A NOTE ON CONVERGENCE OF LOW ENERGY CRITICAL POINTS OF NONLINEAR ELASTICITY FUNCTIONALS, FOR THIN SHELLS OF ARBITRARY GEOMETRY

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Abstract. We prove that the critical points of the 3d nonlinear elasticity functional on shells of small thickness h and around the mid-surface S of arbitrary geometry, converge as $h \to 0$ to the critical points of the von Kármán functional on S, recently proposed in [8]. This result extends the statement in [16], derived for the case of plates when $S \subset \mathbb{R}^2$. The convergence holds provided the elastic energies of the 3d deformations scale like h^4 and the external body forces scale like h^3.

Contents

1. Introduction and statement of the main results

Since the beginning of research in nonlinear elasticity, a major topic has been the derivation of lower dimensional theories, appropriately approximating the three dimensional theory on structures which are thin in one or more directions (such as beams, rods, plates or shells). Recently, the application of variational methods, notably the Γ-convergence [3], lead to many significant and rigorous results in this setting [7, 5]. Roughly speaking, a Γ-limit approach guarantees the convergence of minimizers of a sequence of functionals, to the minimizers of the limit. However, it does not usually imply convergence of the possibly non-minimizing critical points (the equilibria) and hence other tools must be applied to study this problem.

In this note, following works [13, 14, 16] in which beams, rods and plates were analyzed, we study critical points of the 3d nonlinear elasticity functional on a thin shell of arbitrary geometry, in the von Kármán scaling regime. A Γ-convergence result in this framework was recently derived in [8], providing the natural from the minimization point of view generalization of the von Kármán functional [5] to shells. In analogy with the analysis done in [16] for plates, we now proceed to prove convergence of the weak solutions to the static equlibrium equations of nonlinear elasticity (1.13) (the Euler-Lagrange equations associated to the elasticity functional), which in the present setting are the critical points of the 3d energy (1.1), to the critical points of the functional obtained in [8].

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We now introduce the basic framework for our results. We consider a 2-dimensional surface S embedded in $\mathbb{R}^3$, which is compact, connected, oriented, of class $\mathcal{C}^{1,1}$, and with boundary ∂S being the union of finitely many (possibly none) Lipschitz curves. A family $\{S^h\}_{h>0}$ of shells of small thickness h around S is given through:

$$S^h = \{z = x + t\vec{n}(x);\ x \in S,\ -h/2 < t < h/2\}, \qquad 0 < h < h_0.$$

By $\vec{n}(x)$ we denote the unit normal to S and by T_xS the tangent space. With a slight abuse of notation, we shall treat T_xS as a 2d subspace of $\mathbb{R}^3$, so that both 2d and 3d operators can be applied on vectors $\tau \in T_xS$. By $\Pi(x) = \nabla\vec{n}(x)$ we denote the shape operator on S (the negative second fundamental form). Recall that Π is symmetric and $\Pi(x)\tau \in T_xS$ for all $\tau \in T_xS$. Again, we shall view Π as a linear operator from T_xS to $\mathbb{R}^3$, or a linear operator from T_xS to T_xS, or otherwise as a symmtric bilinear form, or as its matric representation, whichever is more convenient.

The projection onto S along $\vec{n}$ is denoted by π, so that:

$$\pi(z) = x \qquad \forall z = x + t\vec{n}(x) \in S^h.$$

We assume that $h < h_0$, with h_0 sufficiently small to have π defined on each S^h.

To a deformation $u \in W^{1,2}(S^h, \mathbb{R}^3)$ we associate its elastic energy (scaled per unit thickness):

$$I^h(u) = \frac{1}{h}\int_{S^h} W(\nabla u). \tag{1.1}$$

The stored energy density $W : \mathbb{R}^{3\times3} \longrightarrow [0, \infty]$ is assumed to be $\mathcal{C}^2$ in a neighborhood of $SO(3)$, and to satisfy the following normalization, frame indifference and nondegeneracy conditions:

$$\begin{aligned} \forall F \in \mathbb{R}^{3\times3} \quad \forall R \in SO(3) \qquad & W(R) = 0, \quad W(RF) = W(F), \\ & W(F) \geq c\ \mathrm{dist}^2(F, SO(3)) \end{aligned} \tag{1.2}$$

(with a uniform constant $c > 0$). Our objective is to describe the limiting behavior, as $h \to 0$, of critical points u^h to the following total energy functionals:

$$J^h(u) = I^h(u) - \frac{1}{h}\int_{S^h} f^h u, \tag{1.3}$$

subject to external forces f^h, where we assume that:

$$f^h(x + t\vec{n}) = h\sqrt{e^h} f(x)\det(\mathrm{Id} + t\Pi)^{-1}, \quad f \in L^2(S, \mathbb{R}^3) \text{ and } \int_S f = 0.$$

Above, e^h is a given sequence of positive numbers obeying a prescribed scaling law. It can be shown [5, 8] that if f^h scale like h^α, then the minimizers u^h of (1.3) satisfy $I^h(u^h) \sim h^\beta$ with $\beta = \alpha$ if $0 \leq \alpha \leq 2$ and $\beta = 2\alpha - 2$ if $\alpha > 2$. Throughout this paper we shall assume that $\beta \geq 4$, or more generally:

$$\lim_{h\to0} e^h/h^4 = \kappa < +\infty, \tag{1.4}$$

which for $S \subset \mathbb{R}^2$ corresponds to the von Kármán and the purely linear theories of plates, derived rigorously in [5].

In our recent paper [8], the Γ-limit of $1/e^hJ^h$ has been identified in the scaling range corresponding to (1.4), and for arbitrary surfaces S. It turns out that the elastic energy scaling $I^h(u^h) \leq Ce^h$ implies that on S the deformations $u^h_{|S}$ must be close to some rigid motion $\bar{Q}x + c$, and that the first order term in the expansion of $\bar{Q}^T(u^h_{|S} - c) - \mathrm{id}$ with respect to h, is an element V of the class

$\mathcal{V}$ of *infinitesimal isometries* on S [17]. The space $\mathcal{V}$ consists of vector fields $V \in W^{2,2}(S,\mathbb{R}^3)$ for whom there exists a matrix field $A \in W^{1,2}(S,\mathbb{R}^{3\times 3})$ so that:

$$\partial_\tau V(x) = A(x)\tau \quad \text{and} \quad A(x)^T = -A(x) \qquad \forall \text{a.e. } x \in S \quad \forall \tau \in T_x S. \tag{1.5}$$

Equivalently, the change of metric on S induced by the deformation $\mathrm{id} + hV$ is at most of order h^2, for each $V \in \mathcal{V}$.

When in (1.4) $\kappa = 0$, the limiting total energy is given by:

$$J(V,\bar{Q}) = \frac{1}{24}\int_S \mathcal{Q}_2\left(x, (\nabla(A\vec{n}) - A\Pi)_{tan}\right)\,\mathrm{d}x - \int_S f\cdot \bar{Q}V\,\mathrm{d}x, \qquad \forall V \in \mathcal{V},\ \bar{Q} \in SO(3). \tag{1.6}$$

The first term above measures the first order change in the second fundamental form Π of S, produced by V. The subscript 'tan' refers to the tangential minor of a given matrix field; given $F \in L^2(S,\mathbb{R}^{3\times 3})$ we put: $F_{tan}(x) = [(F(x)\tau)\eta]_{\tau\eta\in T_x S}$.

The quadratic forms $\mathcal{Q}_2(x,\cdot)$ are given as follows:

$$\mathcal{Q}_2(x, F_{tan}) = \min\{\mathcal{Q}_3(\tilde{F});\ (\tilde{F} - F)_{tan} = 0\}, \qquad \mathcal{Q}_3(F) = D^2W(\mathrm{Id})(F,F).$$

The form $\mathcal{Q}_3$ is defined for all $F \in \mathbb{R}^{3\times 3}$, while $\mathcal{Q}_2(x,\cdot)$ for a given $x \in S$, is defined on tangential minors F_{tan} of such matrices (as explained above). Both forms depend only on the symmetric parts of their arguments and are positive definite on the space of symmetric matrices [4]. In the weak formulation of the Euler-Lagrange equations of (1.6) one naturally encounters the linear operators $\mathcal{L}_3$ and $\mathcal{L}_2(x,\cdot)$, defined on and valued in the matrix spaces $\mathbb{R}^{3\times 3}$ and $\mathbb{R}^{2\times 2}$ (identified here with the space of bilinear forms on $T_x S$), respectively, given by:

$$\forall F \in \mathbb{R}^{3\times 3} \qquad \mathcal{Q}_3(F) = \mathcal{L}_3 F : F \quad \text{and} \quad \mathcal{Q}_2(x, F_{tan}) = \mathcal{L}_2(x, F_{tan}) : F_{tan}.$$

Recall that, for two square matrices F_1 and F_2, of the same dimension, their inner product is $F_1 : F_2 = \mathrm{tr}(F_1^T F_2)$.

For $\kappa > 0$, the Γ-limit (which is the generalization of the von Kármán functional [5] to shells), contains also a stretching term, measuring the total second order change in the metric of S:

$$J^{vK}(V, B_{tan}, \bar{Q}) = \frac{\kappa}{2}\int_S \mathcal{Q}_2\left(x, B_{tan} - \frac{1}{2}(A^2)_{tan}\right) + \frac{1}{24}\int_S \mathcal{Q}_2\left(x, (\nabla(A\vec{n}) - A\Pi)_{tan}\right) - \int_S f\cdot\bar{Q}V. \tag{1.7}$$

It involves a symmetric matrix field B_{tan} on S belonging to the *finite strain space* $\mathcal{B}$. Given a vector field $w \in W^{1,2}(S,\mathbb{R}^3)$, by sym ∇w we mean the bilinear form on $T_x S$, given by: $((\mathrm{sym}\ \nabla w(x))\tau)\eta = \frac{1}{2}[(\partial_\tau V(x))\eta + (\partial_\eta V(x))\tau]$ for all $\tau,\eta \in T_x S$. Then we define [8]:

$$\mathcal{B} = \mathrm{cl}_{L^2(S)}\Big\{\mathrm{sym}\nabla w^h;\ w^h \in W^{1,2}(S,\mathbb{R}^3)\Big\}.$$

The two terms in (1.7) correspond, in appearing order, to the stretching and bending energies of a sequence of deformations $v^h = \mathrm{id} + \epsilon V + \epsilon^2 w^h$ of S (where $\epsilon = \sqrt{e^h}/h$) which is induced by:

(i) a first order displacement $V \in \mathcal{V}$
(ii) the second order displacements w^h satisfying $\lim_{h\to 0}\mathrm{sym}\nabla w^h = B_{tan}$.

In view of the fundamental theorem of calculus of variations, the crucial property of (1.7) is the one-to-one correspondence between the minimizing sequences u^h of the total energies $J^h(u^h)$, and their approximations (modulo rigid motions $\bar{Q}x + c$) given by v^h as above with $(V, B_{tan}, \bar{Q})$ minimizing the Γ-limit J^{vK} of $1/e^h J^h$ (or $(V,\bar{Q})$ minimizing J when $\kappa = 0$).

The purpose of this paper is to show that under the following extra assumption of [16]:

$$\forall F \in \mathbb{R}^{3\times 3} \quad |DW(F)| \le C(|F| + 1). \tag{1.8}$$

also the equilibria (possibly non-minimizing) of (1.1) converge to the equilibria of (1.7) or (1.3).

Theorem 1.1. *Assume (1.2) and (1.8). Let $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ be a sequence of deformations, satisfying:*

(a) *the following equilibrium equations hold, for each h:*

$$\forall \phi^h \in W^{1,2}(S^h, \mathbb{R}^3) \qquad \int_{S^h} DW(\nabla u^h) : \nabla \phi^h = \int_{S^h} f^h \phi^h, \tag{1.9}$$

(b) *$I^h(u^h) \leq C e^h$, where e^h is the scaling with (1.4).*

Then there exist a sequence $Q^h \in SO(3)$, converging (up to a subsequence) to some $\bar{Q} \in SO(3)$, and $c^h \in \mathbb{R}^3$ such that for the normalized rescaled deformations:

$$y^h(x + t\vec{n}) = (Q^h)^T u^h(x + h/h_0 t\vec{n}) - c^h$$

defined on the common domain S^{h_0}, we have:

(i) *y^h converge in $W^{1,2}(S^{h_0})$ to π.*

(ii) *The scaled average displacements:*

$$V^h(x) = \frac{h}{\sqrt{e^h}} \fint_{-h_0/2}^{h_0/2} y^h(x + t\vec{n}) - x \, \mathrm{d}t \tag{1.10}$$

converge (up to a subsequence) in $W^{1,2}(S)$ to some $V \in \mathcal{V}$.

(iii) *$h/\sqrt{e^h}$* sym ∇V^h *converge (up to a subsequence) in $L^2(S)$ to some $B_{tan} \in \mathcal{B}$.*

(iv) *The triple $(V, B_{tan}, \bar{Q})$ satisfies the Euler-Lagrange equations of the functional J^{vK}. That is, for all $\tilde{V} \in \mathcal{V}$ with $\tilde{A} = \nabla \tilde{V}$ given as in the formula (1.5), and all $\tilde{B}_{tan} \in \mathcal{B}$, there holds:*

$$\int_S \mathcal{L}_2 \left(x, B_{tan} - \frac{1}{2}(A^2)_{tan} \right) : \tilde{B}_{tan} = 0, \tag{1.11}$$

$$\begin{aligned} & - \kappa \int_S \mathcal{L}_2 \left(x, B_{tan} - \frac{1}{2}(A^2)_{tan} \right) : (A\tilde{A})_{tan} \\ & \qquad + \frac{1}{12} \int_S \mathcal{L}_2 \left(x, (\nabla(A\vec{n}) - A\Pi)_{tan} \right) : (\nabla(\tilde{A}\vec{n}) - \tilde{A}\Pi)_{tan} = \int_S f \cdot \bar{Q}\tilde{V}, \end{aligned} \tag{1.12}$$

When $\kappa = 0$ then the couple $(V, \bar{Q})$ satisfies (1.12) for all $\tilde{V} \in \mathcal{V}$, which is the Euler-Lagrange equations of the functional (1.6).

We prove Theorem 1.1 in section 2. In section 3 we derive the third Euler-Lagrange equation (after the first two (1.11) and (1.12)), corresponding to variation in $\bar{Q} \in SO(3)$. We first notice that the limiting $\bar{Q}$ necessarily satisfies the constraint of the average torque $\tau(\bar{Q}) = \int_S f \times \bar{Q}x \, \mathrm{d}x$ being 0. The main difficulty arises now from the fact that the variations must be taken inside $SO(3)$ in a way that this constraint remains satisfied. Assuming that such variations exist, we establish the limit equation under the nondegeneracy condition that Q^h approach $\bar{Q}$ along a direction $U \in T_{\bar{Q}}SO(3)$ for which $\partial_U \tau(\bar{Q}) \neq 0$.

Remark 1.2. Integrating by parts we see that (1.9) is the weak formulation of the following fundamental balance law [1]:

$$\text{div } [DW(\nabla u^h)] + f^h = 0 \text{ in } S^h, \qquad DW(\nabla u^h)\vec{n} = 0 \text{ on } \partial S^h, \tag{1.13}$$

where the operator div above is understood as acting on rows of the matrix field $DW(\nabla u^h)$.

The definition of an equilibrium of the 3d energy J^h may be understood in two different manners, corresponding to passing with the scaling ϵ of a variation ϕ to 0 outside or inside the integral sign. Namely, for a fixed $h > 0$, we may either require (1.9) or require that:

$$\forall \phi^h \in W^{1,2}(S^h, \mathbb{R}^3) \qquad \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big(J^h(u^h + \epsilon\phi^h) - J^h(u^h)\Big) = 0. \tag{1.14}$$

Whether (1.14) and (1.9) are equivalent, even for local minimizers (without assuming extra regularity, e.g. their Lipschitz continuity) is an open problem of nonlinear elasticity, listed by Ball as Problem 5 in [1]. However, condition (1.8) readily implies that:

$$\lim_{\epsilon\to 0}\int_{S^h} \frac{1}{\epsilon}[W(\nabla u^h + \epsilon\nabla\phi^h) - W(\nabla u^h)] = \int_{S^h} DW(\nabla u^h) : \nabla\phi^h,$$

because of the pointwise convergence of the integrands and of their boundedness by an L^1 function independent of ϵ:

$$\frac{1}{\epsilon}|W(\nabla u^h+\epsilon\nabla\phi^h)-W(\nabla u^h)| \leq \int_0^1 |DW(\nabla u^h+\epsilon s\nabla\phi^h)|\cdot|\nabla\phi^h|\,\mathrm{d}s \leq C\Big(|\nabla u^h| + |\nabla\phi^h| + 1\Big)|\nabla\phi^h|.$$

Hence in presence of (1.8), the equilibrium conditions (1.14) and (1.9) are equivalent.

Remark 1.3. Notice that, in view of (1.2) resulting in $DW(F) = 0$ for all $F \in SO(3)$, condition (1.8) is equivalent to:

$$\forall F \in \mathbb{R}^{3\times 3} \quad |DW(F)| \leq C\mathrm{dist}(F, SO(3)).$$

Using the last assumption in (1.2), the above implies that: $|DW(F)| \leq CW(F)^{1/2}$ for all $F \in \mathbb{R}^{3\times 3}$. Hence, roughly speaking, W has a quadratic growth and we see that (1.8) is actually very restrictive. Independently, Mora and Scardia [15] have recently proved a complementary result where the requirement (1.8) is relaxed, while the equilibrium condition of (1.3) is understood in a different manner, related to Ball's inner variations and the Cauchy stress balance law [1].

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2. Convergence of weak solutions to the Euler-Lagrange equations (equilibria) of the 3d energies

We first gather the relevant information from [8]:

Lemma 2.1. [8] *Let $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ be a sequence of deformations of shells S^h. Assume (1.4) and let the scaled energies $I^h(u^h)/e^h$ be uniformly bounded. Then there exists a sequence of matrix fields $R^h \in W^{1,2}(S, \mathbb{R}^3)$ with $R^h(x) \in SO(3)$ for a.e. $x \in S$, such that:*

$$\|\nabla u^h - R^h\pi\|_{L^2(S^h)} \leq Ch^{1/2}\sqrt{e^h} \qquad \text{and} \qquad \|\nabla R^h\|_{L^2(S)} \leq Ch^{-1}\sqrt{e^h}$$

and there exists a sequence of matrices $Q^h \in SO(3)$ such that:

(i) $\|(Q^h)^T R^h - \mathrm{Id}\|_{W^{1,2}(S)} \leq C\sqrt{e^h}/h$.

(ii) $h/\sqrt{e^h}((Q^h)^T R^h - \mathrm{Id})$ *converges (up to a subsequence) to a skew-symmetric matrix field A, weakly in $W^{1,2}(S)$.*

Moreover, there exists a sequence $c^h \in \mathbb{R}^3$ such that for the normalized rescaled deformations:

$$y^h(x + t\vec{n}) = (Q^h)^T u^h(x + h/h_0 t\vec{n}) - c^h$$

defined on the common domain S^{h_0}, the following holds.

(iii) y^h *converge in $W^{1,2}(S^{h_0})$ to π.*

(iv) *The scaled average displacements V^h, defined in (1.10) converge (up to a subsequence) in $W^{1,2}(S)$ to some $V \in \mathcal{V}$, whose gradient is given by A, as in (1.5).*

(v) $h/\sqrt{e^h}$ sym ∇V^h *converge (up to a subsequence) in $L^2(S)$ to some $B_{tan} \in \mathcal{B}$.*

The statements in Theorem 1.1 (i), (ii), (iii) are contained in the Lemma above. It therefore suffices to use the extra assumptions (1.9) and (1.8) to recover equations (1.11) and (1.12) as $h \to 0$.

We start by rewriting the equilibrium equation (1.9) in a more convenient form. Clearly, every variation $\phi^h \in W^{1,2}(S^h, \mathbb{R}^3)$ can be by a change of variables expressed as:

$$\phi^h(x + t\vec{n}) = \psi(x + th_0/h\vec{n}), \tag{2.1}$$

for the corresponding $\psi \in W^{1,2}(S^{h_0}, \mathbb{R}^3)$. Then, (1.9) becomes:

$$\begin{aligned} & h^2\sqrt{e^h}\int_S f(x) \fint_{-h_0/2}^{h_0/2} \psi(x + t\vec{n}) \ \mathrm{d}t \ \mathrm{d}x \\ & = h\int_S \fint_{-h_0/2}^{h_0/2} \det(\mathrm{Id} + th/h_0\Pi) DW(\nabla u^h(x + th/h_0\vec{n})) : \nabla\phi^h(x + th/h_0\vec{n}) \ \mathrm{d}t \ \mathrm{d}x. \end{aligned} \tag{2.2}$$

Notice also that:

$$\nabla\phi^h(x + th/h_0\vec{n}) = \nabla\psi(x + t\vec{n})P(x + t\vec{n}), \tag{2.3}$$

where the matrix field $P \in L^\infty(S^{h_0}, \mathbb{R}^{3\times3})$ has the following non-zero entries:

$$P(x + t\vec{n})_{tan} = (\mathrm{Id} + th/h_0\Pi(x))^{-1}(\mathrm{Id} + t\Pi(x)), \qquad \vec{n}^T P(x + t\vec{n})\vec{n} = h_0/h.$$

In view of Lemma 2.1, define the matrix fields $E^h, G^h \in L^2(S^{h_0}, \mathbb{R}^{3\times3})$:

$$E^h = \frac{1}{\sqrt{e^h}} DW(\mathrm{Id} + \sqrt{e^h}G^h), \qquad G^h(x + t\vec{n}) = \frac{1}{\sqrt{e^h}}\left((R^h)^T\nabla u^h(x + th/h_0\vec{n}) - \mathrm{Id}\right).$$

With this notation, recalling the frame invariance of W in (1.2) we get, for every $F \in \mathbb{R}^{3\times3}$:

$$\begin{aligned} \frac{1}{\sqrt{e^h}} DW(\nabla u^h(x + th/h_0\vec{n})) : F &= \frac{1}{\sqrt{e^h}} DW(R^h(\mathrm{Id} + \sqrt{e^h}G^h)) : F \\ &= \frac{1}{\sqrt{e^h}} DW(\mathrm{Id} + \sqrt{e^h}G^h) : (R^h)F = R^hE^h : F. \end{aligned}$$

In particular, (2.2) becomes, after exchanging ψ to $(Q^h)^T\psi$, using (2.3) and dividing both sides by $\sqrt{e^h}$:

(2.4)

$$\begin{aligned} & h^2\int_S f(x) \fint_{-h_0/2}^{h_0/2} Q^h\psi \ \mathrm{d}t\mathrm{d}x \\ & = h\int_S \fint_{-h_0/2}^{h_0/2} \det(\mathrm{Id} + th/h_0\Pi) \ \left[(Q^h)^T R^h(x)E^h(x + t\vec{n})\right] : \nabla\phi^h(x + th/h_0\vec{n}) \ \mathrm{d}t\mathrm{d}x \\ & = h\int_S \fint_{-h_0/2}^{h_0/2} \det(\mathrm{Id} + th/h_0\Pi) \ \left[(Q^h)^T R^h E^h\right]_{TS} : \left[(\nabla_{tan}\psi)(\mathrm{Id} + th/h_0\Pi)^{-1}(\mathrm{Id} + t\Pi)\right] \ \mathrm{d}t\mathrm{d}x \\ & \quad + h_0\int_S \fint_{-h_0/2}^{h_0/2} \det(\mathrm{Id} + th/h_0\Pi) \ ((Q^h)^T R^h E^h\vec{n}) \ \partial_{\vec{n}}\psi(x + t\vec{n}) \ \mathrm{d}t\mathrm{d}x, \end{aligned}$$

where ∇_{tan} denotes gradient in the tangent directions of T_xS. The subscript TS stands for taking the 3×2 minor of the matrix under consideration, for example: $\nabla_{tan}\psi = [\nabla\psi]_{TS}$. Also, with a slight abuse of notation, Id stands for the identity map on $\mathbb{R}^3$ or T_xS, whicheven is appropriate.

Lemma 2.2. *The sequence G^h converges (up to a subsequence), weakly in $L^2(S^{h_0},\mathbb{R}^{3\times 3})$ to a $L^2(S^{h_0})$ matrix field G, whose symmetrized tangential minor has the form:*

$$\mathrm{sym}G(x+t\vec{n})_{tan} = \sqrt{\kappa}\left(B_{tan} - \frac{1}{2}(A^2)_{tan}\right) + \frac{t}{h_0}\left(\nabla(A\vec{n}) - A\Pi\right)_{tan}. \tag{2.5}$$

Moreover, if (1.8) holds, then:

(i) *E^h converges (up to a subsequence) weakly in $L^2(S^{h_0},\mathbb{R}^{3\times 3})$ to the matrix field $E = \mathcal{L}_3 G$.*
(ii) *The sequence $(Q^h)^T R^h(x)E^h(x+t\vec{n})$ converges (up to a subsequence) to E, weakly in $L^2(S^{h_0},\mathbb{R}^{3\times 3})$.*

Proof. The convergence of G^h and the formula (2.5) follow from Lemma 3.6 and Lemma 4.1 in [8]. Convergence in (i) is a consequence of Proposition 2.3 in [16], where the crucial role was played by the following equivalent form of the assumption (1.8):

$$\forall F\in\mathbb{R}^{3\times 3} \qquad |DW(\mathrm{Id}+F)| \leq C|F|.$$

Finally, (ii) is an immediate consequence of (i) in view of Lemma 2.1 (i) and the boundedness of $(Q^h)^T R^h$ in $L^\infty(S^{h_0})$. ■

Lemma 2.3. *The matrix field $E\in L^2(S^{h_0},\mathbb{R}^{3\times 3})$, defined in Lemma 2.2 (i) satisfies the following properties, a.e. in S^{h_0}:*

(i) *$E\vec{n} = 0$.*
(ii) *$E^T = E$, that is: E is symmetric.*
(iii) *$E_{tan}(x+t\vec{n}) = \mathcal{L}_2(x, G_{tan}(x+t\vec{n}))$.*

Proof. To prove (i), one needs to pass $h\to 0$ in (2.4) and use Lemma 2.2 (ii) to obtain:

$$\int_S \fint_{-h_0/2}^{h_0/2} \Big(E(x+t\vec{n})\vec{n}\Big)\, \partial_{\vec{n}}\psi(x+t\vec{n})\ \mathrm{d}t\mathrm{d}x = 0. \tag{2.6}$$

Now, any vector field $\phi\in L^2(S^{h_0},\mathbb{R}^3)$ has the form $\phi = \partial_{\vec{n}}\psi$, where $\psi(x+t\vec{n}) = \int_{-h_0/2}^t \phi(x+s\vec{n})\ \mathrm{d}s$. Therefore (i) follows from (2.6).

By frame indifference (1.2) and the fact that W is minimized at Id, it follows that $DW(F)=0$ for all $F\in SO(3)$. It implies that for all $H\in so(3)$ there holds $\mathcal{L}_3 H = 0$, and so $E : H = \mathcal{L}_3 G : H = \mathcal{L}_3 H : G = 0$, proving (ii). Here $so(3)$ stands for the space of 3×3 skew-symmetric matrices.

The assertion (iii) follows from $E = \mathcal{L}_3 G$ and the reasoning exactly as in the proof of Proposition 3.2 [16]. ■

A more precise information, with respect to that in Lemma 2.3 (ii) is given by:

Lemma 2.4. *There holds:*

(i) $\|\ \mathrm{skew}\ E^h\|_{L^1(S^{h_0})} \leq C\sqrt{e^h}$.
(ii) $\displaystyle\lim_{h\to 0}\frac{1}{h}\|\ \mathrm{skew}\ E^h\|_{L^p(S^{h_0})} = 0$, *for some exponent $p\in(1,2)$.*

Proof. By frame indifference (1.2) one has: $0 = DW(F) : HF = DW(F)F^T : H$, for all $F\in\mathbb{R}^{3\times 3}$ and all $H\in so(3)$ (since HF is a tangent vector to $SO(3)F$ at F). We further obtain that

$DW(F)F^T$ is a symmetric matrix. Apply this statement pointwise to the matrix field $F = \mathrm{Id} + \sqrt{e^h}G^h$:

$$\begin{aligned} 0 &= \frac{1}{\sqrt{e^h}}\Big(DW(\mathrm{Id} + \sqrt{e^h}G^h)\ (\mathrm{Id} + \sqrt{e^h}(G^h)^T) - (\mathrm{Id} + \sqrt{e^h}G^h)\ DW^T(\mathrm{Id} + \sqrt{e^h}G^h)\Big) \\ &= E^h - (E^h)^T + \sqrt{e^h}\left(E^h(G^h)^T - G^h(E^h)^T\right). \end{aligned}$$

Hence the claim in (i) is proved, as by Lemma 2.2:

$$\|\mathrm{sym}\ (E^h(G^h)^T)\|_{L^1(S^{h_0})} \leq C\|E^h\|_{L^2(S^{h_0})}\|G^h\|_{L^2(S^{h_0})} \leq C.$$

Now, (ii) follows from (i) in view of the boundedness of E^h in $L^2(S^{h_0})$, (1.4), and through an interpolation inequality:

$$\frac{1}{h}\|\ \mathrm{skew}\ E^h\|_{L^p(S^{h_0})} \leq \frac{1}{h}\|\ \mathrm{skew}\ E^h\|^{\theta}_{L^1}\|\ \mathrm{skew}\ E^h\|^{1-\theta}_{L^2} \leq C/h\sqrt{e^h}^{\theta} = C\left(\sqrt{e^h}/h^2\right)^{\theta} h^{2\theta-1},$$

where $1/p = \theta + (1-\theta)/2$ and $\theta \in (0,1)$. Clearly, the above converges to 0, when $\theta > 1/2$. ■

Introduce now the two matrix fields $\bar{E},\ \hat{E} \in L^2(S, \mathbb{R}^3)$ given by the 0th and 1st moments of E:

$$\bar{E}(x) = \fint_{-h_0/2}^{h_0/2} E(x + t\vec{n})\ \mathrm{d}t, \qquad \hat{E}(x) = \fint_{-h_0/2}^{h_0/2} tE(x + t\vec{n})\ \mathrm{d}t.$$

It easily follows by Lemma 2.3 (iii), Lemma 2.2 and the fact that $\mathcal{L}_2(x,\cdot)$ depends only on the symmetric part of its argument, that:

$$\bar{E}_{tan}(x) = \fint_{-h_0/2}^{h_0/2} \mathcal{L}_2(x, G_{tan}(x + t\vec{n}))\ \mathrm{d}t = \sqrt{\kappa}\mathcal{L}_2\left(x, B_{tan} - \frac{1}{2}(A^2)_{tan}\right), \tag{2.7}$$

$$\hat{E}_{tan}(x) = \fint_{-h_0/2}^{h_0/2} \mathcal{L}_2(x, tG_{tan}(x + t\vec{n}))\ \mathrm{d}t = \frac{h_0}{12}\mathcal{L}_2\left(x, (\nabla(A\vec{n}) - A\Pi)_{tan}\right). \tag{2.8}$$

We will now use the fundamental balance (2.4) and the above formulas to recover the Euler-Lagrange equations (1.11), (1.12) in the limit as $h \to 0$.

Proof of the first Euler-Lagrange equation (1.11).
Use the variation of the form: $\psi(x + t\vec{n}) = \phi(x)$ in (2.4), divide both sides by h and pass to the limit to obtain:

$$\begin{aligned} 0 &= \lim_{h\to 0}\int_S\fint_{-h_0/2}^{h_0/2} \det(\mathrm{Id} + th/h_0\Pi)\ \left[(Q^h)^T R^h E^h\right]_{TS} : \left[\nabla_{tan}\phi(x)(\mathrm{Id} + th/h_0\Pi)^{-1}\right]\ \mathrm{d}t\mathrm{d}x \\ &= \int_S\fint_{-h_0/2}^{h_0/2} E_{TS} : \nabla_{tan}\psi(x)\ \mathrm{d}t\mathrm{d}x = \sqrt{\kappa}\int_S \mathcal{L}_2\left(x, B_{tan} - \frac{1}{2}(A^2)_{tan}\right) : [\nabla\phi(x)]_{tan}\ \mathrm{d}x \\ &= \sqrt{\kappa}\int_S \mathcal{L}_2\left(x, B_{tan} - \frac{1}{2}(A^2)_{tan}\right) : [\mathrm{sym}\ \nabla\phi(x)]_{tan}\ \mathrm{d}x \end{aligned} \tag{2.9}$$

where we have used Lemma 2.2 (i), Lemma 2.3 and (2.7). Therefore, by density of $\{\mathrm{sym}\ \nabla\phi\}$ in the space $\mathcal{B}$, (1.11) follows immediately.

Proof of the second Euler-Lagrange equation (1.12).
Let $\tilde{V} \in \mathcal{V}$ and denote by $\tilde{A}$ the skew-symmetric matrix field representing $\nabla\tilde{V}$, as in (1.5).

1. We now apply (2.4) to a variation of the form: $\psi(x+t\vec{n}) = t\tilde{A}\vec{n}(x)$. For simplicity, write $\eta = \tilde{A}\vec{n} \in W^{1,2}(S,\mathbb{R}^3)$. Upon dividing (2.4) by h and passing to the limit, we obtain:

(2.10)

$$
\begin{aligned}
0 = \lim_{h\to 0}\Bigg[&\int_S \fint_{-h_0/2}^{h_0/2} \det(\mathrm{Id} + th/h_0\Pi)\ \Big[(Q^h)^T R^h t E^h\Big]_{TS} : \big[\nabla_{tan}\eta(x)(\mathrm{Id}+th/h_0\Pi)^{-1}\big]\ \mathrm{d}t\mathrm{d}x \\
&+ \frac{h_0}{h}\int_S \fint_{-h_0/2}^{h_0/2} ((Q^h)^T R^h E^h \vec{n})\ \eta(x)\ \mathrm{d}t\mathrm{d}x \\
&+ \int_S \fint_{-h_0/2}^{h_0/2} (t \text{ trace } \Pi + t^2 h/h_0 \det \Pi)((Q^h)^T R^h E^h\vec{n})\ \eta(x)\ \mathrm{d}t\mathrm{d}x \Bigg],
\end{aligned}
$$

where we used the identity:

$$\det(\mathrm{Id} + th/h_0\Pi) = 1 + th/h_0 \text{trace } \Pi + t^2h^2/h_0^2 \det \Pi.$$

The first term in (2.10), in view of Lemma 2.2 (ii), Lemma 2.3 and (2.8), converges to:

$$
\begin{aligned}
\int_S \fint_{-h_0/2}^{h_0/2} tE_{TS} : \nabla_{tan}\eta(x)\ \mathrm{d}t\mathrm{d}x &= \int_S \hat{E}_{tan} : (\nabla\eta(x))_{tan}\ \mathrm{d}t\mathrm{d}x \\
&= \frac{h_0}{12}\int_S \mathcal{L}_2\left(x, (\nabla(A\vec{n}) - A\Pi)_{tan}\right) : (\nabla\eta(x))_{tan}\ \mathrm{d}x.
\end{aligned}
$$

In turn, the third term in (2.10) converges to 0. This is because $(Q^h)^T R^h E^h \vec{n}$ converge weakly in $L^2(S^{h_0},\mathbb{R}^3)$ to $E\vec{n} = 0$, by Lemma 2.2 (ii) and Lemma 2.3 (i). Summarizing, (2.10) yields:

(2.11)

$$\lim_{h\to 0}\frac{1}{h}\int_S \fint_{-h_0/2}^{h_0/2} ((Q^h)^T R^h E^h\vec{n})\ \tilde{A}\vec{n}\ \mathrm{d}t\mathrm{d}x = -\frac{1}{12}\int_S \mathcal{L}_2\left(x, (\nabla(A\vec{n}) - A\Pi)_{tan}\right) : (\nabla(\tilde{A}\vec{n}))_{tan}\ \mathrm{d}x.$$

2. Now, apply (2.4) to the variation $\psi(x+t\vec{n}) = \tilde{V}(x)$, and pass to the limit after dividing both sides of (2.4) by h^2:

$$
\begin{aligned}
\int_S f(x)\cdot\bar{Q}\tilde{V}(x)\ \mathrm{d}x &= \lim_{h\to 0}\int_S f(x)\cdot Q^h\tilde{V}(x)\ \mathrm{d}x \\
&= \lim_{h\to 0}\Bigg[\int_S \fint_{-h_0/2}^{h_0/2}\left[\frac{1}{h}((Q^h)^T R^h - \mathrm{Id})E^h\right]_{TS} : \left[\tilde{A}(x)_{TS}(\mathrm{Id} + th/h_0 \text{adj } \Pi)\right]\ \mathrm{d}t\mathrm{d}x \\
&\qquad + \int_S \fint_{-h_0/2}^{h_0/2} \frac{1}{h}E^h_{TS} : \left[\tilde{A}(x)_{TS}(\mathrm{Id} + th/h_0\text{adj } \Pi)\right]\ \mathrm{d}t\mathrm{d}x\Bigg] \\
&:= \lim_{h\to 0}[I_h + II_h],
\end{aligned}
\tag{2.12}
$$

where we used the definition of the adjoint matrix:

$$\det(\mathrm{Id} + th/h_0\Pi)\ (\mathrm{Id} + th/h_0\Pi)^{-1} = \text{adj } (\mathrm{Id} + th/h_0\Pi) = \mathrm{Id} + th/h_0\text{adj } \Pi.$$

Notice that, by Lemma 2.1 (ii) and (1.4), the matrix field:

$$1/h((Q^h)^T R^h - \mathrm{Id}) = (\sqrt{e^h}/h^2)h/\sqrt{e^h}((Q^h)^T R^h - \mathrm{Id})$$

converges to $\sqrt{\kappa}A$, weakly in $W^{1,2}(S)$ and hence strongly in $L^2(S)$. Hence, by the weak convergence of E^h to E and the uniform convergence of $(\mathrm{Id} + th/h_0\text{adj } \Pi)$ to Id, the first term of (2.12) converges to:

$$\begin{aligned}\lim_{h\to 0} I_h &= \sqrt{\kappa}\int_S \fint_{-h_0/2}^{h_0/2} (AE)_{TS} : \tilde A_{TS} = \sqrt{\kappa}\int_S (A\bar E)_{TS} : \tilde A_{TS} = \sqrt{\kappa}\int_S (A\bar E) : \tilde A\\ &= -\sqrt{\kappa}\int_S \bar E : (A\tilde A) = -\sqrt{\kappa}\int_S \bar E_{tan} : (A\tilde A)_{tan}\\ &= -\kappa\int_S \mathcal{L}_2\left(x, B_{tan} - \frac{1}{2}(A^2)_{tan}\right) : (A\tilde A)_{tan}\ \mathrm{d}x,\end{aligned} \tag{2.13}$$

where we also have used Lemma 2.3 and (2.7).

3. Towards finding the limit of II_h in (2.12), consider first the contribution of the tangential minors. By Lemma 2.4 (ii) and since $\tilde A \in L^p(S^{h_0})$ for all $p \geq 1$, one observes that:

$$\lim_{h\to 0}\frac{1}{h}\int_S \fint_{-h_0/2}^{h_0/2} \text{ skew } E^h_{tan} : \tilde A_{tan} = 0. \tag{2.14}$$

Hence:

$$\begin{aligned}&\lim_{h\to 0}\int_S \fint_{-h_0/2}^{h_0/2} \frac{1}{h} E^h_{tan} : \Big[\tilde A(x)_{tan}(\mathrm{Id} + th/h_0 \text{ adj } \Pi)\Big]\ \mathrm{d}t\mathrm{d}x\\ &\qquad = \frac{1}{h_0}\lim_{h\to 0}\int_S \fint_{-h_0/2}^{h_0/2} tE^h_{tan} : \Big[\tilde A_{tan} \text{ adj } \Pi\Big] = \frac{1}{h_0}\lim_{h\to 0}\int_S \hat E_{tan} : \Big[\tilde A_{tan} \text{ adj } \Pi\Big]\\ &\qquad = -\frac{1}{h_0}\lim_{h\to 0}\int_S \hat E_{tan} : (\tilde A_{tan}\Pi)^T = -\frac{1}{12}\int_S \mathcal{L}_2\left(x, (\nabla(A\vec n) - A\Pi)_{tan}\right) : (\tilde A\Pi)_{tan}\ \mathrm{d}x,\end{aligned} \tag{2.15}$$

where we have used (2.8) and Lemma 2.3 (ii), combined with the following formula, which can be easily checked for $\tilde A_{tan} \in so(2)$:

$$\tilde A_{tan} \text{ adj } \Pi = -(\tilde A_{tan}\Pi)^T.$$

Further, by (2.11):

$$\begin{aligned}&\lim_{h\to 0}\int_S \fint_{-h_0/2}^{h_0/2} \frac{1}{h}\Big((E^h)^T\vec n\Big)\Big((\tilde A)^T\vec n\Big)\ \mathrm{d}t\mathrm{d}x\\ &\qquad = -\lim_{h\to 0}\frac{1}{h}\int_S \fint_{-h_0/2}^{h_0/2} ((Q^h)^T R^h E^h\vec n)(\tilde A\vec n)\\ &\qquad\quad + \lim_{h\to 0}\int_S \fint_{-h_0/2}^{h_0/2} \left[\frac{1}{h}((Q^h)^T R^h - \mathrm{Id})(E^h\vec n)\right](\tilde A\vec n)\\ &\qquad\quad + 2\lim_{h\to 0}\int_S \fint_{-h_0/2}^{h_0/2} \left[\frac{1}{h}(\text{ skew } E^h)\vec n\right](\tilde A\vec n)\\ &\qquad = \frac{1}{12}\int_S \mathcal{L}_2\left(x, (\nabla(A\vec n) - A\Pi)_{tan}\right) : (\nabla(\tilde A\vec n))_{tan}\ \mathrm{d}x.\end{aligned} \tag{2.16}$$

Indeed, $1/h((Q^h)^T R^h - \mathrm{Id})$ converges to κA weakly in $L^4(S)$ while $\tilde A\vec n \in L^4(S)$ and $\bar E^h\vec n$ converges to 0 weakly in $L^2(S)$. Therefore the second term in (2.16) converges to 0. The last limiting term there vanishes as well, by Lemma 2.4 (ii) as in (2.14).

Finally, we have:

$$\lim_{h\to 0}\frac{1}{h_0}\int_S \fint_{-h_0/2}^{h_0/2} \Big(\vec n^T tE^h\Big)_{tan}(\text{ adj } \Pi)\Big((\tilde A)^T\vec n\Big)_{tan}\ \mathrm{d}t\mathrm{d}x = 0, \tag{2.17}$$

because $(\hat{E}^h)^T\vec{n}$ converges to 0 weakly in $L^2(S)$ by Lemma 2.2 (i) and Lemma 2.3.

Adding now (2.15), (2.16) and (2.17) we obtain:

$$\lim_{h\to 0} II_h = \frac{1}{12}\int_S \mathcal{L}_2\left(x, (\nabla(A\vec{n}) - A\Pi)_{tan}\right) : (\nabla(\tilde{A}\vec{n}) - \tilde{A}\Pi)_{tan}\ \mathrm{d}x. \tag{2.18}$$

Together with (2.12) and (2.13), the formula (2.18) implies (1.12). ■

3. The limiting rotations $\bar{Q}$

In this section we will derive the third Euler-Lagrange equation (after the first two (1.11) and (1.12)), corresponding to variation in $\bar{Q} \in SO(3)$, and under certain nondegeneracy condition. We first notice that the limiting $\bar{Q}$ necessarily satisfies the constraint of the average torque:

$$\tau(\bar{Q}) = \int_S f \times \bar{Q}x\ \mathrm{d}x = 0. \tag{3.1}$$

The main difficulty arises now from the fact that the variations must be taken inside $SO(3)$ in a way that this constraint remains satisfied. Assuming that such variations exist, we establish the limit equation under the additional condition that Q^h approach $\bar{Q}$ along a direction $U \in T_{\bar{Q}}SO(3)$ for which $\partial_U\tau(\bar{Q}) \neq 0$.

In what follows, the crucial role is played by the function $g(Q) = \int_S f\cdot Qx\ \mathrm{d}x$ defined on $SO(3)$. Let $K \in \mathbb{R}^{3\times 3}$ be such that: $g(Q) = K : Q$, for all $Q \in SO(3)$.

Lemma 3.1. *Assume the hypothesis of Theorem 1.1. Then the limit $\bar{Q} \in SO(3)$ of Q^h must satisfy:*

$$\int_S f\cdot \bar{Q}Fx\ \mathrm{d}x = 0 \qquad \forall F \in so(3), \tag{3.2}$$

or equivalently (3.1). Another equivalent formulation of (3.2) is: $\mathrm{skew}(\bar{Q}^TK) = 0$.

Proof. First, for any given $H \in so(3)$, consider the variation $\phi^h = Hu^h$ in the equilibrium equation (1.9). Recalling that $DW(\nabla u^h)(\nabla u^h)^T$ is symmetric (see the proof of Lemma 2.4) we obtain:

$$\int_{S^h} f^h\cdot Hu^h = \int_{S^h} DW(\nabla u^h) : H\nabla u^h = \int_{S^h}\Big(DW(\nabla u^h)(\nabla u^h)^T\Big) : H = 0. \tag{3.3}$$

Similarly, taking $\phi^h = \frac{1}{\epsilon}(\exp(\epsilon H)u^h - u^h)$ in (1.14), by frame indifference of W we get:

$$\int_{S^h} f^h\cdot Hu^h = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{S^h} f^h\cdot(\exp(\epsilon H)u^h - u^h) = h\lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(J^h(\exp(\epsilon H)u^h) - J^h(u^h)\Big) = 0.$$

Now, for any sequence of skew-symmetric matrices F^h we have:

$$\begin{aligned}\int_S f\cdot Q^hF^hV^h &= \frac{1}{he^h}\int_{S^h} f^h\cdot Q^hF^h\Big((Q^h)^Tu^h - c^h - \mathrm{id}\Big)\\ &= \frac{1}{he^h}\int_{S^h} f^h\cdot\big(Q^hF^h(Q^h)^T\big)u^h - \frac{1}{he^h}\int_{S^h} f^h\ \mathrm{d}z\cdot Q^hF^hc^h - \frac{1}{he^h}\int_{S^h} f^h\cdot Q^hF^hz\ \mathrm{d}z\\ &= -\frac{h}{\sqrt{e^h}}\int_S f\cdot Q^hF^hx\ \mathrm{d}x,\end{aligned} \tag{3.4}$$

where the first two terms in the second line above vanish by taking $H = Q^hF^h(Q^h)^T \in so(3)$ in (3.3), and by the normalization of f^h. Passing to the limit with $h \to 0$ in (3.4), where $F^h = F$, we see that: $-\int_S f\cdot\bar{Q}FV = \lim_{h\to 0} h/\sqrt{e^h}\int_S f\cdot Q^hFx\ \mathrm{d}x$. This implies (3.2).

Clearly, (3.2) is also equivalent to $0 = K : \bar{Q}F = \bar{Q}^T K : F$ for all $F \in so(3)$, which means exactly that $\bar{Q}^T K$ is a symmetric matrix.

To prove the other equivalent formulation of (3.2), notice that:

$$\int_S f\cdot \bar{Q}Fx = \int_S \bar{Q}^T f\cdot Fx = -c_F\cdot \int_S \bar{Q}^T f\times x = -c_F \int_S f\times \bar{Q}x,$$

where $c_F\in\mathbb{R}^3$ is such that $Fx = c_F\times x$ for all $x\in\mathbb{R}^3$. Since there is a one to one correspondence between vectors c_F and skew matrices F, the proof is achieved. ■

Define now the set of the rotation equilibria:

$$\mathcal{M} = \{\bar{Q}\in SO(3);\ \mathrm{skew}\ (\bar{Q}^T K) = 0\}.$$

Our goal is to derive the third Euler-Lagrange equation, with respect to the variations of $\bar{Q}$ in $\mathcal{M}$. For $\bar{Q}\in\mathcal{M}$, let $F\in so(3)$ be such that:

$$\bar{Q}F = \lim_{n\to\infty}\frac{\bar{Q}_n - \bar{Q}}{\|\bar{Q}_n - \bar{Q}\|},$$

for some $\bar{Q}_n\in\mathcal{M}$ converging to $\bar{Q}$. Clearly, the above implies that:

$$\mathrm{skew}\ (F\bar{Q}^T K) = 0. \tag{3.5}$$

Lemma 3.2. *Under the hypothesis of Theorem 1.1, assume moreover that:*

$$\lim_{h\to 0}\frac{Q^h - \bar{Q}}{\|Q^h - \bar{Q}\|} = \bar{Q}H, \qquad \textit{with} \qquad \mathrm{skew}(H\bar{Q}^T K)\neq 0.$$

Then for every $F\in so(3)$ satisfying (3.5) there holds:

$$\int_S f\cdot \bar{Q}FV\ \mathrm{d}x = 0.$$

Proof. We will find a sequence $F^h\in so(3)$, converging to F and such that, for all h:

$$\int_S f\cdot Q^h F^h x\ \mathrm{d}x = 0. \tag{3.6}$$

In view of (3.4) this will prove the lemma. Existence of such approximating sequence F^h is guaranteed by the assumed nondegeneracy condition: skew $(H\bar{Q}^T K)\neq 0$.

Firstly, notice that for $Q^h\in\mathcal{M}$ one can take $F^h = F$. Otherwise, define:

$$F^h = F - \frac{(Q^h)^T K : F}{|\mathrm{skew}\ ((Q^h)^T K)|^2}\ \mathrm{skew}\ ((Q^h)^T K).$$

Then:

$$\begin{aligned}\int_S f\cdot Q^h F^h x\ \mathrm{d}x &= K : Q^h F^h = (Q^h)^T K : F^h\\ &= (Q^h)^T K : F - \frac{(Q^h)^T K : F}{|\mathrm{skew}\ ((Q^h)^T K)|^2}\ (Q^h)^T K : \mathrm{skew}\ \big((Q^h)^T K\big) = 0,\end{aligned}$$

and moreover:

$$\begin{aligned}\lim_{h\to 0}|F^h - F| &= \lim_{h\to 0}\frac{|(Q^h)^T K : F|}{|\mathrm{skew}\ ((Q^h)^T K)|} = \lim_{h\to 0}\frac{|(Q^h)^T K : F - \bar{Q}^T K : F|}{|\mathrm{skew}\ ((Q^h)^T K - \bar{Q}^T K)|}\\ &= \lim_{h\to 0}\left|\left(\frac{Q^h - \bar{Q}}{\|Q^h - \bar{Q}\|}\right)^T K : F\right| \Big/ \left|\mathrm{skew}\ \left(\frac{Q^h - \bar{Q}}{\|Q^h - \bar{Q}\|}\right)^T K\right| = \frac{|H^T\bar{Q}^T K : F|}{|\mathrm{skew}\ (H^T\bar{Q}^T K)|} = 0.\end{aligned}$$

The last expression above equals to 0 because of the nullity of its numerator:

$$H^T\bar{Q}^TK : F = \bar{Q}^TK : HF = \bar{Q}^TK : (HF)^T = \bar{Q}^TK : FH = -F\bar{Q}^TK : H = 0,$$

where we have used that $\bar{Q}^TK$ is symmetric and (3.5). ■

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