THE UNIFORM KORN - POINCARÉ INEQUALITY IN THIN DOMAINS

L'INÉGALITÉ DE KORN - POINCARÉ DANS LES DOMAINES MINCES

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Contents

Abstract. We study the Korn-Poincaré inequality:

$$\|u\|_{W^{1,2}(S^h)} \leq C_h \|D(u)\|_{L^2(S^h)},$$

in domains S^h that are shells of small thickness of order h, around an arbitrary compact, boundaryless and smooth hypersurface S in $\mathbf{R}^n$. By $D(u)$ we denote the symmetric part of the gradient ∇u, and we assume the tangential boundary conditions:

$$u \cdot \vec{n}^h = 0 \quad \text{on } \partial S^h.$$

We prove that C_h remains uniformly bounded as $h \to 0$, for vector fields u in any family of cones (with angle $< \pi/2$, uniform in h) around the orthogonal complement of extensions of Killing vector fields on S.

We show that this condition is optimal, as in turn every Killing field admits a family of extensions u^h, for which the ratio $\|u^h\|_{W^{1,2}(S^h)}/\|D(u^h)\|_{L^2(S^h)}$ blows up as $h \to 0$, even if the domains S^h are not rotationally symmetric.

1991 *Mathematics Subject Classification.* 74B05.

Key words and phrases. Korn inequality, Killing vector fields, thin domains, Poincaré inequality.

Abstract. On étudie l'inégalité de Korn-Poincaré:

$$\|u\|_{W^{1,2}(S^h)} \leq C_h \|D(u)\|_{L^2(S^h)},$$

dans les domaines S^h de type des coques d'épaisseurs d'ordre h autour d'une hypersurface compacte sans bord et regulière S de $\mathbf{R}^n$. Par $D(u)$, on réfère à la partie symétrique du gradient ∇u et on suppose la condition au bord:

$$u \cdot \vec{n}^h = 0 \quad \text{on } \partial S^h.$$

On démontre que C_h reste uniformément borné car $h \to 0$, pour tout champ de vecteurs dans une famille de cônes donnée (faisant un angle $< \pi/2$, uniforme en h) autour du complément orthogonal des extensions de champs de vecteurs de Killing sur S.

On montre que cette condition est optimale comme tout champ de Killling u sur S admet une famille d'extensions u^h sur S^h pour lesquelles le rapport $\|u^h\|_{W^{1,2}(S^h)}/\|D(u^h)\|_{L^2(S^h)}$ tend à l'infini comme $h \to 0$, même si les S^h ne possèdent pas de symmetrie axiale.

1. Introduction

The objective of this paper is to study the Korn-Poincaré inequality:

$$\|u\|_{W^{1,2}(S^h)} \leq C_h \|D(u)\|_{L^2(S^h)}, \tag{1.1}$$

under the tangential boundary conditions:

$$u \cdot \vec{n}^h = 0 \quad \text{on } \partial S^h, \tag{1.2}$$

in domains S^h that are shells of small thickness of order h, around an arbitrary compact, boundaryless and smooth hypersurface S in $\mathbf{R}^n$. By $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ we denote the symmetric part of the gradient ∇u.

Korn's inequality was discovered in the early XXth century, in the context of the boundary value problem of linear elastostatics [15, 16]. There is by now an extensive literature on the subject, relating to various contexts and various boundary conditions (see for example a review [11], and the references therein). If (1.2) is replaced by $u = 0$ on ∂S^h, one can easily prove that $\|\nabla u\|_{L^2} \leq \sqrt{2}\|D(u)\|_{L^2}$, and so (1.1) follows by the Poincaré inequality. In the absence of this boundary condition, or with its weaker versions, the bound (1.1) requires an extra assumption to eliminate pure rotations and translations, when $D(u) = 0$ but $\nabla u \neq 0$. In particular, (1.1) holds for all $W^{1,2}(S^h)$ vector fields u satisfying (1.2), which are L^2- orthogonal to the space of those linear fields on S^h with skew-symmetric gradient that are themselves tangent on the boundary.

We are interested in the behaviour of the constant C_h, as $h \to 0$. It turns out that in general, C_h may blow up, even if S^h are not rotationally symmetric (and so the aforementioned spaces are trivial). The correct way of looking at this problem is to consider the asymptotic inequality as $h \to 0$, i.e. the related Korn inequality on S (see also [2]):

$$\|v\|_{W^{1,2}(S)} \leq C\|D(v)\|_{L^2(S)}. \tag{1.3}$$

This inequality holds true for all tangent vector fields v on S, which are L^2-orthogonal to the space of Killing fields on S. A Killing field v is defined to be a smooth tangent vector field which generates a one-parameter family of isometries on S. It is well known that the space of Killing fields on a given surface is a finite dimensional Lie algebra. An equivalent characterisation is:

$$D(v) = 0, \qquad \text{i.e.: } \tau \nabla v(x) \tau = 0 \quad \forall x \in S \quad \forall \tau \in T_x S. \tag{1.4}$$

In this paper, we first notice that any v satisfying (1.4) admits a family of extensions $v^h : S^h \to \mathbf{R}^n$, such that the boundary conditions (1.2) hold and so that the ratio $\|v^h\|_{W^{1,2}(S^h)}/\|D(v^h)\|_{L^2(S^h)}$ goes to infinity as $h \to 0$. This construction turns out to be the worst case scenario for the possible blow-up of C_h. Our main results state that the constants C_h remain uniformly bounded for vector fields u inside any family of cones (with angle $< \pi/2$, uniform in h) around the orthogonal complement of the space of extensions of all Killing fields on S.

Our main motivation in this work has been its application to dynamics of Navier-Stokes equations in thin 3-dimensional domains. Thin domains are encountered in many problems in solid or fluid mechanics. For example, in ocean dynamics, one is dealing with the fluid regions which are thin compared to the horizontal length scales. Other examples include lubrication, meteorology, blood circulation etc.; they are a part of a broader study of the behaviour of various PDEs on thin n-dimensional domains, where $n \geq 2$ (for a review see [20]).

The study of the global existence and asymptotic properties of solutions to the Navier-Stokes equations in thin 3d domains began with Raugel and Sell in [21]. They proved global existence of strong solutions for large initial data and in presence of large forcing, for the sufficiently thin 3d product domain $\Omega = Q \times (0, \epsilon)$, with the boundary conditions either purely periodical or combined periodic-Dirichlet. Further generalisations to other boundary conditions followed (see the references in [12]). Towards analysing thin domains other than simple product domains, Iftimie, Raugel and Sell [12] treated domains of the type: $\Omega = \{x \in \mathbf{R}^3;\ (x_1, x_2) \in Q,\ 0 < x_3 < \epsilon g(x_1, x_2)\}$, with the mixed boundary conditions: periodic on the lateral boundary and the Navier boundary conditions:

$$D(u)\vec{n}^h || \vec{n}^h \text{ and } u \cdot \vec{n}^h = 0 \quad \text{ on } \partial S^h \tag{1.5}$$

on the top and on the bottom.

The Korn inequality arises naturally when one considers the incompressible flow subject to (1.5), for the following reason. In order to define the relevant Stokes operator one uses the symmetric bilinear form $B(u, v) = \int D(u) : D(v)$ rather than the usual $\int \nabla u : \nabla v$. Hence the energy methods give suitable bounds for $\|D(u^h)\|_{L^2(S^h)}$, for a solution flow u^h in S^h. On the other hand, in order to establish compactness in the limit problem as $h \to 0$, one needs bounds for the $W^{1,2}$ norm of u^h, with constants independent of h. The inequality (1.1) (with uniform constants C_h) provides thus a necessary uniform equivalence of the two norms $\|u^h\|_{W^{1,2}}$ and $\|D(u^h)\|_{L^2}$ on S^h.

It is therefore hoped that we can apply the result of this paper to study the dynamics of the Navier-Stokes equations, under the Navier boundary conditions, in thin shells with various geometries of the reference surface S and of the boundaries of S^h.

Starting with the original papers of Korn [15, 16], Korn's inequality has also been widely used as a basic tool for the existence of solutions of the linearised displacement-traction equations in elasticity [4, 3, 11]. In this context, for a given displacement vector field u, the matrix field $D(u)$ is the linearised strain, which measures the pointwise deviation of the deformation $Id + \epsilon u$ from a rigid motion, up to the first order terms in ϵ. Hence, Korn's inequality can be interpreted as a rigidity estimate for small displacement deformations: they are $W^{1,2}$ close to Id, by the error given in the right hand side of (1.1). A nonlinear version of this rigidity estimate, obtained recently in [5], has been extensively applied to problems in nonlinear elasticity and plate theories (see eg [5, 6]). Earlier, Korn's inequalities in thin neighbourhoods of flat surfaces have been discussed in series of papers by Kohn and Vogelius [14]. They derive an estimate which degenerates as $h \to 0$ for clamped boundary conditions at the side of the plate. An analogous result in our setting is given in Theorem 2.3.

Acknowledgments. M.L. was partially supported by the NSF grant DMS-0707275.

2. The main theorems

Let S be a smooth, closed hypersurface (a compact boundaryless manifold of co-dimension 1) in $\mathbf{R}^n$. Consider a family $\{S^h\}_{h>0}$ of thin shells around S:

$$S^h = \{z = x + t\vec{n}(x);\ x \in S,\ -g_1^h(x) < t < g_2^h(x)\},$$

given by a family of smooth positive functions $g_1^h, g_2^h : S \longrightarrow \mathbf{R}$. We will use the following notation: $\vec{n}^h$ for the outward unit normal to ∂S^h, $\vec{n}(x)$ for the outward unit normal to S (seen as the boundary of some bounded domain in $\mathbf{R}^n$), T_xS for the tangent space to S at a given $x \in S$. The projection onto S along $\vec{n}$ will be denoted by π, so that, for h sufficiently small:

$$\pi(z) = x \qquad \forall z = x + t\vec{n}(x) \in S^h.$$

The standard Korn inequality (see Theorem 9.1 in Appendix A) on bounded Lipschitz domains implies that for each $u \in W^{1,2}(S^h, \mathbf{R}^n)$ satisfying the orthogonality condition:

$$\int_{S^h} u(z) \cdot (Az + b)\ \mathrm{d}z = 0, \quad \forall A \in so(n), \quad \forall b \in \mathbf{R}^n \tag{2.1}$$

one has:

$$\|u\|_{W^{1,2}(S^h)} \leq C_h \|D(u)\|_{L^2(S^h)} \tag{2.2}$$

and the constant C_h depends only on the domain S^h, but not on u. Here, $so(n)$ stands for the linear space of all $n \times n$ skew-symmetric matrices:

$$so(n) = \{A \in M^{n\times n};\ A = -A^T\} = \{A \in M^{n\times n};\ \tau^T A \tau = 0 \quad \forall \tau \in \mathbf{R}^n\}$$

while by $D(u)$ we mean the symmetric part of ∇u:

$$D(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right).$$

The same result is true for u satisfying additionally:

$$u \cdot \vec{n}^h = 0 \ \text{ on } \partial S^h,$$

when in (2.1) we take only linear fields $Az + b \in \mathcal{R}_\partial(S^h)$; with skew-symmetric gradient, and satisfying the same boundary condition as u:

$$\mathcal{R}_\partial(S^h) = \{w = Az + b;\ A \in so(n), b \in \mathbf{R}^n, w \cdot \vec{n}^h = 0 \text{ on } \partial S^h\}.$$

The standard proof by contradiction (see Theorem 9.2 in Appendix A) shows that the constant C_h in (2.2) again does not depend on u but it may depend on the geometry of S^h. In particular, as follows from the example in section 4, C_h may converge to infinity as the thickness of S^h (that is $\|g_1^h + g_2^h\|_{L^\infty(S)}$) converges to 0. Our goal is to investigate the behaviour of C_h in two frameworks, relating to the following hypotheses:

(H1) For some positive constants C_1, C_2 and C_3, and all small $h > 0$ there holds:

$$C_1 h \leq g_i^h(x) \leq C_2 h, \qquad |\nabla g_i^h(x)| \leq C_3 h \qquad \forall x \in S, \quad i = 1, 2.$$

(H2) For some smooth positive functions $g_1, g_2 : S \longrightarrow \mathbf{R}$, there holds:

$$\frac{1}{h} g_i^h \to g_i \quad \text{ in } \mathcal{C}^1(S) \quad \text{ as } h \to 0, \qquad i = 1, 2.$$

Clearly **(H2)** implies **(H1)** with: $C_1 = 1/2 \min\{g_i(x);\ x \in S,\ i = 1, 2\}$, $C_2 = 2 \max_i \|g_i\|_{L^\infty}$, $C_3 = \max_i \|\nabla g_i\|_{L^\infty(S)} + 1$.

Before stating our main results, we need to recall the notion of a Killing vector field. The Lie algebra of smooth Killing fields on S will be denoted by $\mathcal{I}(S)$. That is, $v \in \mathcal{I}(S)$ if and only if:

(i) $v : S \longrightarrow \mathbf{R}^n$ is smooth and $v(x) \in T_x(S)$ for every $x \in S$,
(ii) $\dfrac{\partial v}{\partial \tau}(x) \cdot \tau = 0$ for every $x \in S$ and every $\tau \in T_x S$.

Here $\partial v/\partial \tau(x)$ denotes the derivative of v in the tangent direction τ, i.e. if $\gamma : (-\epsilon, \epsilon) \longrightarrow S$ is a $\mathcal{C}^1$ curve with $\gamma(0) = x$ and $\gamma'(0) = \tau$, then $\partial v/\partial \tau(x) = (v \circ \gamma)'(0)$.

Condition (ii) implies that

$$\frac{\partial v}{\partial \tau}(x) \cdot \eta + \frac{\partial v}{\partial \eta}(x) \cdot \tau = 0 \qquad \forall \tau, \eta \in T_x S \quad \forall x \in S. \tag{2.3}$$

Recall that Killing vector fields are infinitesimal generators of isometries on S, in the sense that if Φ is the flow generated by v:

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi(s,x) = v(\Phi(s,x)), \qquad \Phi(0,x) = x,$$

then for every fixed s the map $S \ni x \mapsto \Phi(s,x) \in S$ is an isometry. The linear space $\mathcal{I}(S)$ has finite dimension [13, 19]. Also, any Killing field of class $W^{1,2}$ is in fact smooth (see Lemma 11.1); we recall these facts in Appendix C.

For $g_1, g_2 : S \longrightarrow \mathbf{R}$, define the subspace of $\mathcal{I}(S)$:

$$\mathcal{I}_{g_1,g_2}(S) = \left\{ v \in \mathcal{I}(S);\ v(x) \cdot \nabla(g_1 + g_2)(x) = 0 \quad \text{for all } x \in S \right\},$$

formed of those Killing fields v which satisfy $\lim_{h\to 0} h^{-1} v \cdot (\vec{n}^h_+ + \vec{n}^h_-) = 0$, where $\vec{n}^h_+$ and $\vec{n}^h_-$ denote, respectively, the outward normals to S^h at its boundary points $x + g_2^h(x)$ and $x - g_1^h(x)$.

Our main results are the following:

Theorem 2.1. *Assume* **(H1)** *and let $\alpha \in [0,1)$. Then, for all $h > 0$ sufficiently small and all $u \in W^{1,2}(S^h, \mathbf{R}^n)$ satisfying one of the following tangency conditions:*

$$u \cdot \vec{n}^h = 0 \quad \text{on } \partial^+ S^h = \{x + g_2^h(x)\vec{n}(x);\ x \in S\},$$

or:

$$u \cdot \vec{n}^h = 0 \quad \text{on } \partial^- S^h = \{x - g_1^h(x)\vec{n}(x);\ x \in S\},$$

together with:

$$\left| \int_{S^h} u(z) v(\pi(z))\ \mathrm{d}z \right| \leq \alpha \|u\|_{L^2(S^h)} \cdot \|v\pi\|_{L^2(S^h)} \qquad \forall v \in \mathcal{I}(S), \tag{2.4}$$

there holds:

$$\|u\|_{W^{1,2}(S^h)} \leq C \|D(u)\|_{L^2(S^h)}, \tag{2.5}$$

where C is independent of u and of h.

Theorem 2.2. *Assume* **(H2)** *and let $\alpha \in [0,1)$. Then for all $h > 0$ sufficiently small and all $u \in W^{1,2}(S^h, \mathbf{R}^n)$ satisfying $u \cdot \vec{n}^h = 0$ on ∂S^h and:*

$$\left| \int_{S^h} u(z) v(\pi(z))\ \mathrm{d}z \right| \leq \alpha \|u\|_{L^2(S^h)} \cdot \|v\pi\|_{L^2(S^h)} \qquad \forall v \in \mathcal{I}_{g_1,g_2}(S), \tag{2.6}$$

there holds (2.5) with C independent of u and of h.

The example constructed in section 4 shows that conditions (2.4) (or (2.6)) are necessary for the bound (2.5). In particular, any Killing field v on S generates a family of vector fields v^h on S^h, satisfying the boundary condition and such that $\|\nabla v^h\|^2_{L^2(S^h)} \geq Ch$ but $\|D(v^h)\|^2_{L^2(S^h)} \leq Ch^3$. Hence, if one naively assumes that u satisfies the angle condition only with the space of generators of appropriate rotations on S, rather than the whole $\mathcal{I}(S)$, the constant C_h has a blow-up rate of at least h^{-1}, as $h \to 0$. The following theorem shows that this is the actual blow-up rate, under the above mentioned conditions.

More precisely, define:

$$\mathcal{R}(S) = \big\{v : S \longrightarrow \mathbf{R}^n;\ v(x) = Ax + b,\ A \in so(n),\ b \in \mathbf{R}^n,\ v \cdot \vec{n} = 0 \text{ on } S\big\} \subset \mathcal{I}(\mathcal{S}),$$

$$\mathcal{R}_{g_1,g_2}(S) = \big\{v \in \mathcal{R}(S);\ v(x) \cdot \nabla(g_1 + g_2)(x) = 0 \text{ for all } x \in S\big\} \subset \mathcal{I}_{g_1,g_2}(S).$$

Theorem 2.3. *Let $\alpha \in [0, 1)$. Then, for all h sufficiently small and all $u \in W^{1,2}(S^h, \mathbf{R}^n)$, there holds:*

$$\|u\|_{W^{1,2}(S^h)} \leq Ch^{-1}\|D(u)\|_{L^2(S^h)}, \tag{2.7}$$

in any of the following situations:

(i) **(H1)** *holds, $u \cdot \vec{n}^h = 0$ on $\partial^+ S^h$ or $u \cdot \vec{n}^h = 0$ on $\partial^- S^h$, and:*

$$\left|\int_{S^h} u(z)v(\pi(z))\ \mathrm{d}z\right| \leq \alpha\|u\|_{L^2(S^h)} \cdot \|v\pi\|_{L^2(S^h)} \qquad \forall v \in \mathcal{R}(S).$$

(ii) **(H2)** *holds, $u \cdot \vec{n}^h = 0$ on ∂S^h, and:*

$$\left|\int_{S^h} u(z)v(\pi(z))\ \mathrm{d}z\right| \leq \alpha\|u\|_{L^2(S^h)} \cdot \|v\pi\|_{L^2(S^h)} \qquad \forall v \in \mathcal{R}_{g_1,g_2}(S).$$

Notice that (i) is implied by the hypotheses of Theorem 2.1 and (ii) by the hypotheses of Theorem 2.2, as the spaces $\mathcal{R}(S)$ and $\mathcal{R}_{g_1,g_2}(S)$ are contained in $\mathcal{I}(S)$ or $\mathcal{I}_{g_1,g_2}(S)$, respectively. The bound (2.7) was obtained also in [14], but in a different context of thin plates with clamped boundary conditions and rapidly varying thickness.

3. Remarks and an outline of proofs

Remark 3.1. Conditions (2.4) and (2.6) may be understood in the following way: the cosine of the angle (in $L^2(S^h)$) between u and its projection onto the linear space $W^h \subset L^2(S^h)$ of 'trivial' extensions $v\pi$ of certain Killing fields $v \in \mathcal{I}(S)$ (or $v \in \mathcal{I}_{g_1,g_2}(S)$) should be smaller than α.

Equivalently, one considers vector fields $u \in W^{1,2}(S^h)$, which for a given constant $\beta \geq 1$ (related to α through: $\beta = (1 - \alpha^2)^{-1/2}$) satisfy:

$$\|u\|_{L^2(S^h)} \leq \beta\|u - v\pi\|_{L^2(S^h)} \qquad \forall v \in \mathcal{I}(S) \quad (\text{or } \forall v \in \mathcal{I}_{g_1,g_2}(S)). \tag{3.1}$$

That is, the distance of u from the space W^h controls (uniformly) the full norm $\|u\|_{L^2(S^h)}$.

By Theorems 2.1 and 2.2, inside each closed cone around $(W^h)^\perp$, of fixed angle $\theta < \pi/2$ in $L^2(S^h)$, the bound (2.5) holds, with a constant C, that is uniform in u and h. One could therefore think that W^h is the kernel for the uniform Korn-Poincaré inequality, in the same manner as the linear maps $Az + b$ with skew gradients $A \in so(n)$ constitute the kernel for the standard Korn inequality (2.1), (2.2). This is not exactly the case, as the uniform Korn inequality is true for the extensions $v\pi$ (see Remark 4.1). The role of the kernel is played by the space $\widetilde{W}^h$ of 'smart' extensions v^h of the Killing fields v (see the formula (4.3)).

Still, with $v\pi$ replaced by v^h in (2.4) or (2.6), both Theorems 2.1 and 2.2 remain true. This is because the spaces W^h and $\widetilde{W}^h$ are asymptotically tangent at $h = 0$:

$$\|v\pi - v^h\|_{L^2(S^h)} \leq Ch\|v\pi\|_{L^2(S^h)} \qquad \forall v \in \mathcal{I}(S).$$

Hence, if $|\langle u, v^h\rangle_{L^2}| \leq \alpha\|u\|_{L^2} \cdot \|v^h\|_{L^2}$ for some $\alpha < 1$, then $|\langle u, v\pi\rangle_{L^2}| \leq (\alpha + Ch)\|u\|_{L^2} \cdot \|v\pi\|_{L^2}$, and the angle conditions in main theorems hold, for h sufficiently small. The fact that we chose to work with 'trivial' extensions, in W^h (giving a simpler condition), instead of the real kernel $\widetilde{W}^h$, is thus not restrictive.

In the particular case when ∂S^h is parallel to S, say $g_i^h = h$, we have

$$\vec{n}^h(x + g_2(x)\vec{n}(x)) = \vec{n}(x), \qquad \vec{n}^h(x - g_1(x)\vec{n}(x)) = -\vec{n}(x),$$
$$\mathcal{I}(S) = \mathcal{I}_{g_1,g_2}(S).$$

If $w \in \mathcal{R}_\partial(S^h)$ then $w_{|S}$ is tangent to S and, as shown in Appendix A (Theorem 9.4) it generates a rotation on S. Actually: $w = (w_{|S})^h \in \widetilde{W}^h$ and so by the preceding comment we see that the condition (2.1) is asymptotically contained in (2.4) (or (2.6)).

Remark 3.2. A natural question is whether $\mathcal{I}(S)$ may contain other vector fields than the restrictions of generators of rigid motions on the whole $\mathbf{R}^n$. This is clearly the case when $n = 2$: any tangent vector field of constant length is a Killing field. The same question for higher dimensions and even for $n = 3$ and general (nonconvex) hypersurfaces is open, to our knowledge. It is closely related to other open problems: whether the class of rotationally symmetric surfaces is closed under intrinsic isometries; or whether every intrinsic isometry on S is actually a restriction of some isometry of $\mathbf{R}^3$. When S is convex, it is well known that the last property holds, while for non-convex surfaces it does not. The answer to the same question, formulated for 1-parameter families of isometries is not known (see [23] vol. 5).

An outline of proofs of Theorems 2.1 and 2.2.

The general strategy is as follows. Suppose that $\|D(u)\|_{L^2(S^h)}$ is small. It is natural to study the map $\bar{u} : S \longrightarrow \mathbf{R}^n$ which is obtained by averaging u in the normal direction:

$$\bar{u}(x) = \fint_{-g_1^h(x)}^{g_2^h(x)} u(x + t\vec{n}(x))\ \mathrm{d}t$$

(see e.g. [20, 21, 12, 8, 9, 10]). By the boundary condition, one has $\bar{u} \cdot \vec{n} \approx 0$, i.e. $\bar{u}$ is almost tangential to S. Moreover, $D(\bar{u})$ is essentially bounded by the average of $D(u)$. Hence if $D(u)$ is small, by Korn's inequality on surfaces, the field $\bar{u}$ must be close to a Killing field v. If v is not small, we will get a contradiction to the angle condition (2.4) or (2.6). If v is small then we get good estimates for $\bar{u}$ and ultimately for u.

More precisely, the proof proceeds as follows. First (see Theorem 5.1), an application of Korn's inequality to cylinders of size h and an interpolation argument yield a smooth field $R : S \longrightarrow so(n)$ such that:

$$\int_{S^h} |\nabla u - R\pi|^2 \leq C \int_{S^h} |D(u)|^2, \tag{3.2}$$

$$\int_S |\nabla R|^2 \leq Ch^{-3} \int_{S^h} |D(u)|^2. \tag{3.3}$$

From this we deduce (see Lemma 6.1):

$$\int_S |\nabla \bar{u} - R_{tan}|^2 \leq Ch^{-1} \int_{S^h} |D(u)|^2 + Ch \int_{S^h} |\nabla u|^2, \tag{3.4}$$

where $R_{tan}\tau = R\tau$ for all tangent fields τ and $R_{tan}\vec{n} = 0$.

Using the boundary conditions it is easy to show that (see Lemma 6.3):

$$\int_S |\bar{u} \cdot \vec{n}|^2 \leq Ch \int_{S^h} |\nabla u|^2. \tag{3.5}$$

It is thus natural to study the tangent field:

$$\bar{u}_{tan} = \bar{u} - (\bar{u}\cdot\vec{n})\vec{n}.$$

Now Korn's inequality on S implies that there exists a Killing field v such that:

$$\|\bar{u}_{tan} - v\|_{W^{1,2}(S)} \leq C\|D(\bar{u}_{tan})\|_{L^2(S)}.$$

By the angle condition, v must be small in $L^2(S)$, and hence in $W^{1,2}(S)$ since the Killing fields form a finite dimensional space. Thus, $\|\bar{u}_{tan}\|_{W^{1,2}(S)}$ is controlled, and by (3.5) $\|\bar{u}\|_{L^2(S)}$ is also controlled. Now the crucial step is to combine (3.3) and (3.4) to deduce that:

$$\int_S |\nabla(\bar{u}\cdot\vec{n})|^2 + |R\vec{n}|^2 \leq Ch^{-3/2}\|D(u)\|_{L^2(S^h)}\cdot\|\bar{u}\cdot\vec{n}\|_{L^2(S)} + \text{ harmless terms} \tag{3.6}$$

(see Lemma 6.4). From (3.6) and (3.5) we obtain control on $\nabla\bar{u}$. By (3.4) this controls R_{tan}, hence R, and finally (3.2) gives the estimate for ∇u.

The actual argument is by contradiction, assuming that $h^{-1/2}\|u^h\|_{W^{1,2}(S^h)} = 1$ and $h^{-1/2}\|D(u^h)\|_{L^h(S^h)} \longrightarrow 0$ (see section 7).

Above and in all subsequent proofs, C denotes an arbitrary positive constant, depending on the geometry of S and constants C_1, C_2, C_3 in **(H1)** or the functions g_1, g_2 in **(H2)**. The constant C may also depend on the choice of α, but it is always independent of u and h.

4. An example where the constant C_h blows up

Let $g_1, g_2 : S \longrightarrow \mathbf{R}$ be some positive and smooth functions, and let $g_i^h = hg_i$, $i = 1, 2$. Assume that on S there exists a nonzero Killing vector field v such that:

$$v \in \mathcal{I}_{g_1,g_2}(S). \tag{4.1}$$

We are going to construct a family $v^h \in W^{1,2}(S^h, \mathbf{R}^n)$ satisfying the boundary condition

$$v^h \cdot \vec{n}^h = 0 \qquad \text{on } \partial S^h, \tag{4.2}$$

for which the uniform bound (2.5) is not valid (after we take $u^h = v^h$).

By $\Pi(x) = \nabla\vec{n}(x)$ we denote the shape operator on S, that is, the (tangential) gradient of $\vec{n}$. For all $x \in S$ and all $t \in (-hg_1(x), hg_2(x))$ define:

$$v^h(x + t\vec{n}(x)) = \Big(\mathrm{Id} + t\Pi(x) + h\vec{n}(x)\otimes\nabla g_2(x)\Big)v(x). \tag{4.3}$$

By (4.1) we obtain:

$$\begin{aligned} v^h(x + t\vec{n}(x)) =& \frac{hg_1(x) + t}{h(g_1(x) + g_2(x))}\cdot\Big(\mathrm{Id} + hg_2(x)\Pi(x) + h\vec{n}(x)\otimes\nabla g_2(x)\Big)v(x) \\ &+ \frac{hg_2(x) - t}{h(g_1(x) + g_2(x))}\cdot\Big(\mathrm{Id} - hg_1(x)\Pi(x) - h\vec{n}(x)\otimes\nabla g_1(x)\Big)v(x), \end{aligned}$$

which means that each v^h is a linear interpolation between the push-forward of the vector field v from S onto the external part ∂^+S^h of the boundary of S^h and the other push-forward onto the internal part ∂^-S^h of ∂S^h (see figure 4.1). Indeed, the derivative of the map:

$$S \ni x \mapsto x \pm hg_i(x)\vec{n}(x)$$

is given through:

$$\mathrm{Id} \pm hg_i(x)\Pi(x) \pm h\vec{n}(x)\otimes\nabla g_i(x).$$

In particular, we see that (4.2) holds.

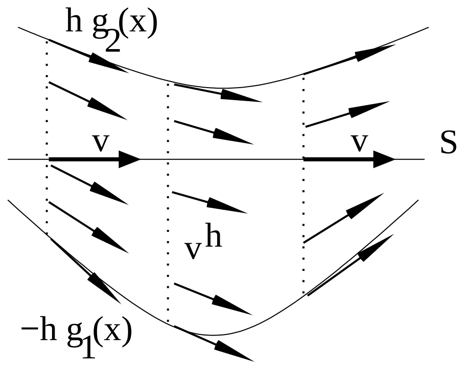

FIGURE 4.1. The vector fields v^h and v.

Write now $v^h = w + (v^h - w)$, with:

$$w(z) = \Big(\mathrm{Id} + t\Pi(x)\Big)v(x), \qquad z = x + t\vec{n}(x).$$

We wish to estimate the order of different coefficients in ∇w and $D(w)$. For every $\tau \in T_xS$, $x \in S$, there holds:

$$\begin{aligned} &\frac{\partial w}{\partial \vec{n}}(z) = \Pi(x)v(x), \\ &\frac{\partial w}{\partial \tau}(z) = t\frac{\partial \Pi}{\partial((\mathrm{Id}+t\Pi(x))^{-1}\tau)}(x)v(x) + (\mathrm{Id}+t\Pi(x))\nabla v(x)(\mathrm{Id}+t\Pi(x))^{-1}\tau. \end{aligned} \tag{4.4}$$

Observe that:

$$\begin{aligned} \left(\frac{\partial w}{\partial \tau}\cdot\vec{n} + \frac{\partial w}{\partial \vec{n}}\cdot\tau\right)(z) &= \left(-\frac{\partial \vec{n}}{\partial \tau}\cdot w + \frac{\partial w}{\partial \vec{n}}\cdot\tau\right)(z) \\ &= -\big(\Pi(x)(\mathrm{Id}+t\Pi(x))^{-1}\tau\big)\cdot(\mathrm{Id}+t\Pi(x))v(x) + \Pi(x)v(x)\cdot\tau \\ &= 0, \end{aligned} \tag{4.5}$$

because $\vec{n}\cdot w = 0$ and the symmetric form $\Pi(x)$ commutes with $(\mathrm{Id}+t\Pi(x))^{-1}$. Likewise:

$$\left(\frac{\partial w}{\partial \vec{n}}\cdot\vec{n}\right)(z) = 0. \tag{4.6}$$

To estimate $\eta^T D(w)(z)\tau$, for $\tau, \eta \in T_xS$, notice that:

$$\begin{aligned} \big|\eta^T&(\mathrm{Id}+t\Pi(x))\nabla v(x)(\mathrm{Id}+t\Pi(x))^{-1}\tau \\ &- \eta^T(\mathrm{Id}+t\Pi(x))^{-1}\nabla v(x)(\mathrm{Id}+t\Pi(x))^{-1}\tau\big| \le Ct|\nabla v(x)|, \end{aligned}$$

because $|(\mathrm{Id}+t\Pi(x)) - (\mathrm{Id}+t\Pi(x))^{-1}| \le Ct$. Above and in the sequel, C denotes any positive constant independent of h. Since $\tau(\mathrm{Id}+t\Pi(x))^{-1} \in T_xS$, by (2.3) and (4.4) we obtain:

$$|\eta^T D(w)(z)\tau| \le Ct(|v(x)| + |\nabla v(x)|). \tag{4.7}$$

We also have: $|\nabla(v^h - w)(z)| \le Ch$ and by (4.5), (4.6) and (4.7): $|D(w)(z)| \le Ch$ for every $z \in S^h$. Hence:

$$\|D(v^h)\|^2_{L^2(S^h)} \le Ch^3.$$

On the other hand, inspecting the terms in (4.4) and recalling that $v \neq 0$ (and therefore $\nabla v \neq 0$ as well) we see that:

$$\|\nabla v^h\|^2_{L^2(S^h)} \ge \frac{1}{2}\|\nabla v\|^2_{L^2(S^h)} - Ch^3 \ge Ch.$$

The two last inequalities imply that the uniform bound (2.5) is not valid, without the restriction (2.6). Even if S has no rotational symmetry, the constants C_h in (2.2) become unbounded as $h \to 0$.

Remark 4.1. The construction (4.3) is crucial for the counterexample to work. Indeed, one cannot simply take 'trivial' extensions $v\pi \in W^{1,2}(S^h)$ for the blow-up of C_h. The reason is that, for any $\tau \in T_xS$, one has:

$$\frac{\partial(v\pi)}{\partial\tau}(z)\cdot\vec{n} = -\frac{\partial(\vec{n}\pi)}{\partial\tau}(z)\cdot(v\pi)(z) = -\Pi(x)(\mathrm{Id}+t\Pi(x))^{-1}\tau\cdot v(x) = \mathcal{O}(1),$$
$$\frac{\partial(v\pi)}{\partial\vec{n}}(z) = 0,$$

and thus both $\nabla(v\pi)(z)$ and $D(v\pi)(z)$ are of the order $\mathcal{O}(1)$. Hence, with a uniform constant C:

$$\|\nabla(v\pi)\|^2_{L^2(S^h)} \leq Ch\|v\|^2_{W^{1,2}(S)} \leq Ch \leq Ch\|v\|^2_{L^2(S)} \leq C\|D(v\pi)\|^2_{L^2(S^h)}.$$

5. An approximation of ∇u

In this section we construct a smooth function R with skew-symmetric matrix values, approximating ∇u on S^h with the error $\|D(u)\|_{L^2(S^h)}$. The construction relies on Appendix B, where for convenience of the reader we analyse the constant in Korn's inequality on a fixed, star-shaped with respect to a ball domain (Theorem 10.1). We apply this estimate locally and then use a mollification argument as in [5]. The same approximation result is independently obtained in [10] Theorem 4.3, in the context of the unfolding method in the linearized elasticity.

As always, C denotes any uniform constant, independent of u and h.

Theorem 5.1. *Assume* **(H1)**. *For every $u \in W^{1,2}(S^h, \mathbf{R}^n)$ there exists a smooth map $R : S \longrightarrow so(n)$ such that:*

(i) $\|\nabla u - R\pi\|_{L^2(S^h)} \leq C\|D(u)\|_{L^2(S^h)}$,

(ii) $\|\nabla R\|_{L^2(S)} \leq Ch^{-3/2}\|D(u)\|_{L^2(S^h)}$.

Proof. **1.** For $x \in S$ consider balls in S and 'cylinders' in S^h defined by:

$$D_{x,h} = B(x,h)\cap S, \qquad B_{x,h} = \pi^{-1}(D_{x,h})\cap S^h.$$

The main observation is that sets $B_{x,h}$ are contained in a ball of radius $(C_2+1)h$ and are star-shaped with respect to a ball of radius $r(C_1, C_2, C_3, S)h$, when h is sufficiently small. Hence, an application of Korn's inequality on $B_{x,h}$ (see Theorem 10.1) yields a skew-symmetric matrix $A_{x,h} \in so(n)$ such that:

$$\int_{B_{x,h}} |\nabla u(z) - A_{x,h}|^2 \,\mathrm{d}z \leq C\int_{B_{x,h}} |D(u)|^2. \tag{5.1}$$

Indeed, recalling the assumption **(H1)** we see that for h sufficiently small, $B_{x,h}$ are star-shaped with respect to x and that both the Lipschitz constants of their boundaries and the ratios of their diameters have common bounds.

Our goal is to replace $A_{x,h}$ by a matrix $R(x)$ which depends smoothly on x. This will allow us to replace $A_{x,h}$ by $R(\pi z)$ in (5.1). The desired estimate on S^h then follows by summing over a suitable family of cylinders. The smoothness of R will also play essential role in the key estimate in Lemma 6.4.

2. To define $R(x)$ consider a cut-off function $\vartheta \in \mathcal{C}_c^\infty([0,1))$, with $\vartheta \geq 0$, ϑ constant in a neighbourhood of 0, and $\int_0^1 \vartheta = 1$. For each $x \in S$ define:

$$\eta_x(z) = \frac{\vartheta(|\pi z - x|/h)}{\int_{S^h} \vartheta(|\pi z - x|/h)\ \mathrm{d}z}.$$

Then $\eta_x(z) = 0$ for $z \notin B_{x,h}$ and:

$$\int_{S^h} \eta_x(z)\ \mathrm{d}z = 1, \qquad |\eta_x| \leq \frac{C}{h^n}, \qquad |\nabla_x \eta_x| \leq \frac{C}{h^{n+1}}$$

Define $R(x)$ as the average:

$$R(x) = \int_{S^h} \eta_x(z)\ \mathrm{skew}(\nabla u(z))\ \mathrm{d}z,$$

where $\mathrm{skew}(F) = (F - F^T)/2$ denotes the skew-symmetric part of a given matrix F. Since $\int \eta_x = 1$, we have:

$$R(x) - A_{x,h} = \int_{S^h} \eta_x(z)\ \mathrm{skew}(\nabla u(z) - A_{x,h})\ \mathrm{d}z,$$

and by the Cauchy-Schwarz inequality, noting that $|\mathrm{skew}(F)| \leq C|F|$ we obtain:

$$|R(x) - A_{x,h}|^2 \leq C\left(\int_{S^h} \eta_x(z)|\nabla u(z) - A_{x,h}|\ \mathrm{d}z\right)^2 \leq \frac{C}{h^n}\int_{B_{x,h}} |D(u)|^2. \tag{5.2}$$

To estimate the derivative of R we use that:

$$\int_{S^h} \nabla_x \eta_x(z)\ \mathrm{d}z = \nabla_x\left(\int_{S^h} \eta_x(z)\ \mathrm{d}z\right) = 0.$$

Thus:

$$\nabla R(x) = \int_{S^h} (\nabla_x \eta_x)\ \ \mathrm{skew}(\nabla u) = \int_{S^h} (\nabla_x \eta_x)\ \ \mathrm{skew}(\nabla u - A_{x,h})$$

and by (5.1):

$$|\nabla R(x)|^2 \leq \int_{B_{x,h}} |\nabla_x \eta_x|^2 \cdot \int_{B_{x,h}} |\nabla u - A_{x,h}|^2 \leq \frac{C}{h^{n+2}}\int_{B_{x,h}} |D(u)|^2. \tag{5.3}$$

Similarly, we get for all $x' \in D_{x,h}$:

$$|\nabla R(x')|^2 \leq \frac{C}{h^{n+2}}\int_{B_{x',h}} |D(u)|^2 \leq \frac{C}{h^{n+2}}\int_{2B_{x,h}} |D(u)|^2, \tag{5.4}$$

where $2B_{x,h} = \pi^{-1}(D_{x,2h}) \cap S^h$. From this, by the fundamental theorem of calculus:

$$|R(x'') - R(x)|^2 \leq \frac{C}{h^n}\int_{2B_{x,h}} |D(u)|^2 \qquad \forall x'' \in D_{x,h}.$$

In combination with (5.1) and (5.2) this yields:

$$\int_{B_{x,h}} |\nabla u(z) - R(\pi z)|^2\ \mathrm{d}z \leq C\int_{2B_{x,h}} |D(u)|^2. \tag{5.5}$$

Now cover S^h with a family $\{B_{x_i,h}\}_{i=1}^{N(h)}$ so that the covering number of $\{2B_{x_i,h}\}_{i=1}^{N(h)}$ is independent of h. A possible argument for the existence of such a covering goes as follows. The surface S is contained in the finite union of balls $\cup_{i=1}^{N(h)} B(x_i, h/2)$ where $k_i \in (\frac{h}{2}\mathbb{Z})^n$. Fix a one-to-one map $k_i \mapsto x_i \in S \cap B(k_i, h/2)$, so that $S^h = \cup_i B_{x_i,h}$. Then, if $z \in 2B_{x_i,h}$ there must be $\pi(z) \in B(x_i, 2h)$, so that $|k_i - \pi(z)| \leq |k_i - x_i| + |\pi(z) - x_i| \leq 5h/2$. Therefore $k_i \in B(x, 5h/2) \cap (\frac{h}{2}\mathbb{Z})^n$. The cardinality of this last set is bounded by 10^n, which must as well be a covering number for the family $\{2B_{x_i,h}\}_{i=1}^{N(h)}$.

Summing (5.5) over $i = 1 \dots N$ proves (i). Finally, integrating (5.4) on $D_{x,h}$ we get;

$$\int_{D_{x,h}} |\nabla R(x')|^2 \ \mathrm{d}x' \leq \frac{C}{h^3} \int_{2B_{x,h}} |D(u)|^2,$$

and using the same covering as before we obtain (ii). ■

Following the same argument, we will prove a uniform Poincaré inequality in thin domains - see Theorem 12.1 in Appendix D.

6. Key estimates

Let $\bar{u} : S \longrightarrow \mathbf{R}^n$ be the average of u in the normal direction:

$$\bar{u}(x) = \fint_{-g_1^h(x)}^{g_2^h(x)} u(x + t\vec{n}(x)) \ \mathrm{d}t \qquad \forall x \in S. \tag{6.1}$$

In this section we will establish four useful estimates on various components of $\bar{u}$ and their derivatives.

The first estimate on $\nabla\bar{u}$, is an extension of the previous Theorem 5.1:

Lemma 6.1. *Assume* **(H1)**. *For every $u \in W^{1,2}(S^h, \mathbf{R}^n)$ there holds:*

$$\|\nabla\bar{u} - R_{tan}\|_{L^2(S)} \leq Ch^{1/2}\|u\|_{W^{1,2}(S^h)} + Ch^{-1/2}\|D(u)\|_{L^2(S^h)},$$

where the subscript 'tan' refers to the tangential components of the appropriate matrix valued function, that is: $R_{tan}(x)\vec{n}(x) = 0$ and $R_{tan}(x)\tau = R(x)\tau$ for all $x \in S$ and $\tau \in T_xS$.

Proof. Through a direct calculation one checks that for every $x \in S$ and $\tau \in T_xS$ there holds:

$$\begin{aligned}&\left|\partial_\tau\bar{u}(x) - \fint_{-g_1^h(x)}^{g_2^h(x)} \nabla u(x + t\vec{n}(x)) \cdot \{\tau + t\partial_\tau\vec{n}(x)\} \ \mathrm{d}t\right| \\ &\qquad \leq \frac{C}{h}\Big(|\partial_\tau g_1^h(x)| + |\partial_\tau g_2^h(x)|\Big) \cdot \int_{-g_1^h(x)}^{g_2^h(x)} |\partial_{\vec{n}} u(x + t\vec{n}(x))| \ \mathrm{d}t \leq C\int_{-g_1^h(x)}^{g_2^h(x)} |\nabla u| \ \mathrm{d}t\end{aligned}$$

and:

$$\begin{aligned}&\fint_{-g_1^h(x)}^{g_2^h(x)} |\nabla u(x + t\vec{n}(x)) \cdot (\tau + t\partial_\tau\vec{n}(x)) - R(x)\tau| \ \mathrm{d}t \\ &\qquad \leq C\int_{-g_1^h(x)}^{g_2^h(x)} |\nabla u| \ \mathrm{d}t + \fint_{-g_1^h(x)}^{g_2^h(x)} |\nabla u(x + t\vec{n}(x)) - R(x)| \ \mathrm{d}t.\end{aligned}$$

Hence, by Theorem 5.1 (i):

$$\begin{aligned}\|\nabla\bar{u} - R_{tan}\|^2_{L^2(S)} &\leq C\int_S \left\{h\int_{-g_1^h(x)}^{g_2^h(x)} |\nabla u|^2 \ \mathrm{d}t + h^{-1}\fint_{-g_1^h(x)}^{g_2^h(x)} |\nabla u - R\pi|^2 \ \mathrm{d}t\right\} \ \mathrm{d}x \\ &\leq Ch\|\nabla u\|^2_{L^2(S^h)} + Ch^{-1}\|D(u)\|^2_{L^2(S^h)}.\end{aligned}$$

■

In order to estimate the normal part $\bar{u}$, we will use the following bounds:

Lemma 6.2. *Recall that* $\partial S^h = \partial^- S^h \cup \partial^+ S^h$, *with:*

$$\begin{aligned}\partial^- S^h &= \{x - g_1^h(x)\vec{n}(x);\ x \in S\},\\ \partial^+ S^h &= \{x + g_2^h(x)\vec{n}(x);\ x \in S\}.\end{aligned} \tag{6.2}$$

(i) *If* **(H1)** *holds then* $|\vec{n}^h(z) - \vec{n}(\pi(z))| \leq Ch$ *for all* $z \in \partial^+ S^h$ *and* $|\vec{n}^h(z) + \vec{n}(\pi(z))| \leq Ch$ *for all* $z \in \partial^- S^h$.

(ii) *If* **(H2)** *holds then:*

$$\begin{aligned}|\vec{n}^h(z) + \vec{n}(\pi(z)) + \nabla g_1^h(\pi(z))| &\leq Ch^2 \qquad \forall z \in \partial^- S^h,\\ |\vec{n}^h(z) - \vec{n}(\pi(z)) + \nabla g_2^h(\pi(z))| &\leq Ch^2 \qquad \forall z \in \partial^+ S^h.\end{aligned}$$

Let now $u \in W^{1,2}(S^h, \mathbf{R}^n)$.

(iii) $|\partial_{\vec{n}}(u \cdot \vec{n})(z)| \leq |D(u)(z)|$ *for all* $z \in S^h$.

(iv) *If* **(H1)** *holds and* $u \cdot \vec{n}^h = 0$ *on* $\partial^+ S^h$, *then:*

$$\|u \cdot \vec{n}\|_{L^2(\partial^+ S^h)} \leq Ch^{1/2}\|u\|_{W^{1,2}(S^h)}.$$

(v) *If* **(H2)** *holds and* $u \cdot \vec{n}^h = 0$ *on* ∂S^h:

$$\begin{aligned}\int_S |u(x - g_1^h(x)\vec{n}(x)) \cdot \nabla g_1^h(x) + u(x + g_2^h(x)\vec{n}(x)) \cdot \nabla g_2^h(x)|^2 \ \mathrm{d}x\\ \leq Ch \int_{S^h} |D(u)|^2 + Ch^3 \|u\|^2_{W^{1,2}(S^h)}.\end{aligned}$$

Proof. (i) is obvious. To prove (ii) observe, for example, that on $\partial^+ S^h$ the normal $\vec{n}^h(z)$ is parallel to $\vec{n}(\pi(z)) - \nabla g_2^h(\pi(z)) + w$, where $|w| \leq C|g_2^h(\pi(x))\nabla g_2^h(\pi(z))| \leq Ch^2$. Normalising this vector we conclude the second inequality in (ii). The first one follows in the same manner.

(iii) follows from: $\partial_{\vec{n}}(u \cdot \vec{n}) = D(u)\vec{n} \cdot \vec{n}$.

To prove (iv), use (i) and the trace theorem in Appendix D:

$$\|u \cdot \vec{n}\|_{L^2(\partial^+ S^h)} = \|u \cdot (\vec{n} - \vec{n}^h)\|_{L^2(\partial^+ S^h)} \leq Ch^{1/2}\|u\|_{W^{1,2}(S^h)}.$$

For (v), use (ii), (iii) and Theorem 12.3:

$$\begin{aligned}&\int_S |u(x + g_2^h(x)\vec{n}(x)) \cdot \nabla g_2^h(x) + u(x - g_1^h(x)\vec{n}(x)) \cdot \nabla g_1^h(x)|^2 \ \mathrm{d}x\\ &\leq \int_S |u(x + g_2^h(x)\vec{n}(x)) \cdot \vec{n}(x) - u(x - g_1^h(x)\vec{n}(x)) \cdot \vec{n}(x)|^2 \ \mathrm{d}x + Ch^4 \int_{\partial S^h} |u|^2\\ &\qquad = \int_S \left| \int_{-g_1^h(x)}^{g_2^h(x)} \partial_{\vec{n}}(u \cdot \vec{n})(x + t\vec{n}(x)) \ \mathrm{d}t \right|^2 \ \mathrm{d}x + Ch^4 \int_{\partial S^h} |u|^2\\ &\qquad \leq Ch \int_{S^h} |D(u)|^2 + Ch^3 \|u\|^2_{W^{1,2}(S^h)}.\end{aligned}$$

■

Lemma 6.3. *Assume* **(H1)** *and let* $u \in W^{1,2}(S^h, \mathbf{R}^n)$ *satisfy* $u \cdot \vec{n}^h = 0$ *on* $\partial^+ S^h$. *Then:*

$$\|\bar{u} \cdot \vec{n}\|_{L^2(S)} \leq Ch^{1/2}\|u\|_{W^{1,2}(S^h)}.$$

Proof. By Lemma 6.2 (iv) and (i), for every $z = x + t\vec{n}(x) \in S^h$ we obtain:

$$\begin{aligned}
|u(x+t\vec{n}(x)) \cdot \vec{n}(x)|^2 &\leq \left(|u(x + g_2^h(x)\vec{n}(x)) \cdot \vec{n}(x)| + \int_{-g_1^h(x)}^{g_2^h(x)} |D(u)| \right)^2 \\
&\leq C \cdot \Big| u(x + g_2^h(x)\vec{n}(x)) \cdot \Big(\vec{n}(x) - \vec{n}^h(x + g_2^h(x)\vec{n}(x)) \Big) \Big|^2 + Ch \int_{-g_1^h(x)}^{g_2^h(x)} |D(u)|^2 \\
&\leq Ch^2 |u(x + g_2^h(x)\vec{n}(x))|^2 + Ch \int_{-g_1^h(x)}^{g_2^h(x)} |D(u)|^2.
\end{aligned}$$

Hence by Theorem 12.3:

$$\begin{aligned}
\|\bar{u} \cdot \vec{n}\|^2_{L^2(S)} &\leq \frac{C}{h} \int_S \int_{-g_1^h(x)}^{g_2^h(x)} |u(x + t\vec{n}(x)) \cdot \vec{n}(x)|^2 \,\mathrm{d}t \,\mathrm{d}x \\
&\leq \frac{C}{h} \left(h^3 \|u\|^2_{L^2(\partial S^h)} + h^2 \|D(u)\|^2_{L^2(S^h)} \right) \leq Ch \|\nabla u\|^2_{L^2(S^h)}.
\end{aligned} \tag{6.3}$$

■

The next, key estimate, is on the gradient of $\bar{u} \cdot \vec{n}$. It is obtained using the divergence theorem on the surface S:

Lemma 6.4. *Assume* **(H1)** *and let* $u \in W^{1,2}(S^h, \mathbf{R}^n)$ *satisfy* $u \cdot \vec{n}^h = 0$ *on* $\partial^+ S^h$*. Then:*

$$\begin{aligned}
\|\nabla(\bar{u} \cdot \vec{n})\|_{L^2(S)} + \|R\vec{n}\|_{L^2(S)} &\leq C \left(\|\bar{u}\|_{L^2(S)} + \|u\|_{W^{1,2}(S^h)} + h^{-1/2} \|D(u)\|_{L^2(S^h)} \right) \\
&\quad + C \left(h^{-1} \|u\|_{W^{1,2}(S^h)} \cdot \|D(u)\|_{L^2(S^h)} \right)^{1/2}.
\end{aligned}$$

Proof. First note that $\|R\vec{n}\|_{L^2(S)} = \|\vec{n}^T R_{tan}\|_{L^2(S)}$, since $\vec{n}R\vec{n} = 0$ and $R \in so(n)$. To prove the desired estimate we use the Hilbert space identity:

$$\|a\|^2 + \|b\|^2 = \|a - b\|^2 + 2\langle a, b \rangle$$

with $a = \nabla(\bar{u} \cdot \vec{n})$ and $b = \vec{n}^T R_{tan}$.

Integration by parts shows that:

$$\begin{aligned}
\langle a, b \rangle &= \left| \int_S (\vec{n} R_{tan}) \cdot \nabla(\bar{u} \cdot \vec{n}) \right| \leq C \|\bar{u} \cdot \vec{n}\|_{L^2(S)} \left(\|R\|_{L^2(S)} + \|\nabla(\vec{n} R_{tan})\|_{L^2(S)} \right) \\
&\leq C \|\bar{u} \cdot \vec{n}\|_{L^2(S)} \|R\|_{W^{1,2}(S)} \\
&\leq C \|\bar{u} \cdot \vec{n}\|_{L^2(S)} \left(h^{-3/2} \|D(u)\|_{L^2(S^h)} + h^{-1/2} \|\nabla u\|_{L^2(S^h)} \right) \\
&\leq Ch^{-1} \|u\|_{W^{1,2}(S^h)} \cdot \|D(u)\|_{L^2(S^h)} + C \|u\|^2_{W^{1,2}(S^h)},
\end{aligned} \tag{6.4}$$

where we applied the divergence theorem, Theorem 5.1 and Lemma 6.3.

On the other hand $a = \vec{n}^T \nabla \bar{u} + \bar{u} \cdot \nabla \vec{n}$, so by Lemma 6.1:

$$\begin{aligned}
\|a - b\| &\leq C \left(\|\bar{u}\|_{L^2(S)} + \|\nabla \bar{u} - R_{tan}\|_{L^2(S)} \right) \\
&\leq C \|\bar{u}\|_{L^2(S)} + Ch^{1/2} \|u\|_{W^{1,2}(S^h)} + Ch^{-1/2} \|D(u)\|_{L^2(S^h)}.
\end{aligned} \tag{6.5}$$

Combining (6.4) and (6.5) proves the result. ■

Finally, in presence of the stronger condition **(H2)**, we have an additional bound:

Lemma 6.5. *Assume* **(H2)** *and let* $u \in W^{1,2}(S^h, \mathbf{R}^n)$, $u \cdot \vec{n}^h = 0$ *on* ∂S^h. *Then:*

$$\frac{1}{h}\int_S |\bar{u}\cdot\nabla(g_1^h + g_2^h)| \leq Ch^{1/2}\|u\|_{W^{1,2}(S^h)} + Ch^{-1/2}\|D(u)\|_{L^2(S^h)}.$$

Proof. We have:

$$\begin{aligned}\frac{1}{h}\int_S &|\bar{u}\cdot\nabla(g_1^h + g_2^h)| \\ &\leq \frac{1}{h}\int_S |u(x - g_1^h(x)\vec{n}(x))\cdot\nabla g_1^h(x) + u(x + g_2^h(x)\vec{n}(x))\cdot\nabla g_2^h(x)|\ \mathrm{d}x \\ &\quad + C\|u - \bar{u}\pi\|_{L^1(\partial S^h)} \\ &\leq Ch^{-1/2}\|D(u)\|_{L^2(S)} + Ch^{1/2}\|u\|_{W^{1,2}(S^h)} + Ch^{1/2}\|\nabla u\|_{L^2(S^h)}.\end{aligned}$$

The last inequality follows from Lemma 6.2 (v) and from an easy bound: $\|u - \bar{u}\pi\|_{L^1(\partial S^h)} \leq Ch^{1/2}\|\nabla u\|_{L^2(S^h)}$. ■

7. A proof of main theorems

In this section we will prove the uniform Korn's estimate:

$$\|u\|_{W^{1,2}(S^h)} \leq C\|D(u)\|_{L^2(S^h)}, \tag{7.1}$$

under the angle constraints (2.4) or (2.6). We argue by contradiction; assume thus that (7.1) is not valid, for any uniform constant C. Hence, there exist sequences $h_n \longrightarrow 0$ and $u^{h_n} \in W^{1,2}(S^{h_n})$ (for simplicity we will write h instead of h_n) such that the assumptions of Theorem 2.1 or 2.2 are satisfied, but:

$$h^{-1/2}\|u^h\|_{W^{1,2}(S^h)} = 1 \quad \text{and} \quad h^{-1/2}\|D(u^h)\|_{L^2(S^h)} \longrightarrow 0 \qquad \text{as } h \longrightarrow 0. \tag{7.2}$$

For the proof of Theorem 2.1 we will assume that $u^h \cdot \vec{n}^h = 0$ on $\partial^+ S^h$. The case of the tangency condition on $\partial^- S^h$ is proved exactly the same.

Notice that (7.2) immediately gives, through Lemmas 6.3, 6.1 and 6.4, that:

$$\lim_{h\to 0}\Big(\|\bar{u}^h\cdot\vec{n}\|_{L^2(S)} + \|\nabla\bar{u}^h - R^h_{tan}\|_{L^2(S)}\Big) = 0, \tag{7.3}$$

$$\lim_{h\to 0}\Big(\|\nabla(\bar{u}^h\cdot\vec{n})\|_{L^2(S)} + \|R^h\vec{n}\|_{L^2(S)}\Big) \leq C\limsup_{h\to 0}\|\bar{u}^h\|_{L^2(S)}. \tag{7.4}$$

Also, Lemma 6.5 implies that under the assumption **(H2)**:

$$\lim_{h\to 0}\int_S |\bar{u}^h\cdot\nabla(g_1 + g_2)| = 0, \tag{7.5}$$

where we used that the sequence $\bar{u}^h$ is bounded in $L^1(S)$, again in view of (7.2).

A contradiction will be derived in several steps. In particular, the tangential component of the average $\bar{u}$:

$$\bar{u}^h_{tan}(x) = \bar{u}^h(x) - (\bar{u}^h\cdot\vec{n})\cdot\vec{n}(x) \in T_xS.$$

will be estimated using the Korn inequality on hypersurfaces (see Appendix C). The conditions (2.4) and (2.6) assumed in Theorems 2.1 and 2.2 will be used in full (not just for rotations as in Theorem 2.3).

Proof of Theorems 2.1 and 2.2.

1. Applying Theorem 11.2 to each tangent vector field $\bar u^h_{tan}$, we obtain a sequence $v_0^h \in \mathcal{I}(S)$ such that:

$$\|\bar u^h_{tan} - v_0^h\|_{W^{1,2}(S)} \leq C\|D(\bar u^h_{tan})\|_{L^2(S)}.$$

For every $x\in S$ and $\tau\in T_xS$ there holds:

$$\begin{aligned}|\partial_\tau \bar u^h_{tan}(x)\cdot\tau| &= |\partial_\tau \bar u^h(x)\cdot\tau - (\bar u^h\cdot\vec n)(x)\cdot\partial_\tau\vec n(x)|\\ &\leq |\partial_\tau\bar u^h(x) - R^h(x)\tau| + C|(\bar u^h\cdot\vec n)(x)|,\end{aligned}$$

as $R^h(x)\in so(n)$. Thus, by (7.3):

$$\|D(\bar u^h_{tan})\|_{L^2(S)} \leq C\left(\|\nabla\bar u^h - R^h_{tan}\|_{L^2(S)} + \|\bar u^h\cdot\vec n\|_{L^2(S)}\right)\longrightarrow 0 \qquad \text{as } h\longrightarrow 0.$$

Therefore:

$$\lim_{h\to 0}\|\bar u^h_{tan} - v_0^h\|_{W^{1,2}(S)} = 0. \tag{7.6}$$

2. Let $\mathbb{P}$ be the orthogonal projection (with respect to the $L^2(S)$ norm) of the space $\mathcal{I}(S)$ onto its subspace V, which we take to be the whole $\mathcal{I}(S)$ in case of Theorem 2.1 and $\mathcal{I}_{g_1,g_2}(S)$ in case of Theorem 2.2. Call $v_1^h = \mathbb{P}v_0^h\in V$ and $v_2^h = v_0^h - v_1^h \in V^\perp$. In both cases (3.1) implies:

$$\|u^h\|_{L^2(S^h)} \leq C\|u^h - v_1^h\pi\|_{L^2(S^h)}. \tag{7.7}$$

We now prove that:

$$\lim_{h\to 0}\|v_2^h\|_{L^2(S)} = 0. \tag{7.8}$$

In case of Theorem 2.1, when $V^\perp = \{0\}$, (7.8) is trivial, so we concentrate on the case of Theorem 2.2. Notice that then, (7.5) and (7.6) yield:

$$\begin{aligned}\int_S |v_2^h\cdot\nabla(g_1+g_2)| &= \int_S |v_0^h\cdot\nabla(g_1+g_2)|\\ &\leq C\|\bar u^h_{tan} - v_0^h\|_{L^1(S)} + C\int_S|\bar u^h\cdot\nabla(g_1+g_2)|\longrightarrow 0 \qquad \text{as } h\longrightarrow 0.\end{aligned} \tag{7.9}$$

Since all norms in the finitely dimensional space $V^\perp$ are equivalent, we have:

$$\|v_2^h\|_{L^2(S)} \leq C\int_S |v_2^h\cdot\nabla(g_1+g_2)|. \tag{7.10}$$

Indeed, the right hand side of (7.10) provides a norm on the space in question. Now, (7.9) and (7.10) clearly imply (7.8).

3. Using the Poincaré inequality on each segment $[-g_1^h(x), g_2^h(x)]$, and by (7.2):

$$h^{-1/2}\|\bar u^h\pi - u^h\|_{L^2(S^h)} \leq Ch^{1/2}\|\nabla u^h\|_{L^2(S^h)}\longrightarrow 0 \qquad \text{as } h\longrightarrow 0. \tag{7.11}$$

We now obtain convergence to 0 of various quantities:

$$\begin{aligned}&h^{-1/2}\|\bar u^h_{tan}\pi - u^h\|_{L^2(S^h)} \leq h^{-1/2}\|\bar u^h\pi - u^h\|_{L^2(S^h)} + C\|\bar u^h\cdot\vec n\|_{L^2(S)}\longrightarrow 0\\ &\qquad\qquad\qquad\qquad \text{by (7.11) and (7.3)},\\ &h^{-1/2}\|v_0^h\pi - v_1^h\pi\|_{L^2(S^h)}\longrightarrow 0 \qquad \text{by (7.8)},\\ &h^{-1/2}\|u^h - v_1^h\pi\|_{L^2(S^h)}\longrightarrow 0 \qquad \text{by (7.6) and convergences above.}\end{aligned}$$

Consequently, by (7.7):

$$\lim_{h\to 0} h^{-1/2}\|u^h\|_{L^2(S^h)} = 0. \tag{7.12}$$

Hence:

$$\|\bar{u}^h\|_{L^2(S)} \longrightx 0, \tag{7.13}$$

$$\|\nabla(\bar{u}^h \cdot \vec{n})\|_{L^2(S)} + \|R^h\vec{n}\|_{L^2(S)} \longrightarrow 0 \qquad \text{by (7.4) and (7.13)}, \tag{7.14}$$

$$\|v_0^h\|_{L^2(S)} \longrightarrow 0 \qquad \text{by (7.6) and (7.13)}. \tag{7.15}$$

Because of the equivalence of all norms on the finitely dimensional space $\mathcal{I}(S)$, (7.15) implies:

$$\lim_{h\to 0} \|v_0^h\|_{W^{1,2}(S)} = 0. \tag{7.16}$$

Now, we may estimate the quantity $h^{-1/2}\|\nabla u^h\|_{L^2(S^h)}$ by the following norms: $h^{-1/2}\|\nabla u^h - R^h\pi\|_{L^2(S^h)}$, $\|R^h\vec{n}\|_{L^2(S)}$, $\|R^h_{tan} - \nabla\bar{u}^h\|_{L^2(S)}$, $\|\nabla(\bar{u}^h\cdot\vec{n})\|_{L^2(S)}$, $\|\nabla\bar{u}^h_{tan} - \nabla v_0^h\|_{L^2(S)}$, $\|\nabla v_0^h\|_{L^2(S)}$, and use Theorem 5.1, (7.14), (7.3), (7.6) and (7.16) to conclude that:

$$\lim_{h\to 0} h^{-1/2}\|\nabla u^h\|_{L^2(S^h)} = 0.$$

Together with (7.12) this contradicts (7.2). ■

8. Estimates without Killing fields

In this section we prove Theorem 2.3. The first step is to give a bound for the distance of u from the generators of rigid motions in $\mathbf{R}^n$. This follows from Theorem 5.1 and the uniform Poincaré inequality in Theorem 12.1:

Lemma 8.1. *Assume* **(H1)**. *For every $u \in W^{1,2}(S^h, \mathbf{R}^n)$ there exists a linear function $v(z) = Az + b$, $A \in so(n)$, $b \in \mathbf{R}^n$, such that:*

$$\|u - v\|_{W^{1,2}(S^h)} \leq Ch^{-1}\|D(u)\|_{L^2(S^h)}.$$

Proof. Recall the results of Theorem 5.1 and define:

$$A = \fint_S R(x)\ \mathrm{d}x \in so(n).$$

By Theorem 5.1 and the Poincaré inequality on S, we obtain:

$$\begin{aligned} \int_{S^h} |\nabla u - A|^2 &\leq C\left\{\int_{S^h} |\nabla u - R\pi|^2 + h\int_S |R(x) - A|^2\ \mathrm{d}x\right\} \\ &\leq C\left\{\int_{S^h} |D(u)|^2 + h\int_S |\nabla R|^2\right\} \leq Ch^{-2}\int_{S^h} |D(u)|^2. \end{aligned} \tag{8.1}$$

We now apply Theorem 12.1 to the function $u(z) - Az$, by which for some $b \in \mathbf{R}^n$ there holds:

$$\int_{S^h} |u(z) - Az - b|^2\ \mathrm{d}z \leq C\int_{S^h} |\nabla u - A|^2 \leq Ch^{-2}\int_{S^h} |D(u)|^2. \tag{8.2}$$

Now (8.1) and (8.2) imply the result. ■

Proof of Theorem 2.3. The proof of part (i) will be carried out assuming that $u\cdot\vec{n}^h = 0$ on ∂^+S^h. For the other case ($u\cdot\vec{n}^h = 0$ on ∂^-S^h) the argument is the same.

1. We argue by contradiction. If (2.7) was not true, then there would be sequences $h_n \longrightarrow 0$ and $u^{h_n} \in W^{1,2}(S^{h_n})$ satisfying the conditions in (i) or (ii) and such that:

$$h^{-1/2}\|u^h\|_{W^{1,2}(S^h)} = 1, \tag{8.3}$$

$$h^{-3/2}\|D(u^h)\|_{L^2(S^h)} \longrightarrow 0 \qquad \text{as } h \longrightarrow 0 \tag{8.4}$$

(to simplify the notation, we write h instead of h_n). By Lemma 8.1, there exists a sequence $v^h(z) = A^h z + b^h$, $A^h \in so(n)$, $b^h \in \mathbf{R}^n$, such that:

$$h^{-1/2}\|u^h - v^h\|_{W^{1,2}(S^h)} \longrightarrow 0 \qquad \text{as } h \longrightarrow 0. \tag{8.5}$$

Because of (8.3), the sequence $h^{-1/2}v^h$ is bounded in $W^{1,2}(S^h)$ and so, without loss of generality, we may assume that:

$$A^h \longrightarrow A \in so(n), \qquad b^h \longrightarrow b \in \mathbf{R}^n \qquad \text{as } h \longrightarrow 0. \tag{8.6}$$

Moreover, by (8.3) and (8.5):

$$\lim_{h\to 0} h^{-1/2}\|v^h\|_{W^{1,2}(S^h)} = \lim_{h\to 0} h^{-1/2}\|u^h\|_{W^{1,2}(S^h)} = 1,$$

and therefore:

$$|A| + |b| \neq 0. \tag{8.7}$$

2. We now prove that if **(H1)** holds together with $u^h \cdot \vec{n}^h = 0$ on $\partial^+ S^h$, then we must have $Ax + b \in \mathcal{R}(S)$. Indeed, by Theorem 12.3 and Lemma 6.2 (iv):

$$\begin{aligned}\|v^h \cdot \vec{n}\|_{L^2(\partial^+ S^h)} &\leq \|u^h - v^h\|_{L^2(\partial^+ S^h)} + \|u^h \cdot \vec{n}\|_{L^2(\partial^+ S^h)}\\ &\leq C\left(h^{-1/2}\|u^h - v^h\|_{W^{1,2}(S^h)} + h^{1/2}\|u^h\|_{W^{1,2}(S^h)}\right) \longrightarrow 0 \quad \text{as } h \longrightarrow 0,\end{aligned}$$

where the convergence above follows from (8.3) and (8.5). Thus:

$$\int_S |(Ax+b)\cdot \vec{n}(x)|^2 \,\mathrm{d}x = \lim_{h\to 0}\int_S |v^h(x)\cdot \vec{n}(x)|^2 \,\mathrm{d}x = \lim_{h\to 0} \|v^h \cdot \vec{n}\pi\|^2_{L^2(\partial^+ S^h)} = 0.$$

We now prove that if **(H2)** holds, together with $u^h \cdot \vec{n}^h = 0$ on ∂S^h, then $Ax + b \in \mathcal{R}_{g_1,g_2}(S)$. By Theorem 12.3 and Lemma 6.2 (v):

$$\begin{aligned}&\frac{1}{h^2}\int_S |v^h(x + g_2^h(x)\vec{n}(x))\cdot \nabla g_2^h(x) + v^h(x - g_1^h(x)\vec{n}(x))\cdot \nabla g_1^h(x)|^2 \,\mathrm{d}x\\ &\leq \frac{1}{h^2}\Big\{Ch^2\|v^h - u^h\|^2_{L^2(\partial S^h)}\\ &\quad + \int_S |u^h(x + g_2^h(x)\vec{n}(x))\cdot \nabla g_2^h(x) + u^h(x - g_1^h(x)\vec{n}(x))\cdot \nabla g_1^h(x)|^2 \,\mathrm{d}x\Big\}\\ &\leq \frac{C}{h}\left\{\|v^h - u^h\|_{W^{1,2}(S^h)} + \|D(u^h)\|^2_{L^2(S^h)} + h^2\|u^h\|^2_{W^{1,2}(S^h)}\right\} \longrightarrow 0 \qquad \text{as } h \longrightarrow 0,\end{aligned} \tag{8.8}$$

where (8.5) with (8.4) justify the convergence. Hence, by (8.8):

$$\int_S |(Ax+b)\cdot \nabla(g_1+g_2)(x)|^2 \,\mathrm{d}x = \lim_{h\to 0}\frac{1}{h^2}\int_S |v^h \cdot \nabla(g_1^h + g_2^h)|^2 = 0$$

3. We see that in both cases (i) and (ii) there holds (using condition (3.1)):

$$\|u^h\|_{L^2(S^h)} \leq C\|u^h - (A\pi(z) + b)\|_{L^2(S^h)}.$$

Thus, by (8.5) and (8.6):

$$\begin{aligned}h^{-1/2}\|u^h\|_{L^2(S^h)} &\leq Ch^{-1/2}\|u^h - (A\pi(z)+b)\|_{L^2(S^h)}\\ &\leq Ch^{-1/2}\|u^h - v^h\|_{L^2(S^h)} + Ch^{-1/2}\|v^h - (A\pi(z)+b)\|_{L^2(S^h)} \longrightarrow 0.\end{aligned}$$

We deduce that $\lim_{h\to 0} h^{-1/2}\|v^h\|_{L^2(S^h)} = 0$ as well, which contradicts (8.7). ■

9. Appendix A - The Korn-Poincaré inequality in a fixed domain

In this section $\Omega \subset \mathbf{R}^n$ is a fixed open, bounded domain with Lipschitz boundary. For $x \in \partial\Omega$, by $\vec{n}_\Omega(x)$ we denote the outward unit normal to $\partial\Omega$ at x. We first recall the standard Korn inequality [3, 7, 5]:

Theorem 9.1. *(i) There holds:*

$$\left\{u \in L^2(\Omega, \mathbf{R}^n);\ D(u) \in L^2(\Omega, M^{n\times n})\right\} = W^{1,2}(\Omega, \mathbf{R}^n),$$

and the following equivalence of norms:

$$\|u\|_{W^{1,2}(\Omega)} \leq C_\Omega \left(\|u\|_{L^2(\Omega)} + \|D(u)\|_{L^2(\Omega)}\right) \leq C_\Omega^2 \|u\|_{W^{1,2}(\Omega)}.$$

(ii) For every $u \in W^{1,2}(\Omega, \mathbf{R}^n)$ there exists $A \in so(n)$ and $b \in \mathbf{R}^n$ so that:

$$\|u - (Ax+b)\|_{W^{1,2}(\Omega)} \leq C_\Omega \|D(u)\|_{L^2(\Omega)}.$$

The constants C_Ω above depend only on the domain Ω and not on u.

Notice that Theorem 9.1 (ii) implies that for each $u \in W^{1,2}(\Omega, \mathbf{R}^n)$ satisfying the orthogonality condition:

$$\int_\Omega u \cdot v = 0 \qquad \forall v \in \mathcal{R}(\Omega) = \{Ax + b;\ A \in so(n), b \in \mathbf{R}^n\}$$

one has:

$$\|u\|_{W^{1,2}(\Omega)} \leq C_\Omega \|D(u)\|_{L^2(\Omega)}.$$

The same is true if we restrict our attention to vector fields tangential on $\partial\Omega$. Define:

$$\mathcal{R}_\partial(\Omega) = \{v \in \mathcal{R}(\Omega);\ v \cdot \vec{n}_\Omega = 0 \text{ on } \partial\Omega\}.$$

Theorem 9.2. *For every $u \in W^{1,2}(\Omega, \mathbf{R}^n)$ such that $u \cdot \vec{n}_\Omega = 0$ on $\partial\Omega$ and:*

$$\int_\Omega u \cdot v = 0 \qquad \forall v \in \mathcal{R}_\partial(\Omega), \tag{9.1}$$

there holds:

$$\|u\|_{W^{1,2}(\Omega)} \leq C_\Omega \|D(u)\|_{L^2(\Omega)},$$

and the constant C_Ω depends only on Ω.

Proof. We argue by contradiction, starting with a sequence $u_n \in W^{1,2}(\Omega)$ satisfying $u_n \cdot \vec{n}_\Omega = 0$ on $\partial\Omega$, (9.1) and:

$$\|u_n\|_{W^{1,2}(\Omega)} = 1, \qquad \|D(u_n)\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{9.2}$$

Without loss of generality, u_n converges hence weakly to some u in $W^{1,2}(\Omega)$, and the convergence is strong in $L^2(\Omega)$. Clearly $u \cdot \vec{n}_\Omega = 0$ on $\partial\Omega$ and (9.1) still holds. By Theorem 9.1 (ii), there exist sequences $A_n \in so(n)$ and $b_n \in \mathbf{R}^n$ so that $u_n - (A_n x + b_n)$ converges to 0 in $W^{1,2}(\Omega)$.

Therefore $A_n x + b_n$ converges weakly to u in $W^{1,2}(\Omega)$ and we see that $u \in \mathcal{R}_\partial(\Omega)$. By (9.1) there hence must be $u = 0$ and u_n converges then (strongly) to 0 in $W^{1,2}(\Omega)$. This contradicts the first condition in (9.2). ■

Example 9.3. Let $\Omega = B_1 \subset \mathbf{R}^3$. Since $A \in so(3)$, there must be $Ax = a \times x$, for some $a \in \mathbf{R}^3$ and we obtain:

$$\mathcal{R}_\partial(B_1) = \{a \times x;\ a \in \mathbf{R}^3.\}$$

Condition (9.1) reads:

$$0 = \int_{B_1} (a \times x) \cdot u(x)\ \mathrm{d}x = a \cdot \int_{B_1} x \times u(x)\ \mathrm{d}x \qquad \forall a \in \mathbf{R}^3.$$

Thus the class of functions u for which the hypotheses of Theorem 9.2 are satisfied is the following:

$$\left\{u \in W^{1,2}(\Omega);\ u\cdot\vec{n}_\Omega = 0 \text{ on } \partial B_1,\ \int_{B_1} x\times u(x)\ \mathrm{d}x = 0\right\}.$$

As observed in the next result, condition (9.1) is not void if and only if our bounded domain Ω is rotationally symmetric.

Theorem 9.4. *If $\mathcal{R}_\partial(\Omega)\neq\{0\}$ then Ω must be rotationally symmetric.*

Proof. Let $v(x) = Ax + b \in \mathcal{R}_\partial(\Omega)$. We will prove that the flow generated by the tangent vector field $v_{|\partial\Omega}$ is a rotation.

Since $A\in so(n)$ we have that $\mathbf{R}^n = Ker(A)\oplus Im(A)$ is an orthogonal decomposition of $\mathbf{R}^n$. Write $b = b^{ker} + Ab_0$, $b^{ker}\in Ker(A)$, and consider the translated domain $\Omega_0 = \Omega + b_0$. Now:

$$Ax + b = A(x+b_0) + b^{ker} \qquad \forall x\in\Omega,$$

so $y\mapsto Ay + b^{ker}$ is a tangent vector field on $\partial\Omega_0$. Consider the flow α which this field generates in $\mathbf{R}^n$:

$$\left\{\begin{array}{l}\alpha'(t) = A\alpha(t) + b^{ker}\\ \alpha(0)\in\partial\Omega_0.\end{array}\right.$$

Then $\alpha(t) = \beta(t)+\delta(t)$, where:

$$\left\{\begin{array}{lll}\beta'(t) = A\beta(t), & \beta(0)\in Im(A) & \\ \delta'(t) = b^{ker}, & \delta(0)\in Ker(A), & \beta(0)+\delta(0) = \alpha(0).\end{array}\right.$$

Notice that:

$$\frac{d}{dt}|\beta(t)|^2 = 2\beta(t)\cdot A\beta(t) = 0,$$

so $\beta(t)$ remains bounded, while $\delta(t) = \delta(0) + tb^{ker}$ is unbounded for $b^{ker}\neq 0$. Since $\alpha(t)\in\partial\Omega_0$ for all $t\geq 0$, there must be $b^{ker} = 0$. Hence the flow α is a rotation (generated by $A\in so(n)$) on $\partial\Omega_0$, which proves the claim. ■

From the proof above it follows that each $v\in\mathcal{R}_\partial(\Omega)$ has the form $v(x) = A(x+b_0)$, $A\in so(n)$, $b_0\in\mathbf{R}^n$. We thus obtain the following characterisation when $\Omega\subset\mathbf{R}^3$:

$$\mathcal{R}_\partial(\Omega) = \left\{\begin{array}{ll}\{0\} & \text{if } \Omega \text{ has no rotational symmetry}\\ \text{a 1-parameter family} & \text{if } \Omega \text{ has one rotational symmetry}\\ \text{a 3-parameter family} & \text{if } \Omega = B_r.\end{array}\right.$$

10. Appendix B - The uniform Korn inequality

Throughout this section we will make the following assumptions on Ω:

(ΩH)

(i) Ω is an open, bounded subset of $\mathbf{R}^n$, star-shaped with respect to the origin.

(ii) There exists $L>0$ such that the following holds. For every $x\in\Omega\setminus\{0\}$, denote by $p(x)$ the unique point on $\partial\Omega$, with the property that the segment $[0,p(x)]$ contains x. Then:

$$|p(x)-x|\leq L\mathrm{dist}\ (x,\partial\Omega).$$

Our goal is to prove:

Theorem 10.1. *For every $u\in W^{1,2}(\Omega,\mathbf{R}^n)$ there exists $A\in so(n)$ such that:*

$$\|\nabla u - A\|_{L^2(\Omega)}\leq C_{n,L}\|D(u)\|_{L^2(\Omega)}, \tag{10.1}$$

and the constant $C_{n,L}$ depends only on n and (in nondecreasing manner) on L.

Our proof is essentially a combination of the arguments in [18, 17], where we need to keep track of the magnitude of various constants, and of [5]. In [5], the L^2 distance of ∇u from a single proper rotation is estimated in terms of the L^2 norm of the pointwise distance of ∇u from the space of proper rotations $SO(n)$. Note that $so(n)$ is the tangent space to $SO(n)$ at Id. Hence (10.1) can be seen as the "linear" version of the result in [5].

For convenience of the reader, we present the proof of Theorem 10.1. A similar line of proof was adopted in [8] where the constant $C_{n,L}$ in (10.1) (or, more recently, its L^p counterpart in [10]) has been calculated explicitly, for domains which are star-shaped with respect to a ball.

Lemma 10.2. *Let Ω be an open, bounded subset of $\mathbf{R}^n$.*

(i) *If $B_r \subset \Omega \subset B_R$ and Ω is star-shaped with respect to B_r, then $(\mathbf{\Omega H})$ holds with $L = R/r$.*
(ii) *Conversely, if Ω satisfies $(\mathbf{\Omega H})$ then it is star-shaped with respect to a ball B_r such that, calling $R = \min\{\tilde{R};\ \Omega \subset B_{\tilde{R}}\}$, the ratio R/r depends only on L, in nondecreasing manner.*

Proof. (i) is immediate. To prove (ii), fix L sufficiently large. For each $x \in \mathbf{R}^n \setminus \{0\}$ define the 'diamond' D_x obtained by rotating the right triangle with vertexes $0, a, x$ and angle $\angle a0x = \alpha$, so that $|x|/|a| = L$, around its hypotenuse $[0, x]$. The property $(\mathbf{\Omega H})(ii)$ can then be translated to: $D_x \subset \Omega$ for every $x \in \Omega$.

Let $x_1 \in \partial\Omega$ be such that $|x_1| = R$. Then $D_{x_1} \subset \Omega$. Let $x_2 = a$ from the construction of D_{x_1}. Clearly $x_2 \in \Omega$ and hence $D_{x_2} \subset \Omega$. Proceed in this manner, constructing diamonds $\{D_{x_i}\}_{i=1}^N$, with equal angles at the origin and all x_i in the same 2d subspace of $\mathbb{R}^n$. After finitely many steps of this procedure we will have $x_N \in D_{x_1}$ and $B_{\tilde{r}} \subset \Omega$ with $\tilde{r} = |x_N| = R/L^{N-1}$, where $N = [2\pi/\alpha]$. An easy argument now shows that Ω is star-shaped with respect to B_r for any $r \leq \tilde{r}/L$. Namely, taking $x \in \partial\Omega$, the convex hull of $B_r \cup \{x\}$ is contained in $D_x \cup B_{\tilde{r}} \subset \Omega$. Therefore, one can take $r = R/L^{(2\pi/\alpha)}$, so:

$$\frac{R}{r} = L^{\frac{2\pi}{\arccos(1/L)}},$$

which is a non-decreasing function of L. ■

Lemma 10.3. *For every $\phi \in W^{1,2}(\Omega)$ there holds:*

$$\int_\Omega |\phi|^2 \leq C_{n,L} \left(\int_{B_r} |\phi|^2 + \int_\Omega |\nabla\phi|^2 \mathrm{dist}^2(x, \partial\Omega)\ \mathrm{d}x \right).$$

Proof. Without loss of generality we may assume that $\phi \in \mathcal{C}^\infty(\mathbf{R}^n)$. We adopt the proof of Theorem 8.2. in [17]. Let R and r be as in Lemma 10.2.

Let $\theta : [0, \infty) \longrightarrow [0, 1]$ be a smooth non-decreasing function satisfying:

$$\theta(s) = 0 \quad \text{for } s \leq \frac{r}{4}, \qquad \theta(s) = 1 \quad \text{for } s \geq \frac{r}{2},$$
$$|\theta'(s)| \leq \frac{8}{r} \quad \text{for } s \geq 0.$$

Fix a point $p \in \partial\Omega$ and consider the function $\theta\phi$ on the segment $[0, p]$ joining the origin and p. Using Hardy's inequality [17] and condition ($\mathbf{\Omega H}$) we obtain:

$$\begin{aligned}\int_{r/2}^{|p|} |\phi|^2 \, \mathrm{d}|x| &\leq \int_0^{|p|} |\theta\phi|^2 \, \mathrm{d}|x| \leq 4 \int_0^{|p|} \left|\frac{\partial(\theta\phi)}{\partial|x|}\right|^2 \cdot \Big||p| - |x|\Big|^2 \, \mathrm{d}|x| \\ &\leq 8L^2 \left(\int_{r/4}^{r/2} |\theta'|^2 |\phi|^2 \mathrm{dist}^2(x, \partial\Omega) + \int_{r/4}^{|p|} |\nabla\phi|^2 \mathrm{dist}^2(x, \partial\Omega) \, \mathrm{d}|x|\right) \\ &\leq C_{n,L,R/r} \left(\int_{r/4}^{r/2} |\phi|^2 + \int_{r/4}^{|p|} |\nabla\phi|^2 \mathrm{dist}^2(x, \partial\Omega) \, \mathrm{d}|x|\right).\end{aligned}$$

Hence, also:

$$\int_{r/2}^{|p|} |x|^{n-1} |\phi|^2 \leq C_{n,L,R/r} \left(\int_{r/4}^{r/2} |x|^{n-1} |\phi|^2 + \int_{r/4}^{|p|} |x|^{n-1} |\nabla\phi|^2 \mathrm{dist}^2(x, \partial\Omega)\right),$$

which after integration in spherical coordinates gives:

$$\begin{aligned}\int_{\Omega \setminus B_{r/2}} |\phi|^2 \, \mathrm{d}x \leq C_{n,L,R/r} \Big(& \int_{B_{r/2} \setminus B_{r/4}} |\phi|^2 \\ & + \int_{\Omega \setminus B_{r/4}} |\nabla\phi|^2 \mathrm{dist}^2(x, \partial\Omega) \, \mathrm{d}x \Big).\end{aligned} \tag{10.2}$$

Since $C_{n,L,R/r} = C_{n,L}$ in view of Lemma 10.2, the result follows by (10.2). ■

Theorem 10.4. *For every $\phi \in W^{1,2}(\Omega)$ there exists $a \in \mathbf{R}$ such that:*

$$\int_\Omega |\phi - a|^2 \leq C_{n,L} \int_\Omega |\nabla\phi|^2 \mathrm{dist}^2(x, \partial\Omega) \, \mathrm{d}x.$$

Proof. We adopt the method of proof from Theorem 3.1. in [5]. Again, let R and r be as in Lemma 10.2.

By the Poincaré inequality we obtain:

$$\int_{B_{r/2}} \left|\phi - \fint_{B_{r/2}} \phi\right|^2 \leq C_n r^2 \int_{B_{r/2}} |\nabla\phi|^2 \leq C_n \int_{B_{r/2}} |\nabla\phi|^2 \mathrm{dist}^2(x, \partial\Omega) \, \mathrm{d}x. \tag{10.3}$$

Applying Lemma 10.3 to the function $\phi - \fint_{B_{r/2}} \phi$ on Ω we therefore get:

$$\begin{aligned}\int_\Omega \left|\phi - \fint_{B_{r/2}} \phi\right|^2 &\leq C_{n,L} \left(\int_{B_{r/2}} \left|\phi - \fint_{B_{r/2}} \phi\right|^2 + \int_\Omega |\nabla\phi|^2 \mathrm{dist}^2(x, \partial\Omega) \, \mathrm{d}x\right) \\ &\leq C_{n,L} \int_\Omega |\nabla\phi|^2 \mathrm{dist}^2(x, \partial\Omega) \, \mathrm{d}x,\end{aligned}$$

where the last inequality follows from (10.3). ■

We now recall the following result from [18]. For convenience of the reader, we reproduce its short proof.

Lemma 10.5. *Let $\phi \in W^{1,2}(\Omega)$ be such that $\Delta\phi = 0$ in $\mathcal{D}'(\Omega)$. Then:*

$$\int_\Omega |\nabla\phi|^2 \mathrm{dist}^2(x, \partial\Omega) \, \mathrm{d}x \leq 4 \int_\Omega |\phi|^2.$$

Proof. Fix a small $\epsilon > 0$ and integrate the equation $\Delta\phi = 0$ against the scalar function $(\mathrm{dist}(x, \partial\Omega) - \epsilon)^2\phi$, over the set $\Omega_\epsilon = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > \epsilon\}$. Integrating by parts we obtain:

$$\begin{aligned}\int_{\Omega_\epsilon} (\mathrm{dist}(x, \partial\Omega) - \epsilon)^2 |\nabla\phi|^2 \ \mathrm{d}x &= -\int_{\Omega_\epsilon} 2\mathrm{dist}(x, \partial\Omega) - \epsilon)\phi(x)\nabla\mathrm{dist}(\cdot, \partial\Omega)\cdot\nabla\phi \ \mathrm{d}x \\ &\leq 2\int_{\Omega_\epsilon} |\phi|^2 + \frac{1}{2}\int_{\Omega_\epsilon} (\mathrm{dist}(x, \partial\Omega) - \epsilon)^2|\nabla\phi|^2 \ \mathrm{d}x.\end{aligned}$$

where we have used the binomial formula and the fact that $|\nabla\mathrm{dist}(\cdot, \partial\Omega)| \leq 1$. The above implies:

$$\int_{\Omega_\epsilon} (\mathrm{dist}(x, \partial\Omega) - \epsilon)^2 |\nabla\phi|^2 \ \mathrm{d}x \leq 4\int_\Omega |\phi|^2,$$

and proves the lemma upon passing $\epsilon \to 0$. ∎

Proof of Theorem 10.1.
The left hand side of (10.1) represents the distance in $L^2(\Omega)$ of ∇u from the closed subspace of constant functions $A \in so(n)$. Since the distance function is continuous, we may without loss of generality assume that $u \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R}^n)$.

1. Consider the problem:

$$\begin{cases} \Delta v = \Delta u & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since:

$$\Delta u = 2 \ \mathrm{div}\left\{D(u) - \frac{1}{2}(\mathrm{tr}\ D(u))\cdot \mathrm{Id}\right\}, \tag{10.4}$$

we see that:

$$\int_\Omega |\nabla v|^2 = 2\int_\Omega \nabla v : \left(D(u) - \frac{1}{2}(\mathrm{tr}\ D(u))\cdot\mathrm{Id}\right) \leq 4\|\nabla v\|_{L^2(\Omega)}\|D(u)\|_{L^2(\Omega)}.$$

Therefore:

$$\|\nabla v\|_{L^2(\Omega)} \leq 4\|D(u)\|_{L^2(\Omega)}. \tag{10.5}$$

2. The remaining part $w = u - v$ is harmonic: $\Delta w = 0$ in Ω. Hence, the components of $D(w)$ are also harmonic, and Lemma 10.5 implies:

$$\int_\Omega |\nabla D(w)|^2 \mathrm{dist}^2(x, \partial\Omega) \ \mathrm{d}x \leq 4\int_\Omega |D(w)|^2. \tag{10.6}$$

Notice that the components of $\nabla^2 w$ are linear combinations of components of $\nabla D(w)$, namely: $[\nabla^2 w^k]_{ls} = \frac{\partial}{\partial x_l}[D(w)]_{ks} + \frac{\partial}{\partial x_s}[D(w)]_{kl} - \frac{\partial}{\partial x_k}[D(w)]_{ls}$. Applying now Theorem 10.4 to the components of ∇w, we obtain $B \in M^{n\times n}$ so that, in view of (10.6):

$$\int_\Omega |\nabla w - B|^2 \leq C_{n,L}\int_\Omega |\nabla^2 w|^2 \mathrm{dist}^2(x, \partial\Omega) \ \mathrm{d}x \leq C_{n,L}\int_\Omega |D(w)|^2. \tag{10.7}$$

Define $A = (B - B^T)/2 \in so(n)$ and notice that for every $x \in \Omega$ there holds:

$$\begin{aligned}|B - A| = \mathrm{dist}_{M^{n\times n}}(B, so(n)) &\leq |B - \nabla w(x)| + \mathrm{dist}_{M^{n\times n}}(\nabla w(x), so(n)) \\ &= |B - \nabla w(x)| + |D(w)(x)|.\end{aligned}$$

Therefore:

$$\int_\Omega |B - A|^2 \leq C_{n,L}\int_\Omega |D(w)|^2. \tag{10.8}$$

Now by (10.5), (10.7) and (10.8):

$$\begin{aligned}\|\nabla u - A\|_{L^2(\Omega)} &\leq \|\nabla v\|_{L^2(\Omega)} + \|\nabla w - B\|_{L^2(\Omega)} + \|B - A\|_{L^2(\Omega)}\\ &\leq C_{n,L}\left(\|D(u)\|_{L^2(\Omega)} + \|D(w)\|_{L^2(\Omega)}\right)\\ &\leq C_{n,L}\|D(u)\|_{L^2(\Omega)},\end{aligned} \tag{10.9}$$

the last inequality following from $D(w) = D(u) - D(v)$ and the bound (10.5). ■

11. Appendix C - The Killing fields and the Korn inequality on hypersurfaces

For a tangent vector field $u \in W^{1,2}(S, \mathbf{R}^n)$, define $D(u)$ as the symmetric part of its tangential gradient:

$$D(u) = \frac{1}{2}\left[(\nabla u)_{tan} + (\nabla u)_{tan}^T\right].$$

That is, for $x \in S$, $D(u)(x)$ is a symmetric bilinear form given through:

$$\tau^T D(u)(x)\eta = \frac{1}{2}\big(\tau \cdot \partial_\eta u(x) + \eta \cdot \partial_\tau u(x)\big) \qquad \forall \tau, \eta \in T_x S.$$

Recall that a smooth vector field u as above is a Killing field, provided that $D(u) = 0$ on S. We first prove that in presence of this last condition, the regularity $u \in W^{1,2}(S)$ actually implies that u is smooth. Further, we directly recover a generalisation of Theorem 9.1 (ii) to the non-flat setting (Theorem 11.2). Actually, the bound in Theorem 9.1 (i) remains true also in the more general framework of Riemannian manifolds [2].

The following extension of u on the neighbourhood of S will be useful in the sequel:

$$\tilde{u}(x + t\vec{n}(x)) = (\mathrm{Id} + t\Pi(x))^{-1}u(x) \qquad \forall x \in S \quad \forall t \in (-h_0, h_0) \tag{11.1}$$

for some small $h_0 > 0$. Here $\Pi(x) = \nabla\vec{n}(x)$ is the shape operator on S. We have $\tilde{u} \in W^{1,2}(\tilde{S}, \mathbf{R}^n)$ where $\tilde{S} = S^{h_0}$ is open in $\mathbf{R}^n$. Notice that for each $z = x + t\vec{n}(x) \in \tilde{S}$ and $\tau_1 \in T_x S$ there holds:

$$\begin{aligned}\partial_{\tau_1}\tilde{u}(z) = \Big\{\nabla\left[(\mathrm{Id} + t\Pi(x))^{-1}\right](\mathrm{Id} + t\Pi(x))^{-1}\tau_1\Big\}u(x)&\\ + (\mathrm{Id} + t\Pi(x))^{-1}\nabla u(x)(\mathrm{Id} + t\Pi(x))^{-1}\tau_1&.\end{aligned}$$

The first component above is bounded by $C|tu(x)|$. Taking the scalar product of the second component with any $\tau_2 \in T_x S$ gives:

$$\big((\mathrm{Id} + t\Pi(x))^{-1}\tau_2\big) \cdot \nabla u(x)(\mathrm{Id} + t\Pi(x))^{-1}\tau_1.$$

Since $(\mathrm{Id} + t\Pi(x))(T_x S) = T_x S$ we obtain:

$$\begin{aligned}\tau_2^T D(\tilde{u})(z)\tau_1 &= \big((\mathrm{Id} + t\Pi(x))^{-1}\tau_2\big) \cdot D(u)(x)(\mathrm{Id} + t\Pi(x))^{-1}\tau_1\\ &\quad + Z(t,x) \cdot u(x),\\ |Z(t,x)| &\leq C.\end{aligned} \tag{11.2}$$

On the other hand, $\vec{n}(x) \cdot \tilde{u}(z) = 0$, so for any $\tau \in T_x S$:

$$\begin{aligned}\vec{n} \cdot \partial_\tau\tilde{u}(z) &= -\Big(\Pi(x)(\mathrm{Id} + t\Pi(x))^{-1}\tau\Big) \cdot \tilde{u}(z)\\ &= -\Big((\mathrm{Id} + t\Pi(x))^{-1}\Pi(x)(\mathrm{Id} + t\Pi(x))^{-1}u(x)\Big) \cdot \tau = \tau \cdot \partial_{\vec{n}}\tilde{u}(z).\end{aligned}$$

Hence:

$$
\begin{aligned}
&\vec{n}^T D(\tilde{u})(z)\tau = -\Big((\mathrm{Id} + t\Pi(x))^{-1}\Pi(x)(\mathrm{Id} + t\Pi(x))^{-1}u(x)\Big)\cdot\tau, \\
&\vec{n}^T D(\tilde{u})(z)\vec{n} = 0.
\end{aligned} \tag{11.3}
$$

Lemma 11.1. *Let $u \in W^{1,2}(S, \mathbf{R}^n)$ be a tangent vector field such that $D(u) = 0$ almost everywhere on S. Then $u \in \mathcal{I}(S)$.*

Proof. We only need to prove that u is smooth. Consider the extension $\tilde{u} \in W^{1,2}(\tilde{S}, \mathbf{R}^n)$ as above. By (11.2), (11.3) and the formula (10.4) we see that $D(\tilde{u}) \in W^{1,2}(\tilde{S})$ and hence:

$$\Delta\tilde{u} \in L^2(\tilde{S}).$$

The result follows now by the elliptic regularity and a bootstrap argument. ■

Theorem 11.2. *For every tangent vector field $u \in W^{1,2}(S, \mathbf{R}^n)$ there exists $v \in \mathcal{I}(S)$ such that:*

$$\|u - v\|_{W^{1,2}(S)} \leq C_S\|D(u)\|_{L^2(S)}$$

and the constant C_S depends only on the surface S.

Proof. Since $\mathcal{I}(S)$ is a finitely dimensional subspace of the Banach space E of all $W^{1,2}(S)$ tangent vector fields, its orthogonal complement $\mathcal{I}(S)^\perp$ is a closed subspace of E. We will prove that:

$$\|u\|_{W^{1,2}(S)} \leq C_S\|D(u)\|_{L^2(S)} \qquad \forall w \in \mathcal{I}(S)^\perp \tag{11.4}$$

which implies the Theorem.

If (11.4) was not true, there would be a sequence $u_n \in \mathcal{I}(S)^\perp$ such that:

$$\|u_n\|_{W^{1,2}(S)} = 1, \qquad \|D(u_n)\|_{L^2(S)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

Without loss of generality, u_n converge weakly in $W^{1,2}(S)$ to some $u \in \mathcal{I}(S)^\perp$. Moreover $D(u) = 0$ by the second condition above, so by Lemma 11.1 we obtain that $u \in \mathcal{I}(S)$.

As the spaces $\mathcal{I}(S)$ and $\mathcal{I}(S)^\perp$ are orthogonal, there must be $u = 0$, and hence the sequence u_n converges to 0 (strongly) in $L^2(S)$. This contradicts $\|u_n\|_{W^{1,2}(S)} = 1$, because:

$$\|u_n\|_{W^{1,2}(S)} \leq C_S\left(\|u_n\|_{L^2(S)} + \|D(u_n)\|_{L^2(S)}\right).$$

The last inequality follows from Theorem 9.1 (i) applied to the extensions $\tilde{u}_n \in W^{1,2}(\tilde{S})$ as in (11.1). Indeed, by (11.2) and (11.3) it follows that:

$$
\begin{aligned}
\|\tilde{u}_n\|_{L^2(\tilde{S})} &\approx h_0^{1/2}\|u_n\|_{L^2(S)}, \\
\|\nabla u_n\|_{L^2(S)} &\leq Ch_0^{-1/2}\|\tilde{u}_n\|_{W^{1,2}(\tilde{S})}, \\
\|D(\tilde{u}_n)\|_{L^2(\tilde{S})} &\leq Ch_0^{1/2}\big(\|u_n\|_{L^2(S)} + \|D(u_n)\|_{L^2(S)}\big).
\end{aligned}
$$

■

We now want to gather a few remarks relating to the fact that the linear space $\mathcal{I}(S)$ of all Killing fields on S is of finite dimension. This is a classical result [13], and it implies that in $\mathcal{I}(S)$ all norms are equivalent. In particular, one has:

$$\forall u \in \mathcal{I}(S) \qquad \|\nabla u\|_{L^2(S)} \leq C_S\|u\|_{L^2(S)}, \tag{11.5}$$

for some constant C_S depending only on the hypersurface S.

The bound (11.5), together with an estimate of C_S, can also be recovered directly, using the following identity [19], valid for Killing vector fields u:

$$\Delta_S\left(\frac{1}{2}|u|^2\right) = \left|\tilde{\nabla} u\right|^2 - \mathrm{Ric}\ (u,u). \tag{11.6}$$

Here Δ_S is the Laplace-Beltrami operator on S, $\tilde{\nabla} u = (\nabla u)_{tan}$ is the covariant derivative of u on S, and Ric stands for the Ricci curvature form on S.

To calculate Ric (u,u) in our particular setting, notice that by Gauss' Teorema Egregium ([23], vol 3), the Riemann curvature 4-tensor on S satisfies:

$$\forall x\in S \quad \forall \tau,\eta,\xi,\vartheta\in T_xS \qquad R(\tau,\eta)\xi\cdot\vartheta = (\Pi(x)\tau\cdot\vartheta)(\Pi(x)\eta\cdot\xi) - (\Pi(x)\tau\cdot\xi)(\Pi(x)\eta\cdot\vartheta).$$

Thus, seeing the Ricci curvature 2-tensor as an appropriate trace of R, we obtain:

$$\begin{aligned}\forall x\in S \quad \forall \eta,\xi\in T_xS \qquad \mathrm{Ric}\ (\eta,\xi) &= \mathrm{tr}\ (\tau\mapsto R(\tau,\eta)\xi)\\ &= (\mathrm{tr}\ \Pi(x))\Pi(x)\eta\cdot\xi - \Pi(x)\xi\cdot\Pi(x)\eta\\ &= \left((\mathrm{tr}\ \Pi(x))\Pi(x) - \Pi(x)^2\right)\eta\cdot\xi.\end{aligned} \tag{11.7}$$

Integrating (11.6) on S and using (11.7) we arrive at:

$$\|\tilde{\nabla} u\|^2_{L^2(S)} = \int_S \Big((\mathrm{tr}\ \Pi(x))\Pi(x) - \Pi(x)^2\Big)u(x)\cdot u(x). \tag{11.8}$$

Notice that in the special case of a 2×2 matrix Π, that is when $n=3$ and S is a 2-d surface in $\mathbf{R}^3$, the Cayley-Hamilton theorem implies:

$$(\mathrm{tr}\ \Pi)\Pi - \Pi^2 = (\det \Pi)\cdot \mathrm{Id},$$

and so:

$$\|\tilde{\nabla} u\|^2_{L^2(S)} = \int_S \det\ \Pi(x)|u|^2.$$

In this case $\det \Pi(x)$ is the Gaussian curvature of S at x (see [19]).

To calculate the L^2 norm of the full gradient ∇u on S, notice that:

$$\|\nabla u\|^2_{L^2(S)} - \|\tilde{\nabla} u\|^2_{L^2(S)} = \int_S \sum_{i=1}^{n-1}\left|\vec{n}\cdot\frac{\partial}{\partial\tau_i}u\right|^2 = \int_S\sum_{i=1}^{n-1}|u\cdot\Pi(x)\tau_i|^2 = \int_S |\Pi(x)u|^2.$$

Hence we arrive at:

$$\|\nabla u\|^2_{L^2(S)} = \int_S (\mathrm{tr}\ \Pi(x))\Pi(x)u(x)\cdot u(x), \tag{11.9}$$

which clearly implies (11.5).

Remark 11.3. An equivalent way of obtaining the formula (11.9), but without using the language of Riemannian geometry, is to look at 'trivial' extension of u:

$$w(x+t\vec{n}(x)) = u(x) \qquad \forall x\in S \quad \forall t\in(-h_0,h_0).$$

Since $\partial_{\vec{n}}w = 0$ and $w\cdot\vec{n}=0$ on the boundary of $\tilde{S} = S^{h_0}$, by (10.4) one has:

$$\|\nabla w\|^2_{L^2(\tilde{S})} = -2\int_{\tilde{S}} \mathrm{div}\ D(w)\cdot w - \|\mathrm{div}\ w\|^2_{L^2(\tilde{S})}. \tag{11.10}$$

Calculating $\int \mathrm{div}\ D(w)\cdot w$ in terms of $\Pi(x)$, dividing both sides of (11.10) by $2h$ and passing to the limit with $h\longrightarrow 0$, one may recover (11.9) directly.

Remark 11.4. From the equivalence of the L^2 and the $W^{1,2}$ norms on $\mathcal{I}(S)$, proved in (11.9), it follows that the linear space $\mathcal{I}(S)$ is finitely dimensional.

For otherwise the space $(\mathcal{I}(S), \|\cdot\|_{W^{1,2}(S)})$ would have a countable Hilbertian (orthonormal) base $\{e_i\}_{i=1}^\infty$ and thus necessarily the sequence $\{e_i\}$ would converge to 0, weakly in $W^{1,2}(S)$. But this implies that $\lim_{h\to 0}\|e_i\|_{L^2(S)} = 0$, which by the norms equivalence gives the same convergence in $W^{1,2}(S)$, and a contradiction.

12. Appendix D - The uniform Poincaré inequality and the trace theorem in thin domains

Theorem 12.1. *Assume* **(H1)** *and let $h > 0$ be sufficiently small. For every $u \in W^{1,2}(S^h, \mathbf{R})$ there exists a constant $a \in \mathbf{R}$ so that:*

$$\|u - a\|_{L^2(S^h)} \leq C\|\nabla u\|_{L^2(S^h)}$$

and C is independent of h, a or u.

Proof. The argument is a combination of the proof of Theorem 5.1 and the Poincaré inequality on fixed surface S. Let $D_{x,h}$, $B_{x,h}$, η_x be as in the proof of Theorem 5.1. Define a smooth function $\tilde{a} : S \longrightarrow \mathbf{R}$:

$$\tilde{a}(x) = \int_{S^h} \eta_x(z)u(z)\ \mathrm{d}z.$$

We will prove the theorem for $a = \fint_S \tilde{a}(x)\ \mathrm{d}x$.

First, by Theorem 10.4, we see that the local estimate (5.1) can be in our new setting replaced by:

$$\int_{B_{x,h}} |u - a_{x,h}|^2 \leq Ch^2 \int_{B_{x,h}} |\nabla u|^2,$$

with C, as usual, a uniform constant. Repeating the calculations leading to (5.2) and (5.3), we thus obtain:

$$|\tilde{a}(x) - a_{x,h}|^2 \leq Ch^{2-n} \int_{B_{x,h}} |\nabla u|^2,$$

$$|\nabla \tilde{a}(x')|^2 \leq Ch^{-n} \int_{2B_{x,h}} |\nabla u|^2 \qquad \forall x' \in D_{x,h},$$

which imply, exactly as in (5.5):

$$\int_{S^h} |u - \tilde{a}\pi|^2 \leq Ch^2 \int_{S^h} |\nabla u|^2, \qquad \int_S |\nabla \tilde{a}|^2 \leq Ch^{-1} \int_{S^h} |\nabla u|^2.$$

By the above inequalities and the standard Poincaré inequality on surfaces, it follows:

$$\begin{aligned}\int_{S^h} |u - a|^2 &\leq C\left\{\int_{S^h} |u - \tilde{a}\pi|^2 + h\int_S |\tilde{a}(x) - a|^2\ \mathrm{d}x\right\} \\ &\leq C\left\{h^2 \int_{S^h} |\nabla u|^2 + h\int_S |\nabla \tilde{a}|^2\right\} \leq C\int_{S^h} |\nabla u|^2,\end{aligned}$$

proving the result. ■

Remark 12.2. Theorem 12.1 provides a Poincaré inequality for sets Ω enjoying properties as in section 10. The following is a more general result. Assume that $\Omega \subset \mathbf{R}^n$ is open, star-shaped with respect to the origin and such that:

$$B_r \subset \Omega \subset B_R.$$

Then for every $u \in W^{1,2}(\Omega, \mathbf{R})$ there holds:

$$\left\| u - \fint_\Omega u \right\|_{L^2(\Omega)} \leq C_{n,R/r} R \cdot \|\nabla u\|_{L^2(\Omega)},$$

where the constant $C_{n,R/r}$ depends only on the upper bound of the quantities n and R/r.

The proof follows from [1] where the first nonzero eigenvalue α_1 of the Neumann problem for $-\Delta$ on Ω is estimated from below by $C_n \cdot \frac{r^n}{R^{n+2}}$, the constant C_n depending on n only. Recalling that the best Poincaré constant equals to $\alpha_1^{-1/2}$, we obtain the result.

Theorem 12.3. *Assume* **(H1)**. *For every $u \in W^{1,2}(S^h, \mathbf{R})$ there holds:*

$$\|u\|_{L^2(S)} \leq Ch^{-1/2}\|u\|_{L^2(S^h)} + Ch^{1/2}\|\nabla u\|_{L^2(S^h)}, \tag{12.1}$$

$$\|u\|_{L^2(\partial S^h)} \leq Ch^{-1/2}\|u\|_{L^2(S^h)} + Ch^{1/2}\|\nabla u\|_{L^2(S^h)}, \tag{12.2}$$

where in the left hand side we have norms of traces of u on S and ∂S^h, respectively. The constant C is independent of u or h.

Proof. Since $|g_i^h(x)| \geq Ch$, (12.1) will be implied by the same inequality for S^h with $g_1^h = g_2^h = Ch$. The latter one can be obtained covering S^h with the cylinders $B_{x,h}$ of size h and applying the scaled version of the usual trace theorem to $B_{x,h}$.

Notice, that the constant C in (12.1) depends only on n and the Lipschitz constant of S. Since $|\nabla g_i^h(x)| \leq Ch$ and $|g_i^h(x)| \geq Ch$ for each $s \in S$, we may use the same argument as before on $\{x - t\vec{n}^h(x);\ x \in \partial S^h,\ t \in (0, Ch)\} \subset S^h$ to prove (12.2). ■

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