 + \mathcal{O}(h^{2\delta}).
$$

Then we get:

$$q^h \nabla u^h (A^h)^{-1}(x', hx_3) = Id_3 + h^\delta \begin{bmatrix} \frac{1}{2}(\nabla v_\varepsilon)^{\otimes 2} + \nabla w_\varepsilon - S_{2\times 2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$+ h^{3\delta/2} \begin{bmatrix} \text{skew}(\nabla v_\varepsilon \otimes s) & (\nabla w_\varepsilon)^T \nabla v_\varepsilon \\ -(\nabla w_\varepsilon)^T \nabla v_\varepsilon & 0 \end{bmatrix} + \frac{1}{3}(P^h)^3$$

$$+ x_3 \Big(h^{1+\delta/2} \begin{bmatrix} -\nabla^2 v_\varepsilon & 0 \\ 0 & 0 \end{bmatrix} - h^{1+\gamma/2} B \Big)$$

$$+ \mathcal{O}\big(h^{1+\delta} + h^{2\delta} + h^{1+\delta/2+\gamma/2} + h^{2+\gamma}\big) + \mathcal{O}\big(h^{1+\delta}(\|\nabla^2 v_\varepsilon\| + \|\nabla^2 w_\varepsilon\|)\big).$$

Finally, we apply a rotation field $\bar{q}^h = \exp(-\bar{P}^h) = Id_3 - \bar{P}^h + \mathcal{O}(h^{2\delta}) \in \mathcal{C}^\infty(\bar{\omega}, SO(3))$, with:

$$\bar{P}^h = \begin{bmatrix} \text{skew}\big(h^\delta \nabla w_\varepsilon + h^{3\delta/2} \nabla v_\varepsilon \otimes s\big) & h^{3\delta/2}(\nabla w_\varepsilon)^T \nabla v_\varepsilon \\ -h^{3\delta/2}(\nabla w_\varepsilon)^T \nabla v_\varepsilon & 0 \end{bmatrix} + \frac{1}{3}(P^h)^3,$$

to get:

$$\bar{q}^h q^h \nabla u^h (A^h)^{-1}(x', hx_3) = Id_3 + h^\delta \begin{bmatrix} \frac{1}{2}(\nabla v_\varepsilon)^{\otimes 2} + \text{sym}\nabla w_\varepsilon - S_{2\times 2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$+ x_3 \Big(h^{1+\delta/2} \begin{bmatrix} -\nabla^2 v_\varepsilon & 0 \\ 0 & 0 \end{bmatrix} - h^{1+\gamma/2} B \Big)$$

$$+ \mathcal{O}\big(h^{1+\delta} + h^{2\delta} + h^{1+\delta/2+\gamma/2} + h^{2+\gamma}\big) + \mathcal{O}\big(h^{1+\delta}(\|\nabla^2 v_\varepsilon\| + \|\nabla^2 w_\varepsilon\|)\big).$$

- In conclusion, we obtain the following energy bound, valid provided that we may use Taylor's expansion of W, which here requires that $h^{1+\delta/2}(\|\nabla^2 v_\varepsilon\| + \|\nabla^2 w_\varepsilon\|) \to 0$ as $h \to 0$:

$$\inf I^h \le I^h(u^h) = \int_{\Omega^1} W\Big(\bar{q}^h q^h \nabla u^h (A^h)^{-1}(x', hx_3)\Big) \, \mathrm{d}(x', x_3)$$

$$\le C \int_{\Omega^1} \Big(h^{2\delta} \big| \frac{1}{2}(\nabla v_\varepsilon)^{\otimes 2} + \text{sym}\nabla w_\varepsilon - S_{2\times 2} \big|^2 + h^{2+\delta}(\|\nabla^2 v_\varepsilon\|^2 + \|\nabla^2 w_\varepsilon\|^2)$$

$$+ h^{2+\gamma} + h^{4\delta} \Big) \, \mathrm{d}(x', x_3)$$

Recalling (3.2), (3.3) this leads to:

$$\inf I^h \le C\big(h^{2\delta} \varepsilon^{4a} + h^{2+\delta} \varepsilon^{2a-2} + h^{2+\gamma} + h^{4\delta}\big) = C\big(h^{2\delta+4at} + h^{2+\delta+(2a-2)t} + h^{2+\gamma} + h^{4\delta}\big).$$

Minimizing the right hand side above is equivalent to maximizing the minimal of the four displayed exponents. We hence choose t in $\varepsilon = h^t$ so that $2\delta + 4at = 2 + \delta + (2a-2)t$, namely $t = \frac{2-\delta}{2a+2}$. We then obtain:

$$\inf I^h \le C\big(h^{2\frac{\delta+2a}{a+1}} + h^{2+\gamma} + h^{4\delta}\big) \le C\big(h^{(5\delta/3+2/3)-} + h^{2+\gamma} + h^{4\delta}\big)$$

upon recalling that $a \in (0, 1/5)$. The conclusion of Theorem 1.4 follows by a direct inspection.

3.2. **Case $\alpha \ge 4, \gamma \ge 2$.** Let $\delta \in [2, \gamma]$ and $\alpha \ge 2 + \delta$. Given $v \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R})$, $w \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^2)$, and $d, \bar{d} \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^3)$, we define $u^h \in \mathcal{C}^\infty(\bar{\Omega}^h, \mathbb{R}^3)$ by:

$$u^h = id_3 + h^{\delta/2} \begin{bmatrix} 0 \\ v \end{bmatrix} + h^{1+\delta/2} \begin{bmatrix} w \\ 0 \end{bmatrix} - h^{\delta/2} x_3 \begin{bmatrix} \nabla v \\ 0 \end{bmatrix} + h^{1+\delta/2} x_3 d + \frac{1}{2} h^{\delta/2} x_3^2 \bar{d}. \qquad (3.4)$$

Consider the rotation fields $q^h \in \mathcal{C}^\infty(\bar{\omega}, SO(3))$:

$$q^h = \exp\left(h^{\delta/2} \begin{bmatrix} 0 & \nabla v \\ -\nabla v & 0 \end{bmatrix}\right) = Id_3 + h^{\delta/2}\begin{bmatrix} 0 & \nabla v \\ -\nabla v & 0 \end{bmatrix} - \frac{1}{2}h^\delta \begin{bmatrix} (\nabla v)^{\otimes 2} & 0 \\ 0 & |\nabla v|^2 \end{bmatrix} + \mathcal{O}(h^{\frac{3}{2}\delta}).$$

Above, the constants on $\mathcal{O}$ depend on $\|\nabla v\|_{L^\infty}$. Thus, we obtain for all $(x', x_3) \in \Omega^1$:

$$q^h \nabla u^h(x', hx_3) = Id_3 + \frac{1}{2}h^\delta \begin{bmatrix} (\nabla v)^{\otimes 2} & 0 \\ 0 & |\nabla v|^2 \end{bmatrix} + h^{1+\delta/2}\begin{bmatrix} \nabla w & 0 \\ 0 & 0 \end{bmatrix} - h^{1+\delta/2}x_3 \begin{bmatrix} \nabla^2 v & 0 \\ 0 & 0 \end{bmatrix}$$
$$+ h^{1+\delta/2}\begin{bmatrix} 0 & d + x_3\bar{d} \end{bmatrix} + \mathcal{O}(h^{2+\delta/2}),$$

where the bound in $\mathcal{O}$ depends on the $L^\infty(\omega)$ norms of: $\nabla v, \nabla^2 v, \nabla w, d, \bar{d}, \nabla d, \nabla \bar{d}$.

Recall that: $(A^h(x', hx_3))^{-1} = Id_3 - h^{\alpha/2}S - h^{1+\gamma/2}x_3 B + \mathcal{O}(h^{\alpha \wedge (2+\gamma)})$. Consequently:

$$\left(q^h (\nabla u^h)(A^h)^{-1}\right)(x', hx_3) = Id_3 - h^{\alpha/2}S - h^{1+\gamma/2}x_3 B + \frac{1}{2}h^\delta \begin{bmatrix} (\nabla v)^{\otimes 2} & 0 \\ 0 & |\nabla v|^2 \end{bmatrix}$$
$$+ h^{1+\delta/2}\begin{bmatrix} \nabla w & d \\ 0 & \end{bmatrix} - h^{1+\delta/2}x_3 \begin{bmatrix} \nabla^2 v & -\bar{d} \\ 0 & \end{bmatrix} + \mathcal{O}(h^{2+\delta/2}).$$

It follows that the integrand $W\left(q^h(\nabla u^h)(A^h)^{-1}\right)$ equals:

$$\mathcal{Q}_3\left(-h^{\alpha/2}S + \frac{1}{2}h^\delta \begin{bmatrix} (\nabla v)^{\otimes 2} & 0 \\ 0 & |\nabla v|^2 \end{bmatrix} + h^{1+\delta/2}\begin{bmatrix} \nabla w & d \\ 0 & \end{bmatrix}\right.$$
$$\left. - h^{1+\gamma/2}x_3 B - h^{1+\delta/2}x_3 \begin{bmatrix} \nabla^2 v & -\bar{d} \\ 0 & \end{bmatrix}\right) + o(h^{2+\delta}),$$

which yields:

$$\lim_{h \to 0}\frac{1}{h^{2+\delta}}I^h(u^h) = \frac{1}{24}\lim_{h \to 0}\int_\omega \mathcal{Q}_3\left(h^{\gamma/2-\delta/2}B + \begin{bmatrix} \nabla^2 v & -\bar{d} \\ 0 & \end{bmatrix}\right) \mathrm{d}x'$$
$$+ \frac{1}{2}\lim_{h \to 0}\int_\omega \mathcal{Q}_3\left(h^{\alpha/2-(1+\delta/2)}S - \frac{1}{2}h^{\delta/2-1}\begin{bmatrix} (\nabla v)^{\otimes 2} & 0 \\ 0 & |\nabla v|^2 \end{bmatrix} - \begin{bmatrix} \nabla w & d \\ 0 & \end{bmatrix}\right) \mathrm{d}x'.$$

Setting d and $\bar{d}$ to be affine functions of $\nabla w, (\nabla v)^{\otimes 2}, \nabla^2 v$ and S, B, so that $\mathcal{Q}_3$ above gets replaced by $\mathcal{Q}_2$ evaluated on the principal 2×2 minors of the respective arguments, we obtain the claimed convergence to the energy functional in (2.11):

$$\lim_{h \to 0}\frac{1}{h^{2+\delta}}I^h(u^h) = I(v, w). \tag{3.5}$$

Finally, we observe that given $v \in W^{2,2}(\omega, \mathbb{R})$ and $w \in W^{1,2}(\omega, \mathbb{R}^2)$, one can first find their smooth approximations v_n, w_n and construct the recovery sequence $\{u^h\}_{h \to 0}$ as the diagonal sequence given by formula in (3.4) with v, w replaced by v_n, w_n. Taking $n = n(h) \to \infty$ as $h \to 0$ at a sufficiently slow rate, guarantees the same limit as in (3.5).

3.3. **Case $S_{2\times 2} \equiv 0$ and $\alpha, \gamma \geq 2$.** Let $\delta \in [2, \alpha \wedge \gamma]$. Given $v \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R})$, $w \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^2)$, $d, \bar{d} \in \mathcal{C}^\infty(\bar{\omega}, \mathbb{R}^3)$, and denoting $s = (S_{31}, S_{32})$, we define $u^h \in \mathcal{C}^\infty(\bar{\Omega}^h, \mathbb{R}^3)$ by:

$$u^h = id_3 + h^{\alpha/2}x_3 \begin{bmatrix} 2s \\ S_{33} \end{bmatrix}$$
$$+ h^{\delta/2}\begin{bmatrix} 0 \\ v \end{bmatrix} + h^{1+\delta/2}\begin{bmatrix} w \\ 0 \end{bmatrix} - h^{\delta/2}x_3 \begin{bmatrix} \nabla v \\ 0 \end{bmatrix} + h^{1+\delta/2}x_3 d + \frac{1}{2}h^{\delta/2}x_3^2 \bar{d}. \tag{3.6}$$

As in subsection 3.2, we apply rotations $q^h = \exp\left(h^{\delta/2}\begin{bmatrix} 0 & \nabla v \\ -\nabla v & 0 \end{bmatrix}\right)$ and obtain on Ω^1:

$$q^h \nabla u^h(x', hx_3) = Id_3 + \frac{1}{2}h^\delta \begin{bmatrix} (\nabla v)^{\otimes 2} & 0 \\ 0 & |\nabla v|^2 \end{bmatrix} + h^{1+\delta/2}\begin{bmatrix} \nabla w & 0 \\ 0 & 0 \end{bmatrix} - h^{1+\delta/2}x_3\begin{bmatrix} \nabla^2 v & 0 \\ 0 & 0 \end{bmatrix}$$
$$+ h^{\alpha/2}\begin{bmatrix} 0 & 2s \\ & S_{33} \end{bmatrix} + h^{1+\alpha/2}x_3\begin{bmatrix} 2\nabla s & 0 \\ \nabla S_{33} & 0 \end{bmatrix} - h^{\delta/2+\alpha/2}\begin{bmatrix} 0 & S_{33}\nabla v \\ & -2\langle s, \nabla v\rangle \end{bmatrix}$$
$$+ h^{1+\delta/2}\begin{bmatrix} 0 & d+x_3\bar{d} \end{bmatrix} + \mathcal{O}(h^{2+\delta/2}),$$

where the bound in $\mathcal{O}$ depends on the $L^\infty(\omega)$ norms of: $\nabla v, \nabla^2 v, \nabla w, d, \bar{d}, \nabla d, \nabla\bar{d}$. We now apply a further rotation $\bar{q}^h = \exp\left(h^{\alpha/2}\begin{bmatrix} 0 & -s \\ s & 0 \end{bmatrix}\right)$ to get:

$$\bar{q}^h q^h \nabla u^h(x', hx_3) = Id_3 + \frac{1}{2}h^\delta\begin{bmatrix} (\nabla v)^{\otimes 2} & 0 \\ 0 & |\nabla v|^2 \end{bmatrix} + h^{1+\delta/2}\begin{bmatrix} \nabla w & 0 \\ 0 & 0 \end{bmatrix} - h^{1+\delta/2}x_3\begin{bmatrix} \nabla^2 v & 0 \\ 0 & 0 \end{bmatrix}$$
$$+ h^{\alpha/2}S + \frac{1}{2}h^{\alpha/2}\begin{bmatrix} 2s^{\otimes 2} & -2S_{33}s \\ 0 & 3|s|^2 \end{bmatrix} + h^{1+\alpha/2}x_3\begin{bmatrix} 2\nabla s & 0 \\ \nabla S_{33} & 0 \end{bmatrix} + h^{\delta/2+\alpha/2}\begin{bmatrix} 0 & S_{33}\nabla v \\ & -2\langle s, \nabla v\rangle \end{bmatrix}$$
$$+ h^{1+\delta/2}\begin{bmatrix} 0 & d+x_3\bar{d} \end{bmatrix} + \mathcal{O}(h^{2+\delta/2}),$$

By the second order expansion of the inverse: $(A^h(x', hx_3))^{-1} = Id_3 - h^{\alpha/2}S - h^{1+\gamma/2}x_3 B + h^\alpha S^2 + \mathcal{O}(h^{1+\delta})$, we obtain:

$$\left(\bar{q}^h q^h(\nabla u^h)(A^h)^{-1}\right)(x', hx_3) = Id_3 - h^{1+\gamma/2}x_3 B + \frac{1}{2}h^\delta\begin{bmatrix} (\nabla v)^{\otimes 2} & 0 \\ 0 & |\nabla v|^2 \end{bmatrix}$$
$$- \frac{1}{2}h^\alpha\begin{bmatrix} s^{\otimes 2} & 2S_{33}s \\ 0 & -2S_{33}^2 \end{bmatrix} + h^{1+\alpha/2}x_3\begin{bmatrix} 2\nabla s & 0 \\ \nabla S_{33} & 0 \end{bmatrix} + h^{\delta/2+\alpha/2}\begin{bmatrix} 0 & S_{33}\nabla v \\ 0 & -2\langle s, \nabla v\rangle \end{bmatrix}$$
$$+ h^{1+\delta/2}\begin{bmatrix} \nabla w & d \\ 0 & \end{bmatrix} - h^{1+\delta/2}x_3\begin{bmatrix} \nabla^2 v & -\bar{d} \\ 0 & \end{bmatrix} + \mathcal{O}(h^{2+\delta/2}).$$

It follows that the integrand $W\left(\bar{q}^h q^h(\nabla u^h)(A^h)^{-1}\right)$ equals:

$$\mathcal{Q}_3\Bigg(-\frac{1}{2}h^\alpha\begin{bmatrix} s^{\otimes 2} & S_{33}s \\ S_{33}s & -2S_{33}^2 \end{bmatrix} + h^{\delta/2+\alpha/2}\begin{bmatrix} 0 & \frac{1}{2}S_{33}\nabla v \\ \frac{1}{2}S_{33}\nabla v & -2\langle s, \nabla v\rangle \end{bmatrix}$$
$$+ \frac{1}{2}h^\delta\begin{bmatrix} (\nabla v)^{\otimes 2} & 0 \\ 0 & |\nabla v|^2 \end{bmatrix} + h^{1+\delta/2}\begin{bmatrix} \nabla w & d \\ 0 & \end{bmatrix}$$
$$+ h^{1+\alpha/2}x_3\begin{bmatrix} 2\,\text{sym}\nabla s & \frac{1}{2}\nabla S_{33} \\ \frac{1}{2}\nabla S_{33} & 0 \end{bmatrix} - h^{1+\gamma/2}x_3 B + h^{1+\delta/2}x_3\begin{bmatrix} -\nabla^2 v & \bar{d} \\ 0 & \end{bmatrix}\Bigg) + o\left(h^{2+\delta}\right),$$

which yields:

$$\lim_{h\to 0}\frac{1}{h^{2+\delta}}I^h(u^h) =$$
$$\frac{1}{24}\lim_{h\to 0}\int_\omega \mathcal{Q}_3\Bigg(h^{\alpha/2-\delta/2}\begin{bmatrix} 2\,\text{sym}\nabla s & \frac{1}{2}\nabla S_{33} \\ \frac{1}{2}\nabla S_{33} & 0 \end{bmatrix} + h^{\gamma/2-\delta/2}B + \begin{bmatrix} -\nabla^2 v & \bar{d} \\ 0 & \end{bmatrix}\Bigg)\,\mathrm{d}x'$$
$$+ \frac{1}{2}\lim_{h\to 0}\int_\omega \mathcal{Q}_3\Bigg(-\frac{1}{2}h^{\alpha-(1+\delta/2)}\begin{bmatrix} s^{\otimes 2} & S_{33}s \\ S_{33}s & -2S_{33}^2 \end{bmatrix} + h^{\alpha/2-1}\begin{bmatrix} 0 & \frac{1}{2}S_{33}\nabla v \\ \frac{1}{2}S_{33}\nabla v & -2\langle s, \nabla v\rangle \end{bmatrix}$$
$$+ \frac{1}{2}h^{\delta/2-1}\begin{bmatrix} (\nabla v)^{\otimes 2} & 0 \\ 0 & |\nabla v|^2 \end{bmatrix} + \begin{bmatrix} \nabla w & d \\ 0 & \end{bmatrix}\Bigg)\,\mathrm{d}x'.$$

In each of the cases of ordering α, γ and δ, we may set d and $\bar{d}$ to be affine functions of ∇w, $(\nabla v)^{\otimes 2}$, $\nabla^2 v$ and S, B, so that $\mathcal{Q}_3$ above gets replaced by $\mathcal{Q}_2$ evaluated on the principal 2×2 minors of the respective arguments. Thus, we obtain the claimed convergence to the energy functional in (2.13) in case when the displacements v and w are smooth. The general case $v \in W^{2,2}(\omega, \mathbb{R})$ and $w \in W^{1,2}(\omega, \mathbb{R}^2)$ follows by a diagonal argument as in subsection 3.2.

4. Discussion of $\inf I$ and the optimality conditions

We start by proving two decomposition results for matrices. Recall that given $F \in \mathbb{R}^{2\times 2}$, its cofactor matrix is: $\operatorname{cof} F = \begin{bmatrix} F_{22} & -F_{12} \\ -F_{21} & F_{11} \end{bmatrix}$, while for fields $w : \omega \to \mathbb{R}^2$ and $\alpha : \omega \to \mathbb{R}$ we denote: $\operatorname{curl} w = \partial_1 w_2 - \partial_2 w_1$ and $\nabla^\perp \alpha = (-\partial_2 \alpha, \partial_1 \alpha)$. The congruency symbol in: $a \cong b$ means that $a \leq Cb$ and $b \leq Ca$ with a constant C depending only on ω.

Lemma 4.1. *Let $\omega \subset \mathbb{R}^2$ be an open, bounded, simply connected domain with Lipschitz boundary. For every $F \in L^2(\omega, \mathbb{R}^{2\times 2}_{sym})$ there exist unique $v \in W^{2,2}(\omega, \mathbb{R})$, $\phi \in W^{1,2}_0(\omega, \mathbb{R}^2)$ satisfying:*

$$F = \nabla^2 v + \operatorname{cof} \operatorname{sym} \nabla \phi.$$

Moreover, there hold the following equivalences with $\operatorname{curl} F$ (taken row-wise):

$$\|F - \nabla^2 v\|_{L^2} = \operatorname{dist}_{L^2}\big(F, \{\nabla^2 r;\ r \in W^{2,2}(\omega, \mathbb{R})\}\big)$$
$$\cong \|\operatorname{curl} F\|_{H^{-1}(\omega)} = \operatorname{dist}_{L^2}\big(F, \{\nabla w;\ w \in W^{1,2}(\omega, \mathbb{R}^2)\}\big).$$

Proof. Since the linear space $\{\operatorname{cof} \operatorname{sym} \nabla \phi;\ \phi \in W^{1,2}_0(\omega, \mathbb{R}^2)\}$ is a closed subspace of $L^2(\omega, \mathbb{R}^{2\times 2})$, the following minimization problem has the unique solution:

$$\text{minimize} \Big\{ \int_\omega |F - \operatorname{cof} \operatorname{sym} \nabla \phi|^2 \, \mathrm{d}x';\ \phi \in W^{1,2}_0(\omega, \mathbb{R}^2)\Big\},$$

identified as the solution to the Euler-Lagrange equation:

$$\int_\omega \langle F : \operatorname{cof} \operatorname{sym} \nabla \alpha \rangle \, \mathrm{d}x' = \int_\omega \langle \operatorname{sym} \nabla \phi : \operatorname{sym} \nabla \alpha \rangle \, \mathrm{d}x' \qquad \text{for all }\ \alpha \in W^{1,2}_0(\omega, \mathbb{R}^2). \qquad (4.1)$$

By Korn's inequality, the right hand side above is a scalar product on $W^{1,2}_0(\omega, \mathbb{R}^2)$. A basic application of the Riesz representation theorem yields existence of the unique solution ϕ, and:

$$\|\operatorname{sym} \nabla \phi\|_{L^2} = \sup \Big\{ \int_\omega \langle F : \operatorname{cof} \operatorname{sym} \nabla \alpha \rangle \, \mathrm{d}x';\ \alpha \in W^{1,2}_0(\omega, \mathbb{R}^2),\ \|\operatorname{sym} \nabla \alpha\|_{L^2} \leq 1\Big\}.$$

Observing that $\langle F : \operatorname{cof} \operatorname{sym} \nabla \alpha \rangle = \langle F : \operatorname{cof} \nabla \alpha \rangle = \big\langle F : \begin{bmatrix} \nabla^\perp(-\alpha_2) \\ \nabla^\perp \alpha_1 \end{bmatrix} \big\rangle$ and using Korn's inequality again, we get:

$$\|\operatorname{sym} \nabla \phi\|_{L^2} \cong \sup \Big\{ \int_\omega \big\langle F : \begin{bmatrix} \nabla^\perp \alpha_1 \\ \nabla^\perp \alpha_2 \end{bmatrix} \big\rangle \, \mathrm{d}x';\ \alpha \in W^{1,2}_0(\omega, \mathbb{R}^2),\ \|\nabla \alpha\|_{L^2} \leq 1\Big\}$$
$$\cong \sup \Big\{ \int_\omega \langle (F_{11}, F_{12}), \nabla^\perp \alpha_1 \rangle;\ \alpha_1 \in W^{1,2}_0(\omega, \mathbb{R}),\ \|\nabla \alpha_1\|_{L^2} \leq 1\Big\}$$
$$+ \sup \Big\{ \int_\omega \langle (F_{21}, F_{22}), \nabla^\perp \alpha_2 \rangle;\ \alpha_2 \in W^{1,2}_0(\omega, \mathbb{R}),\ \|\nabla \alpha_2\|_{L^2} \leq 1\Big\}$$
$$\cong \|\operatorname{curl}(F_{11}, F_{12})\|_{H^{-1}(\omega)} + \|\operatorname{curl}(F_{21}, F_{22})\|_{H^{-1}(\omega)}$$
$$\cong \|\operatorname{curl} F\|_{H^{-1}(\omega)} \cong \operatorname{dist}_{L^2}\big(F, \{\nabla w;\ w \in W^{1,2}(\omega, \mathbb{R}^2)\}\big).$$

Finally, from (4.1) we deduce that:

$$\int_\omega \left\langle F - \operatorname{cof} \operatorname{sym} \nabla\phi : \left[\begin{array}{c} \nabla^\perp \alpha_1 \\ \nabla^\perp \alpha_2 \end{array} \right] \right\rangle \, \mathrm{d}x' = 0 \qquad \text{for all } \alpha_1, \alpha_2 \in W_0^{1,2}(\omega, \mathbb{R}).$$

Hence, by de Rham's theorem there must be: $F - \operatorname{cof} \operatorname{sym} \nabla\phi = \nabla w$ for some $w \in W^{1,2}(\omega, \mathbb{R}^2)$. Since $F - \operatorname{cof} \operatorname{sym} \nabla\phi$ is symmetric, we get that $\nabla w = \nabla^2 v$ for some $v \in W^{2,2}(\omega, \mathbb{R})$. ∎

Lemma 4.2. *Let* $\omega \subset \mathbb{R}^2$ *be open, bounded, simply connected, with Lipschitz boundary. For every* $F \in L^2(\omega, \mathbb{R}_{sym}^{2\times 2})$, *there exist unique* $r \in W_0^{2,2}(\omega, \mathbb{R})$ *and* $w \in W^{1,2}(\omega, \mathbb{R}^2)$ *such that:*

$$F = \operatorname{cof} \nabla^2 r + \operatorname{sym} \nabla w.$$

Moreover, there hold the following equivalences with the scalar field $\operatorname{curl}^T \operatorname{curl} F$:

$$\|F - \operatorname{sym} \nabla w\|_{L^2} = \operatorname{dist}_{L^2}\big(F, \{\operatorname{sym} \nabla\phi; \ \phi \in W^{1,2}(\omega, \mathbb{R}^2)\}\big) \cong \|\operatorname{curl}^T \operatorname{curl} F\|_{H^{-2}(\omega)}.$$

Proof. Similarly to the proof of Lemma 4.1, we consider the minimization problem:

$$\text{minimize} \Big\{ \int_\omega |F - \operatorname{cof} \nabla^2 r|^2 \, \mathrm{d}x'; \ r \in W_0^{2,2}(\omega, \mathbb{R}) \Big\},$$

whose unique solution is given through the orthogonal projection on the closed subspace $\{\operatorname{cof} \nabla^2 r; \ r \in W_0^{2,2}(\omega, \mathbb{R})\}$ of $L^2(\omega, \mathbb{R}_{sym}^{2\times 2})$. Equivalently, the solution r satisfies:

$$\int_\omega \langle F : \operatorname{cof} \nabla^2 \alpha \rangle \, \mathrm{d}x' = \int_\omega \langle \nabla^2 r : \nabla^2 \alpha \rangle \, \mathrm{d}x' \qquad \text{for all } \alpha \in W_0^{2,2}(\omega, \mathbb{R}),$$

and we get:

$$\|\nabla r\|_{L^2}^2 = \sup \Big\{ \int_\omega \langle F : \operatorname{cof} \nabla^2 \alpha \rangle \, \mathrm{d}x'; \ \alpha \in W_0^{2,2}(\omega, \mathbb{R}), \ \|\nabla^2 \alpha\|_{L^2} \leq 1 \Big\} = \|\operatorname{curl}^T \operatorname{curl} F\|_{H^{-2}(\omega)}.$$

The last equality above follows by observing that for all $\alpha \in \mathcal{C}_c^\infty(\omega)$ there holds:

$$\int_\omega \big(\operatorname{curl}^T \operatorname{curl} F\big)\alpha \, \mathrm{d}x' = -\int_\omega \langle \operatorname{curl} F, \nabla^\perp \alpha \rangle \, \mathrm{d}x'$$

$$= \int_\omega \big\langle (F_{11}, F_{12}), \nabla^\perp(-\partial_2 \alpha) \big\rangle \, \mathrm{d}x' + \int_\omega \big\langle (F_{21}, F_{22}), \nabla^\perp(\partial_1 \alpha) \big\rangle \, \mathrm{d}x' \qquad (4.2)$$

$$= \int_\omega \big\langle F : \operatorname{cof} \nabla^2 \alpha \big\rangle \, \mathrm{d}x'.$$

Denoting $\bar{F} = F - \operatorname{cof} \nabla^2 r \in L^2(\omega, \mathbb{R}_{sym}^{2\times 2})$, it thus follows that:

$$\int_\omega \big\langle \bar{F} : \operatorname{cof} \nabla^2 \alpha \big\rangle \, \mathrm{d}x' = 0 \qquad \text{for all } \alpha \in W_0^{2,2}(\omega, \mathbb{R}).$$

As in (4.2), we deduce that: $\operatorname{curl}^T \operatorname{curl} \bar{F} = 0$ in distributions. Hence, for some $v \in L^2(\omega, \mathbb{R})$:

$$\operatorname{curl} \bar{F} = \nabla v = \operatorname{curl}(\operatorname{skew}_2 v), \qquad \text{where: } \operatorname{skew}_2 v = \left[\begin{array}{cc} 0 & v \\ -v & 0 \end{array} \right].$$

Consequently: $\operatorname{curl}\big(\bar{F} + \operatorname{skew}_2 v\big) = 0$, so there exists $w \in W^{1,2}(\omega, \mathbb{R}^2)$ satisfying:

$$\bar{F} + \operatorname{skew}_2 v = \nabla w.$$

Since $\bar{F}$ is symmetric, this yields: $\bar{F} = \operatorname{sym} \nabla w$. The proof is done. ∎

Proof of Proposition 1.3

1. For $v \in W^{2,2}(\omega, \mathbb{R})$ we have: $\det \nabla^2 v = \det(\nabla^2 v + \bar{B}) + \det \bar{B} - \langle \text{cof} \bar{B} : \nabla^2 v + \bar{B} \rangle$, so:

$$\| \det \nabla^2 v + \text{curl}^T \text{curl} \, \bar{S} \|_{H^{-2}(\omega)} \leq \| \det \bar{B} + \text{curl}^T \text{curl} \, \bar{S} \|_{H^{-2}(\omega)}$$
$$+ \| \det(\nabla^2 v + \bar{B}) \|_{H^{-2}(\omega)} + \| \langle \text{cof} \bar{B} : \nabla^2 v + \bar{B} \rangle \|_{H^{-2}(\omega)}.$$

Observe that:

$$\| \det(\nabla^2 v + \bar{B}) \|_{H^{-2}(\omega)} \leq C \| \nabla^2 v + \bar{B} \|_{L^2(\omega)}^2,$$
$$\| \langle \text{cof} \bar{B} : \nabla^2 v + \bar{B} \rangle \|_{H^{-2}(\omega)} \leq C \| \bar{B} \|_{L^\infty(\omega)} \cdot \| \nabla^2 v + \bar{B} \|_{L^2(\omega)}, \tag{4.3}$$

which implies:

$$\bar{I}_0(v) \leq C \| \nabla^2 v + \bar{B} \|_{L^2(\omega)}^2 \Big(1 + \| \bar{B} \|_{L^\infty(\omega)}^2 + \| \nabla^2 v + \bar{B} \|_{L^2(\omega)}^2 \Big) + C \| \det \bar{B} + \text{curl}^T \text{curl} \, \bar{S} \|_{H^{-2}(\omega)}^2.$$

The upper bound on $\inf \bar{I}_0$ follows now by infimizing the right hand side expression with respect to v, and applying Lemma 4.1.

2. For the lower bound, we use (4.3) to get:

$$\| \det \nabla^2 v + \text{curl}^T \text{curl} \, \bar{S} \|_{H^{-2}(\omega)}^2 \geq \frac{1}{2} \| \det \bar{B} + \text{curl}^T \text{curl} \, \bar{S} \|_{H^{-2}(\omega)}$$
$$- c \| \nabla^2 v + \bar{B} \|_{L^2(\omega)}^2 \big(\| \bar{B} \|_{L^\infty(\omega)}^2 + \| \nabla^2 v + \bar{B} \|_{L^2(\omega)}^2 \big).$$

The established upper bound yields along a minimizing sequence $\{v_n\}_{n \to \infty}$ of $\bar{I}_0$:

$$\| \nabla^2 v_n + \bar{B} \|_{L^2(\omega)}^2 \leq C \| \text{curl} \, \bar{B} \|_{H^{-1}(\omega)}^2 \big(1 + \| \bar{B} \|_{L^\infty(\omega)}^2 + \| \text{curl} \, \bar{B} \|_{H^{-1}(\omega)}^2 \big)$$
$$+ C \| \det \bar{B} + \text{curl}^T \text{curl} \, \bar{S} \|_{H^{-2}(\omega)}^2,$$

so consequently, for every $\varepsilon \leq 1$ there holds:

$$\inf \bar{I}_0 = \lim_{n \to \infty} \bar{I}_0(v_n)$$
$$\geq \| \nabla^2 v_n + \bar{B} \|_{L^2(\omega)}^2 \Big(1 - \varepsilon \big(\| \bar{B} \|_{L^\infty(\omega)}^2 + \| \det \bar{B} + \text{curl}^T \text{curl} \, \bar{S} \|_{H^{-2}(\omega)}^2$$
$$+ \| \text{curl} \, \bar{B} \|_{H^{-1}(\omega)}^2 \big(1 + \| \bar{B} \|_{L^\infty(\omega)}^2 + \| \text{curl} \, \bar{B} \|_{H^{-1}(\omega)}^2 \big) \big) \Big)$$
$$+ c\varepsilon \| \det \bar{B} + \text{curl}^T \text{curl} \, \bar{S} \|_{H^{-2}(\omega)}^2,$$

The result now follows by invoking Lemma 4.1 and taking 2ε to be the minimum of 1 and the inverse of the expression: $1/a = \| \bar{B} \|_{L^\infty(\omega)}^2 + \| \det \bar{B} + \text{curl}^T \text{curl} \, \bar{S} \|_{H^{-2}(\omega)}^2 + \| \text{curl} \, \bar{B} \|_{H^{-1}(\omega)}^2 \big(1 + \| \bar{B} \|_{L^\infty(\omega)}^2 + \| \text{curl} \, \bar{B} \|_{H^{-1}(\omega)}^2 \big)$. This completes the proof. ∎

5. Connection to curvature

In this section, we compute the Riemann curvatures in (1.6) and (1.7). Recall that the Riemannian metrics on Ω^h, induced by the prestrain tensors A^h, are:

$$\mathcal{G}^h = (A^h)^T A^h = Id_3 + 2h^{\alpha/2} S + h^\alpha S^2 + x_3 \big(2h^{\gamma/2} B + 2h^{(\alpha+\gamma)/2} \text{sym}(SB) \big) + x_3^2 h^\gamma B^2.$$

The Christoffel symbols of $\mathcal{G}^h$ are gathered in matrices $\{\Gamma_a\}_{a=1...3}$, where:

$$\Gamma_a = [\Gamma_{ac}^b]_{b,c=1...3}, \qquad \Gamma_{ac}^b = \frac{1}{2} \mathcal{G}^{bm} \big(\partial_a \mathcal{G}_{mc} + \partial_c \mathcal{G}_{ma} - \partial_m \mathcal{G}_{ac} \big).$$

Writing the Riemann curvatures as $\{R_{\cdot,cd}\}_{c,d=1\ldots3}$ and $\{R_{\cdot\cdot,cd}\}_{c,d=1\ldots3}$, so that $R_{\cdot,cd} = [R^a_{b,cd}]_{a,b=1\ldots3}$ and $R_{\cdot\cdot,cd} = [R_{ab,cd}]_{a,b=1\ldots3}$, recall that:

$$R_{\cdot,cd} = \big(\partial_c\Gamma_d - \partial_d\Gamma_c\big) + \big(\Gamma_c\Gamma_d - \Gamma_d\Gamma_c\big), \qquad R_{\cdot\cdot,cd} = \mathcal{G}^h R_{\cdot,cd}$$

We now compute $R_{\cdot\cdot,cd}$ at $x_3 = 0$, in the two cases studied in this paper.

5.1. **Case $\alpha \geq 4$, $\gamma \geq 2$.** We directly compute for all $a, b, c = 1\ldots2$:

$$\Gamma^b_{ac} = h^{\alpha/2}\big(\partial_a S_{bc} + \partial_c S_{ab} - \partial_b S_{ac}\big) + h^{\gamma/2}x_3\big(\partial_a B_{bc} + \partial_c B_{ab} - \partial_b B_{ac}\big) + 2h^\gamma x_3 B_{b3}B_{ac} + e,$$

$$\Gamma^3_{ac} = h^{\alpha/2}\big(\partial_a S_{c3} + \partial_c S_{a3}\big) - h^{\gamma/2}B_{ac} + h^{\gamma/2}x_3\big(\partial_a B_{c3} + \partial_c B_{a3}\big)$$
$$- h^\gamma x_3\big((B^2)_{ac} - 2B_{33}B_{ac}\big) + e,$$

$$\Gamma^b_{a3} = h^{\alpha/2}\big(\partial_a S_{3b} - \partial_b S_{a3}\big) + h^{\gamma/2}B_{ab} + h^{\gamma/2}x_3\big(\partial_a B_{b3} - \partial_b B_{a3}\big)$$
$$- h^\gamma x_3\big((B^2)_{ab} - 2B_{b3}B_{a3}\big) + e,$$

$$\Gamma^3_{a3} = h^{\alpha/2}\partial_a S_{33} + h^{\gamma/2}\partial_a B_{33} - h^\alpha x_3\big((B^2)_{a3} - B_{33}B_{a3}\big) + e,$$

$$\Gamma^b_{33} = 2h^{\gamma/2}B_{b3} - h^{\alpha/2}\partial_b S_{33} - h^{\gamma/2}x_3\partial_b B_{33} - 2h^\gamma x_3\big((B^2)_{b3} - B_{33}B_{b3}\big) + e,$$

$$\Gamma^3_{33} = h^{\gamma/2}B_{33} - h^\gamma x_3\big(3(B^2)_{33} - 2B^2_{33}\big) + e,$$

where e denotes the error terms of order $\mathcal{O}\big(h^{\alpha\wedge(\alpha+\gamma)/2} + h^{(\alpha+\gamma)/2}x_3 + h^\gamma x_3^2\big)$.

- Consequently, we obtain the following skew-symmetric matrix fields at $x_3 = 0$, whose lowest order terms are:

$$\partial_1\Gamma_2 - \partial_2\Gamma_1 \simeq \begin{bmatrix} 0 & -h^{\gamma/2}\mathrm{curl}^T\mathrm{curl}\, S_{2\times2} & -h^{\alpha/2}\partial_1\mathrm{curl}(S_{13}, S_{23}) + h^{\gamma/2}(\mathrm{curl}B_{2\times2})_1 \\ \cdot & 0 & -h^{\alpha/2}\partial_2\mathrm{curl}(S_{13}, S_{23}) + h^{\gamma/2}(\mathrm{curl}B_{2\times2})_2 \\ \cdot & \cdot & 0 \end{bmatrix},$$

$$\Gamma_1\Gamma_2 - \Gamma_2\Gamma_1 \simeq \begin{bmatrix} 0 & -h^\gamma\det B_{2\times2} & 0 \\ \cdot & 0 & 0 \\ \cdot & \cdot & 0 \end{bmatrix}.$$

Thus, the lowest order terms of the curvatures computed at $x_3 = 0$ are:

$$R_{\cdot,12} \simeq \begin{bmatrix} \begin{bmatrix} 0 & -h^{\alpha/2}\mathrm{curl}^T\mathrm{curl}\, S_{2\times2} - h^\gamma\det B_{2\times2}) \\ \cdot & 0 \end{bmatrix} & -h^{\alpha/2}\nabla'\mathrm{curl}(S_{13}, S_{23}) + h^{\gamma/2}\mathrm{curl}B_{2\times2} \\ \cdot & 0 \end{bmatrix},$$

where we denoted: $\nabla' = (\partial_1, \partial_2)$.

- For every $a = 1\ldots2$ we have, at $x_3 = 0$, with the same notation as above:

$$\big(\partial_a\Gamma_3 - \partial_3\Gamma_a\big)_{2\times2} \simeq -2h^\gamma\begin{bmatrix} B_{13}B_{a1} & B_{13}B_{a2} \\ B_{23}B_{a1} & B_{23}B_{a2} \end{bmatrix} + \begin{bmatrix} 0 & -h^{\alpha/2}\partial_a\mathrm{curl}(S_{13}, S_{23}) + h^{\gamma/2}(\mathrm{curl}B_{2\times2})_a \\ \cdot & 0 \end{bmatrix},$$

$$\big((\partial_a\Gamma_3 - \partial_3\Gamma_a)_{13}, (\partial_a\Gamma_3 - \partial_3\Gamma_a)_{23}\big) \simeq h^\gamma\big(((B^2)_{a1}, (B^2)_{a2}) - 2B_{a3}(B_{13}, B_{23})\big),$$
$$- h^{\alpha/2}\partial_a\nabla'S_{33} + h^{\gamma/2}\big(\nabla'B_{a3} + \partial_a(B_{13}, B_{23})\big),$$

$$\big((\partial_a\Gamma_3 - \partial_3\Gamma_a)_{31}, (\partial_a\Gamma_3 - \partial_3\Gamma_a)_{32}\big) \simeq h^\gamma\big(((B^2)_{a1}, (B^2)_{a2}) - 2B_{33}(B_{a1}, B_{a2})\big),$$
$$+ h^{\alpha/2}\partial_a\nabla'S_{33} - h^{\gamma/2}\big(\nabla'B_{a3} + \partial_a(B_{13}, B_{23})\big)$$

$$\big(\partial_a\Gamma_3 - \partial_3\Gamma_a\big)_{33} \simeq 2h^\gamma\big(B_{13}B_{a1} + B_{23}B_{a2}\big).$$

We further have:

$$\left(\Gamma_a\Gamma_3 - \Gamma_3\Gamma_a\right)_{2\times2} \simeq 2h^\gamma \begin{bmatrix} B_{13}B_{a1} & B_{13}B_{a2} \\ B_{23}B_{a1} & B_{23}B_{a2} \end{bmatrix},$$

$$\left(\left(\Gamma_a\Gamma_3 - \Gamma_3\Gamma_a\right)_{13}, \left(\Gamma_a\Gamma_3 - \Gamma_3\Gamma_a\right)_{23}\right) = \left(\left(\Gamma_a\Gamma_3 - \Gamma_3\Gamma_a\right)_{31}, \left(\Gamma_a\Gamma_3 - \Gamma_3\Gamma_a\right)_{32}\right)$$
$$\simeq h^\gamma\left(B_{33}(B_{a1}, B_{a2}) - B_{2\times2}(B_{a1}, B_{a2})\right)$$

$$\left(\Gamma_a\Gamma_3 - \Gamma_3\Gamma_a\right)_{33} \simeq -2h^\gamma\left(B_{13}B_{a1} + B_{23}B_{a2}\right).$$

We thus obtain the corresponding lowest order terms of curvatures, computed at $x_3 = 0$:

$$R_{\cdot,a3}^{\cdot} \simeq \begin{bmatrix} \begin{bmatrix} 0 & -h^{\alpha/2}\partial_a\mathrm{curl}\,S_{2\times2} + h^{\gamma/2}\left(\mathrm{curl}B_{2\times2}\right)_a \\ \cdot & 0 \end{bmatrix} & v \\ \cdot & 0 \end{bmatrix}, \quad \text{where:}$$
$$v = -h^{\alpha/2}\nabla'\partial_a S_{33} + h^{\gamma/2}\left(\nabla'B_{a3} + \partial_a(B_{13}, B_{23})\right).$$

Finally, we observe that $R_{ab,cd} \simeq R^a_{b,cd}$ at $x_3 = 0$.

5.2. **Case $\mathbf{S_{2\times 2} \equiv 0}$ and $\alpha, \gamma \geq 2$.** We compute for all $a, b, c = 1 \ldots 2$:

$$\Gamma^b_{ac} = h^\alpha\Big(\frac{1}{2}S_{3c}(\partial_a S_{3b} - \partial_b S_{3a}) + \frac{1}{2}S_{3a}(\partial_c S_{3b} - \partial_b S_{3c}) - \frac{3}{2}S_{3b}(\partial_a S_{3c} + \partial_c S_{3a})\Big) + 2h^{(\alpha+\gamma)/2}S_{b3}B_{ac}$$
$$+ h^{\gamma/2}x_3\big(\partial_a B_{bc} + \partial_c B_{ab} - \partial_b B_{ac}\big) + 2h^\gamma x_3 B_{b3}B_{ac}$$
$$+ h^{(\alpha+\gamma)/2}x_3\Big(\partial_a\mathrm{sym}(SB)_{bc} + \partial_c\mathrm{sym}(SB)_{ab} - \partial_b\mathrm{sym}(SB)_{ac}$$
$$- 2S_{b3}(\partial_a B_{3c} + \partial_c B_{a3}) - 2B_{b3}(\partial_a S_{3c} + \partial_c S_{a3})\Big) + e,$$

$$\Gamma^3_{ac} = h^{\alpha/2}\big(\partial_a S_{c3} + \partial_c S_{a3}\big) + h^\alpha\Big(\frac{1}{2}S_{3c}\partial_a S_{33} + S_{3a}\partial_c S_{33} - \frac{3}{2}S_{33}(\partial_a S_{3c} + \partial_c S_{3a})\Big)$$
$$- h^{\gamma/2}B_{ac} + h^{(\alpha+\gamma)/2}\big(-\mathrm{sym}(SB)_{ac} + B_{33}B_{ac}\big)$$
$$+ h^{\gamma/2}x_3\big(\partial + aB_{3c} + \partial_c B_{3a}\big) + h^\gamma x_3\big(-(B^2)_{ac} + B_{33}B_{ac}\big)$$
$$+ h^{(\alpha+\gamma)/2}x_3\Big(\partial_a\mathrm{sym}(SB)_{3c} + \partial_c\mathrm{sym}(SB)_{3a} - 2(S\partial_a B)_{3c} - 2(A\partial_c B)_{3a}$$
$$+ 2\langle(S_{31}, S_{32}), \nabla'B_{ac}\rangle - 2B_{33}(\partial_a S_{3c} + \partial_c S_{3a})\Big) + e,$$

$$\Gamma^b_{a3} = h^{\alpha/2}\big(\partial_a S_{3b} - \partial_b S_{a3}\big) + h^{\gamma/2}B_{ab}$$
$$+ h^\alpha\Big(\frac{1}{2}S_{33}(\partial_a S_{b3} - \partial_b S_{a3}) - \frac{3}{2}S_{b3}\partial_a S_{33} - \frac{1}{2}S_{a3}\partial_b S_{33}\Big) + h^{(\alpha+\gamma)/2}\mathrm{sym}(SB)_{ab}$$
$$+ h^{\gamma/2}x_3\big(\partial_a B_{b3} - \partial_b B_{a3}\big) - h^\gamma x_3\big((B^2)_{ab} - 2B_{b3}B_{a3}\big)$$
$$+ h^{(\alpha+\gamma)/2}x_3\Big(\partial_a\mathrm{sym}(SB)_{b3} - \partial_b\mathrm{sym}(SB)_{a3}$$
$$- 2(S\partial_a B)_{b3} - 2B_{b3}\partial_a S_{33} + 2\langle(B_{b1}, B_{b2}), \nabla'S_{a3}\rangle\Big) + e,$$

$$\Gamma^3_{a3} = h^{\alpha/2}\partial_a S_{33} + h^\alpha\Big(\frac{1}{2}((\partial_a S)S)_{33} - \frac{3}{2}(S\partial_a S)_{33} + 2\langle(S_{31}, S_{32}), \partial'S_{a3}\rangle\Big)$$
$$+ 2h^{(\alpha+\gamma)/2}\big(-(SB)_{3a} + S_{33}B_{3a}\big)$$
$$+ h^{\gamma/2}x_3\partial_a B_{33} + 2h^\gamma x_3\big(-(B^2)_{3a} + B_{33}B_{3a}\big)$$
$$+ h^{(\alpha+\gamma)/2}x_3\Big(\partial_a\mathrm{sym}(SB)_{33} - 2(S\partial_a B)_{33}$$
$$- 2(B\partial_a S)_{33} + 2\langle(S_{31}, S_{32}), \nabla'B_{a3}\rangle + \langle(B_{31}, B_{32}), \nabla'S_{a3}\rangle\Big) + e,$$

$$\Gamma^b_{33} = 2h^{\gamma/2}B_{b3} - h^{\alpha/2}\partial_b S_{33} - \frac{1}{2}h^\alpha\partial_b(S^2)_{33} + 2h^{(\alpha+\gamma)/2}\big(\mathrm{sym}(SB)_{b3} - S_{b3}B_{33}\big)$$
$$- h^{\gamma/2}x_3\partial_b B_{33} - 2h^\gamma x_3\big((B^2)_{b3} - B_{33}B_{b3}\big)$$
$$+ h^{(\alpha+\gamma)/2}x_3\Big(-\partial_b\mathrm{sym}(SB)_{33} + \langle(B_{31}, B_{32}), \nabla'S_{33}\rangle\Big) + e,$$

$$\Gamma^3_{33} = h^{\gamma/2}B_{33} + 2h^\alpha\langle(S_{31}, S_{32}), \nabla'S_{33}\rangle + h^{(\alpha+\gamma)/2}\big(\mathrm{sym}(SB)_{33} - 4(SB)_{33} + 2S_{33}B_{33}\big)$$
$$+ h^\gamma x_3\big(-3(B^2)_{33} - 2B_{33}^2\big)$$
$$+ 2h^{(\alpha+\gamma)/2}x_3\Big(\langle(S_{31}, S_{32}), \nabla'B_{33}\rangle + \langle(B_{31}, B_{32}), \nabla'S_{33}\rangle\Big) + e,$$

where e denotes the error terms of order: $\mathcal{O}\big(h^{\alpha+(\alpha\wedge\gamma)/2} + h^{(\alpha+\gamma)/2+(\alpha\wedge\gamma)/2}x_3 + h^\gamma x_3^2\big)$.

- Proceeding as in subsection 5.1, we obtain the following expressions for the lowest order terms of tangential curvatures at $x_3 = 0$, where we denote $s = (S_{31}, S_{32})$:

$$R_{12,12} \simeq h^\alpha \big(-3 \det \nabla' s + \langle \nabla' \operatorname{curl} s, s^\perp \rangle \big) - h^\gamma \det B_{2\times 2} + 2h^{(\alpha+\gamma)/2} \langle B_{2\times 2} : \operatorname{cof} \nabla' s \rangle,$$

$$(R_{13,12}, R_{23,12}) \simeq -h^{\alpha/2} \nabla' \operatorname{curl} s + h^{\gamma/2} \operatorname{curl} B_{2\times 2}.$$

It is also instructive to directly check that:

$$4 \det \big(\operatorname{sym} \nabla' s \big) + \frac{1}{2} \operatorname{curl}^T \operatorname{curl} (S_{31}, S_{32})^{\otimes 2} = 3 \det \nabla' s - \langle \nabla' \operatorname{curl} s, s^\perp \rangle,$$

which justifies the equivalence of the two conditions in Theorem 1.8 (i) when $\alpha = \gamma = 2$.

- The lowest order terms of the remaining curvatures are contained in the following skew-symmetric matrix field at $x_3 = 0$, where $a = 1 \dots 2$:

$$R_{\cdots,a3} \simeq \left[\begin{array}{cc} \left[\begin{array}{cc} 0 & -h^{\alpha/2}\partial_a \operatorname{curl}(S_{31}, S_{32}) + h^{\gamma/2} \big(\operatorname{curl} B_{2\times 2} \big)_a \\ \cdot & 0 \end{array} \right] & \begin{array}{c} v \\ \end{array} \\ \cdot & 0 \end{array} \right], \quad \text{where:}$$

$$v = -h^{\alpha/2} \nabla' \partial_a S_{33} + h^{\gamma/2} \big(\nabla' B_{a3} + \partial_a (B_{13}, B_{23}) \big).$$

We observe that the non-tangential curvatures above are the same as in subsection 5.1.

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