SPECTRAL STABILITY CONDITIONS FOR SHOCK WAVE PATTERNS

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Abstract. We compare inviscid stability conditions obtained by Lewicka for large-amplitude shock wave patterns with "slow eigenvalue", or low-frequency, stability conditions obtained by Lin and Schecter through a vanishing viscosity analysis of the Dafermos regularization. Under the structural condition that scattering coefficients for each component wave are positive, we show that BV and L^1 inviscid stability are equivalent to respective versions of low-frequency Dafermos-regularized stability. When scattering coefficients appear with different signs, the conditions are in general distinct. We give various examples demonstrating this phenomenon and indicating the subtle role of cancellation in linearized behavior in the presence of negative scattering coefficients.

1. Setting of the problem

The purpose of this study is to establish the correspondence between the two stability conditions for patterns of large noninteracting shocks. The patterns we have in mind arise as solutions to Riemann problems for a hyperbolic system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0. \tag{1.1}$$

In the study of the inviscid stability [BM, Le1] one formulates stability conditions ensuring that the flow of (1.1) is a contraction, with respect to the distance in u which depends on the position of a wave in the (t, x) plane.

On the other hand, the viscous stability as studied in [LS] introduces conditions on the eigenvalues of the Dafermos operator; the so-called fast eigenvalues are connected with the stability of viscous shock profiles and the slow eigenvalues correspond to the shifts and amplifications of the traveling waves.

For the discussion of the stability conditions and examples of their validation in various physically interesting systems (eg 1-dimensional gas dynamics) we refer to [BM, LS, Le3]. Here, we are solely interested in studying the relations between the two conditions; indeed our main result claims their equivalence, under some structural assumptions.

We now explain the structure of the wave patterns whose stability one wants to study. Consider the following piecewise constant (Riemann) solution to (1.1):

$$\bar{u}(t,x) = \begin{cases} \bar{u}_0 & \text{for } x/t < \Lambda^1, \\ \bar{u}_i & \text{for } \Lambda^i < x/t < \Lambda^{i+1}, \quad i : 1 \dots n-1, \\ \bar{u}_n & \text{for } x/t > \Lambda^n. \end{cases} \tag{1.2}$$

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Each discontinuity $(\bar{u}_{i-1}, \bar{u}_i)$, $i : 1 \dots n$ connects two disctinct states $\bar{u}_{i-1}$ and $\bar{u}_i$ in $\mathbf{R}^n$. We assume that in a sufficiently small neighbourhood of the states $\bar{u}_0 \dots \bar{u}_n$ the system (1.1) is strictly hyperbolic and genuinely nonlinear. We call the eigenvalues of the Jacobian matrix $\mathrm{D}f(u)$ by $\lambda_1(u) < \dots < \lambda_n(u)$, and the corresponding right eigenvectors of the Jacobian are $\{r_i(u)\}_{i=1}^n$. With this notation we assume that $\nabla\lambda_i \cdot r_i > 0$ for every $i : 1 \dots n$.

Finally, each $(\bar{u}_{i-1}, \bar{u}_i)$ is a Lax compressive, Majda stable shock having speed Λ^i:

$$\Lambda^i \cdot (\bar{u}_i - \bar{u}_{i-1}) = f(\bar{u}_i) - f(\bar{u}_{i-1}), \tag{1.3}$$

$$\lambda_{i-1}(\bar{u}_{i-1}) < \Lambda^i < \lambda_i(\bar{u}_{i-1}) \quad \text{and} \quad \lambda_i(\bar{u}_i) < \Lambda^i < \lambda_{i+1}(\bar{u}_i), \tag{1.4}$$

$$\det\left[r_1(\bar{u}_{i-1}) \dots r_{i-1}(\bar{u}_{i-1}), \bar{u}_i - \bar{u}_{i-1}, r_{i+1}(\bar{u}_i) \dots r_n(\bar{u}_i)\right] \neq 0. \tag{1.5}$$

In the next section we recall the stability conditions from [Le2, LS] and introduce our main result which states their equivalence (under some assumptions). Section 3 contains the proof of this theorem; in section 4 we discuss when its assumptions may be relaxed, while in section 5 we show that in general they are optimal. Another example is given in section 4, together with its more detailed analysis.

2. The main result

In studying the linearized stability of (1.2) one considers the following system describing the evolution of the first order perturbation $v(t,x)$ of the solution $\bar{u}$:

$$v_t + \mathrm{D}f(\bar{u}_i)v_x = 0 \quad \text{for} \quad \Lambda^i < x/t < \Lambda^{i+1}, \quad i : 0 \dots n. \tag{2.1}$$

Above $\Lambda^0 = -\infty$, $\Lambda^{n+1} = +\infty$. This linear PDE is supplemented by the boundary conditions obtained by linearizing the Rankine-Hugoniot equations along the shocks:

$$\begin{aligned}(\mathrm{D}f(\bar{u}_i) - \Lambda^i\mathrm{Id})v(t, \Lambda^i t+) - &(\mathrm{D}f(\bar{u}_{i-1}) - \Lambda^i\mathrm{Id})v(t, \Lambda^i t-) \\ &= S^i(t)(\bar{u}_i - \bar{u}_{i-1}) \qquad \forall i : 1 \dots n,\end{aligned} \tag{2.2}$$

where $S^i(t)$ is yet another unknown. Indeed, writing the Taylor expansions for $f(u)$ and $\Lambda^i(t)$ with respect to perturbation v, we obtain:

$$\begin{aligned} f(\bar{u}_i + v(t,\Lambda^i t+)) &- f(\bar{u}_{i-1} + v(t, \Lambda^i t-)) \\ &= \mathrm{D}f(\bar{u}_i)v(t, \Lambda^i t+) - \mathrm{D}f(\bar{u}_{i-1})v(t, \Lambda^i t-) \\ &\quad + \Lambda^i \cdot (\bar{u}_i - \bar{u}_{i-1}) + \mathcal{O}(|v(t, \Lambda^i t-)|^2 + |v(t, \Lambda^i t+)|^2), \\ \Lambda^i(t) \cdot \big(\bar{u}_i &+ v(t, \Lambda^i t+) - \bar{u}_{i-1} - v(t, \Lambda^i t-)\big) \\ &= \Lambda_i \cdot \big(\bar{u}_i - \bar{u}_{i-1} + v(t, \Lambda^i t+) - v(t, \Lambda^i t-)\big) \\ &\quad + S^i(t) \cdot (\bar{u}_i - \bar{u}_{i-1}) + \mathcal{O}(|v(t, \Lambda^i t-)|^2 + |v(t, \Lambda^i t+)|^2). \end{aligned}$$

Equating the right hand sides of the above formulas gives (2.2).

In the study of inviscid stability of (1.2), the system (2.1) (2.2) describes the scattering of incoming small waves by the large shocks. It has been shown in [Le1] that for the stability of these solutions of the nonlinear system (1.1) which remain in the vicinity of (1.2), it is sufficient to require that, in some norm, the total amount of the scattered waves is smaller than the total weight of the incoming waves.

For each $i : 1 \dots n$ let v_i be the i-th component of v in the basis $\{r_1 \dots r_n\}$ so that $v = \sum_i v_i r_i$. The basic stability assumption in this context is the existence of positive weights $w_i = w_i(x/t)$ such that the TV or the L^1 norm of the function $x \mapsto \sum_i w_i(t,x) \cdot |v_i(t,x)|$ is non-increasing in time, for every solution of (2.1) (2.2).

We now recall another, equivalent version of these stability conditions. First of all, for each $i : 1 \dots n$ let V_i be the $(n-1) \times n$ matrix whose rows span the orthogonal complement of the vector $\bar{u}_i - \bar{u}_{i-1}$. Call F_i the $(n-1)\times(n-1)$ matrix of the form:

$$F_i = -\Bigg[V_i \cdot \Big[\mathrm{D}f(\bar{u}_{i-1}) - \Lambda^i \mathrm{Id}\Big] \cdot \Big[\{r_j(\bar{u}_{i-1})\}_{j=1}^{i-1}\Big], \quad V_i \cdot \Big[\mathrm{D}f(\bar{u}_i) - \Lambda^i \mathrm{Id}\Big] \cdot \Big[\{r_j(\bar{u}_i)\}_{j=i+1}^{n}\Big]\Bigg] \tag{2.3}$$

$$= -V_i \cdot \left[\Big\{(\lambda_j(\bar{u}_{i-1}) - \Lambda^i) \cdot r_j(\bar{u}_{i-1})\Big\}_{j=1}^{i-1}, \Big\{(\lambda_j(\bar{u}_i) - \Lambda^i) \cdot r_j(\bar{u}_i)\Big\}_{j=i+1}^{n}\right]$$

By (1.4) (1.5) each F_i is invertible and therefore we may define further matrices:

$$M_i^{left} = -F_i^{-1} \cdot V_i \cdot \Big[\mathrm{D}f(\bar{u}_{i-1}) - \Lambda^i \mathrm{Id}\Big] \cdot \Big[\{r_j(\bar{u}_{i-1})\}_{j=i}^{n}\Big], \tag{2.4}$$

$$M_i^{right} = -F_i^{-1} \cdot V_i \cdot \Big[\mathrm{D}f(\bar{u}_i) - \Lambda^i \mathrm{Id}\Big] \cdot \Big[\{r_j(\bar{u}_i)\}_{j=1}^{i}\Big]. \tag{2.5}$$

As shown in [Le2], the elements of M_i^{left} and M_i^{right} are the ratios of strengths of the small outgoing waves and the small incoming waves at their left/right interaction with $(\bar{u}_{i-1}, \bar{u}_i)$, respectively.

Define the square $n \cdot (n-1)$ dimensional matrix:

$$M = \begin{bmatrix} [\Theta] & M_1^{right} & & & \\ M_2^{left} & [\Theta] & M_2^{right} & & \\ & M_3^{left} & [\Theta] & M_3^{right} & \\ & & \ddots & \ddots & \\ & & & M_n^{left} & [\Theta] \end{bmatrix}.$$

Above $[\Theta]$ stands for the $(n-1)\times(n-1)$ zero matrix. Let M_1 be defined in the same manner as M, but with the submatrices M_i^{right}, $i : 1 \dots n-1$ replaced by:

$$\mathrm{diag}\Big\{\{\Lambda^i - \lambda_j(\bar{u}_{i-1})\}_{j=1}^{i-1}, \{\lambda_j(\bar{u}_i) - \Lambda^i\}_{j=i+1}^{n}\Big\} \cdot M_i^{right} \cdot \mathrm{diag}\Big\{\big(\Lambda^i - \lambda_j(\bar{u}_i)\big)^{-1}\Big\}_{j=1}^{i}, \tag{2.6}$$

and the submatrices M_i^{left}, $i : 2 \dots n$ replaced by:

$$\mathrm{diag}\Big\{\{\Lambda^i - \lambda_j(\bar{u}_{i-1})\}_{j=1}^{i-1}, \{\lambda_j(\bar{u}_i) - \Lambda^i\}_{j=i+1}^{n}\Big\} \cdot M_i^{left} \cdot \mathrm{diag}\Big\{\big(\lambda_j(\bar{u}_{i-1}) - \Lambda^i\big)^{-1}\Big\}_{j=i+1}^{n}. \tag{2.7}$$

The BV and the L^1 stability conditions in [Le1, Le2] read:

$$\mathrm{specRad}\, |M| < 1, \tag{BV}$$

$$\mathrm{specRad}\, |M_1| < 1, \tag{L1}$$

where specRad stands for the spectral radius and $|\cdot|$ denotes taking the absolute values of the entries of a given matrix.

Another approach to study the stability of the Riemann solution (1.2) was proposed in [LS]. After changing the variables

$$X = x/t, \quad T = \ln t, \tag{2.8}$$

the viscous regularization of (1.1) takes the form:

$$u_T + (\mathrm{D}f(u) - X\mathrm{Id})u_X = \epsilon u_{XX}, \tag{2.9}$$

with $\epsilon = e^{-T}$. Freezing a small ϵ, the solution to (2.9) of the form $u = u_\epsilon(X)$ and satisfying $u(-\infty) = \bar{u}_0,\ u(+\infty) = \bar{u}_n$, approximates well the solution to (2.9) with variable ϵ; and it converges to $\bar{u}(t,x)$ as $T \to +\infty$. In this regard, one wants to study the spectrum of the linearization of the Dafermos system (2.9) at the Riemann-Dafermos solution u_ϵ. The leading coefficients of the Taylor expansions of the eigenvalues and eigenfunctions of the Dafermos operator, computed at the shock positions Λ^i turn out to be related to the eigenvalues and eigenfunctions of (2.1) (2.2).

More precisely, after the change of variables (2.8), the solution to (2.1) (2.2) of the form $u(T,X) = e^{\lambda T}U(X)$, $S^i(T) = e^{\lambda T}S^i$ satisfies:

$$\lambda U + (\mathrm{D}f(\bar{u}_i) - X\mathrm{Id})U_X = 0 \quad \text{for} \quad \Lambda^i < X < \Lambda^{i+1}, \quad \forall i : 0 \ldots n, \tag{2.10}$$

$$\begin{aligned} &(\mathrm{D}f(\bar{u}_i) - \Lambda^i\mathrm{Id})U(\Lambda^i+) - (\mathrm{D}f(\bar{u}_{i-1}) - \Lambda^i\mathrm{Id})U(\Lambda^i-) \\ &\qquad\qquad\qquad = S^i \cdot (\bar{u}_i - \bar{u}_{i-1}), \quad \forall i : 1 \ldots n. \end{aligned} \tag{2.11}$$

Complement (2.10) (2.11) by the condition:

$$U(X) = 0 \quad \text{for } X < \Lambda^1 \text{ and } X > \Lambda^n. \tag{2.12}$$

In this setting, we will prove:

Main Theorem. *Assume that the entries of the matrix M are all nonnegative. Then:*

(i) *Condition (BV) is equivalent to: all solutions to (2.10) (2.11) (2.12) have $Re\ \lambda < 0$.*

(ii) *Condition (L1) is equivalent to: all solutions to (2.10) (2.11) (2.12) have $Re\ \lambda < -1$.*

3. A proof of the Main Theorem

We will look for a solution of (2.10) (2.11) (2.12) having the form:

$$U(X) = \sum_{j=1}^{n} \alpha_j(X) \cdot r_j(\bar{u}_i) \quad \text{for } \Lambda^i < X < \Lambda^{i+1}, \quad \forall i : 0 \ldots n, \tag{3.1}$$

assuming the convention $\Lambda^0 = -\infty,\ \Lambda^{n+1} = +\infty$. By (2.11) (2.12) we have:

$$V_1 \cdot \sum_{j=1}^{n} (\lambda_j(\bar{u}_1) - \Lambda^1) \cdot \alpha_j(\Lambda^1+) \cdot r_j(\bar{u}_1) = 0. \tag{3.2}$$

Using (2.5), the equality (3.2) becomes:

$$0 = -\alpha_1(\Lambda^1+) \cdot F_1 \cdot M_1^{right} - F_1 \cdot \begin{bmatrix} \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} (\Lambda^1+)$$

and therefore we obtain:

$$0 = \begin{bmatrix} \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} (\Lambda^1+) + \alpha_1(\Lambda^1+) \cdot M_1^{right}. \tag{3.3}$$

The solution to (2.10) on $[\Lambda^1, \Lambda^2]$ with initial data given at Λ^1 by $\{\alpha_j(\Lambda^1+)\}_{j=1}^n$ is easily computed to be:

$$\alpha_j(X) = \alpha_j(\Lambda^1+) \cdot \left(\frac{X - \lambda_j(\bar{u}_1)}{\Lambda^1 - \lambda_j(\bar{u}_1)} \right)^\lambda.$$

Note that since $\lambda_1(\bar{u}_1) < \Lambda^1 \leq X \leq \Lambda^2 < \lambda_2(\bar{u}_1)$ (by virtue of (1.4)) the expressions $X - \lambda_j(\bar{u}_1)$ and $\Lambda^1 - \lambda_j(\bar{u}_1)$ have always the same sign. Therefore $\alpha_j(X)$ are well defined. We then have:

$$\alpha_j(\Lambda^2-) = \alpha_j(\Lambda^1+) \cdot \left(\frac{\Lambda^1 - \lambda_j(\bar{u}_1)}{\Lambda^2 - \lambda_j(\bar{u}_1)} \right)^\lambda \quad \forall j : 1 \ldots n. \tag{3.4}$$

We now wish to solve (2.11) (3.4). We obtain:

$$\begin{aligned}
0 &= V_2 \cdot \left[\big(\mathrm{D}f(\bar{u}_2) - \Lambda^2 \mathrm{Id}\big)\, U(\Lambda^2+) - \big(\mathrm{D}f(\bar{u}_1) - \Lambda^2 \mathrm{Id}\big)\, U(\Lambda^2+) \right] \\
&= V_2 \cdot \left[\sum_{j=1}^n (\lambda_j(\bar{u}_2) - \Lambda^2) \cdot \alpha_j(\Lambda^2+) \cdot r_j(\bar{u}_2) - \sum_{j=1}^n (\lambda_j(\bar{u}_1) - \Lambda^2) \cdot \alpha_j(\Lambda^2-) \cdot r_j(\bar{u}_1) \right] \\
&= V_2 \cdot \Bigg[\sum_{j>2} (\lambda_j(\bar{u}_2) - \Lambda^2) \cdot \alpha_j(\Lambda^2+) \cdot r_j(\bar{u}_2) \\
&\qquad\qquad - \sum_{j<2} (\lambda_j(\bar{u}_1) - \Lambda^2) \cdot \alpha_j(\Lambda^1+) \cdot \left(\frac{\Lambda^2 - \lambda_j(\bar{u}_1)}{\Lambda^1 - \lambda_j(\bar{u}_1)} \right)^\lambda \cdot r_j(\bar{u}_1) \Bigg] \\
&\quad + V_2 \cdot \Bigg[\sum_{j\leq 2} (\lambda_j(\bar{u}_2) - \Lambda^2) \cdot \alpha_j(\Lambda^2+) \cdot r_j(\bar{u}_2) \\
&\qquad\qquad - \sum_{j\geq 2} (\lambda_j(\bar{u}_1) - \Lambda^2) \cdot \alpha_j(\Lambda^1+) \cdot \left(\frac{\Lambda^2 - \lambda_j(\bar{u}_1)}{\Lambda^1 - \lambda_j(\bar{u}_1)} \right)^\lambda \cdot r_j(\bar{u}_1) \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= V_2 \cdot \left[(\lambda_1(\bar u_1) - \Lambda^2)\cdot r_1(\bar u_1), \Big\{ (\lambda_j(\bar u_2) - \Lambda^2)\cdot r_j(\bar u_2)\Big\}_{j>2} \right] \\
&\qquad\qquad\qquad \cdot \operatorname{diag}\left\{ \left(\frac{\Lambda^2 - \lambda_1(\bar u_1)}{\Lambda^1 - \lambda_1(\bar u_1)}\right)^{\lambda}, 1, \dots, 1 \right\} \cdot \begin{bmatrix} -\alpha_1(\Lambda^1+) \\ \alpha_3(\Lambda^2+) \\ \vdots \\ \alpha_n(\Lambda^2+) \end{bmatrix} \\
&\quad - V_2 \cdot \left[\Big\{ (\lambda_j(\bar u_1) - \Lambda^2) r_j(\bar u_1)\Big\}_{j\geq 2} \right] \\
&\qquad\qquad\qquad \cdot \operatorname{diag}\left\{ \left(\frac{\Lambda^2 - \lambda_j(\bar u_1)}{\Lambda^1 - \lambda_j(\bar u_1)}\right)^{\lambda} \right\}_{j\geq 2} \cdot \begin{bmatrix} \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} (\Lambda^1+) \\
&\quad + V_2 \cdot \left[\Big\{ (\lambda_j(\bar u_2) - \Lambda^2) r_j(\bar u_2)\Big\}_{j\leq 2} \right] \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} (\Lambda^2+).
\end{aligned}
$$

Noting (2.3) (2.4) (2.5) and recalling the invertibility of F_2, we arrive at:

$$
\begin{aligned}
0 =& \operatorname{diag}\left\{ \left(\frac{\Lambda^2 - \lambda_1(\bar u_1)}{\Lambda^1 - \lambda_1(\bar u_1)}\right)^{\lambda}, 1, \dots, 1 \right\} \cdot \begin{bmatrix} -\alpha_1(\Lambda^1+) \\ \alpha_3(\Lambda^2+) \\ \vdots \\ \alpha_n(\Lambda^2+) \end{bmatrix} \\
& - M_2^{left} \cdot \operatorname{diag}\left\{ \left(\frac{\Lambda^2 - \lambda_j(\bar u_1)}{\Lambda^1 - \lambda_j(\bar u_1)}\right)^{\lambda} \right\}_{j=2}^{n} \cdot \begin{bmatrix} \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} (\Lambda^1+) \\
& + M_2^{right} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} (\Lambda^2+).
\end{aligned} \tag{3.5}
$$

Now similarily to (3.4) we calculate:

$$
\alpha_j(\Lambda^3-) = \alpha_j(\Lambda^2+) \cdot \left(\frac{\Lambda^3 - \lambda_j(\bar u_2)}{\Lambda^2 - \lambda_j(\bar u_2)}\right)^{\lambda} \quad \forall j : 1 \dots n.
$$

Proceeding in the same manner, we obtain the corresponding relations involving $\{\alpha_j\}_{j=1}^n$ across every shock $(\bar u_i, \bar u_{i+1})$. Eventually, considering the n-th shock $(\bar u_{n-1}, \bar u_n)$ we arrive at:

$$
\alpha_j(\Lambda^n-) = \alpha_j(\Lambda^{n-1}+) \cdot \left(\frac{\Lambda^n - \lambda_j(\bar u_{n-1})}{\Lambda^{n-1} - \lambda_j(\bar u_{n-1})}\right)^{\lambda} \quad \forall j : 1 \dots n.
$$

Since by (2.12) all $\alpha_j(\Lambda^n+) = 0,\ j : 1 \dots n$, (2.11) implies:

$$
\begin{aligned}
0 =& \operatorname{diag}\left\{ \left(\frac{\Lambda^n - \lambda_j(\bar u_{n-1})}{\Lambda^{n-1} - \lambda_j(\bar u_{n-1})}\right)^{\lambda} \right\}_{j=1}^{n-1} \cdot \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} (\Lambda^{n-1}+) \\
& + M_n^{left} \cdot \left(\frac{\Lambda^n - \lambda_n(\bar u_{n-1})}{\Lambda^{n-1} - \lambda_n(\bar u_{n-1})}\right)^{\lambda} \cdot \alpha_n(\Lambda^{n-1}+).
\end{aligned} \tag{3.6}
$$

Summarizing, we have obtained the following vector equality:

$$\begin{bmatrix} N_1 & M_1^{right} & & & \\ \tilde{M}_2^{left} & N_2 & M_2^{right} & & \\ & \tilde{M}_3^{left} & N_3 & M_3^{right} & \\ & & \ddots & \ddots & \\ & & & \tilde{M}_n^{left} & N_n \end{bmatrix} \cdot \begin{bmatrix} \chi_1 \\ \vdots \\ \vdots \\ \chi_n \end{bmatrix} = 0, \tag{3.7}$$

where each χ_i is a (complex) $n-1$ dimensional vector:

$$\chi_i = \begin{bmatrix} \alpha_1(\Lambda^{i-1}+) \\ \vdots \\ \alpha_{i-1}(\Lambda^{i-1}+) \\ \alpha_{i+1}(\Lambda^{i}+) \\ \vdots \\ \alpha_n(\Lambda^{i}+) \end{bmatrix}, \qquad \forall i : 1 \dots n$$

and the matrices $N_i, \tilde{M}_i^{left}$ (each of them having $n-1$ columns) are given by:

$$\begin{aligned} N_i &= \operatorname{diag}\left\{ \left\{ \left(\frac{\Lambda^i - \lambda_j(\bar{u}_{i-1})}{\Lambda^{i-1} - \lambda_j(\bar{u}_{i-1})} \right)^{\lambda} \right\}_{j=1}^{i-1}, 1, \dots, 1 \right\}, \\ \tilde{M}_i^{left} &= -M_i^{left} \cdot \operatorname{diag}\left\{ \left(\frac{\Lambda^i - \lambda_j(\bar{u}_{i-1})}{\Lambda^{i-1} - \lambda_j(\bar{u}_{i-1})} \right)^{\lambda} \right\}_{j=i}^{n}. \end{aligned}$$

We thus see that for $\lambda \in \mathbf{C}$ to be a (complex) eigenvalue of the problem (2.10) (2.11) (2.12), we need that the matrix in (3.7) is singular. This is in turn equivalent to the non-invertibility of the following matrix:

$$P_\lambda = M - (D)^\lambda, \tag{3.8}$$

where $D = \operatorname{diag}(D_1 \dots D_n)$ and:

(3.9)

$$D_i = \operatorname{diag}\left\{ \left\{ \left(\frac{\Lambda^i - \lambda_j(\bar{u}_{i-1})}{\Lambda^{i-1} - \lambda_j(\bar{u}_{i-1})} \right) \right\}_{j<i}, \left\{ \left(\frac{\Lambda^i - \lambda_j(\bar{u}_{i})}{\Lambda^{i+1} - \lambda_j(\bar{u}_{i})} \right) \right\}_{j>i} \right\} \quad \forall i : 1 \dots n.$$

Notice that D is a real diagonal matrix whose all diagonal elements are bigger than 1 (by (1.4)). Assuming all entries of M to be nonnegative, we have $|M| = M$. The proof of Main Theorem (i) is now a direct consequence of the following lemma.

Lemma 3.1. *Let Q be a $n \times n$ matrix with real nonnegative elements. Let S be a diagonal matrix of the same dimension with each diagonal element real and > 1. Then the following two conditions are equivalent:*

(i) specRad $Q < 1$.

(ii) *For every complex number λ with* Re $\lambda \geq 0$ *the matrix $P_\lambda = Q - (S)^\lambda$ is invertible.*

Proof. **(i)** $\implies$ **(ii)**. Since specRad $Q < 1$, then for some diagonal matrix W with positive diagonal elements there holds:

$$||WQW^{-1}||_1 < 1. \tag{3.10}$$

Here $||A||_1$ denotes the maximum of the sums of absolute values of A's entries in columns. The proof of this fact is elementary; it follows by Gershgorin's theorem and may be also found in [LY, Theorem 1 in Appendix 1]. Since

$$WP_\lambda W^{-1} = WQW^{-1} - (WSW^{-1})^\lambda,$$

is invertible iff P_λ is invertible, we may without loss of generality assume that already $||Q||_1 < 1$. Assume that for some complex λ with $Re\ \lambda \geq 0$ and a nonzero vector $v = [v_1 \dots v_n] \in \mathbf{C}^n$ we have $v \cdot P_\lambda = 0$. Let $|v_i| = \max_{j:1\dots n} |v_j| \neq 0$. Then:

$$|v_i \cdot (S_{ii})^\lambda| = \left| \sum_{j=1}^n Q_{ji} \cdot v_j \right| \leq |v_i| \cdot \sum_{j=1}^n Q_{ji}.$$

Thus

$$(S_{ii})^{Re\ \lambda} = |(S_{ii})^\lambda| \leq \sum_{j=1}^n Q_{ji} < 1,$$

which is a contradiction with $S_{ii} > 1$ for all $i : 1 \dots n$.

(**ii**) $\Longrightarrow$ (**i**) (suggested by prof. Denis Serre). Consider the continuous function $g : [0, +\infty) \longrightarrow \mathbf{R}$, $g(\lambda) = \text{specRad } (S^{-\lambda}Q)$. Since all diagonal entries of S are > 1, we have $\lim_{\lambda \to +\infty} g(\lambda) = 0$.

Now if (i) is not satisfied then $g(0) \geq 1$ so by the mean value theorem $g(\lambda_0) = 1$ for some $\lambda_0 \geq 0$. By the nonnegativity of the matrices $S^{-\lambda}Q$ and the Perron-Frobenius theorem this implies $1 \in \text{spec } S^{-\lambda_0}Q$. This is equivalent to $P_{\lambda_0} = Q - S^{\lambda_0}$ is noninvertible and contradicts (ii). ∎

This ends the proof of Main Theorem (i). In order to deduce (ii), define for $i : 1 \dots n$:

$$\begin{aligned} D_i^{left} &= \text{diag}\left\{ \left\{ \Lambda^i - \lambda_j(\bar{u}_{i-1}) \right\}_{j<i}, \left\{ \lambda_j(\bar{u}_i) - \Lambda^i \right\}_{j>i} \right\}, \\ D_i^{right} &= \text{diag}\left\{ \left\{ \left(\Lambda^{i-1} - \lambda_j(\bar{u}_{i-1}) \right)^{-1} \right\}_{j<i}, \left\{ \left(\lambda_j(\bar{u}_i) - \Lambda^{i+1} \right)^{-1} \right\}_{j>i} \right\}, \end{aligned}$$

$$D^{left} = \text{diag}\left\{ D_1^{left} \dots D_n^{left} \right\}, \qquad D^{right} = \text{diag}\left\{ D_1^{right} \dots D_n^{right} \right\},$$

Note that by (2.6) (2.7) and (3.9) we have:

$$M_1 = D^{left} \cdot M \cdot D^{right}, \qquad D^{left} \cdot D^{right} = D.$$

Since the matrices D, D^{left}, D^{right} commute, we arrive at:

$$D^{left} \cdot P_\lambda \cdot D^{right} = M_1 - (D)^{\lambda+1}. \tag{3.11}$$

Now recalling that λ is an eigenvalue of (2.10) (2.11) (2.12) iff the left hand side of (3.11) is singular, we conclude (ii) in Main Theorem directly from Lemma 3.1.

4. Further remarks and an example

Note that in Lemma 3.1, specRad $|Q| < 1$ implies (ii) even without assuming the nonnegativity of the elements of Q. Therefore both (BV) and (L1) still imply the spectral properties of (2.10) (2.11) (2.12) as in the statement of Main Theorem, without assuming that $M = |M|$ (or equivalently $M_1 = |M_1|$).

On the other hand, the converse implications are not true in general for signed matrices M. If all diagonal entries of S are equal to the same number $s > 1$, then $P_\lambda = Q - (S)^\lambda = Q - s^\lambda \mathrm{Id}$ and it is invertible iff $s^\lambda \notin \mathrm{spec}\ Q$. Since $|s^\lambda| = s^{Re\ \lambda}$, (i) and (ii) in Lemma 3.1 are equivalent without any assumption on Q. Since (BV) may be violated even if specRad $M < 1$, we see that the spectral conditions for (2.10) (2.11) (2.12) are in general weaker than the corresponding (BV) or (L1) conditions.

We should note however, that for $n = 2$ we have specRad $M = \mathrm{specRad}|M| = (|M_{12} \cdot M_{21}|)^{1/2}$. For two given real numbers $s > 1$ and $a \in \mathbf{R}$, one always has $|a| < 1$ iff $a \neq s^\lambda$ for every complex λ with $Re\ \lambda \geq 0$. Therefore the assertion of Main Theorem holds true for $n = 2$ without any restrictions on M, as already noticed in [LS].

In the remaining part of this section, we will derive and discuss an example showing that the (sufficient) bounded-variation stability condition (BV) is indeed stronger than the spectral stability criterion as in the Main Theorem (i).

Example 1. Notice first that given a state $u_0 \in \mathbf{R}^3$, a nonsingular 3×3 matrix R and numbers $\lambda_1 < \lambda_2 < \lambda_3$, the function f defined on a neighbourhood of u_0:

$$f(u) = f(u_0) + R \cdot \mathrm{diag}\Big\{ \{\lambda_i + \langle R^{-1} \cdot (u - u_0), e_i \rangle\}_{i=1}^3 \Big\} \cdot R^{-1} \cdot (u - u_0) \tag{4.1}$$

is strictly hyperbolic and genuinely nonlinear. The eigenvalues are given by $\lambda_i(u) = \lambda_i + 2 \langle R^{-1} \cdot (u - u_0), e_i \rangle$ and the corresponding eigenvalues are constant vectors $r_i(u) = R \cdot e_i$.

Let:

$$\begin{gathered} \bar{u}_0 = 0, \quad \bar{u}_1 = (1,0,0), \quad \bar{u}_2 = (0,1,0), \quad \bar{u}_3 = (0,0,1), \\ f(\bar{u}_0) = 0,\ f(\bar{u}_1) = (-1,0,0),\ f(\bar{u}_2) = (-1,0,0),\ f(\bar{u}_3) = (-1,-1,1), \end{gathered} \tag{4.2}$$

and take:

$$\begin{gathered} r_1(\bar{u}_1) = \left[-\frac{\gamma^2+1}{2\gamma^4-1}, 0, \frac{\gamma^2}{2\gamma^4-1} \right]^T,\ r_3(\bar{u}_1) = \left[\gamma\frac{2\gamma^2+1}{2\gamma^4-1}, 0, -\frac{\gamma}{2\gamma^4-1} \right]^T, \\ r_1(\bar{u}_2) = \left[-\frac{\gamma}{2\gamma^4-1}, 0, \gamma\frac{2\gamma^2-1}{2\gamma^4-1} \right]^T,\ r_3(\bar{u}_2) = \left[\frac{\gamma^2}{2\gamma^4-1}, 0, \frac{\gamma^2-1}{2\gamma^4-1} \right]^T, \\ r_2(\bar{u}_1) = r_2(\bar{u}_2) = [-1,1,0]^T, \end{gathered} \tag{4.3}$$

Above $\gamma = 2\sqrt{2/3}$. Notice that the following identities hold:

$$\begin{gathered} -\gamma r_3(\bar{u}_1) + e_1 = r_1(\bar{u}_1), \qquad -\gamma r_1(\bar{u}_1) + \gamma r_3(\bar{u}_2) = r_3(\bar{u}_1), \\ \gamma r_1(\bar{u}_1) + \gamma r_3(\bar{u}_2) = r_1(\bar{u}_2), \qquad -\gamma r_1(\bar{u}_2) + e_3 = r_3(\bar{u}_2). \end{gathered} \tag{4.4}$$

Now taking:

$$\lambda_1(\bar{u}_1) = -2,\ \lambda_3(\bar{u}_1) = 1,\ \lambda_1(\bar{u}_2) = -1,\ \lambda_3(\bar{u}_2) = 2, \tag{4.5}$$

and noticing that by (4.2):

$$\Lambda^1 = -1,\ \Lambda^2 = 0,\ \Lambda^3 = 1, \tag{4.6}$$

we may actually calculate the coefficients of the stability matrix M by means of (2.3), (2.4), (2.5) and (4.5):

$$M = \frac{\gamma}{2} \cdot \begin{bmatrix} 0 & 0 & & & & \\ 0 & 0 & 1 & & & \\ & 1 & 0 & 0 & 1 & \\ & 1 & 0 & 0 & -1 & \\ & & & 1 & 0 & 0 \\ & & & & 0 & 0 \end{bmatrix} \tag{4.7}$$

where the only nonzero entries of M are these indicated above. We also see that the diagonal matrix D in (3.8) given by means of (3.9) has four intermediate diagonal elements equal to 2. Thus (compare Lemma 3.1 and the previous discussion in this section) we have that specRad $M < 1$ iff all solutions to (2.10) (2.11) (2.12) have negative real part. On the other hand, the condition (BV) reads: specRad $|M| < 1$.

It is easy to calculate that specRad $|M| = 2/\sqrt{3} > 1$ while specRad $M = 2^{3/4}/\sqrt{3} < 1$. We thus obtained:

Theorem 4.1. *Let f be a smooth function defined in the neighbourhood of states $\{\bar{u}_i\}_{i=1}^3$ as in (4.2), by means of (4.1). Let the choice of the eigenvalues and eigenvectors close to $\bar{u}_1$ and $\bar{u}_2$ be determined by (4.3) (4.5) and $\lambda_2(\bar{u}_1) = 1/2$, $\lambda_2(\bar{u}_2) = -1/2$. The choice of the linearly independent eigenvectors and distinct eigenvalues at $\bar{u}_0$ (all eigenvalues > -1) and $\bar{u}_3$ (all eigenvalues < 1) may be arbitrary. Then the conditions (1.3) (1.4) (1.5) are satisfied and we have:*

(i) *The BV stability condition (BV) does not hold.*
(ii) *All solutions to (2.10) (2.11) (2.12) have their real parts negative.*

An eigenvector of the transmission matrix $|M|$ corresponding to its maximal eigenvalue $2/\sqrt{3}$ is: $[0, 1, \sqrt{2}, \sqrt{2}, 1, 0]^T$. We may thus expect that if the pattern in (1.2) is perturbed by: a 1-shock of strength $\delta\sqrt{2}$ and a 3-shock of strength δ in the region between the first and the second large shock, plus a 1-shock of strength δ and a 3-shock of strength $\delta\sqrt{2}$ located in the region between the second and the third large shock, then this perturbation may grow indefinitely in time.

Indeed, at the linearised level we have the following observation which clarifies the matter.

Remark 4.2. If the various scattered 1- and 3-wave perturbations, corresponding to the eigenvector of $|M|$ as described above, have the property that they never hit simultaneously any large shock, then they never interact and never cancel. Thus, choosing a time at which these at most countable waves are all distinct, we find that the BV norm of perturbation is the sum of the absolute values of their strengths, which is the same as would be obtained from a modified problem with all interaction coefficients nonnegative. Therefore the BV norm of perturbation will grow without bound.

For simplicity, consider now the 'shifted in time' pattern (1.2) in which at time $t = 0$ the shock $(\bar{u}_0, \bar{u}_1)$ is located at $x = -x_0$ (with $x_0 > 0$), while the shocks $(\bar{u}_1, \bar{u}_2)$ and $(\bar{u}_2, \bar{u}_3)$ have respectful locations at $x = 0$ and $x = 1$. Moreover, let a small single 1-wave be initally located at $x = 1$, $t = 0$. We claim that for x_0

irrational, no 1- or 3-wave signals born from this initial configuration, will coincide at any large shock.

As we read from the coefficients of M in (4.7), a 3-wave hitting the right shock is reflected back as a 1-wave, a 1-wave hitting the left shock is reflected back as a 3-wave, and thus the only splitting of signals that occurs is through interactions at the middle shock, in which a single incoming 1- or 3-wave results in a pair of outgoing waves in the 1- and 3-families. Thus, we may conveniently index waves of various generations by words formed with letters R and L, corresponding to the directions taken by the wave in successive interactions with the middle shock.

A wave moving left after interacting with $(\bar{u}_1, \bar{u}_2)$ at time t is easily calculated to return next to the same shock at time

$$3x_0 + 4t,$$

while a wave moving to the right returns at time

$$3 + 4t.$$

Iterating, we find that the intersection with the middle shock after a series of directions represented by L/R word of length $|\alpha|$ is:

$$4^{|\alpha|}t + \alpha_1 x_0 + \alpha_2 = (4^{|\alpha|} + \alpha_2) + \alpha_1 x_0,$$

where α_j are base 4 numbers with $\alpha_1 + \alpha_2 = 4^{|\alpha|} - 1$, distinct for each distinct word (that is, uniquely indexing the word; these are obtained by putting a one in each place where L, respectively R appeares). We see that two such can agree only if $jx_0 + k = 0$ for j, k integer, which cannot occur for x_0 irrational.

This shows that the signals indeed never interact, and therefore grow without bound (as the vector $[0,0,0,0,1,0]^t$ is not orthogonal to the eigenvector

$$[0, 1, \sqrt{2}, \sqrt{2}, 1, 0]^t$$

of $|M|$), in contrast to the cancellation observed in the Dafermos - regularized viscous problem.

Remark 4.3. The example of Remark 4.2 suggests interesting further questions about both inviscid hyperbolic model and the Dafermos regularization. Namely, the lack of cancellation for the linearized inviscid problem, due to noncrossing of characteristic paths, was a consequence of the asymmetric arrangement of the background shocks; for a centered pattern, on the other hand, it is easily seen by equality of wave and characteristic speeds for the different families that total cancellation would occur, yielding stability. Since the Lin–Schecter result for the Dafermos regular system concerns a centered background pattern, by assumption, one may ask whether it is this fact alone that distinguishes the two results. We conjecture that this is not the case, but rather that the spreading effect of diffusion permits efficient cancellation whether or not inviscid characteristic paths intersect, and even for the standard case of constant (hence weaker) rather than time-growing Dafermos viscosity. Likewise, for the inviscid problem, one may ask whether nonlinear effects, in particular convergence of characteristic paths due to genuine nonlinearity, might enforce sufficient cancellation to restore stability at the nonlinear level of the asymmetric pattern in the example. We hope to address both of these problems in future work.

5. Two more examples

In this section we presnt two more examples with the purpose of showing that conditions (i) and (ii) in Lemma 3.1 do not imply each other, for general signed matrices Q. The matrices Q and S will be prepared so that they can be seen as the transmission matrices M and D of some three-shock pattern of a genuinely nonlinear system.

Define the 6×6 matrix:

$$S = \mathrm{diag}\ \left(s_1, \frac{5}{2}, 10, 2, \frac{5}{2}, s_6\right),$$

where s_1, s_6 are positive numbers to be chosen in the sequel. Notice that S comes from the shock pattern defined through (4.1) (4.2) (4.6), with

$$\lambda_1(\bar{u}_1) = -10/9, \quad \lambda_2(\bar{u}_1) = 1/3, \quad \lambda_3(\bar{u}_1) = 2/3,$$

$$\lambda_1(\bar{u}_2) = -2/3, \quad \lambda_2(\bar{u}_2) = -1/3, \quad \lambda_3(\bar{u}_2) = 2.$$

Indeed, the diagonal elements of S agree with this choice through (3.9).

Example 2. We will define the transmission matrix Q_1 so that condition in Lemma 3.1 (i) is not satisfied: specRad Q_1 = specRad $|Q_1| > 1$, but condition (ii) is satisfied. Take:

$$Q_1 = \begin{bmatrix} 0 & 0 & & & & \\ 0 & 0 & -4/9 & & & \\ & 1 & 0 & 0 & -1 & \\ & -1 & 0 & 0 & 1 & \\ & & & -4/3 & 0 & 0 \\ & & & & 0 & 0 \end{bmatrix}.$$

It is easy to see that both spectral radii of Q_1 and $|Q_1|$ equal $4/3$. To check condition (ii), calculate:

$$\det\ (Q_1 - S^\lambda) = (s_1 s_6)^\lambda \cdot \left(5^{3\lambda} + \frac{4}{3}5^{2\lambda} + \frac{4}{9}5^\lambda\right).$$

Thus $Q_1 - S^\lambda$ is singular if and only if $5^{2\lambda} + \frac{4}{3}5^\lambda + \frac{4}{9} = 0$, that is when $5^\lambda = -2/3$. This is impossible for $\lambda \in \mathbf{C}$ such that Re $\lambda \geq 0$.

We now need to see that Q_1 complements S as the transition matrix M given through (2.3) - (2.5), for some suitable choice of eigenvectors $\{r_i(\bar{u}_j)\}_{i=1..3}^{j=1..2}$. By an elementary calculation, this is equivalent to having:

$$\begin{aligned} -\frac{20}{3} \cdot r_3(\bar{u}_1) + r_1(\bar{u}_1) &\parallel [1,0,0]^t, \\ r_2(\bar{u}_1), r_2(\bar{u}_2) &\parallel [-1,1,0]^t, \\ -\frac{20}{9} \cdot r_1(\bar{u}_2) + r_3(\bar{u}_2) &\parallel [0,1,-1]^t, \\ -\frac{5}{3} \cdot r_1(\bar{u}_1) - 3r_3(\bar{u}_2) - r_3(\bar{u}_1) &\parallel [-1,1,0]^t, \\ -\frac{5}{3} \cdot r_1(\bar{u}_1) - 3r_3(\bar{u}_2) - r_1(\bar{u}_2) &\parallel [-1,1,0]^t. \end{aligned}$$

The above is implied by:

$$\begin{aligned} r_2(\bar u_1) = r_2(\bar u_2) &= [-1,1,0]^t, \\ -\frac{20}{3}\cdot r_3(\bar u_1) + r_1(\bar u_1) &= [1,0,0]^t, \\ -\frac{20}{9}\cdot r_1(\bar u_2) + r_3(\bar u_2) &= [0,1,-1]^t, \\ -\frac{5}{3}\cdot r_1(\bar u_1) - 3r_3(\bar u_2) - r_3(\bar u_1) &= 0, \\ -\frac{5}{3}\cdot r_1(\bar u_1) - 3r_3(\bar u_2) - r_1(\bar u_2) &= 0. \end{aligned}$$

One can see that this system of equations has a solution and that both matrices $R_i = [r_1, r_2, r_3](\bar u_i)$, $i = 1, 2$ are nonsingular.

Example 3. We now define a matrix Q_2 so that condition (i) of Lemma 3.1 is satisfied: specRad $Q_2 < 1$ but specRad $|Q_2| > 1$, and condition (ii) is violated. Take:

$$Q_2 = \begin{bmatrix} 0 & 0 & & & & \\ 0 & 0 & 1 & & & \\ & 1 & 0 & 0 & 1 & \\ & 1 & 0 & 0 & 1 & \\ & & & -1 & 0 & 0 \\ & & & & 0 & 0 \end{bmatrix}.$$

We then have: specRad $Q_2 = 0$ and specRad $|Q_2| = \sqrt{2}$. To check condition (ii), calculate:

$$\det\,(Q_2 - S^\lambda) = (s_1 s_6)^\lambda \cdot \left(5^{3\lambda} + 5^{2\lambda} - 5^\lambda\right).$$

Thus $Q_2 - S^\lambda$ is singular provided that $5^{2\lambda} + 5^\lambda - 1 = 0$, which holds, for example if $5^\lambda = -(1+\sqrt{5})/2$. Since the right hand side of this expression is ≤ -1, there exists λ solving this equation and Re $\lambda \geq 0$.

To see that Q_2 is a transmission matrix, we proceed as in Example 2. The eigenvectors' equations may be taken as:

$$\begin{aligned} r_2(\bar u_1) = r_2(\bar u_2) &= [-1,1,0]^t, \\ 15\cdot r_3(\bar u_1) + r_1(\bar u_1) &= [1,0,0]^t, \\ -\frac{5}{3}\cdot r_1(\bar u_2) + r_3(\bar u_2) &= [0,1,-1]^t, \\ -\frac{5}{3}\cdot r_1(\bar u_1) + 3r_3(\bar u_2) - r_3(\bar u_1) &= 0, \\ -\frac{5}{3}\cdot r_1(\bar u_1) + 3r_3(\bar u_2) + r_1(\bar u_2) &= 0. \end{aligned}$$

Again, the above system has a solution such that both matrices $R_i = [r_1, r_2, r_3](\bar u_i)$, $i = 1, 2$ are nonsingular.

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