ON THE WELL POSEDNESS OF A SYSTEM OF BALANCE LAWS WITH $\mathbf{L}^\infty$ DATA

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Introduction

In their paper [3], Bressan and Shen consider a class of (strictly hyperbolic) 2×2 systems of the form:

$$u_t + f(u)_x = 0, \qquad u(0,\cdot) = \bar{u}, \tag{1}$$

$$\theta_t + h(u)\theta_x = 0, \qquad \theta(0,\cdot) = \bar{\theta}, \tag{2}$$

where $f \in \mathcal{C}^2$ is strictly convex (that is $f''(x) > 0$ for any $x \in \mathbf{R}$), h is Lipschitz continuous, $\bar{u} \in \mathbf{L}^1 \cap \mathbf{L}^\infty$, and $\bar{\theta} \in \mathcal{C}^0$.

As proved in a classical paper of Kruzkov [7], there exists exactly one weak entropy admissible solution of (1), which depends in a Lipschitz continuous way on $\bar{u}$. Namely:

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{\mathbf{L}^1} \leq \|\bar{u}_1 - \bar{u}_2\|_{\mathbf{L}^1} \quad \text{for any } t \geq 0,$$

where u_1 and u_2 are solutions of (1) with the initial data $\bar{u}_1$ and $\bar{u}_2$, respectively.

As soon as the function u is determined from (1), a solution of (2) can be constructed by the standard method of characteristics. Indeed, the function θ must be constant along the integral curves of the ODE

$$\dot{x} = h(u(t,x)). \tag{3}$$

Uniqueness and continuous dependence of solutions of (2) can thus be derived from the well-posedness of the Cauchy problem for (3).

We remark that the genuine nonlinearity of (1) implies that the total variation of $u(t_0,\cdot)$ is locally bounded, for each $t_0 > 0$. Hence the well posedness of the Cauchy problem (3) with initial data $x(t_0) = x_0$ follows from [2].

It is worth noting that in [3] the well posedness of (3) follows from a more general result on the well posedness of ODE's of the form

$$\dot{x} = F(t,x), \tag{4}$$

where $F : [0,T] \times \mathbf{R} \longrightarrow \mathbf{R}$ is measurable and such that:

(**A1**) For every point $(\bar{t},\bar{x}) \in (0,T] \times \mathbf{R}$, there exists a slope $\lambda_{(\bar{t},\bar{x})}$ such that the function F is constant along the segment $I_{(\bar{t},\bar{x})} = \{(t,x) : \ t \in (0,\bar{t}), \ x - \bar{x} = \lambda_{(\bar{t},\bar{x})}(t - \bar{t})\}$. Moreover, $(t,x) \in I_{(\bar{t},\bar{x})}$ implies that $\lambda_{(t,x)} = \lambda_{(\bar{t},\bar{x})}$ and hence $I_{(t,x)} \subset I_{(\bar{t},\bar{x})}$.

(**A2**) There exist disjoint intervals $[a,b]$ and $[c,d]$ such that $F(t,x) \in [a,b]$ and $\lambda_{(t,x)} \in [c,d]$ for all $(t,x) \in (0,T] \times \mathbf{R}$.

It is clear that since u is constant along the backward characteristics of (1), which are the straight lines with corresponding slopes $f'(u)$, the composite function $h \circ u$ satisfies (**A1**), (**A2**).

A natural question is whether a similar result holds for the perturbed system

$$(5)\qquad \begin{aligned} u_t + f(u)_x = g(u), &\qquad u(0,\cdot) = \bar{u}, \\ \theta_t + h(u)\theta_x = 0, &\qquad \theta(0,\cdot) = \bar{\theta}. \end{aligned}$$

We will assume that $g \in \mathcal{C}^1$ and g' is bounded. It is known [5], that (5) has then the unique weak entropy admissible solution, acquiring (after possibly a modification on a set of measure zero) the following properties. For each fixed $(\bar{t}, \bar{x}) \in (0,\infty) \times \mathbf{R}$ the one-sided limits $u(\bar{t}, \bar{x}\pm)$ exist and $u(\bar{t}, \bar{x}-) \geq u(\bar{t}, \bar{x}+) = u(\bar{t}, \bar{x})$. Moreover, the minimal and maximal backward characteristics $y_-(\cdot; \bar{t}, \bar{x})$ and $y_+(\cdot; \bar{t}, \bar{x})$ through $(\bar{t}, \bar{x})$ are determined by solving

$$(6)\qquad \left\{ \begin{array}{rcl} \dot{y} & = & f'(v) \\ \dot{v} & = & g(v), \end{array} \right.$$

with initial data $(y(\bar{t}), v(\bar{t})) = (\bar{x}, u(\bar{t}, \bar{x}-))$ and $(y(\bar{t}), v(\bar{t})) = (\bar{x}, u(\bar{t}, \bar{x}+))$, respectively. Along those characteristics u coincides with the corresponding function v; that is $u(t, y(t)) = v(t)$ for any $t \in (0, \bar{t})$.

Consequently, the composite function $F = h \circ u$ satisfies:

(**A1'**) For every point $(\bar{t}, \bar{x}) \in (0, T] \times \mathbf{R}$, there exists a $\mathcal{C}^2$ curve $y_{(\bar{t},\bar{x})} : (0, \bar{t}) \longrightarrow \mathbf{R}$ along which the function F is Lipschitz continuous. Moreover, if $y_{(\bar{t},\bar{x})}(t) = x$, then $y_{(t,x)} = y_{(\bar{t},\bar{x})}$ on $(0, t)$. Finally, the first and the second derivatives of the curves $y_{(\bar{t},\bar{x})}$ are uniformly bounded.

The purpose of this paper is to discuss the following two questions:

(**I**) Let $F : [0, T] \times \mathbf{R} \longrightarrow \mathbf{R}$ be measurable and satisfy (**A1'**) and (**A2**) (where the slope $\lambda_{(t,x)}$ is replaced by the derivative $\dot{y}_{(t,x)}$). Is the problem (4) well posed?

(**II**) Assuming that the system (5)(2) is strictly hyperbolic, that is the ranges of the two functions f' and h are disjoint, is this system well posed?

The answer to the first question is negative, as is shown in the first section. Nevertheless, the answer to (**II**) is positive. This is shown by the main theorem of the paper, in section 2. In the third section we present another example, showing that if f is not convex, the problem (1)(2) may not be well posed. Indeed, in this case the corresponding ODE (3) may have multiple solutions. The last section contains the technical details of the proof of our main theorem.

1. A counterexample

We give an example of a measurable function $F : [0,1] \times \mathbf{R} \longrightarrow \mathbf{R}$, satisfying (**A1'**) and (**A2**) (with the slope $\lambda_{(t,x)}$ replaced by the derivative $\dot{y}_{(t,x)}$), such that there exist two solutions $x_1,\ x_2 : [0,1] \longrightarrow \mathbf{R}$ of the Cauchy problem

$$\dot{x} = F(t, x), \quad x(0) = 0.$$

Let $y : [0,1] \longrightarrow \mathbf{R}$ be a smooth function such that

- $y(t) = 1 - t$ for $t \in [0, 1/3]$,
- $y(1) = 1/3$,
- y is decreasing, convex and $1 - t < y(t) < 4/3 - t$ for $t \in (1/3, 1)$.

Define a sequence of functions $y_n : [0, t_n] \longrightarrow \mathbf{R}$ in the following way. For $n = 1$ set $t_1 = 1$ and $y_1 = y$. For $n > 1$ let $t_n = \frac{3}{4} y_{n-1}(0)$. The graph of y_n is constructed by shifting the graph of y_{n-1} in a way that:

- $y_n(t_n) = y_{n-1}(0) - t_n = t_n/3$,
- $y_n(t) = y_n(0) - t$ for $t \in [0, y_n(0)/3]$,
- $y_n(t) > y_n(0) - t$ for $t \in (y_n(0)/3, t_n]$.

F is then described by:

$$F(t,x) = \begin{cases} 1/3 & \text{for} \quad t \in [0,1] \text{ and } x \leq t/3, \\ 2 & \text{for} \quad t \in [0,1] \text{ and } x > y_1(t), \\ 1/3 & \text{for} \quad t \in [0,t_n],\ x \in [y_n(0) - t, y_n(t)] \text{ and } n \geq 1, \\ 2 & \text{for} \quad t \in [0,t_n],\ x \in (y_n(t), y_{n-1}(0) - t) \text{ and } n > 1. \end{cases}$$

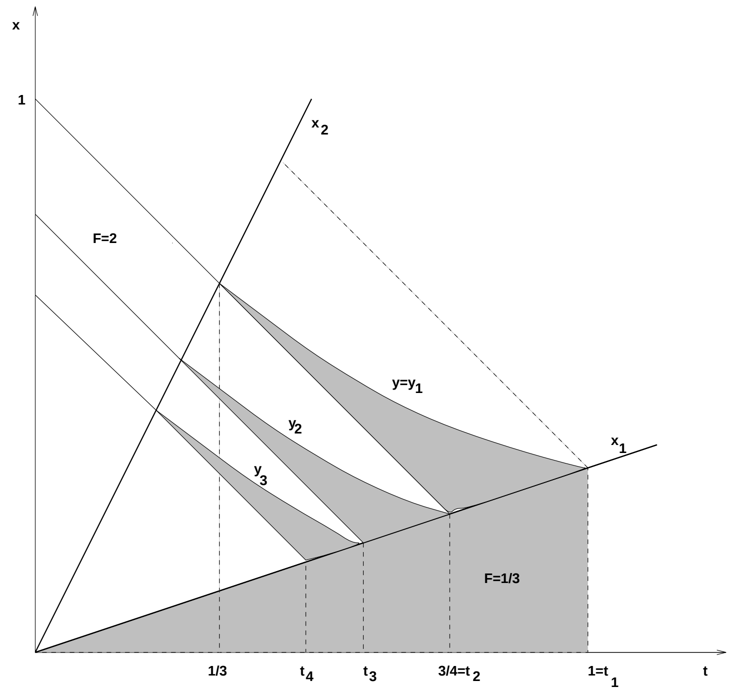

FIGURE 1

F is thus constant along appropriate smooth curves $y_{(t,x)}$ whose first derivatives are uniformly bounded and negative, while the values of F belong to the interval $[1/3, 2]$. However, $x_1(t) = t/3$ and $x_2(t) = 2t$ are two solutions of the given Cauchy problem.

2. The main result

Theorem 1. *Let $f \in \mathcal{C}^2$ with $f''(x) > 0$ for all $x \in \mathbf{R}$, $g \in \mathcal{C}^1$, with g' bounded, $\bar{u} \in \mathbf{L}^1 \cap \mathbf{L}^\infty$, $\bar{\theta} \in \mathcal{C}^0$. Fix $T > 0$ and define the constants*

$$C_1 = K + T \exp(T \|g'\|_{\mathbf{L}^\infty}) \max_{[-K,K]} |g| \,,$$

$$C_2 = C_1 + T \exp(T\|g'\|_{\mathbf{L}^\infty}) \max_{[-C_1,C_1]} |g|\,,$$

for some fixed $K \geq \|\bar{u}\|_{\mathbf{L}^\infty}$. *Let* $h : \mathbf{R} \longrightarrow \mathbf{R}$ *be Lipschitzian on an open neighbourhood of* $[-C_2, C_2]$ *and assume that there exist two disjoint intervals* $[a,b]$ *and* $[c,d]$ *such that* $h(x) \in [a,b]$ *and* $f'(x) \in [c,d]$ *for any* $x \in [-C_2, C_2]$.

Then the system (5)(2) has a unique admissible solution, that is the weak entropy admissible solution of (5) and the broad solution of (2) (which is in fact continuous), defined on $[0,T] \times \mathbf{R}$, *that depends continuously on the initial data. More precisely: if* $\bar{u}_n \to \bar{u}$ *in* $\mathbf{L}^1$, $\|\bar{u}_n\|_{\mathbf{L}^\infty} \leq K$, *for each* n, *and* $\bar{\theta}_n \to \bar{\theta}$ *uniformly, then* $u_n(t,\cdot) \to u(t,\cdot)$ *in* $\mathbf{L}^1$, *for any* $t \in [0,T]$ *and* $\theta_n \to \theta$ *uniformly on compact subsets of* $[0,T]\times\mathbf{R}$ *(here* (u_n, θ_n) *stands for the solution of (5)(2) with the initial data* $(\bar{u}_n, \bar{\theta}_n)$*).*

Proof. For the convenience of the reader we divided the proof into five steps, containing several lemmas, whose proofs will be given in the last section. Also, for future considerations we assume that $d < a$, the other case beeing treated similarily. By $L > 0$ we denote the Lipschitz constant of h on a neighbourhood of $[-C_2, C_2]$.

STEP 1. Note first, that the unique weak entropy admissible solution of (5) satisfies

$$|u(t,x)| \leq C_1 \quad \text{for a.a. } (t,x) \in [0,T] \times \mathbf{R}. \tag{7}$$

Lemma 1. *Let* $v : [0,T] \longrightarrow \mathbf{R}$ *be Lipschitzian, with* $\dot{v}(t) \in [a,b]$ *for a.a.* $t \in [0,T]$. *Then the composition* $t \mapsto u(t, v(t))$ *is measurable.*

Consider the equation (3) with the initial data $x(0) = x_0$. Lemma 1 guarantees that the Picard operator $\mathcal{P}$ for this problem

$$\mathcal{P} : \mathcal{U} \longrightarrow \mathcal{U}, \quad \mathcal{P}(v)(t) = x_0 + \int_0^t h(u(\tau, v(\tau)))\mathrm{d}\tau,$$

$$\mathcal{U} = \{v : [0,T] \longrightarrow \mathbf{R} : \quad v \text{ is Lipschitzian and } \dot{v}(t) \in [a,b] \text{ for a.a. } t \in [0,T]\},$$

is well defined.

Our first goal will be to show that $\mathcal{P}$ is continuous and has a unique fixed point. To do this, we will approximate $\mathcal{P}$ with the Picard operator of an another ODE, whose right hand side will be the composition of h and a suitable approximation of the discontinuous function u. □

STEP 2. Fix $t_0 \in [0,T]$. Let $\psi \in \mathbf{L}^1$ be a piecewise constant function with finite number of jumps located at points x_i, and assume that $\|\psi\|_{\mathbf{L}^\infty} \leq C_1$. For each x_i, let $\xi_i : [t_0, T] \longrightarrow \mathbf{R}$ be the unique forward characteristic of (5), originating from (t_0, x_i). Without loss of generality we may assume that each x_i is a continuity point of the function $u(t_0, \cdot)$, so each ξ_i can be prolonged along the unique backward characteristic emanating from (t_0, x_i). Thus the functions ξ_i are defined on $[0,T]$ (note that each ξ_i is differentiable at all but a countable number of points and there holds $\dot{\xi}_i \in [c,d]$) and divide the stripe $[0,T]\times\mathbf{R}$ into finite number of regions $\overline{\mathcal{R}_i}$. $\mathcal{R}_i$ is the (open) region with the property: $\mathcal{R}_i \cap \{(t_0, x) : \ x \in \mathbf{R}\} = \{t_0\} \times (x_{i-1}, x_i)$. We also have two unbounded $\mathcal{R}_i$'s, defined in an obvious way.

Let $\alpha_i : [0,T] \longrightarrow \mathbf{R}$ be the solution of

$$\dot{\alpha_i} = g(\alpha_i), \quad \alpha_i(t_0) = \psi(\frac{x_i + x_{i-1}}{2}).$$

Define a measurable function $w : [0,T] \times \mathbf{R} \longrightarrow \mathbf{R}$, by

$$w(t,x) = \alpha_i(t) \ \text{ for } (t,x) \in \mathcal{R}_i.$$

Note that since $\|\psi\|_{\mathbf{L}^\infty} \le C_1$, $|w(t,x)| \le C_2$ for all $(t,x) \in \cup_i \mathcal{R}_i$. Moreover, in each $\mathcal{R}_i$ the function w is $\mathcal{C}^2$ and Lipschitzian with the constant

$$L_2 = \max_{[-C_2,C_2]} |g|\,.$$

Lemma 2. *Fix $\varepsilon > 0$, then there exist $t_0 \in (0,T)$ and a number $\delta > 0$ such that if $\|\psi - u(t_0,\cdot)\|_{\mathbf{L}^1} < \delta$ then for any $v \in \mathcal{U}$*

$$\int_0^T |h(w(\tau, v(\tau)) - h(u(\tau, v(\tau)))|\, d\tau < \varepsilon.$$

□

STEP 3. We will discuss the ODE

$$\dot{x} = h(w(t,x)), \tag{8}$$

(where the piecewise continuous function w is constructed as in step 2). Since the slopes of discontinuities and the values of the composite function $h \circ w$ belong to disjoint intervals, (8) is well posed.

Let x be any solution of (8), which crosses the curves ξ_i only at their differentiability points. Define $V : [0,T] \longrightarrow \mathbf{R}$

$$V(t) = \lim_{\varepsilon \to 0} \frac{x_\varepsilon(t) - x(t)}{\varepsilon},$$

where x_ε is the solution of

$$\dot{x_\varepsilon} = h(w(t,x_\varepsilon)), \quad x_\varepsilon(t_0) = x(t_0) + \varepsilon.$$

V is well defined and continuous in the intervals where x remains in the same region $\mathcal{R}_i$. The standard computations [6] show also that there holds

$$V(t_2) = V(t_1) \exp\left(\int_{t_1}^{t_2} \mathrm{D}_x (h \circ w)(\tau, x(\tau))\, \mathrm{d}\tau \right), \quad t_1 \le t_2.$$

On the other hand, at every point t in which $x(t) = \xi_i(t)$, V has a jump described by the formula

$$\frac{V(t+)}{V(t-)} = \frac{h(w(t,x(t)+)) - \dot{\xi}_i(t)}{h(w(t,x(t)-)) - \dot{\xi}_i(t)}.$$

Define the functions $z, \varphi, \lambda : [0,T] \longrightarrow \mathbf{R}$

$$\begin{aligned} z(t) &= \varphi(t) V(t), \\ \varphi(t) &= \frac{b - \lambda(t)}{h(w(t,x(t))) - \lambda(t)}. \end{aligned}$$

The function λ will be defined separately on each interval $[t_{i-1}, t_i]$ (where $\xi_{i-1}(t_{i-1}) = x(t_{i-1})$ and $\xi_i(t_i) = x(t_i)$) and have the following properties:

(**P1**) $\lambda(t_{i-1}) = \dot{\xi}_{i-1}(t_{i-1})$, $\lambda(t_i) = \dot{\xi}_i(t_i)$,

(**P2**) $\lambda(t) \in [c,d]$ for any $t \in [0,T]$,

(**P3**) λ is piecewise $\mathcal{C}^1$ and has only downward jumps,

(**P4**) for a.a. $t \in [0,T]$ one has

$$\dot{\lambda}(t) \le \beta\, \frac{h(w(t,x(t))) - \lambda(t)}{t} + Q,$$

where the constants $Q \ge 0$ and $\beta \in [0, \frac{b-d}{b-a})$ are uniform, that is they do not depend on a particular approximation w or a solution x of (8).

Lemma 3. *There exists a function $\lambda : (0,T) \longrightarrow \mathbf{R}$ with the properties **(P1)**-**(P4)**.*

Compute the derivative of φ in the regularity intervals of λ

$$\begin{aligned}
\dot\varphi(t) &= \frac{-\dot\lambda(t)[h(w(t,x(t))) - \lambda(t)]}{[h(w(t,x(t))) - \lambda(t)]^2} \\
&\quad - \frac{[b-\lambda(t)][\mathrm{D}_t(h\circ w)(t,x(t)) + h(w(t,x(t)))\mathrm{D}_x(h\circ w)(t,x(t)) - \dot\lambda(t)]}{[h(w(t,x(t))) - \lambda(t)]^2} \\
&= \frac{b - h(w(t,x(t)))}{[h(w(t,x(t))) - \lambda(t)]^2}\dot\lambda(t) \\
&\quad - \frac{b-\lambda(t)}{[h(w(t,x(t))) - \lambda(t)]^2}[\mathrm{D}_t(h\circ w)(t,x(t)) + h(w(t,x(t)))\mathrm{D}_x(h\circ w)(t,x(t))] \\
&= \varphi(t)\ \frac{b - h(w(t,x(t)))}{b-\lambda(t)}\frac{\dot\lambda(t)}{h(w(t,x(t))) - \lambda(t)} \\
&\quad -\varphi(t)\ \frac{\mathrm{D}_t(h\circ w)(t,x(t)) + h(w(t,x(t)))\mathrm{D}_x(h\circ w)(t,x(t))}{h(w(t,x(t))) - \lambda(t)}.
\end{aligned}$$

Thus

$$\frac{\dot\varphi(t)}{\varphi(t)} \leq \beta\ \frac{b-a}{b-d}\frac{1}{t} + \frac{b-a}{b-d}\frac{Q}{a-d} + \frac{LL_2 + LL_2\max(|a|,|b|)}{a-d} \leq \gamma\frac{1}{t} + C_3,$$

where the constants $\gamma \in (0,1)$ and $C_3 > 0$ depend only on the system (5)(2).

Compute now the derivative of z in the regularity intervals of λ

$$\dot z(t) = \dot\varphi(t)V(t) + \varphi(t)V(t)\mathrm{D}_x(h\circ w)(t,x(t)) = \frac{\dot\varphi(t) + \varphi(t)\mathrm{D}_2 h(w(t,x(t)))}{\varphi(t)}\ z(t).$$

Hence

$$\frac{\dot z(t)}{z(t)} \leq \gamma\frac{1}{t} + C_3,$$

and finally, for any t_1, t_2 which are in the same regularity interval of λ and $t_1 \leq t_2$

$$\frac{z(t_2)}{z(t_1)} \leq \exp(C_3(t_2 - t_1))\left(\frac{t_2}{t_1}\right)^\gamma. \tag{9}$$

Note that z is continuous at the points t where $x(t) = \xi_i(t)$, as

$$z(t-) = \frac{b - \dot\xi_i(t)}{h(w(t,x(t)-)) - \dot\xi_i(t)}\ V(t-) = \frac{b-\dot\xi_i(t)}{h(w(t,x(t)+)) - \dot\xi_i(t)}\ V(t+) = z(t+).$$

Moreover, z has only downward jumps, since the same is true for φ (in each interval (t_{i-1}, t_i)).

Concluding, the formula (9) holds for all $t_1, t_2 \in [0,T]$ such that $t_1 \leq t_2$ and that $V(t_1), V(t_2)$ are defined. This yields

$$\frac{V(t_2)}{V(t_1)} = \frac{z(t_2)}{z(t_1)}\frac{\varphi(t_1)}{\varphi(t_2)} \leq \frac{b-c}{a-d}\frac{b-c}{b-d}\exp\left(C_3(t_2-t_1)\right)\left(\frac{t_2}{t_1}\right)^\gamma \leq M\left(\frac{t_2}{t_1}\right)^\gamma, \tag{10}$$

for all t_1, t_2 as above, where the constant $M > 0$ depends only on the system (5)(2) Later on, by integrating the formula (10) one will be able to obtain an estimate on continuous dependence of solutions to (8) on initial data. □

STEP 4. Now we conclude the proof of the well posedness of (3). By step 2, the Picard operator $\mathcal{P}$ of (3) with initial data $x(0) = x_0$ is a uniform limit of the corresponding Picard operators $\mathcal{P}_n$, written for (8) with the approximating function w constructed according to Lemma 2, and initial data $x(0) = x_0$. Hence $\mathcal{P}$ is continuous. $\mathcal{P}$ has also the relatively compact image, contained in $\mathcal{U}$ (note that $\mathcal{U}$ is a closed, convex subset of the Banach space $\mathcal{C}^0([0,T],\mathbf{R})$). By Schauder fixed point theorem, $\mathcal{P}$ must thus have a fixed point, which is a solution of (3) with the initial data $x(0) = x_0$.

To prove the uniqueness of solutions of (3) and the continuous dependence on x_0, we first note

Lemma 4. *Let $e : [0,T] \longrightarrow \mathrm{R}$ be measurable and bounded by $b-a$. If v is a solution of the problem*

$$\dot{v}(t) = \Pi_{[a,b]}[h(w(t,v(t)) + e(t)], \quad v(0) = x_0 \tag{11}$$

(here $\Pi_{[a,b]} : \mathbf{R} \longrightarrow [a,b]$ is the projection of $\mathbf{R}$ onto $[a,b]$), then for any $t \in [0,T]$

$$|v(t) - x(t)| \leq \frac{Mt^\gamma (b-a)^\gamma}{1-\gamma} \left(\int_0^t |e(\tau)| \, d\tau \right)^{1-\gamma}$$

(x here stands for the unique fixed point of $\mathcal{P}$).

Having once the estimate (10) established, one can see that Lemma 4 is proved in the same way as Lemma 1 in [3].

Let x_1, x_2 be two solutions of (3). For $i = 1,2$ set $e_i(t) = h(u(t,x_i(t))) - h(w(t,x_i(t)))$ (where w is constructed as in step 3). Note that x_i is a solution of (11) with initial data $v(0) = x_i(0)$. By Lemma 4 we can thus estimate the differences $\|x_i - y_i\|_{\mathbf{L}^\infty}$, where y_i is the solution of (8) with $y_i(0) = x_i(0)$. On the other hand, the difference $\|y_1 - y_2\|_{\mathbf{L}^\infty}$ is estimated by means of the formula (10)

$$|y_1(t) - y_2(t)| \leq (1+b-a) \max\{Mt^\gamma \left|y_1(0) - y_2(0)\right|^{1-\gamma}, \ |y_1(0) - y_2(0)|\}.$$

Since by Lemma 2 $\int_0^T |e_i(\tau)| \, \mathrm{d}\tau$ can be arbitrarily small, provided that the approximating function w is choosen suitably, we obtain

$$\|x_1 - x_2\|_{\mathbf{L}^\infty} \leq (1+b-a) \max\{MT^\gamma \left|x_1(0) - x_2(0)\right|^{1-\gamma}, \ |x_1(0) - x_2(0)|\}, \tag{12}$$

which proves the well posedness of (3). (For the details, see the proof of Theorem 1 in [3].) □

STEP 5. The uniqueness and existence of the admissible solution of (5)(2) is clear in view of step 4. To justify the continuous dependence on the initial data, note that by [4] $u_n(t,\cdot) \to u(t,\cdot)$ in $\mathbf{L}^1$, for any $t \in [0,T]$ (we use the notation introduced in the statement of the theorem).

The convergence $\theta_n \to \theta$ is proved exactly as in [3]. □ ■

3. The case of nonconvex flux

In this section we show that the convexity of the flux function f in (1) is crucial for the well posedness of (3) and thus also for the well posedness of (1)(2). To do this, we shall define two smooth functions $f, h : \mathbf{R} \longrightarrow \mathbf{R}$ such that $f'(x) \in [-1/2, 1]$ and $h(x) \in [3,5]$ for all $x \in \mathbf{R}$, h beeing Lipschitzian, and a piecewise constant function $\bar{u} \in \mathbf{L}^1 \cap \mathbf{L}^\infty$, with $\bar{u}(x) \in [0,2]$ for all $x \in \mathbf{R}$, such that there exist two solutions $x_1, x_2 : [0, 2/9] \longrightarrow \mathbf{R}$ of (3) with the initial data $x(0) = 0$. Note that in view of Theorem 1 or Theorem 2 in [3], f'' must change sign in the interval $[0,2]$.

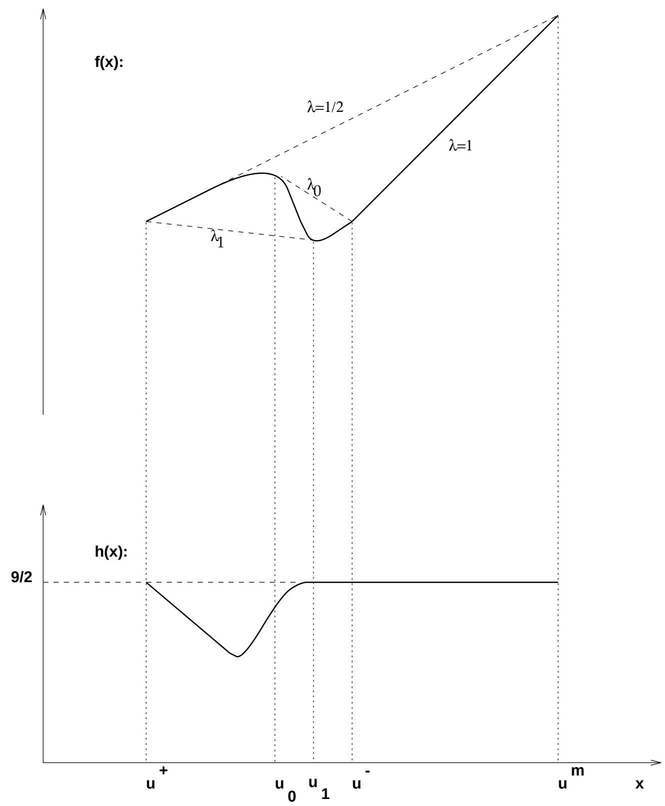

FIGURE 2

Define $u^+ = 0,\ u^- = 1,\ u^m = 2$. The function f, shaped as in Fig. 3 should have the following features:

- f is smooth and $f'(x) \in [-1/2, 1]$ for all $x \in \mathbf{R}$,
- $f(x) = x$ for all $x \in [u^-, u^m]$,
- the upper concave envelope of f on $[u^+, u^m]$ is a straight line with the slope $1/2$,
- the lower convex envelope of f on $[u^+, u^-]$ is a straight line with the slope λ_1 on $[u^+, u_1]$ and coincides with f on $[u_1, u^-]$, for some point $u_1 \in (u^+, u^-)$, which is close to u^-,
- the upper concave envelope of f on $[u^+, u^-]$ coincides with f on $[u^+, u^0]$ and is a straight line with the slope λ_0 on $[u_0, u^-]$, where $u_0 \in (u^+, u_1)$.

The fuction h, as in Fig. 3 should satisfy:

- h is smooth and $h(x) \in [3, 5]$ for all $x \in \mathbf{R}$,
- $h(x) = 9/2$ for $x \in [u_1, u^m] \cup \{u^+\}$,
- $h(x) < 9/2$ in (u^+, u_1).

The initial data $\bar{u}$ is uniquely defined by:

- $\bar{u}(x) = 0$ for $x \in (-\infty, 0] \cup [1, +\infty)$,

- $\bar{u}(x) = U(2^n x)$ for $x \in (2^{-n-1}, 2^{-n}]$, $n \geq 0$, where

$$U(x) = \begin{cases} u^+ & \text{for} \quad x \in (1/2, 7/10) \\ u^m & \text{for} \quad x \in (7/10, 8/10) \\ u^- & \text{for} \quad x \in (8/10, 1). \end{cases}$$

The solution of (1) is shown in Fig. 3 below.

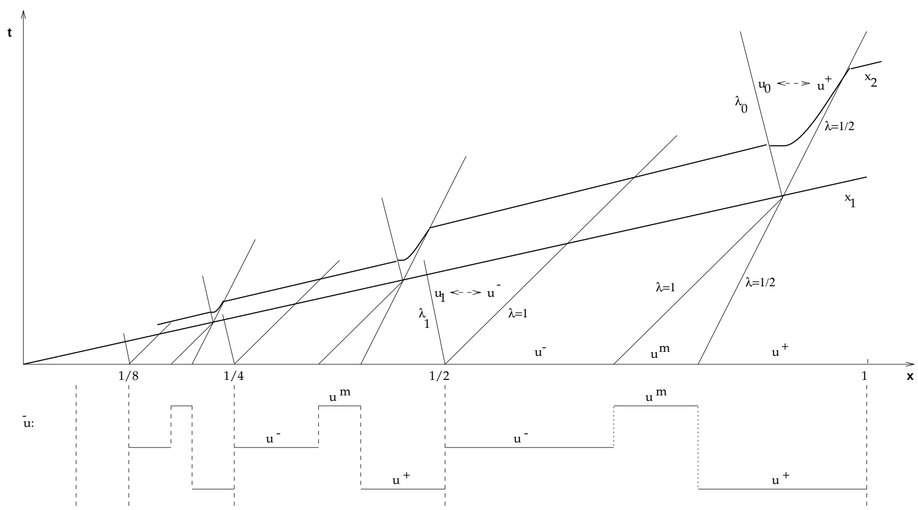

Figure 3

Note that the initial data U yields two shocks (of opposite signs), whose interaction gives in turn a centered rarefaction wave. This pattern is reapeated in a self-similar way, namely $u(t,x) = u(t/2, x/2)$ for $x \in [0,1]$ and $t \in [0, 2/9]$. Consider now the ODE (3) with $x(0) = 0$. Certainly $x_1(t) = \frac{9}{2}t$ is a solution of this problem. Defining h appropriately on $[u^+, u_0]$, one can find another solution x_2, such that $x_1(t) - x_2(t) = 2(x_1(t/2) - x_2(t/2))$ for all $t \in [0, 2/9]$. The distance between $x_1(t)$ and $x_2(t)$ increases rapidly when $x_2(t)$ lies inside the rarefaction waves.

Note that the above result gives rise to the ill-posedness of (2) with $\bar{\theta}(\cdot) = 0$. In fact, for any $\alpha \in \mathbf{R}$ the function

$$\theta(t,x) = \begin{cases} 0 & \text{for} \quad x < x_1(t) \text{ or } x > x_2(t) \\ \alpha & \text{for} \quad x \in [x_2(t), x_1(t)] \end{cases}$$

defined on $[0, 2/9] \times \mathbf{R}$ is then a solution of (2).

4. The details of the proofs

Proof of Lemma 1. Fix any $t_0 \in (0,T)$. We will show that the function $t \mapsto u(t, v(t))$ is measurable on $[t_0, T]$. Let $y_t : [0,t] \longrightarrow \mathbf{R}$ be the maximal backward characteristic for (5), emanating from $(t, v(t))$,

$$\begin{cases} \dot{y}_t = f'(w_t), & y_t(t) = v(t) \\ \dot{w}_t = g(w_t), & w_t(t_0) = u(t_0, z(t)). \end{cases}$$

For $t \in [t_0, T]$ define $Z(t) = y_t(t_0)$. Since $u(t, v(t)) = w_t(t)$, it is enough to prove that the counterimage of any measurable set, under the function Z, remains measurable. To do this, we will show that there exists a constant $C_{t_0} > 0$ such that

$$Z(t) - Z(s) \geq C_{t_0}(t - s), \quad \text{for} \quad t_0 \leq s < t \leq T. \tag{13}$$

Fix s, t as above. For $\tau \in [t_0, s]$ define $\Delta(\tau) = y_t(\tau) - y_s(\tau)$. We have

$$\begin{aligned} \frac{Z(t) - Z(s)}{t - s} &= \frac{\Delta(t_0)}{\Delta(s)} \frac{\Delta(s)}{t - s} = \frac{\Delta(t_0)}{\Delta(s)} \left[\frac{y_t(s) - v(t)}{t - s} + \frac{v(t) - v(s)}{t - s} \right] \\ &\geq \frac{\Delta(t_0)}{\Delta(s)} (a - d). \end{aligned} \tag{14}$$

We need to estimate $\frac{\Delta(s)}{\Delta(t_0)}$.

CASE 1. $w_t \leq w_s$ in $[t_0, s]$. Then $\dot{y}_t \leq \dot{y}_s$ in $[t_0, s]$, so

$$\Delta(s) = \Delta(t_0) + \int_{t_0}^{s} \dot{y}_t(\tau) - \dot{y}_s(\tau) \mathrm{d}\tau \leq \Delta(t_0),$$

and thus $\frac{\Delta(s)}{\Delta(t_0)} \leq 1$. □

CASE 2. $w_t > w_s$ in $[t_0, s]$. In this case, for any $\tau \in [t_0, s]$ there holds

$$\begin{aligned} y_t(\tau) - y_s(\tau) &= y_t(0) - y_s(0) + \int_0^{\tau} \dot{y}_t(\zeta) - \dot{y}_s(\zeta) \mathrm{d}\zeta \\ &\geq y_t(0) - y_s(0) + \int_0^{\tau} w_t(\zeta) - w_s(\zeta) \mathrm{d}\zeta \min_{[-C_1, C_1]} f'' \\ &\geq \tau \, (w_t(\tau) - w_s(\tau)) \exp(-T \|g'\|_{\mathbf{L}^\infty}) \min_{[-C_1, C_1]} f''. \end{aligned}$$

Since

$$\dot{y}_t(\tau) - \dot{y}_s(\tau) \leq (w_t(\tau) - w_s(\tau)) \max_{[-C_1, C_1]} f'',$$

we obtain

$$\frac{\dot{y}_t(\tau) - \dot{y}_s(\tau)}{y_t(\tau) - y_s(\tau)} \leq \tilde{C}_{t_0} = \frac{\max_{[-C_1, C_1]} f''}{t_0 \min_{[-C_1, C_1]} f''}.$$

Note that

$$\ln\left(\frac{\Delta(s)}{\Delta(t_0)}\right) = \int_{t_0}^{s} \frac{\Delta'(\tau)}{\Delta(\tau)} \mathrm{d}\tau = \int_{t_0}^{s} \frac{\dot{y}_t(\tau) - \dot{y}_s(\tau)}{y_t(\tau) - y_s(\tau)} \mathrm{d}\tau.$$

Finally

$$\frac{\Delta(s)}{\Delta(t_0)} \leq \exp\left((s - t_0) \max\left\{\frac{\dot{y}_t(\tau) - \dot{y}_s(\tau)}{y_t(\tau) - y_s(\tau)}; \quad \tau \in [t_0, s]\right\}\right) \leq \exp(T\tilde{C}_{t_0}).$$

□

Combining the above estimates with (14), one can see that (13) holds with

$$C_{t_0} = (a - d) \exp(-T\tilde{C}_{t_0}).$$

■

Proof of Lemma 2. We will use the notions of the functions Z, w_t and constants C_{t_0}, introduced in the proof of Lemma 1.

We will show that the estimate of Lemma 2 is true if $t_0 \in (0,T)$, $\delta > 0$ and

$$t_0 \leq \frac{\varepsilon}{4C_2L,} \qquad \delta < \frac{\varepsilon C_{t_0}}{2L\exp(T\|g'\|_{\mathbf{L}^\infty})}.$$

Estimate:

$$\begin{aligned}\int_0^T |h(w(\tau,v(\tau))) - h(u(\tau,v(\tau)))|\,\mathrm{d}\tau &\leq L\int_0^T |w(\tau,v(\tau)) - u(\tau,v(\tau))|\,\mathrm{d}\tau \\ &\leq L\left(2C_2t_0 + \int_{t_0}^T |w(\tau,v(\tau)) - u(\tau,v(\tau))|\,\mathrm{d}\tau\right) \\ &\leq \frac{\varepsilon}{2} + L\sum_i \int_{t_{i-1}}^{t_i} |\alpha_i(\tau) - w_\tau(\tau)|\,\mathrm{d}\tau,\end{aligned}$$

where $\xi_{i-1}(t_{i-1}) = v(t_{i-1})$ and $\xi_i(t_i) = v(t_i)$. By the Gronwall inequality

$$\begin{aligned}\int_{t_{i-1}}^{t_i} |\alpha_i(\tau) - w_\tau(\tau)|\,\mathrm{d}\tau &\leq \int_{t_{i-1}}^{t_i} |\psi(Z(\tau)) - u(t_0,Z(\tau))|\exp(T\|g'\|_{\mathbf{L}^\infty})\mathrm{d}\tau \\ &\leq \frac{\exp(T\|g'\|_{\mathbf{L}^\infty})}{C_{t_0}}\int_{Z(t_{i-1})}^{Z(t_i)} |\psi(x) - u(t_0,x)|\,\mathrm{d}x.\end{aligned}$$

Finally

$$\int_0^T |h(w(\tau,v(\tau))) - h(u(\tau,v(\tau)))|\,\mathrm{d}\tau \leq \frac{\varepsilon}{2} + L\frac{\exp(T\|g'\|_{\mathbf{L}^\infty})}{C_{t_0}}\delta < \varepsilon.$$

■

Proof of Lemma 3. Fix the interval $[t_{i-1},t_i]$. The function λ will be constructed in two different ways, according to if $t_i < T_1$ or $t_{i-1} \geq T_1$. The constant $T_1 \in (0,T)$, sufficiently small and depending only on the equation (5) will be determined later (in CASE 1B below).

Let $y_{i-1} : [0,t_{i-1}] \longrightarrow \mathbf{R}$ be the maximal backward characteristic, emanating from the point $(t_{i-1},x(t_{i-1}))$ and let $y_i : [0,t_i] \longrightarrow \mathbf{R}$ be the minimal backward characteristic, emanating from $(t_i,x(t_i))$,

$$\begin{cases} \dot v_{i-1} = g(v_{i-1}) & v_{i-1}(t_{i-1}) = u(t_{i-1},x(t_{i-1})+) \\ \dot y_{i-1} = f'(v_{i-1}) & y_{i-1}(t_{i-1}) = x(t_{i-1}), \end{cases}$$

$$\begin{cases} \dot v_i = g(v_i) & v_i(t_i) = u(t_i,x(t_i)-) \\ \dot y_i = f'(v_i) & y_i(t_i) = x(t_i). \end{cases}$$

Note that the condition (**P1**) can be replaced by another condition

(**P1**)' $\lambda(t_{i-1}) = \dot y_{i-1}(t_{i-1})$, $\lambda(t_i) = \dot y_i(t_i)$,

as any function λ with the properties (**P1**)' (**P2**) (**P3**) (**P4**) (on $[t_{i-1},t_i]$) can be modified in a way that it satisfies (**P1**) – (**P4**).

CASE 1A. $t_i < T_1$. Assume additionally that $v_i(0) \leq v_{i-1}(0)$. Then also $v_{i-1}(t_{i-1}) \geq v_i(t_{i-1})$. Define

$$\lambda(t) = \frac{t - t_{i-1}}{t_i - t_{i-1}}\ \dot y_i(t_i) + \frac{t_i - t}{t_i - t_{i-1}}\ \dot y_{i-1}(t_{i-1}).$$

The conditions (**P1**)', (**P2**) and (**P3**) are fulfilled. We check (**P4**):

$$\begin{aligned}\dot\lambda(t) &= \frac{\dot y_i(t_i) - \dot y_{i-1}(t_{i-1})}{t_i - t_{i-1}}\\ &= \frac{f'(v_i(t_i)) - f'(v_i(t_{i-1}))}{t_i - t_{i-1}} + \frac{f'(v_i(t_{i-1})) - f'(v_{i-1}(t_{i-1}))}{t_i - t_{i-1}}\\ &\leq \max_{[-C_1,C_1]} |g| \ \max_{[-C_1,C_1]} f'' \leq \beta\ \frac{h(w(t,x(t)) - \lambda(t)}{t} + Q,\end{aligned}$$

provided that $\beta \geq 0$ and $Q \geq \max_{[-C_1,C_1]} |g|\ \max_{[-C_1,C_1]} f''$. □

Before we consider the case $v_{i-1}(0) < v_i(0)$ we need some more computations. Denote $v_0 = v(0)$ and $y_0 = y_{i-1}(0)$. For $\varepsilon > 0$ define $y^\varepsilon, v^\varepsilon : [0,T] \longrightarrow \mathbf{R}$ to be the solutions of

$$\begin{cases} \dot v^\varepsilon = g(v^\varepsilon) & v^\varepsilon(0) = v_0 + \varepsilon \\ \dot y^\varepsilon = f'(v^\varepsilon) & y^\varepsilon(0) = y_0. \end{cases}$$

For a fixed $\varepsilon > 0$ let ε' be such that $y^\varepsilon(t_{i-1} + \varepsilon') = x(t_{i-1} + \varepsilon')$. It is clear that the function $\varepsilon' \mapsto \varepsilon$ is strictly increasing and continuous in its domain $[0, \varepsilon_0)$. Define

$$\lambda(t_{i-1} + \varepsilon') = \dot y^\varepsilon(t_{i-1} + \varepsilon') = f'(v^\varepsilon(t_{i-1} + \varepsilon')). \tag{15}$$

We will compute the derivative of λ at t_{i-1}. This will give us also the formula for $\dot\lambda(t)$, with $t \in [t_{i-1}, t_{i-1} + \varepsilon_0)$.

$$\begin{aligned}\int_{t_{i-1}}^{t_{i-1}+\varepsilon'} h(w(\tau, x(\tau)))\mathrm{d}\tau &= x(t_{i-1} + \varepsilon') - x(t_{i-1})\\ &= y^\varepsilon(t_{i-1} + \varepsilon') - y_{i-1}(t_{i-1})\\ &= y_0 - y_{i-1}(t_{i-1}) + \int_0^{t_{i-1}+\varepsilon'} f'(v^\varepsilon(\tau))\mathrm{d}\tau\\ &= \int_0^{t_{i-1}+\varepsilon'} f'(v^\varepsilon(\tau))\mathrm{d}\tau - \int_0^{t_{i-1}} f'(v(\tau))\mathrm{d}\tau\\ &= \int_0^{t_{i-1}} f'(v^\varepsilon(\tau)) - f'(v(s))\mathrm{d}\tau + \int_{t_{i-1}}^{t_{i-1}+\varepsilon'} f'(v^\varepsilon(\tau))\mathrm{d}\tau.\end{aligned}$$

Note that

$$\begin{aligned}&\lim_{\varepsilon'\to 0} \frac{1}{\varepsilon'} \int_{t_{i-1}}^{t_{i-1}+\varepsilon'} h(w(\tau, x(\tau)))\mathrm{d}\tau = h(w(t_{i-1}, x(t_{i-1}))),\\ &\lim_{\varepsilon'\to 0} \frac{1}{\varepsilon'} \int_{t_{i-1}}^{t_{i-1}+\varepsilon'} f'(v^\varepsilon(\tau))\mathrm{d}\tau = f'(v(t_{i-1})).\end{aligned}$$

Hence

$$h(w(t_{i-1}, x(t_{i-1}))) - f'(v(t_{i-1})) = \lim_{\varepsilon'\to 0} \frac{1}{\varepsilon'} \int_0^{t_{i-1}} f'(v^\varepsilon(\tau)) - f'(v(\tau))\mathrm{d}\tau$$

$$= \lim_{\varepsilon'\to 0} \frac{\varepsilon}{\varepsilon'} \int_0^{t_{i-1}} \frac{f'(v^\varepsilon(\tau)) - f'(v(\tau))}{v^\varepsilon(\tau) - v(\tau)}\ \frac{v^\varepsilon(\tau) - v(\tau)}{\varepsilon}\mathrm{d}\tau$$

$$= \left(\lim_{\varepsilon'\to 0} \frac{\varepsilon}{\varepsilon'}\right) \int_0^{t_{i-1}} f''(v(\tau))C(\tau)\mathrm{d}\tau,$$

where
$$C(\tau)=\lim_{\varepsilon\to 0}\frac{v^{\varepsilon}(\tau)-v(\tau)}{\varepsilon}=\exp\left(\int_0^{\tau}g'(v(s))\mathrm{d}s\right).$$
Consequently
$$\lim_{\varepsilon'\to 0}\frac{\varepsilon}{\varepsilon'}=\frac{h(w(t_{i-1},x(t_{i-1})))-f'(v(t_{i-1}))}{\int_0^{t_{i-1}}f''(v(\tau))C(\tau)\mathrm{d}\tau}.$$
Finally
$$\begin{aligned}(16)\qquad \dot\lambda(t_{i-1}) &=\\ &= f''(v(t_{i-1}))\ \lim_{\varepsilon'\to 0}\frac{v^{\varepsilon}(t_{i-1}+\varepsilon')-v(t_{i-1})}{\varepsilon'}\\ &= f''(v(t_{i-1}))\ \left[\lim_{\varepsilon'\to 0}\frac{v^{\varepsilon}(t_{i-1}+\varepsilon')-v^{\varepsilon}(t_{i-1})}{\varepsilon'}+\lim_{\varepsilon'\to 0}\frac{v^{\varepsilon}(t_{i-1})-v(t_{i-1})}{\varepsilon'}\right]\\ &= f''(v(t_{i-1}))\left[g(v(t_{i-1}))+\left(\lim_{\varepsilon'\to 0}\frac{\varepsilon}{\varepsilon'}\right)C(t_{i-1})\right]\\ &= f''(v(t_{i-1}))g(v(t_{i-1}))\\ &\qquad +\frac{t_{i-1}f''(v(t_{i-1}))C(t_{i-1})}{\int_0^{t_{i-1}}f''(v(\tau))C(\tau)\mathrm{d}\tau}\ \frac{h(w(t_{i-1},x(t_{i-1})))-f'(v(t_{i-1}))}{t_{i-1}}.\end{aligned}$$

CASE 1B. $t_i<T_1$ and $v_{i-1}(0)<v_i(0)$. Define λ by formula (15). By (16) we have
$$\begin{aligned}\dot\lambda(t_{i-1}+\varepsilon') &= f''(v^{\varepsilon}(t_{i-1}+\varepsilon'))g(v^{\varepsilon}(t_{i-1}+\varepsilon'))\\ &+ \frac{(t_{i-1}+\varepsilon')f''(v^{\varepsilon}(t_{i-1}+\varepsilon'))C(t_{i-1}+\varepsilon')}{\int_0^{t_{i-1}+\varepsilon'}f''(v^{\varepsilon}(\tau))C(\tau)\mathrm{d}\tau}\cdot\\ &\qquad\cdot\frac{h(w(t_{i-1}+\varepsilon',x(t_{i-1}+\varepsilon')))-\lambda(t_{i-1}+\varepsilon')}{t_{i-1}+\varepsilon'}.\end{aligned}$$
If $\varepsilon\in[0,(v_i(0)-v_{i-1}(0))/2]$, then
$$\begin{aligned}\dot\lambda(t_{i-1}+\varepsilon') &\le Q+\frac{(t_{i-1}+\varepsilon')f''(v^{\varepsilon}(t_{i-1}+\varepsilon'))C(t_{i-1}+\varepsilon')}{\int_0^{t_{i-1}+\varepsilon'}f''(v^{\varepsilon}(\tau))C(\tau)\mathrm{d}\tau}\cdot\\ &\qquad\cdot\frac{h(w(t_{i-1}+\varepsilon',x(t_{i-1}+\varepsilon')))-\lambda(t_{i-1}+\varepsilon')}{t_{i-1}+\varepsilon'},\end{aligned}$$
for $Q\ge\max_{[-C_2,C_2]}|g|\ \max_{[-C_2,C_2]}f''$. Note that
$$\lim_{t_{i-1}+\varepsilon'\to 0}\ \frac{(t_{i-1}+\varepsilon')f''(v^{\varepsilon}(t_{i-1}+\varepsilon'))C(t_{i-1}+\varepsilon')}{\int_0^{t_{i-1}+\varepsilon'}f''(v^{\varepsilon}(\tau))C(\tau)\mathrm{d}\tau}=1,$$
and the convergence is uniform in ε. Thus (**P4**) is satisfied, for some $\beta\in[1,(b-a)/(b-d))$, if T_1 is small enough. In this way λ is defined on some interval $[t_{i-i},t_{i-1}+\varepsilon'_{i-1}]$.

In the similar way (taking $\varepsilon\in[-(v_i(0)-v_{i-1}(0))/2,0]$), one can define λ on some interval $[t_i-\varepsilon'_i,t_i]$.

If $\varepsilon'_{i-1}+\varepsilon'_i\ge t_i-t_{i-1}$, then for some $t'\in(t_{i-1},t_{i-1}+\varepsilon'_{i-1}]\cap[t_i-\varepsilon'_i,t_i)$, our function λ, defined as above separately on $[t_{i-1},t')$ and $(t',t_i]$ must have a downward jump at t', since $y_i(t)>y_{i-1}(t)$ for $t\in(0,t_{i-1}]$. Such jumps are allowed by (**P3**).

On the other hand, if $\varepsilon'_{i-1}+\varepsilon'_i<t_i-t_{i-1}$, then in the 'missing' interval $[t_{i-1}+\varepsilon'_{i-1},t_i-\varepsilon'_i]$ we define λ linearly (as in CASE 1A). The estimates similar to those

of CASE 1A are valid because for the corresponding numbers $\varepsilon_{i-1}, \varepsilon_i$ there holds $v_{i-1}(0) + \varepsilon_{i-1} = v_i(0) - \varepsilon_i$. □

CASE 2. $t_{i-1} \geq T_1$. Let $\lambda_1, \lambda_2 : [t_{i-1}, t_i] \longrightarrow \mathbf{R}$ be the solutions of the following problems:

$$\dot\lambda_1(t) = \frac{h(w(t,x(t))) - \lambda_1(t)}{t} + Q, \qquad \lambda_1(t_{i-1}) = \dot y_{i-1}(t_{i-1}),$$
$$\dot\lambda_2(t) = \frac{h(w(t,x(t))) - \lambda_2(t)}{t} + Q, \qquad \lambda_2(t_i) = \dot y_i(t_i).$$

More explicitely:

$$\begin{aligned} \lambda_1(t) &= \frac{t_{i-1}\dot y_{i-1}(t_{i-1})}{t} + \frac{x(t) - x(t_{i-1})}{t} + \frac{Q}{2}\, \frac{t^2 - t_{i-1}^2}{t}, \\ \lambda_2(t) &= \frac{t_i \dot y_i(t_i)}{t} + \frac{x(t) - x(t_i)}{t} + \frac{Q}{2}\, \frac{t^2 - t_i^2}{t}. \end{aligned}$$

We will show that if Q is large enough then $\lambda_1(t) \geq \lambda_2(t)$ for some $t \in (t_{i-1}, t_i)$. This will justify the definition of the function λ as $\lambda = \lambda_1$ on (t_{i-1}, t) and $\lambda = \lambda_2$ on (t, t_i).

Note that $\lambda_1(t) \geq \lambda_2(t)$ if and only if

$$[x(t_{i-1}) - t_{i-1}\dot y_{i-1}(t_{i-1})] - [x(t_i) - t_i \dot y_i(t_i)] \leq \frac{Q}{2}\left[t_i^2 - t_{i-1}^2\right].$$

We have

$$y_i(t_{i-1}) = x(t_i) - \int_{t_{i-1}}^{t_i} \dot y_i(\tau)\mathrm{d}\tau = x(t_i) - (t_i - t_{i-1})\dot y_i(t_i) + \int_{t_{i-1}}^{t_i} \dot y_i(t_i) - \dot y_i(\tau)\mathrm{d}\tau$$

and since $y_{i-1}(t_{i-1}) = x(t_{i-1})$,

$$y_i(t_{i-1}) - y_{i-1}(t_{i-1}) = x(t_i) - x(t_{i-1}) - (t_i - t_{i-1})\dot y_i(t_i) + \int_{t_{i-1}}^{t_i} \dot y_i(t_i) - \dot y_i(\tau)\mathrm{d}\tau$$

$$\leq x(t_i) - x(t_{i-1}) - \dot y_i(t_i)(t_i - t_{i-1}) + \frac{1}{2}\left(\max_{[-C_1,C_1]} |g| \max_{[-C_1,C_1]} f''\right)(t_i - t_{i-1})^2.$$

Hence

$$\begin{aligned} &[x(t_{i-1}) - t_{i-1}\dot y_{i-1}(t_{i-1})] - [x(t_i) - t_i\dot y_i(t_i)] \\ &\leq t_i(\dot y_i(t_i) - \dot y_i(t_{i-1})) + t_i(\dot y_i(t_{i-1}) - \dot y_{i-1}(t_{i-1})) + C(t_i - t_{i-1})^2 \\ &\leq C\left[t_i(t_i - t_{i-1}) + t_i(u(t_{i-1}, y_i(t_{i-1})) - u(t_{i-1}, x(t_{i-1}))) + (t_i - t_{i-1})^2\right], \end{aligned}$$

where $C > 0$ is a constant depending only on the equation (5). By Oleinik inequality ([8]), the last estimate yields the desired

$$\begin{aligned} &[x(t_{i-1}) - t_{i-1}\dot y_{i-1}(t_{i-1})] - [x(t_i) - t_i\dot y_i(t_i)] \\ &\leq C\left[t_i(t_i - t_{i-1}) + \frac{t_i}{T_1}(y_i(t_{i-1}) - x(t_{i-1})) + (t_i - t_{i-1})^2\right] \leq C(t_i^2 - t_{i-1}^2). \end{aligned}$$

□ ■

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