Shift Differentials of Maps in BV Spaces.

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1 Introduction

Aim of this note is to provide a brief outline of the theory of shift-differentials, introduced in [2], and show how their construction can be extended to the case of vector valued functions.

In the following, we consider the space BV of scalar integrable functions having bounded variation, endowed with the $\mathbf{L}^1$ norm. We recall that, given a map $\Phi : X \mapsto Y$ between normed linear spaces, its differential at a point x_0 is the linear map $\Lambda : X \mapsto Y$ such that

$$\lim_{h\to 0} \frac{\|\Phi(x_0+h) - \Phi(x_0) - \Lambda(h)\|_Y}{\|h\|_X} = 0. \tag{1.1}$$

This concept of differential (see for example [6]) is one of the cornerstone of mathematical analysis, providing a basic tool in the study of regular maps. For maps which do not admit a first-order linear approximation, various concepts of weak or generalized differential can be found in the literature [4] [7] [9] [11]. The present paper intends to provide some further contribution in this direction.

The primary motivation for the introduction of shift differentials comes from the theory of hyperbolic conservation laws [8] [10]. As a simple example, consider Burgers' equation

$$u_t + [u^2/2]_x = 0 \tag{1.2}$$

with the family of initial conditions

$$u^\theta(0,x) = \theta x \cdot \chi_{[0,1]}(x). \tag{1.3}$$

By $\chi_{_I}$ we denote here the characteristic function of the interval I. Assuming $\theta > 0$, the corresponding solution of (1.2)-(1.3) is

$$u^\theta(t,x) = \frac{\theta x}{1+\theta t} \cdot \chi_{_{[0,\ \sqrt{1+\theta t}]}}(x). \tag{1.4}$$

Observe that the map $\theta \mapsto u^\theta(0,\cdot)$ describes a smooth curve in $\mathbf{L}^1$, namely a segment. However, for $t > 0$, the map $\theta \mapsto u^\theta(t,\cdot)$ is Lipschitz continuous but nowhere differentiable because the location $x^\theta(t) = \sqrt{1+\theta t}$ of the shock varies with θ. Therefore, the limit

$$\lim_{\Delta\theta \to 0} \frac{u^{\theta+\Delta\theta} - u^\theta}{\Delta\theta}$$

is not well defined as an element of the space $\mathbf{L}^1$.

More generally, call $S : \mathbf{L}^1 \times [0,\infty[\mapsto \mathbf{L}^1$ the semigroup [5] [8] generated by the nonlinear conservation law

$$u_t + f(u)_x = 0. \tag{1.5}$$

In other words, let $t \mapsto u(t,\cdot) = S_t\bar u$ be the unique entropic solution of (1.5) with initial condition $u(0,x) = \bar u(x)$. Then for each $t > 0$ the flow map $\bar u \mapsto S_t \bar u$ is a contraction in the space $\mathbf{L}^1$, in general non differentiable in the usual sense.

To cope with this situation, a new differential structure on the space BV of integrable functions with bounded variation can be introduced. Let a function $u \in BV$ be given. In order to define a "tangent space" at u, we follow a procedure which is now standard in differential geometry. On the family of continuous maps $\gamma : [0,\theta^*] \mapsto \mathbf{L}^1$, with $\gamma(0) = u$ and $\theta^* > 0$ (possibly depending on γ), consider the equivalence relation

$$\gamma \sim \gamma' \qquad \text{iff} \qquad \lim_{\theta\to 0} \frac{1}{\theta} \|\gamma(\theta) - \gamma'(\theta)\|_{_{\mathbf{L}^1}} = 0. \tag{1.6}$$

Every equivalence class can be regarded as a "first-order tangent vector" at the point u. Call $\widehat T_u$ the set of all these equivalence classes. Observe that, if Φ is any Lipschitz continuous map from $\mathbf{L}^1$ into itself, then $\gamma \sim \gamma'$ implies $\Phi\circ\gamma \sim \Phi\circ\gamma'$. Therefore Φ induces a well defined map $D\Phi(u) : \widehat T_u \mapsto \widehat T_{\Phi(u)}$. In principle, this map $D\Phi$ could be regarded as an abstract differential of Φ at the point u. However, this does not seem a fruitful point of view. Indeed, the set $\widehat T_u$ of all equivalence classes is extremely large and cannot be adequately described. Therefore, one usually works with a particular subset $T_u \subset \widehat T_u$ of tangent vectors which admit a suitable representative.

The standard choice at this stage is to consider a family T_u of tangent vectors which can be put in a one-to-one correspondence with $\mathbf{L}^1(I\!R)$. More precisely, T_u is defined as the family of all equivalence classes of the maps

$$\theta \mapsto \gamma_v(\theta) \doteq u + \theta v, \qquad v \in \mathbf{L}^1(I\!R). \tag{1.7}$$

As we saw earlier, this choice is not adequate for describing a first-order variation of the flow map S_t for (1.2). For example, taking

$$\bar{u}(x) \doteq x \cdot \chi_{[0,1]} \,,$$

the path $\vartheta \mapsto \gamma(\vartheta) \doteq u^{1+\vartheta}(0,\cdot)$ in (1.3) determines an equivalence class in $T_{\bar{u}}$. On the other hand, the equivalence class of the path $\vartheta \mapsto S_t \circ \gamma(\vartheta) = u^{1+\vartheta}(t,\cdot)$ defined at (1.4) determines some tangent vector which is not in the space $T_{S_t\bar{u}} \doteq \mathbf{L}^1$.

In this paper, given $u \in BV$, we study a different space T_u of tangent vectors, which can be put into a one-to-one correspondence with $\mathbf{L}^1(\mathrm{D}u)$. Here $\mathrm{D}u$ denotes the (signed) Radon measure corresponding to the distributional derivative of u. The basic idea is the following. In the special case where v is continuously differentiable with compact support, to v we associate the equivalence class of the map $\theta \mapsto u^\theta$, where u^θ is implicitly defined as

$$u^\theta(x + \theta v(x)) = u(x) \tag{1.8}$$

for all $\theta \geq 0$ sufficiently small. We then show that this correspondence can be uniquely extended to the whole space $\mathbf{L}^1(\mathrm{D}u)$. Observe that in (1.7) the graph of u^θ is obtained by lifting the graph of u vertically by θv. On the other hand, in (1.8), the graph of u is shifted horizontally by θv. This motivates the term "shift-differential" used in the sequel.

In Section 2 of this paper we review the definition and the basic properties of shift differentials, for maps within the space of scalar BV functions. All results are taken from [2], to which we refer for details of the proofs.
In Section 3 we extend the notion of shift differential to the case of vector-valued BV functions. If $u : I\!R \mapsto I\!R^n$ is a BV function, a shift tangent vector is defined in terms of:

(i) A decomposition of u into n scalar components,

(ii) A n-tuple of functions $(v_1, \ldots, v_n) \in \mathbf{L}^1(\mathrm{D}u)$, determining the rate at which each component of u is shifted.

This approach is strongly motivated by the study of hyperbolic systems of conservation laws. The main result in [2] shows that the flow generated by a single conservation law is differentiable "almost everywhere". We conjecture that a similar result holds for $n \times n$ strictly hyperbolic systems of conservation laws, with the above definition of shift differential. For piecewise Lipschitz solutions to general $n \times n$ systems of conservation laws, a detailed analysis of first order variations can be found in [3]. Shift differentials of an approximate flow, defined within a class of piecewise constant functions, played a key role in [1] to construct the semigroup generated by a 2×2 system of conservation laws.

2 Shift Tangent Vectors

Throughout the following, our basic function space is the normed space

$$X \doteq (\mathbf{L}^1(I\!\!R) \cap \mathbf{BV}(I\!\!R);\ \|\cdot\|_{\mathbf{L}^1(I\!\!R)}). \tag{2.1}$$

Here $\mathbf{L}^1(I\!\!R)$ refers to the standard Lebesgue measure. The elements of X are thus equivalence classes of functions. For sake of definiteness, we shall always deal with left continuous representatives. By $\mathbf{Lip}(I\!\!R)$ we denote the space of Lipschitz continuous functions. The Lipschitz constant and the $\mathbf{L}^\infty$ norm of v are denoted by $\mathrm{Lip}(v)$ and $\|v\|_{\mathbf{L}^\infty}$, respectively. For $u \in \mathbf{BV}(I\!\!R)$, by $\mathbf{L}^1(\mathrm{D}u)$ we denote the space of functions integrable w.r.t. the Radon measure $\mathrm{D}u$.
Given $u \in X$, for every $v \in \mathbf{L}^1(\mathrm{D}u)$ we shall construct a first-order variation $\theta \mapsto u^\theta$ of u, consistent with the definition (1.8) in the smooth case. The idea is to approximate v by a family of Lipschitz continuous functions v^θ and define u^θ implicitly by

$$u^\theta(x + \theta v^\theta(x)) = u(x). \tag{2.2}$$

Observe that the condition $\mathrm{Lip}(\theta v^\theta) < 1$ is essential in order that the function u^θ in (2.2) be well defined. Some key properties of the approximating functions v^θ are singled out by the next definition.

Definition 1. Let $u \in X$ and $v \in \mathbf{L}^1(\mathrm{D}u)$. Consider a family of functions $v^\theta \in \mathbf{L}^1(\mathrm{D}u) \cap \mathbf{Lip}(I\!R)$, with $\theta \in (0, \theta^*]$. We say that v^θ *hat-converges* to v, and write $v^\theta \overset{\wedge}{\longrightarrow} v$, provided that

$$\lim_{\theta \to 0} \int |v^\theta(x) - v(x)| \, d\mu_u(x) = 0, \qquad \limsup_{\theta \to 0} \mathrm{Lip}(\theta v^\theta) < 1, \qquad \lim_{\theta \to 0} \|\theta v^\theta\|_{\mathbf{L}^\infty} = 0. \tag{2.3}$$

Here and in the sequel, $\mu_u \doteq |\mathrm{D}u|$ denotes the Radon measure of total variation of u. Clearly, $\mathbf{L}^1(\mu_u) \approx \mathbf{L}^1(\mathrm{D}u)$. Observe that the last two conditions in (2.3) essentially depend on the parametrization of the family $\{v^\theta\}$. Indeed, from any family of Lipschitz functions $v^\theta \to v$, it is always possible to recover a hat-convergent family by a simple reparametrization:

Lemma 1. *For any $u \in X$ and $v \in \mathbf{L}^1(\mathrm{D}u)$ there exists a family of functions $v^\theta \in \mathbf{C}_c^\infty(I\!R)$, $\theta \in (0, \theta^*]$ satisfying*

$$\mathrm{Lip}(\theta v^\theta) \le \sqrt{\theta}, \qquad \|\theta v^\theta\|_{\mathbf{L}^\infty} \le \sqrt{\theta}, \qquad \lim_{\theta \to 0} v^\theta = v \quad in \ \ \mathbf{L}^1(\mathrm{D}u). \tag{2.4}$$

In particular $v^\theta \overset{\wedge}{\longrightarrow} v$.

If $\mathrm{Lip}(\theta v^\theta) \le \alpha < 1$, then the function $y_\theta(x) \doteq x + \theta v^\theta(x)$ is Lipschitz continuous with $\mathrm{Lip}(y_\theta) \le 1 + \alpha$. Its inverse $x_\theta(y)$ is also Lipschitz continuous, with constant $\mathrm{Lip}(x_\theta) \le 1/(1-\alpha)$. Hence, the function u^θ in (2.2) is well defined. In particular, if $v^\theta \overset{\wedge}{\longrightarrow} v$, then for all $\theta > 0$ sufficiently small the definition (2.2) is meaningful. We now introduce a convenient notation for the functions u^θ, obtained by shifting horizontally the graph of u.

Definition 2. If $u \in X$ and $\mathrm{Lip}(\theta v^\theta) \le \alpha < 1$, we denote by $v^\theta \star u$ the function implicitly defined by

$$(v^\theta \star u)(x + \theta v^\theta(x)) = u(x). \tag{2.5}$$

With the above notations, the basic definition of shift tangent vector can now be introduced.

Definition 3. Fix $u \in X$ and consider a path $\theta \mapsto u^\theta \in \mathbf{L}^1(I\!R)$, defined on some interval $\theta \in (0, \theta^*]$. We say that the path u^θ *generates the shift tangent vector*

$v \in \mathbf{L}^1(\mathrm{D}u)$ if, for some functions $v^\theta \xrightarrow{\wedge} v$, one has

$$\lim_{\theta\to 0} \frac{1}{\theta} \|u^\theta - v^\theta \star u\|_{\mathbf{L}^1(I\!R)} = 0. \tag{2.6}$$

Roughly speaking, this means that in first approximation the functions u^θ are obtained by shifting the graph of u horizontally by the amount θv. For the above definition to be meaningful, there must be a one-to-one correspondence between shift tangent vectors and functions in $\mathbf{L}^1(\mathrm{D}u)$. In other words, all families v^θ hat-converging to the same function v should determine paths belonging to the same equivalence class. This is indeed the content of the following theorem.

Theorem 1. *Let $u \in X$, $v, \tilde{v} \in \mathbf{L}^1(\mathrm{D}u)$ and assume $v^\theta \xrightarrow{\wedge} v$, $\tilde{v}^\theta \xrightarrow{\wedge} \tilde{v}$. Then*

$$\lim_{\theta\to 0} \frac{1}{\theta} \int |(v^\theta \star u)(y) - (\tilde{v}^\theta \star u)(y)|\, dy = \int |v(x) - \tilde{v}(x)|\, d\mu_u(x). \tag{2.7}$$

Theorem 1 yields the uniqueness of shift tangent vectors. Indeed, assume that the path $\theta \mapsto u^\theta$ generates the shift tangent vectors $v, \tilde{v} \in \mathbf{L}^1(\mathrm{D}u)$. Choosing functions $v^\theta \xrightarrow{\wedge} v$, $\tilde{v}^\theta \xrightarrow{\wedge} \tilde{v}$, from Theorem 1 it now follows

$$\begin{aligned} \|v - \tilde{v}\|_{\mathbf{L}^1(\mathrm{D}u)} &= \lim_{\theta\to 0} \frac{1}{\theta} \|v^\theta \star u - \tilde{v}^\theta \star u\|_{\mathbf{L}^1(I\!R)} \\ &\le \lim_{\theta\to 0} \frac{1}{\theta} \|v^\theta \star u - u^\theta\|_{\mathbf{L}^1(I\!R)} + \lim_{\theta\to 0} \frac{1}{\theta} \|u^\theta - \tilde{v}^\theta \star u\|_{\mathbf{L}^1(I\!R)} \\ &= 0. \end{aligned}$$

Therefore, v and $\tilde{v}$ must coincide almost everywhere w.r.t. the measure $\mathrm{D}u$.

On the other hand, let $v^\theta \xrightarrow{\wedge} v$, $\tilde{v}^\theta \xrightarrow{\wedge} v$. Then by (2.7) the paths $\gamma(\theta) \doteq v^\theta \star u$, $\tilde{\gamma}(\theta) \doteq \tilde{v}^\theta \star u$ are equivalent. This shows that the construction of a tangent vector does not depend on the choice of the family v^θ in (2.3).

Example 1. Let $u(x) = x \cdot \chi_{[0,1]}(x)$ and define the parametrized curve

$$u^\theta(x) = (1+\theta)^2 x \cdot \chi_{[0,1+\theta]}(x).$$

Then the curve $\theta \mapsto u^\theta$ generates the shift tangent vector $v \in \mathbf{L}^1(\mathrm{D}u)$, with

$$v(x) = \begin{cases} -2x & \text{if} \quad x \in [0,1)\,, \\ 1 & \text{if} \quad x = 1. \end{cases}$$

Indeed, one can check that (2.4) holds, choosing for example $v^\theta(x) = \max\{-2x,\ 1+(x-1)/2\theta\}$.

It is interesting to relate the standard notion of differentiability to the existence of a shift tangent vector. Recall that a continuous path $\theta \mapsto u^\theta$ with $u^0 = u \in X$ is shift differentiable if there exists $v \in \mathbf{L}^1(\mathrm{D}u)$ such that (2.6) holds for some $v^\theta \overset{\wedge}{\longrightarrow} v$. On the other hand, the path is differentiable in the usual sense if there exists $w \in \mathbf{L}^1(\mathbb{R})$ such that

$$\lim_{\theta\to 0}\left\|\frac{u^\theta - u}{\theta} - w\right\|_{\mathbf{L}^1(\mathbb{R})} = 0. \tag{2.8}$$

Theorem 2. *Consider a map $\theta \mapsto u^\theta \in X$, with $u^0 = u$, generating the shift tangent vector $v \in \mathbf{L}^1(\mathrm{D}u)$. Then this map is differentiable at $\theta = 0$ in the usual sense if and only if $v \cdot \mathrm{D}u$ is absolutely continuous w.r.t. Lebesgue measure. In this case we have $v \cdot \mathrm{D}u = -w \cdot dx$ with w defined by (2.8).*
Viceversa, let (2.8) hold. Then the curve $\theta \mapsto u^\theta$ generates a shift tangent vector $v \in \mathbf{L}^1(\mathrm{D}u)$ if and only if the measure $w \cdot dx$ is absolutely continuous w.r.t. the measure $\mathrm{D}u$, and the equality $w \cdot dx = -v \cdot \mathrm{D}u$ holds.

Starting from the definition of shift tangent vector, one can introduce the notion of shift differentiability for a map with values in the space X as in (2.1). In the following, we consider a locally Lipschitz continuous operator Φ mapping the function space X into itself.

Definition 4. We say that Φ is *shift differentiable at the point $u \in X$ along the direction $v \in \mathbf{L}^1(\mathrm{D}u)$* if there exists $w \in \mathbf{L}^1(\mathrm{D}\Phi(u))$ such that

$$\lim_{\theta\to 0}\frac{1}{\theta}\left\|w^\theta \star \Phi(u) - \Phi(v^\theta \star u)\right\|_{\mathbf{L}^1(\mathbb{R})} = 0 \tag{2.9}$$

for some $v^\theta \overset{\wedge}{\longrightarrow} v$, $w^\theta \overset{\wedge}{\longrightarrow} w$. In this case, we call w the shift derivative of Φ at the point u along v and write $w = \overleftrightarrow{\nabla}_v \Phi(u)$.

Remark 1. The Lipschitz continuity of Φ and Remark 2 imply that, if Φ is shift differentiable at u along the direction v, then (2.9) holds for all $\tilde{v}^\theta, \tilde{w}^\theta$ hat-converging to v and $\overleftrightarrow{\nabla}_v \Phi(u)$ respectively. Moreover, Remark 3 shows that, if a shift derivative exists, it is necessarily unique.

Definition 5. We say that Φ is *shift-differentiable* at u if there exists a continuous linear map $\Lambda\colon \mathbf{L}^1(\mathrm{D}u) \to \mathbf{L}^1(\mathrm{D}\Phi(u))$ such that the following holds. For all $v \in \mathbf{L}^1(\mathrm{D}u)$ there exist $v^\theta \xrightarrow{\wedge} v$ and $w^\theta \xrightarrow{\wedge} \Lambda v$ such that

$$\lim_{\theta\to 0} \frac{1}{\theta} \| w^\theta \star \Phi(u) - \Phi(v^\theta \star u) \|_{\mathbf{L}^1(I\!\!R)} = 0. \tag{2.10}$$

In this case we say that Λ is the *shift differential* of Φ at u and write $\Lambda = \overset{\leftrightarrow}{\nabla}\Phi(u)$.

In other words Φ is shift differentiable at u if it is shift differentiable along each direction $v \in \mathbf{L}^1(\mathrm{D}u)$ and the map $v \mapsto \overset{\leftrightarrow}{\nabla}_v \Phi(u)$ is linear.

Remark 2. By Theorem 1, the shift differential of Φ is a linear continuous map Λ whose norm is bounded by the local Lipschitz constant of Φ.

3 The vector valued case

In this section we consider the notion of shift tangent vector at a point $u \in X$, where X is now the space of all vector valued, integrable functions $u : I\!\!R \mapsto I\!\!R^n$ of bounded variation, endowed with the $\mathbf{L}^1$ topology. As mentioned in the Introduction, to define such a tangent vector we first need to split the vector u into n scalar components, then we have to specify the rate at which each component shifts.
The first task is accomplished by assigning, for each $x \in I\!\!R$, a (positively oriented) basis of unit vectors in $I\!\!R^n$, say $r_1(x), \ldots, r_n(x)$. Calling $l_1(x), \ldots, l_n(x)$ the dual basis, for every $x \in I\!\!R$ we thus have

$$r_1(x) \wedge \cdots \wedge r_n(x) > 0, \tag{3.1}$$

$$|r_i(x)| = 1, \qquad \langle l_i(x),\ r_i(x)\rangle = \delta_{ij} \qquad \text{for all } i, j. \tag{3.2}$$

We then assign n scalar functions $v_1, \ldots, v_n$ determining the shifts. In this way, we thus determine a matrix valued function $A : I\!\!R \mapsto I\!\!R^{n\times n}$ having the r_i, l_i as right and left eigenvectors, and the v_i as eigenvalues. In other words:

$$A(x)w = \sum_{i=1}^{n} v_i(x) \langle l_i(x), w\rangle r_i(x), \qquad x \in I\!\!R,\ w \in I\!\!R^n. \tag{3.3}$$

In the following we shall assume that, for every $i = 1, \ldots, n$,

(A1) the functions r_i, $l_i : I\!\!R \mapsto I\!\!R^n$ are Borel measurable and uniformly bounded,

(A2) $v_i \in \mathbf{L}^1(\mu_u)$.

We recall that $\mu_u \doteq |\mathrm{D}u|$ denotes the Radon measure of total variation of u. In analogy with Definition 1, we introduce

Definition 5. We say that the family of matrix valued functions $A^\theta : I\!\!R \mapsto I\!\!R^{n\times n}$, parametrized by $\theta \in (0, \theta^*]$, *hat-converges* to A (and write $A^\theta \stackrel{\wedge}{\longrightarrow} A$), provided that:

(i) for each $\theta \in (0, \theta^*]$, A^θ has the form:

$$A^\theta(x)w = \sum_{i=1}^{n} v_i^\theta(x)\langle l_i^\theta(x), w\rangle r_i^\theta(x), \;\; x \in I\!\!R, \;\; w \in I\!\!R^n,$$

where $\{r_i^\theta(x)\}_{i=1}^n$ and $\{l_i^\theta(x)\}_{i=1}^n$ are the corresponding bases of right and left eigenvectors of $A^\theta(x)$, normalized as in (3.2),

(ii) For each given θ, the functions r_i^θ, $l_i^\theta : I\!\!R \mapsto I\!\!R^n$ and $v_i^\theta : I\!\!R \mapsto I\!\!R$ are Lipschitz continuous and bounded. Moreover, they remain constant outside a compact interval K^θ,

(iii) for each $i = 1 \ldots n$ one has

$$\lim_{\theta\to 0} \int_{I\!\!R} \Big\{ |r_i^\theta(x) - r_i(x)| + |l_i^\theta(x) - l_i(x)| \; + |v_i^\theta(x) - v_i(x)| \Big\} \, d\mu_u(x) = 0, \quad (3.4)$$

$$\lim_{\theta\to 0} \theta^{1/4} \left(\mathrm{Lip}(r_i^\theta) + \mathrm{Lip}(l_i^\theta) + \mathrm{Lip}(v_i^\theta) + \|v_i^\theta\|_{\mathbf{L}^\infty} \right) = 0, \quad (3.5)$$

$$\lim_{\theta\to 0} \max_j \{ \|v_j^\theta\|_{\mathbf{L}^\infty} \} \int_{I\!\!R} \Big(|r_i^\theta(x) - r_i(x)| + |l_i^\theta(x) - l_i(x)| \Big) d\mu_u(x) = 0, \quad (3.6)$$

$$\|r_i^\theta\|_{\mathbf{L}^\infty} \equiv 1, \qquad \sup_{i,\theta} \|l_i^\theta\|_{\mathbf{L}^\infty} < \infty. \quad (3.7)$$

The next lemma ensures that some hat-convergent family always exists.

Lemma 2. *Let the matrix valued funtion A in (3.3) satisfy* **(A1)-(A2)** *Then there exists $\theta^* > 0$ and a family of matrix valued functions A^θ, with $\theta \in (0, \theta^*]$, such that $A^\theta \stackrel{\wedge}{\longrightarrow} A$.*

Proof. Denote by R the matrix valued function, having $r_1, \ldots, r_n$ as column vectors. By (3.2), its inverse R^{-1} is the matrix having $l_1, \ldots, l_n$ as row vectors. By (3.1) and the assumption **(A1)**, there exists some $Q > 0$ such that $\det R(x) \in [1/Q,\ Q]$ for all x. Fix $\theta \in (0,1)$. By Lusin's theorem, there exists a compact set B_θ such that

$$\mu_u(I\!R \setminus B_\theta) < \theta \tag{3.8}$$

and such that the restriction of R to B_θ is continuous. The complementary set $I\!R \setminus B_\theta$ consists of two open half-lines $(-\infty, b_0)$ and (a_0, ∞) and a countable number of open intervals (a_k, b_k), $k \geq 1$. For each such k, let $\widetilde{R}^k : [a_k, b_k] \mapsto I\!R^{n\times n}$ be a continuous matrix valued function with the following properties:

$$\begin{array}{l} \widetilde{R}^k(a_k) = R(a_k), \qquad \widetilde{R}^k(b_k) = R(b_k), \\ \text{each column vector } \ \tilde{r}_i^k(x) \ \text{ has unit length }, \\ \|\widetilde{R}^k(x) - R(a_k)\| \leq C\|R(b_k) - R(a_k)\|, \\ \det \widetilde{R}^k(x) \in [1/Q_1,\ Q_1], \end{array} \tag{3.9}$$

for some constants $Q_1 > Q$ and $C > 0$ depending only on Q, and for all $x \in [a_k, b_k]$.
We now define

$$\widetilde{R}^\theta(x) \doteq \begin{cases} R(x) & \text{if} \quad x \in B_\theta, \\ R(b_0) & \text{if} \quad x \in (-\infty,\ b_0), \\ R(a_0) & \text{if} \quad x \in (a_0,\ \infty), \\ \widetilde{R}^k & \text{if} \quad x \in (a_k, b_k). \end{cases}$$

Let $\widetilde{L}^\theta \doteq \left(\widetilde{R}^\theta\right)^{-1}$. Then $\widetilde{R}^\theta$ and $\widetilde{L}^\theta$ are continuous, constant outside the compact interval $[b_0, a_0]$, bounded by a number which does not depend on θ, and such that each column vector $\tilde{r}_i^\theta(x)$ has unit length. Repeating this construction for every $\theta > 0$, by (3.8) and (3.9) we thus obtain functions $\tilde{r}_i^\theta$, $\tilde{l}_i^\theta$, $i = 1, \ldots, n$, normalized as in (3.2), such that

$$\lim_{\theta\to 0} \left(\|\tilde{r}_i^\theta - r_i\|_{\mathbf{L}^1(|\mathrm{D}u|)} + \|\tilde{l}_i^\theta - l_i\|_{\mathbf{L}^1(|\mathrm{D}u|)} \right) = 0. \tag{3.10}$$

By a mollification and renormalization procedure, we can also assume that the all functions $x \mapsto \tilde{r}_i^\theta(x)$ and $x \mapsto \tilde{l}^\theta(x)$ are smooth.
Next, we approximate each v_i in $\mathbf{L}^1(\mu_u)$ by a sequence $\{\tilde{v}_i^k\}_{k\geq 1}$ of smooth functions with compact support.
Define a sequence $\{\theta_k\}_{k\geq 1}$, strictly decreasing to 0 and such that, for each k,

$$\max_j \{\mathrm{Lip}(\tilde{r}_j^{1/k})\} + \max_j \{\mathrm{Lip}(\tilde{l}_j^{1/k})\} \leq \theta_k^{-1/8}. \tag{3.11}$$

Moreover, define

$$r_i^\theta(x) \doteq \tilde r_i^{1/k}, \qquad l_i^\theta(x) \doteq \tilde l_i^{1/k}, \quad \text{for } x \in (\theta_{k-1}, \theta_k].$$

After having constructed the eigenvector functions r_i^θ, l_i^θ, we now determine a second sequence $\{\theta'_k\}_{k\geq 1}$, strictly decreasing to 0, with the property that for each k,

$$\max_j \{\mathrm{Lip}(\tilde v_j^k)\} + \max_j \{\|\tilde v_j^k\|_{\mathbf{L}^\infty}\} \leq \Big(\theta^{1/4} + \max_j \{\|\tilde r_j^\theta - r_j\|_{\mathbf{L}^1(\mathrm{D}u)}\} + \max_j \{\|\tilde l_j^\theta - l_j\|_{\mathbf{L}^1(\mathrm{D}u)}\}\Big)^{-1/2} \quad \text{for all } \theta \in (0, \theta'_k]. \tag{3.12}$$

One can now define:

$$v_i^\theta(x) = \tilde v_i^k \ \text{ for } x \in (\theta'_{k-1},\ \theta'_k].$$

This completes the construction of the approximating family A^θ. All requirements of Definition 5 are then satisfied. Indeed, (3.4) follows from (3.10) and the convergence $\tilde v_i^k \to v_i$ in $\mathbf{L}^1(\mu_u)$. The bounds (3.11) and (3.12) together imply (3.5), while (3.6) is also a consequence of (3.12). By construction, $|r_i^\theta(x)| = 1$ for all i, θ, x. This property, together with the uniform boundedness of the matrices $\widetilde L^\theta$, yields (3.7).

Let now A be a matrix valued function, described in terms of its eigenvectors and eigenvalues as in (3.3). Let the conditions **(A1)-(A2)** be satisfied and assume $A^\theta \overset{\wedge}{\longrightarrow} A$. We wish to construct a path $\theta \mapsto u^\theta \in \mathbf{L}^1(I\!R, I\!R^n)$, originating from $u^0 = u$, corresponding to the shift tangent vector determined by A. Roughly speaking, u^θ should thus be obtained from u by shifting each component $u_i \doteq \langle l_i,\, u\rangle r_i$ in the amount θv_i. Some notation must first be introduced. For a fixed $\theta > 0$ consider the points $P_k^\theta \doteq \frac{k}{2}\theta^{3/4}$, $k \in Z\!\!\!Z$. Moreover, for each integer k let I_k^θ and J_k^θ be the open intervals centered at P_k^θ and with lenghts $\frac{1}{2}\theta^{3/4}$ and $\theta^{3/4}$, respectively.

Definition 6. Let $A^\theta \overset{\wedge}{\longrightarrow} A$, $\quad \theta \in (0, \theta^*]$. Then for each $\theta \in [0, \theta^*]$ we define the function $u^\theta \doteq A^\theta \star u$ by the formula:

$$\begin{aligned} u^\theta(x) &= \sum_{i=1}^n \Big\langle l_i^\theta(P_k^\theta),\ u(x - \theta v_i^\theta(P_k^\theta))\Big\rangle r_i^\theta(P_k^\theta), \qquad \text{for } x \in I_k^\theta \text{ and } \theta > 0, \\ u^0(x) &= u(x), \qquad \text{for } x \in I\!R. \end{aligned} \tag{3.13}$$

Note that, restricted to each interval I_k^θ, one has $u^\theta = w_k^\theta(\theta, \cdot)$, where w_k^θ is the solution to the linear hyperbolic system with constant coefficients:

$$w_t + A^\theta(P_k^\theta) w_x = 0, \quad w(0, \cdot) = u.$$

The values of u^θ on I_k^θ depend only on the values of u on J_k^θ. From the assumptions (ii) and (iii) in Definition 5 it follows:

Lemma 3. *The path $\theta \mapsto u^\theta \doteq A^\theta \star u$ is continuous at $\theta = 0$, namely*

$$\lim_{\theta \to 0} \int_{I\!R} |u^\theta - u| \, dx = 0.$$

The following is a natural generalization of Definition 3.

Definition 7. Let $A : I\!R \mapsto I\!R^{n\times n}$ satisfy **(A1)-(A2)**. We say that a path $\theta \mapsto w^\theta \in \mathbf{L}^1(I\!R, I\!R^n)$, defined on some interval $[0, \theta^*]$, with $w^0 = u$, generates the shift tangent vector determined by the matrix valued function A, if for some hat-converging family $A^\theta \stackrel{\wedge}{\longrightarrow} A$ one has

$$\lim_{\theta \to 0} \frac{1}{\theta} \int_{I\!R} |w^\theta - A^\theta \star u| \, dx = 0. \tag{3.14}$$

To justify the above definition, one needs to check that the equivalence class of the path $A^\theta \star u$ does not depend on the choice of the approximating family A^θ. This is the content of the next theorem.

Theorem 3. *Let A be a matrix valued function, satisfying (A1) and (A2). Let $A^\theta \stackrel{\wedge}{\longrightarrow} A$ and $\widetilde{A}^\theta \stackrel{\wedge}{\longrightarrow} A$. Then the paths $u^\theta = A^\theta \star u$ and $\tilde{u}^\theta = \widetilde{A}^\theta \star u$ satisfy*

$$\lim_{\theta \to 0} \frac{1}{\theta} \int_{I\!R} |u^\theta - \tilde{u}^\theta| \, dx = 0. \tag{3.15}$$

Toward a proof of Theorem 3, two technical lemmas will be used.

Lemma 4. *Let $a, b, y, z \in I\!R$ and $a < b$. Then*

$$\int_a^b |u(x-y) - u(x-z)| \, dx \le |y-z| \cdot \text{Tot.Var.}\Big\{u;\ [a - \max\{|y|, |z|\},\ b + \max\{|y|, |z|\}]\Big\}.$$

Lemma 5. *Let $\{r_i\}_{i=1}^n$ and $\{\tilde{r}_i\}_{i=1}^n$ be two bases of $I\!R^n$ and $\{l_i\}_{i=1}^n$, $\{\tilde{l}_i\}_{i=1}^n$ their corresponding dual bases, normalized as in (3.2) and bounded by a number M. Let $\lambda_1, \ldots, \lambda_n$ be real numbers. Fix an open interval I and let J be a second open* *interval, with the same center as* I *but twice as long. Then for every* $t \geq 0$ *such that* $|I| > 2t \max_i |\lambda_i|$ *one has*

$$\int_I \left| \sum_{i=1}^n \langle l_i,\ u(x - t\lambda_i)\rangle r_i - \sum_{i=1}^n \langle \tilde{l}_i,\ u(x - t\lambda_i)\rangle \tilde{r}_i \right| dx$$
$$\leq 2Mt \cdot \Big(\max_i |\lambda_i| \Big) \cdot \left(\sum_{i=1}^n |l_i - \tilde{l}_i| + \sum_{i=1}^n |r_i - \tilde{r}_i| \right) \cdot \text{Tot.Var.}\{u;\ J\}.$$

Proof of Theorem 3. Note first that $\int_{I\!R} |u^\theta - \tilde{u}^\theta| dx \leq \mathrm{I}_1 + \mathrm{I}_2$, where

$$\mathrm{I}_1 \doteq \sum_k \int_{I_k^\theta} \left| \sum_{i=1}^n \langle l_i^\theta(P_k^\theta),\ u(x - \theta v_i^\theta(P_k^\theta))\rangle r_i^\theta(P_k^\theta) \right.$$
$$\left. - \sum_{i=1}^n \langle l_i^\theta(P_k^\theta),\ u(x - \theta \tilde{v}_i^\theta(P_k^\theta))\rangle r_i^\theta(P_k^\theta) \right| dx,$$

$$\mathrm{I}_2 \doteq \sum_k \int_{I_k^\theta} \left| \sum_{i=1}^n \langle l_i^\theta(P_k^\theta),\ u(x - \theta \tilde{v}_i^\theta(P_k^\theta))\rangle r_i^\theta(P_k^\theta) \right.$$
$$\left. - \sum_{i=1}^n \langle \tilde{l}_i^\theta(P_k^\theta), u(x - \theta \tilde{v}_i^\theta(P_k^\theta))\rangle \tilde{r}_i^\theta(P_k^\theta) \right| dx.$$

Let M be a common bound for all functions $l_i^\theta,\quad r_i^\theta$. Using Lemma 4 we estimate the first integral:

$$\begin{aligned}
\mathrm{I}_1 \ &\leq\ M \sum_{i=1}^n \sum_k \int_{I_k^\theta} |u(x - \theta v_i^\theta(P_k^\theta)) - u(x - \theta \tilde{v}_i^\theta(P_k^\theta))|\, dx \\
&\leq\ M\theta \sum_{i=1}^n \sum_k |v_i^\theta(P_k^\theta) - \tilde{v}_i^\theta(P_k^\theta)| \cdot \text{Tot.Var.}\{u; J_k^\theta\} \\
&\leq\ M\theta \sum_{i=1}^n \left\{ \sum_k \int_{J_k^\theta} |J_k^\theta| \Big(\text{Lip}(v_i^\theta) + \text{Lip}(\tilde{v}_i^\theta) \Big)\, d\mu_u(x) \right. \\
&\qquad \left. + \sum_k \int_{J_k^\theta} |v_i^\theta(x) - \tilde{v}_i^\theta(x)|\, d\mu_u(x) \right\} \\
&\leq\ 2M\theta \sum_{i=1}^n \left\{ |J_k^\theta| \Big(\text{Lip}(v_i^\theta) + \text{Lip}(\tilde{v}_i^\theta) \Big) \cdot \text{Tot.Var.}\{u; I\!R\} \right. \\
&\qquad \left. + \int_{I\!R} |v_i^\theta(x) - v_i(x)|\, d\mu_u(x) + \int_{I\!R} |\tilde{v}_i^\theta(x) - v_i(x)|\, d\mu_u(x) \right\}
\end{aligned}$$

Concerning the second integral, using Lemma 5 we obtain

$$
\begin{aligned}
\mathrm{I}_2 \quad \leq \quad & 2M\theta \cdot \max_i \{\|v_i^\theta\|_{\mathbf{L}^\infty}\} \\
& \cdot \sum_k \left(\sum_{i=1}^n |r_i^\theta(P_k^\theta) - \tilde{r}_i^\theta(P_k^\theta)| + \sum_{i=1}^n |l_i^\theta(P_k^\theta) - \tilde{l}_i^\theta(P_k^\theta)| \right) \cdot \text{Tot.Var.}\{u;\ J_k^\theta\} \\
\leq \quad & 2M\theta \cdot \max_i \{\|v_i^\theta\|_{\mathbf{L}^\infty}\} \\
& \cdot \sum_{i=1}^n \sum_k \left\{ \int_{J_k^\theta} |J_k^\theta| \Big(\text{Lip}(r_i^\theta) + \text{Lip}(\tilde{r}_i^\theta) + \text{Lip}(l_i^\theta) + \text{Lip}(\tilde{l}_i^\theta) \Big) d\mu_u(x) \right. \\
& \qquad \left. + \int_{J_k^\theta} \Big(|r_i^\theta(x) - \tilde{r}_i^\theta(x)| + |l_i^\theta(x) - \tilde{l}_i^\theta(x)| \Big)\, d\mu_u(x) \right\} \\
\leq \quad & 4M\theta \cdot \max_i \{\|v_i^\theta\|_{\mathbf{L}^\infty}\} \\
& \cdot \sum_{i=1}^n \left\{ |J_k^\theta| \Big(\text{Lip}(r_i^\theta) + \text{Lip}(\tilde{r}_i^\theta) + \text{Lip}(l_i^\theta) + \text{Lip}(\tilde{l}_i^\theta) \Big) \cdot \text{Tot.Var.}\{u;\ I\!R\} \right. \\
& \qquad + \int_{I\!R} |r_i^\theta(x) - r_i(x)|\, d\mu_u(x) + \int_{I\!R} |l_i^\theta(x) - l_i(x)|\, d\mu_u(x) \\
& \qquad \left. + \int_{I\!R} |\tilde{r}_i^\theta(x) - r_i(x)|\, d\mu_u(x) + \int_{I\!R} |\tilde{l}_i^\theta(x) - l_i(x)|\, d\mu_u(x) \right\}
\end{aligned}
$$

The above estimates together yield

$$
\lim_{\theta \to 0} \frac{1}{\theta} (\mathrm{I}_1 + \mathrm{I}_2) = 0,
$$

proving Theorem 3.

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