TRAVELING WAVES IN 2D REACTIVE BOUSSINESQ SYSTEMS WITH NO-SLIP BOUNDARY CONDITIONS

PETER CONSTANTIN, MARTA LEWICKA AND LENYA RYZHIK

ABSTRACT. We consider systems of reactive Boussinesq equations in two dimensional strips that are not aligned with gravity's direction. We prove that for any width of such strips and for arbitrary Rayleigh and Prandtl numbers, the systems admit smooth, non-planar traveling wave solutions with the fluid's velocity satisfying no-slip boundary conditions.

1. INTRODUCTION

In this note we establish existence of non-planar traveling wave solutions of reactive Boussinesq systems with no-slip boundary conditions (the fluid flow vanishes at the boundary). Much of the analysis is similar to the no-stress case [5], but there are several difficulties due to the no-slip boundary conditions that require new ideas. These are explained below.

Existence of non-planar traveling waves for reaction-diffusion equations in a prescribed flow has been a subject of active study in the last decade – we refer to [4] and [17] for excellent overviews. When u is an imposed flow of shear type, then the study of existence and stability of the multidimensional traveling waves for the single temperature equation can be found, for example, in [6, 7, 13].

Recently, traveling waves have been also shown to exist in a reactive system with a Boussinesq-type coupling between the temperature field and the fluid flow [3, 5, 14, 15]. In non-dimensional variables, this system has the form

$$\begin{aligned} T_t + u \cdot \nabla T - \Delta T &= f(T) \\ \frac{1}{\sigma}(u_t + u \cdot \nabla u) - \Delta u + \nabla p &= T\rho\vec{g} \\ \text{div } u &= 0. \end{aligned} \tag{1.1}$$

Here T is the temperature, u the fluid velocity and p the pressure. The parameters ρ and σ are, respectively, the Rayleigh and Prandtl numbers, and $\vec{g}$ is the unit vector in the direction of the gravity force. The Lipschitz continuous nonlinearity f is of ignition type, that is,

$$f(T) = 0 \text{ on } (-\infty, \theta_0] \cup \{1\}, \qquad f(T) > 0 \text{ on } (\theta_0, 1), \qquad f(T) < 0 \text{ on } (1, \infty), \tag{1.2}$$

where the ignition takes place at some temperature $T = \theta_0 \in (0, 1)$.

The reactive Boussinesq system has been considered in the references mentioned above in a two-dimensional strip, with either periodic or no-stress boundary conditions: the normal component of the flow and the flow vorticity vanish at the boundary. When the strip is vertical (gravity is aligned with the strip's direction) it is not difficult to see [9] that non-planar fronts do not exist for small Rayleigh numbers but planar fronts with $u = 0$ exist for all values of parameters. On the other hand, for large Rayleigh numbers, the planar fronts become unstable: there is a bifurcation at some critical ρ_c [14, 15] so that non-planar fronts exist for ρ close to ρ_c. It turns out that the situation is quite different when the domain is not vertical [3, 5]. In particular, it has been observed in [3] that planar fronts cannot exist in a slanted strip and that a non-planar traveling wave exists for small $\rho > 0$. This result has been extended to all $\rho > 0$ in [5]. Reaction-diffusion-convection systems with the Lewis number different but close to one have been considered in [10] by perturbative techniques.

We consider (1.1) in an infinite strip $D \subset \mathbb{R}^2$. The vector $\vec{g} = (g_1, g_2)$ representing the gravity direction is assumed to be non-parallel to the unbounded direction of D. By an elementary change of variables, without loss of generality, we may restrict our attention to the horizontal strip

$$D = \{(x, y);\ x \in (-\infty, \infty),\ y \in [0, h]\} \tag{1.3}$$

and assume that

$$g_2 \neq 0. \tag{1.4}$$

We are interested in the existence of a smooth traveling front solution $T = T(x - ct, y),\ u = u(x - ct, y)$ of (1.1). Naturally, it satisfies:

$$\begin{aligned} -cT_x + u \cdot \nabla T - \Delta T &= f(T) \\ \frac{1}{\sigma}(-cu_x + u \cdot \nabla u) - \Delta u + \nabla p &= T\rho\vec{g} \\ \operatorname{div} u &= 0. \end{aligned} \tag{1.5}$$

We impose front-like boundary conditions for temperature and no-slip boundary conditions for the flow:

$$\begin{aligned} &T_x \to 0 \text{ as } x \to -\infty,\ T \to 0 \text{ as } x \to \infty,\ \frac{\partial T}{\partial y} = 0 \text{ at } y = 0, h, \\ &u = 0 \text{ at } y = 0, h,\ |u| \to 0 \text{ as } x \to \pm\infty. \end{aligned} \tag{1.6}$$

Notice that by the maximum principle any solution to (1.5), (1.6) satisfies:

$$0 \leq T(x, y) \leq 1 \qquad \text{for all } (x, y) \in D.$$

The main result of this note is the following:

Theorem 1.1. *Assume that gravity satisfies (1.4) and the nonlinearity f satisfies (1.2). Then there exists a smooth traveling front solution T, u, c of (1.5), (1.6) in* *the two-dimensional strip (1.3). In particular, this traveling wave satisfies: the speed* $c > 0$, *the functions* $T, u \in \mathcal{C}^{2,\alpha}(D)$, $p \in \mathcal{C}^{1,\alpha}_{loc}(D)$ *(for some* $\alpha > 0$*),* $\nabla T \in L^2(D)$, $u \in W^{3,2}(D)$ *and the boundary conditions at infinity are approached uniformly in* $y \in [0, h]$*:*

(1.7)
$$\lim_{x\to\pm\infty} ||u(x,\cdot)||_{L^\infty([0,h])} = \lim_{x\to\pm\infty} ||\nabla u(x,\cdot)||_{L^\infty([0,h])} = \lim_{x\to\pm\infty} ||\nabla T(x,\cdot)||_{L^\infty([0,h])} = 0.$$

$$\lim_{x\to+\infty} ||T(x,\cdot)||_{L^\infty([0,h])} = \lim_{x\to-\infty} ||T(x,\cdot) - \theta_-||_{L^\infty([0,h])} = 0, \tag{1.8}$$

for some $\theta_- \in (0, \theta_0] \cup \{1\}$. *If, in addition, we have*

$$f(T) \le [(T - \theta_0)_+]^2/h^2 \quad \forall T \in [0, 1] \tag{1.9}$$

then $\theta_- = 1$.

The paper is organized as follows. In Section 2, following the method of [8], we introduce compactified versions of the problem (1.1) on truncated domains (rectangles) of increasing lengths $2a > 0$, and prove that they have solutions that are uniformly bounded in a. In particular, the constants in the elliptic regularity inequalities depend only on the local uniform properties of the embedding of the boundary of the domain in the ambient space and they are independent of a. These and other uniform bounds are contained in Theorem 2.1 and are proved in Section 3. The crucial estimate can be found in Lemma 3.5 whose proof is new and does not follow from the analysis in [5]. The difficulty is due to the non-slip boundary conditions (see Remark 3.6). In Section 4 we conclude the proof of Theorem 1.1 by obtaining the traveling wave solution as the limit of a subsequence of solutions to the compactified problems as $a \to +\infty$. The main novelty in this part of the argument is in the proof of Lemma 4.5, which replaces the analysis of the inner and outer solutions from [5].

Acknowledgment. We thank the referee for bringing to our attention the paper [10]. PC was supported by NSF grant DMS-0504213, LR was supported by an Alfred P. Sloan fellowship and by NSF grant DMS-0604687.

2. The compact domain problem

In this section, following the approach of [5, 8], for each large $a > 0$ we consider the problem in a finite rectangle $R_a = [-a, a] \times [0, h]$ and obtain bounds on the solutions that are uniform in a. In order to avoid dealing with questions of regularity at the corners, the Navier-Stokes part of the problem (1.1) will be considered on a larger smooth convex domain D_a, such that $R_a \subset D_a \subset R_{a+1}$ (see figure 1).

The temperature T satisfies an approximate problem in the rectangle R_a:

$$-cT_x + u \cdot \nabla T - \Delta T = f(T) \text{ in } R_a, \tag{2.1}$$

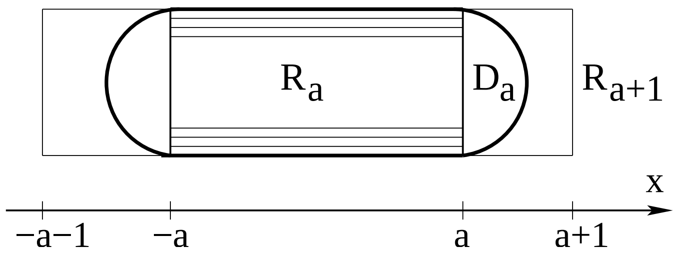

FIGURE 1

with the boundary conditions from (1.6), replacing the conditions at $\pm\infty$ by prescribing the temperature on the vertical boundary of R_a:

$$\begin{aligned} &T(-a,y)=1,\ T(a,y)=0 \text{ for } y\in[0,h] \\ &\frac{\partial T}{\partial y}(x,0)=\frac{\partial T}{\partial y}(x,h)=0 \text{ for } x\in[-a,a]. \end{aligned} \tag{2.2}$$

In order to formulate the problem for u in the larger domain D_a we extend the function $T(x,y)$ from R_a to D_a by the odd reflection across the vertical boundary of R_a:

$$T(x,y)=\begin{cases} -T(-2a-x,y)+2T(-a,y) & \text{for } x<-a \\ -T(2a-x,y)+2T(a,y) & \text{for } x>a. \end{cases} \tag{2.3}$$

Note that

$$||T||_{\mathcal{C}^{1,\alpha}(D_a)}\le 2||T||_{\mathcal{C}^{1,\alpha}(R_a)}, \quad ||\nabla T||_{L^2(D_a)}\le 2||\nabla T||_{L^2(R_a)}. \tag{2.4}$$

The flow u satisfies the following problem in D_a:

$$\begin{aligned} \frac{1}{\sigma}(-cu_x+u\cdot\nabla u)-\Delta u+\nabla p &= T\vec{\rho} \quad \text{in } D_a, \\ \text{div } u &= 0 \quad \text{in } D_a, \end{aligned} \tag{2.5}$$

with the no-slip boundary conditions

$$u=0 \text{ on } \partial D_a. \tag{2.6}$$

We denoted by $\vec{\rho}$ the vector $\rho\vec{g}$ in (2.5).

Finally, we introduce the following normalization condition (see [6]):

$$\max\{T(x,y);\ x\in[0,a],\ y\in[0,h]\}=\theta_0. \tag{2.7}$$

Therefore, $f(T)\equiv 0$ for $x\ge 0$ and thus the maximum principle implies that:

$$\max\{T(x,y);\ x\in[0,a],\ y\in[0,h]\}=\max\{T(0,y);\ y\in[0,h]\}=\theta_0. \tag{2.8}$$

The flow equations (2.5)-(2.6) may be also written in terms of a non-linear elliptic system for two unknowns: the stream-function ψ_1 such that $u=\text{curl }\psi_1=$

$(\psi_{1,y}, -\psi_{1,x})$, and the vorticity function $\psi_2 = \Delta\psi_1$:

$$
\begin{aligned}
&\Delta\psi_1 - \psi_2 = 0 && \text{in } D_a\\
&\Delta\psi_2 = \left(-\mathrm{curl}(T\vec{\rho}) + \frac{1}{\sigma}\mathrm{curl}(-cu_x + u\cdot\nabla u)\right) && \text{in } D_a\\
&\psi_1 = 0,\ \nabla\psi_1 = 0 && \text{on } \partial D_a.
\end{aligned}
\tag{2.9}
$$

The aim of this section is to prove existence of a regular solution to (2.1)-(2.2), (2.5)-(2.7) and obtain uniform in a bounds for (c, u, T). More precisely, we have the following result.

Theorem 2.1. *There exists $a_0 > 0$ so that for all $a > a_0$ the problem (2.1)-(2.2), (2.5)-(2.7) has a solution $c \in \mathbb{R}$, $u \in \mathcal{C}^{2,\alpha}(D_a)$, $T \in \mathcal{C}^{1,\alpha}(R_a)$, $p \in \mathcal{C}^{1,\alpha}(D_a)$. In addition, we have the following bounds:*

$$
|c| + ||T||_{\mathcal{C}^{1,\alpha}(R_a)} + ||u||_{\mathcal{C}^{2,\alpha}(D_a)} + ||\nabla T||_{L^2(R_a)} + ||u||_{W^{3,2}(D_a)} + \int_{R_a} f(T) \le C,
\tag{2.10}
$$

with a constant C which is independent of $a > a_0$.

Proof. We postpone the proof of the uniform bound (2.10) to Section 3. Assuming that the a priori bound (2.10) holds for any (smooth) solution of (2.1)-(2.2), (2.5)-(2.7), we now prove the existence part of Theorem 2.1 by a degree argument. In order to set-up a fixed point problem we let $c \in \mathbb{R}$, $u \in \mathcal{C}^{2,\alpha}(D_a)$, $T \in \mathcal{C}^{1,\alpha}(R_a)$ be given, and take the homotopy parameter $\tau \in [0, 1]$. We extend T to D_a as in (2.3) and solve the following system of elliptic equations for the unknowns ψ_1 and ψ_2 with a given right side:

$$
\begin{aligned}
&\Delta\psi_1 - \psi_2 = 0 && \text{in } D_a\\
&\Delta\psi_2 = \tau\left(-\mathrm{curl}(T\vec{\rho}) + \frac{1}{\sigma}\mathrm{curl}(-cu_x + u\cdot\nabla u)\right) && \text{in } D_a\\
&\psi_1 = 0,\ \nabla\psi_1 = 0 && \text{on } \partial D_a
\end{aligned}
\tag{2.11}
$$

and define $v = \nabla^\perp\psi_1$. Having constructed v as above we let Z be the solution of the linear problem

$$
\begin{aligned}
&-cZ_x + \tau v\cdot\nabla Z - \Delta Z = \tau f(T) && \text{in } R_a\\
&Z(-a, y) = 1,\ Z(a, y) = 0 && \text{for } y \in [0, h]\\
&\frac{\partial Z}{\partial y}(x, 0) = \frac{\partial Z}{\partial y}(x, h) = 0 && \text{for } x \in [-a, a].
\end{aligned}
\tag{2.12}
$$

We now set the mapping:

$$
K(c, u, T, \tau) = (c - \theta_0 + \max\{T(x, y);\ x \in [0, a],\ y \in [0, h]\}, v, Z).
\tag{2.13}
$$

By construction a fixed point of $K_1 := K(\cdot, \cdot, \cdot, 1)$ is a solution of (2.1)-(2.2), (2.5)-(2.7).

The classical regularity results in [11, 12] imply that the operator

$$
K : \mathbb{R} \times \mathcal{C}^{2,\alpha}(D_a) \times \mathcal{C}^{1,\alpha}(R_a) \times [0, 1] \longrightarrow \mathbb{R} \times \mathcal{C}^{2,\alpha}(D_a) \times \mathcal{C}^{1,\alpha}(R_a)
$$

is well defined, continuous and compact. If we now calculate the Leray-Schauder degree $\deg(\mathrm{Id} - K(\cdot,\cdot,\cdot,0), B_R(0), 0)$ and find it to be different from 0 (for some sufficiently large radius R) then, in view of the uniform bounds (2.10) and the homotopy invariance of the Leray-Schauder degree, we conclude that K_1 must have a fixed point.

To this end, note that at $\tau = 0$ we may solve both (2.11) and (2.12) explicitly: $v = 0$ and the function Z is given by

$$Z(x,y) = \phi^c(x) := \frac{e^{-cx} - e^{-ca}}{e^{ca} - e^{-ca}}.$$

When $c \neq 0$ the function $\phi^c(x)$ satisfies

$$\phi^c_{xx} + c\phi^c_x = 0 \quad \text{ in } [-a,a], \qquad \phi^c(-a) = 1,\ \phi^c(a) = 0.$$

Therefore, the last two components of the operator $K(c,u,T,0)$ do not depend either on u or on T and K has then an explicit form:

$$K_0(c,u,T) := K(c,u,T,0) = (c - \theta_0 + M_T, 0, Z),$$

where

$$M_T = \max\{T(x,y);\ x \in [0,a],\ y \in [0,h]\}.$$

As in [5], we note that K_0 is homotopic to the map

$$\mathcal{F}(c,u,T) = (c - \theta_0 + \max_{x\in[0,a]} \phi^c(x), 0, \phi^{c^*}),$$

where c^* is the unique number such that

$$\max_{x\in[0,a]} \phi^{c^*}(x) = \phi^{c^*}(0) = \theta_0.$$

The a priori bound (2.10) implies that homotopy can be taken to be compact and without fixed points on the boundary of a sufficiently large ball $B_R(0)$ in $\mathbb{R}\times\mathcal{C}^{2,\alpha}(D_a)\times\mathcal{C}^{1,\alpha}(R_a)$ with the radius R independent of a.

By the homotopy invariance of the Leray-Schauder degree of compact perturbations of identity we conclude that

$$\deg(\mathrm{Id} - K_0, B_R(0), 0) = \deg(\mathrm{Id} - \mathcal{F}, B_R(0), 0) = 1,$$

as the degree of the last map $(\mathrm{Id} - \mathcal{F})(c,u,T) = (\theta_0 - \phi^c(0), u, T - \phi^{c^*})$ is the product of degrees of each component, all three of them being equal to 1. Hence the operator K_1 has to have a fixed point and the existence part of the proof of Theorem 2.1 is complete. ∎

3. A proof of the uniform bound (2.10).

Let $K(c,u,T,\tau)=(c,u,T)$, where K is defined as in (2.13). The proof of (2.10) will be achieved through a sequence of lemmas estimating norms of the quantities T,u,c. The main point is as in [5]: from the reaction-diffusion equation we deduce an upper bound for $\|\nabla T\|_{L^2}^2$ that is linear in the $\|u\|_{L^\infty}$; from the flow equation we deduce an upper bound for $\|u\|_{L^\infty}$ that is sub-quadratic, $\|\nabla T\|_{L^2}^\alpha$ with an exponent $\alpha\in(0,2)$. These two bounds allow us to close the argument and show that both quantities are bounded. The second upper bound is more difficult to obtain than in [5] because of the no-slip boundary conditions. Throughout the rest of the paper we denote by C any uniform constant, not depending on a,c,T,u.

Lemma 3.1. *We have:*

$$\|\psi_1\|_{W^{2,2}(D_a)}\le C\|\nabla T\|_{L^2(R_a)}. \tag{3.1}$$

Proof. By the standard elliptic estimates [12], we have

$$\|\psi_1\|_{W^{2,2}(D_a)}\le C\left(||\psi_1||_{L^2(D_a)}+||\Delta\psi_1||_{L^2(D_a)}\right).$$

Using Poincaré's inequality on $[0,h]$ we obtain:

$$||\psi_1||_{L^2(D_a)}\le C||\nabla\psi_1||_{L^2(D_a)}.$$

As

$$||\nabla\psi_1||^2_{L^2(D_a)}\le||\Delta\psi_1||_{L^2(D_a)}\cdot||\psi_1||_{L^2(D_a)},$$

we conclude that

$$||\psi_1||_{W^{2,2}(D_a)}\le C||\Delta\psi_1||_{L^2(D_a)}. \tag{3.2}$$

We multiply now the second equation in (2.11) by ψ_1 to obtain

$$\left|\int_{D_a}\Delta\psi_2\cdot\psi_1+\frac{\tau c}{\sigma}(\psi_2)_x\psi_1-\frac{\tau}{\sigma}\langle u,\nabla\psi_2\rangle\cdot\psi_1\right|\le||\psi_1||_{L^2(D_a)}\cdot||\nabla T||_{L^2(D_a)}. \tag{3.3}$$

Furthermore, using the boundary conditions for ψ_1 and $\Delta\psi_1=\psi_2$ we have

$$\int_{D_a}\Delta\psi_2\cdot\psi_1=-\int_{D_a}\nabla\psi_2\cdot\nabla\psi_1=\int_{D_a}\psi_2\cdot\Delta\psi_1=||\Delta\psi_1||^2_{L^2(D_a)},$$

$$\int_{D_a}(\psi_2)_x\psi_1=-\int_{D_a}(\nabla\psi_1)_x\cdot\nabla\psi_1=-\frac{1}{2}\int_{D_a}(|\nabla\psi_1|^2)_x=0,$$

$$\int_{D_a}\langle u,\nabla\psi_2\rangle\cdot\psi_1=\frac{1}{2}\int_{D_a}\langle\nabla^\perp(\psi_1^2),\nabla\psi_2\rangle=\frac{1}{2}\int_{\partial D_a}\psi_2\langle\nabla^\perp(\psi_1^2),\vec n\rangle=0.$$

Therefore, (3.3) becomes

$$\|\Delta\psi_1\|^2_{L^2(D_a)}\le\|\psi_1\|_{L^2(D_a)}\cdot\|\nabla T||_{L^2(D_a)}.$$

Finally, in view of (2.4) and (3.2) we see that (3.1) follows. ∎

We denote by $g \in \mathcal{C}^{0,\alpha}(D_a)$ the right hand side of the second equation in (2.11):

$$g = \tau \left(-\mathrm{curl}(T\vec{\rho}) + \frac{1}{\sigma}\mathrm{curl}(-cu_x + u \cdot \nabla u) \right). \tag{3.4}$$

The following bound follows from a standard local a-priori estimate on the regular solutions of the Stokes problem, which can be found for example in [11]. For completeness, we sketch a possible proof.

Lemma 3.2. *We have*

$$||\psi_1||_{W^{4,2}(D_a)} \leq C \left(||g||_{L^2(D_a)} + ||\psi_1||_{L^2(D_a)} \right). \tag{3.5}$$

Proof. The proof is based on the general regularity theory for elliptic systems [2]. Using the notation of this fundamental paper, the system (2.11) is described by the matrix L, and the boundary conditions (after local flattening of the boundary) are given by the matrix operator B:

$$L = \begin{bmatrix} \Delta & -1 \\ 0 & \Delta \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ \partial_z & 0 \end{bmatrix},$$

where by z we denote the variable transversal to ∂D_a. These operators act on the column vector $(\psi_1, \psi_2)^t$.

We assign the weights $t_1 = 4$, $t_2 = 2$, $s_1 = -2$, $s_2 = 0$, $r_1 = -4$, $r_2 = 3$. To obtain the local a-priori internal and up to the boundary estimate:

$$\|\psi_1\|_{W^{t_1,2}} \leq C\left[\|g\|_{W^{-s_2,2}} + \|\psi_1\|_{L^2} + \|\psi_2\|_{L^2}\right]. \tag{3.6}$$

one needs to check the related analytic Lopatinsky-Shapiro condition. To this end, consider a single mode of the Fourier transform of the solution on the hyperplanes parallel to the 'flattened' boundary of D_a. That is, for a fixed $\xi \in \mathbf{R} \setminus \{0\}$ consider a solution of the form: $(\psi_1, \psi_2) = (e^{ix\xi}\psi_1(z), e^{ix\xi}\psi_2(z))$. It satisfies the following system of ODEs (in $z \geq 0$):

$$\begin{array}{lll} -|\xi|^2\psi_1(z) + \psi_1''(z) & = \psi_2(t), & \psi_1(0) = 0 \\ -|\xi|^2\psi_2(z) + \psi_2''(z) & = 0, & \psi_1'(0) = 0. \end{array}$$

It is easy to see that the only bounded solution $(\psi_1(z), \psi_2(z))$ is $\psi_1, \psi_2 = 0$.

Consequently Theorems 10.4 and 10.5 in [2] imply (3.6). Recalling that $\psi_2 = \Delta\psi_1$ and using an interpolation inequality [12], the uniform global bound in (3.5) follows. ■

An upper bound for various quantities in terms of $\|u\|_{L^\infty}$ is provided by the following lemma.

Lemma 3.3. *We have*

$$|c| + ||\nabla T||^2_{L^2(R_a)} + \int_{R_a} f(T) \leq C(1 + ||u||_{L^\infty(D_a)}). \tag{3.7}$$

This estimate can be proved exactly as in [5] (see Lemmas 2.2, 2.5 and 2.6 of this reference), using the reaction-diffusion equation for T. In particular, it is here that the normalization condition (2.7) is used in order to obtain an upper bound for $|c|$ in (3.7), using the maximum principle and the sliding method. Then the bound for $f(T)$ follows immediately after integration of the 2.1 over R_a. The estimate for $\|\nabla T\|_{L^2}$ is somewhat more involved – we refer the reader to [5] for details.

We will also need the following elementary lemma:

Lemma 3.4. *Let $\alpha, \beta, \gamma \geq 0$. Then,*

(i) $\alpha \leq \beta + \gamma\alpha^{1/2}$ *implies* $\alpha \leq \beta^2 + (1+\gamma)^2$,

(ii) $\alpha \leq 1 + \beta^{5/4} + \beta\alpha^{1/4}$ *implies* $\alpha \leq C(1+\beta^{4/3})$, *and the constant C does not depend on α, β.*

Proof. In (i) it is enough to consider $\alpha \geq \beta^2$. In this case however, $\alpha \leq (1+\gamma)\alpha^{1/2}$ and the result follows.

In the same manner, the condition in (ii) implies $\alpha \leq C$ for $\alpha \geq (2\beta)^{4/3}$. ∎

Lemma 3.5. *We have the following uniform bound:*

$$||\nabla T||_{L^2(R_a)} \leq C.$$

Proof. We start by estimating $||\psi_1||_{W^{3,2}(D_a)}$. Using an interpolation inequality [1] we obtain

$$||\psi_1||_{W^{3,2}(D_a)} \leq C||\psi_1||^{1/2}_{W^{4,2}(D_a)} \cdot ||\psi_1||^{1/2}_{W^{2,2}(D_a)}.$$

where the constant C does not depend on a. Therefore by Lemma 3.2 and Lemma 3.1 we obtain, using the definition of the function g:

$$\begin{aligned} ||\psi_1||_{W^{3,2}(D_a)} &\leq C\Big(||\nabla T||_{L^2(D_a)} + \big(c + ||u||_{L^\infty(D_a)}\big)||\psi_1||_{W^{3,2}(D_a)}\Big)^{1/2} \cdot ||\psi_1||^{1/2}_{W^{2,2}(D_a)} \\ &\leq C\left(||\nabla T||^2_{L^2(R_a)} + \big(1 + ||u||_{L^\infty(D_a)}\big)\, ||\psi_1||_{W^{3,2}(D_a)}||\nabla T||_{L^2(R_a)}\right)^{1/2} \\ &\leq C\Big(||\nabla T||_{L^2(R_a)} + \big(1 + ||u||_{L^\infty(D_a)}\big)^{1/2}||\nabla T||^{1/2}_{L^2(R_a)}||\psi_1||^{1/2}_{W^{3,2}(D_a)}\Big), \end{aligned}$$

where we used $\mathrm{curl}(u \cdot \nabla u) = u \cdot \nabla(\mathrm{curl}\ u)$ and Lemma 3.3 to bound $|c|$.

Applying now Lemma 3.4 (i) with

$$\alpha = ||\psi_1||_{W^{3,2}(D_a)},\ \beta = C||\nabla T||_{L^2(R_a)},\ \gamma = C(1 + ||u||_{L^\infty(D_a)})^{1/2}||\nabla T||^{1/2}_{L^2(R_a)},$$

we arrive at

$$||\psi_1||_{W^{3,2}(D_a)} \leq C\Big(1 + ||\nabla T||^2_{L^2(R_a)} + ||u||_{L^\infty(D_a)}||\nabla T||_{L^2(R_a)}\Big). \tag{3.8}$$

On the other hand, applying twice the interpolation inequality of Theorem 5.8 [1], we obtain

$$||u||_{L^\infty(D_a)} \leq C||u||^{1/2}_{W^{1,4}(D_a)}||u||^{1/2}_{L^4(D_a)} \leq C||u||^{1/4}_{W^{2,2}(D_a)}||u||^{3/4}_{W^{1,2}(D_a)}. \tag{3.9}$$

Combining (3.8), (3.9) and recalling Lemma 3.1 we deduce

$$\begin{aligned}
||u||_{L^\infty(D_a)} &\leq C||\psi_1||^{1/4}_{W^{3,2}(D_a)}||\psi_1||^{3/4}_{W^{2,2}(D_a)}\\
&\leq C\left(1+||\nabla T||^2_{L^2(R_a)}+||u||_{L^\infty(D_a)}||\nabla T||_{L^2(R_a)}\right)^{1/4}\|\nabla T\|^{3/4}_{L^2(R_a)}\\
&\leq C\Big(1+||\nabla T||^{5/4}_{L^2(R_a)}+||u||^{1/4}_{L^\infty(D_a)}||\nabla T||_{L^2(R_a)}\Big).
\end{aligned}$$

We use now Lemma 3.4 (ii) to arrive at

$$||u||_{L^\infty(D_a)} \leq C(1+||\nabla T||^{4/3}_{L^2(R_a)}). \tag{3.10}$$

By Lemma 3.3 this yields the desired uniform estimate on $||\nabla T||_{L^2(R_a)}$. ■

Note now that Lemma 3.5 and the estimate (3.10) imply that

$$||u||_{L^\infty(D_a)} \leq C.$$

Other uniform bounds in (2.10) follow now from Lemma 3.3, Lemma 3.2, the bound (3.8) and from standard elliptic estimates for T in R_a. Also, the same bound as in Lemma 3.2 holds if we replace L^2 (and $W^{4,2}$) by L^p ($W^{4,p}$ respectively) for any $p \geq 2$. This ends the proof of (2.10) and that of Theorem 2.1.

Remark 3.6. In order to obtain uniform bounds on the quantities in (2.10), we needed to prove that $||u||_{L^\infty(D_a)} \leq C$. In the situation of [5], the stress-free boundary conditions imply that $\psi_2 = \text{curl } u$ is zero on ∂D_a. Multiplying the second equation in (2.11) by ψ_2 one obtains then $||\nabla\psi_2||_{L^2(D_a)} \leq C||\nabla T||_{L^2(R_a)}$. This, in view of Lemma 3.3 implies $||u||_{L^\infty(D_a)} \leq C||\psi_1||_{W^{3,2}(D_a)} \leq C||\nabla T||_{L^2(R_a)}$. In our situation, when ψ_2 is free on ∂D_a, one cannot proceed in a similar manner. A simpler case is when g does not depend on u, which happens at the infinite Prandtl number, $1/\sigma = 0$. In this particular case the calculations towards Lemma 3.5 are much simpler.

4. Identification of the limit and a proof of Theorem 1.1

For a proof of Theorem 1.1, we denote by c^a, T^a, u^a, p^a a solution of (2.1)-(2.2), (2.5)-(2.7). Note that by a bootstrap argument we also have

$$||T^a||_{\mathcal{C}^{2,\alpha}(D_{a-1})} \leq C.$$

Therefore we may choose a sequence $a_n \to \infty$ such that $c_n := c^{a_n}$ converges to some $c \in \mathbb{R}$ and $T_n := T^{a_n}$, $u_n := u^{a_n}$ converge in $\mathcal{C}^{2,\alpha}_{loc}(D)$ to some $T, u \in \mathcal{C}^{2,\alpha}(D)$. Also, $p_n := p^{a_n}$ converges in $\mathcal{C}^{1,\alpha}_{loc}(D)$ to some $p \in \mathcal{C}^{1,\alpha}_{loc}(D)$. Obviously, c, T, u, p must satisfy (1.1) and the correct boundary conditions at $x = 0$ and $x = h$ in (1.6). The normalization (2.7) and its consequence (2.8) imply that

$$\max\{T(x,y);\ x \geq 0,\ y \in [0,h]\} = \theta_0.$$

Since $\nabla T \in L^2(D) \cap \mathcal{C}^{1,\alpha}(D)$ and $u \in W^{3,2}(D) \cap \mathcal{C}^{2,\alpha}(D)$, we obtain (1.7). Note that by (2.10), we also have a bound for the total reaction rate:

$$\int_D f(T) < +\infty. \tag{4.1}$$

As in [5] (Lemma 3.2), we can prove the following lemma: we leave the minor modifications to the reader.

Lemma 4.1. *The propagation speed is positive: $c > 0$.*

Next, we show that temperature approaches constant limits at the two ends of the domain.

Lemma 4.2. *There exist $\theta_-, \theta_+ \in [0,1]$ such that*

$$\lim_{x\to\pm\infty} ||T(x,\cdot) - \theta_\pm||_{L^\infty([0,h])} = 0.$$

Proof. We argue by contradiction. Assume, for example, that $\lim_{n\to\infty} T(x_n, y_n) = \theta_1$ and let $\lim_{n\to\infty} T(\tilde{x}_n, \tilde{y}_n) = \theta_2$ with some $\theta_1 \neq \theta_2$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \tilde{x}_n = +\infty$. Integrating the first equation in (1.1) on $S_n := [x_n, \tilde{x}_n] \times [0,h]$ and using the boundary conditions we obtain

$$\begin{aligned} &-c\int_0^h [T(\tilde{x}_n, y) - T(x_n, y)]\,\mathrm{d}y \\ &\qquad = -\int_0^h [(Tu^1 - T_x)](\tilde{x}_n, y) - (Tu^1 - T_x)](x_n, y)]\,\mathrm{d}y + \int_{S_n} f(T), \end{aligned} \tag{4.2}$$

where u^1 refers to the horizontal component of the velocity vector u. Note that by (1.7) and (4.1), the right hand side in (4.2) converges to 0 as $n \to \infty$. At the same time, because of our assumptions on (x_n, y_n) and $(\tilde{x}_n, \tilde{y}_n)$, together with (1.7), the left hand side converges to $-ch(\theta_2 - \theta_1)$. In view of Lemma 4.1 we conclude that $\theta_1 = \theta_2$. ∎

Now, by (4.1), Lemma 4.2 implies that

$$f(\theta_-) = f(\theta_+) = 0.$$

As in [5] (Lemmas 3.3 and 3.4, respectively), one also has the following two results:

Lemma 4.3. *The temperature limits satisfy $\theta_- > \theta_+$.*

Lemma 4.4. *For every $\epsilon > 0$ there exists A such that for all sufficiently large n:*

$$|\nabla T_n(x,y)| \le \epsilon \qquad \forall (x,y) \in [A, a_n] \times [0,h].$$

We now deduce:

Lemma 4.5. *For every $\epsilon > 0$ there exists A such that for all sufficiently large n:*

$$|T_n(x,y)| \le \epsilon \qquad \forall (x,y) \in [A, a_n] \times [0,h].$$

Proof. If the claim is not true then there is some $\epsilon_0 > 0$ and a sequence A_n converging to $+\infty$ such that:

$$T_n(A_n, y_n) \geq \epsilon_0, \tag{4.3}$$

for some $y_n \in [0, h]$ and possibly passing to a subsequence in T_n. Notice that by Lemma 4.4 it must happen that

$$\lim_{n\to+\infty}(a_n - A_n) = +\infty. \tag{4.4}$$

Now, define $\Phi_n(x, y) = T_n(x + A_n, y)$ and $\zeta_n(x, y) = u_n(x + A_n, y)$ on domains R_{a_n} and D_{a_n} "shifted" to the left by the distance A_n. Using the normalization (2.8), we have on $\tilde{R}_n = [0, a_n - A_n] \times [0, h]$

$$-c_n(\Phi_n)_x + \zeta_n \cdot \nabla\Phi_n - \Delta\Phi_n = 0.$$

Multiplying the last equation by Φ_n and integrating over $\tilde{R}_n$ we obtain

$$c_n \int_{\tilde{R}_n} \partial_x|\Phi_n|^2 - \int_{\partial\tilde{R}_n} |\Phi_n|^2\zeta_n \cdot \nu + 2\int_{\partial\tilde{R}_n} \Phi_n\partial_\nu\Phi_n = 2\int_{\tilde{R}_n} |\nabla\Phi_n|^2,$$

where ν is the outward normal to $\partial\tilde{R}_n$. Using the boundary conditions, this yields

$$\int_0^h \Big[(\zeta_n^1 - c_n)|\Phi_n|^2 - 2\Phi_n\partial_x\Phi_n\Big](0, y)\ \mathrm{d}y \geq 0 \tag{4.5}$$

where the superscript in ζ_n^1 refers to the horizontal component of the vector ζ_n.

On the other hand, Φ_n and ζ_n obviously satisfy the same uniform bounds as T_n and u_n in (2.10) and therefore they converge (uniformly on compact sets, together with their derivatives) to some Φ and ζ defined by virtue of (4.4) on whole D. Moreover, Φ is a solution of

$$-c\Phi_x + \zeta \cdot \nabla\Phi - \Delta\Phi = 0 \tag{4.6}$$

and it converges to some $\Phi_\pm$ as $x \to \pm\infty$ (the argument is as in the proof of Lemma 4.2). Integrating (4.6) on D and using boundary conditions we obtain

$$c(\Phi_- - \Phi_+) = 0,$$

which by Lemma 4.1 implies $\Phi_- = \Phi_+$. Thus, by the maximum principle Φ must be constant and, say, equal to Φ_0. Obviously, by (4.3), $\Phi_0 \geq \epsilon_0 > 0$. Passing to the limit in (4.5), we obtain

$$|\Phi_0|^2 \int_0^h (\zeta^1 - c)(0, y)\ \mathrm{d}y \geq 0. \tag{4.7}$$

On the other hand, since div $u_n = 0$ and $u_n = 0$ on ∂D_{a_n}, we have $\int_0^h \zeta_n^1(0, y)\ \mathrm{d}y = 0$ which implies that $\int_0^h \zeta^1(0, y)\ \mathrm{d}y = 0$. Finally, (4.7) becomes

$$-hc|\Phi_0|^2 \geq 0,$$

which contradicts the positivity of c in Lemma 4.1. ■

In view of Lemma 4.5 we obtain $\theta_+ = 0$, which by Lemma 4.2 proves (1.8). The final statement, asserting that (1.9) is a sufficient condition for $\theta_- = 1$ is obtained exactly as in Lemma 3.7 of [5]. This completes the proof of Theorem 1.1.

References

[1] R.A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, **65**, Academic Press, New York-London, 1975.

[2] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*, Comm. Pure Appl. Math., **17** (1964), 35–92.

[3] M. Belk, B. Kazmierczak and V. Volpert, *Existence of reaction-diffusion-convection waves in unbounded strips*, Int. J. Math. Math. Sci. 2005, no. 2, 169–193.

[4] H. Berestycki, The influence of advection on the propagation of fronts in reaction-diffusion equations, in *Nonlinear PDEs in Condensed Matter and Reactive Flows*, NATO Science Series C, 569, H. Berestycki and Y. Pomeau eds, Kluwer, Doordrecht, 2003.

[5] H. Berestycki, P. Constantin and L. Ryzhik, *Non-planar fronts in Boussinesq reactive flows*, Ann. Inst. Henri Poincare, Analyse non Linéaire, **23** (2006), 407–437.

[6] H. Berestycki and B. Larrouturou, *A semi-linear elliptic equation in a strip arising in a two-dimensional flame propagation model*, J. Reine Angew. Math., **396** (1989), 14–40.

[7] H. Berestycki, B. Larrouturou and P.L. Lions, *Multi-dimensional traveling wave solutions of a flame propagation model*, Arch. Rational Mech. Anal., **111** (1990), 33–49.

[8] H. Berestycki, B. Nicolaenco and B. Scheurer, Traveling wave solutions to combustion models and their singular limits, SIAM Jour. Math. Anal., **16**, 1983, 1207-1242.

[9] P. Constantin, A. Kiselev and L. Ryzhik, *Fronts in reactive convection: bounds, stability and instability*, Comm. Pure Appl. Math., **56** (2003), 1781–1803.

[10] A. Ducrot, M. Marion and V. Volpert, *Reaction-diffusion-convection problems with non Fredholm operators*, Int. J. Pure Appl. Math., **27** (2006), no 2, 179–204.

[11] G.P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*, Springer Tracts in Natural Philosophy, **38, 39**, Springer, 1998.

[12] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 2001.

[13] J.M. Roquejoffre, *Eventual monotonicity and convergence to traveling fronts for the solutions of parabolic equations in cylinders*, Ann. Inst. Henri Poincare, **14** (1997), no 4, 499–552.

[14] R. Texier-Picard and V. Volpert, *Problemes de reaction-diffusion-convection dans des cylindres non bornes*, C. R. Acad. Sci. Paris Sr. I Math., **333** (2001), 1077–1082.

[15] R. Texier-Picard and V. Volpert, Reaction-diffusion-convection problems in unbounded cylinders. Rev. Mat. Complut., **16** (2003), 233–276.

[16] N. Vladimirova and R. Rosner, *Model flames in the Boussinesq limit: the effects of feedback*, Phys. Rev. E., **67** (2003), 066305.

[17] J. Xin, *Front propagation in heterogeneous media*, SIAM Review **42** (2000), no 2, 161–230.

P.C. and L.R., University of Chicago, Department of Mathematics, 5734 S. University Ave., Chicago, IL 60637

M.L., University of Minnesota, Department of Mathematics, 127 Vincent Hall, 206 Church St. S.E., Minneapolis, MN 55455