ON THE GENERICITY OF THE MULTIPLICITY RESULTS FOR FORCED OSCILLATIONS ON COMPACT MANIFOLDS

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1. Introduction

In [3], some multiplicity results for the forced oscillations of a mass point constrained on a sphere have been obtained. In particular, it was proved that a small periodic perturbation of a gravitation-like tangent vector field induces at least two forced oscillations. Such results depend strictly on the strong geometric properties of the sphere and cannot be (easily) extended to the general setting of a second order ODE on an arbitrary compact manifold. However, we will show that such multiplicity results are, in some sense, "generic".

Let $M \subset \mathbf{R}^k$ be a compact, boundaryless m-dimesional smooth manifold. Let $h : TM \longrightarrow \mathbf{R}^k$ be continuous and tangent to M (i.e. $h(p,v) \in T_pM$ for any $p \in M$ and $v \in T_pM$), and let T be a fixed positive real number.

We will be concerned with periodic solutions of the equation

$$\ddot{x}_\pi = h(x, \dot{x}) + \lambda f(t, x, \dot{x}) \ , \ \lambda \geq 0, \tag{1}$$

where the perturbing function $f : \mathbf{R} \times TM \longrightarrow \mathbf{R}^k$ has the following properties:

(**P1**) (Carathéodory, T-periodicity in t)
- for any $(p,v) \in TM$, $f(\cdot, p, v) : \mathbf{R} \longrightarrow \mathbf{R}^k$ is measurable and T-periodic,
- for a.a. $t \in \mathbf{R}$, $f(t, \cdot, \cdot) : TM \longrightarrow \mathbf{R}^k$ is continuous,

(**P2**) (tangency)
- for any $(p,v) \in TM$ for a.a. $t \in \mathbf{R}$, $f(t,p,v) \in T_pM$,

(**P3**) (admissibility)
- for any compact $K \subset TM$ there exists a function $\gamma_K \in L^1([0,T], \mathbf{R})$ such that for a.a. $t \in [0,T]$, for any $(p,v) \in K$,
$$|f(t,p,v)| < \gamma_K(t).$$

Following [4], where (1) was studied for f continuous, we first establish a result on existence of an unbounded branch of the set of (T-periodic) solution pairs (λ, x) for (1) (Theorem 3.3 below). Then, by introducing the notion of second order non-T-resonancy, we study the equation

$$\ddot{x}_\pi = g(x) + f(t, x, \dot{x}) \tag{2}$$

and we prove that the set of C^r $(r \geq 0)$, autonomous vector fields g, having the property that any "small enough" perturbation f induces at least $|\chi(M)|$ forced oscillations (that is, T-periodic solutions of (2)), is open and dense in the space of C^r tangent vector fields on M (Theorem 5.2).

Finally, we restrict our attention to the gradient vector fields on M and prove a corresponding "qualitative" multiplicity result with a sharper estimate of the number of forced oscillations (Theorem 5.5).

The key step of our proof is Theorem 4.3 below. Although a similar result could be gained (in the case of C^1 perturbing funtions) by an appropriate use of the implicit function theorem, we prefer to use a more topological approach, based on a continuation principle (Corollary 3.4), which stresses the geometrical aspects of the problem and allows a greater generality.

2. Preliminaries and notation

Throughout all the paper $M,\ h,\ T,\ f$ will be as in the introduction, and by TM we will mean the tangent bundle to the manifold M, that is the set

$$TM = \big\{(p,v) \in \mathbf{R}^k \times \mathbf{R}^k : p \in M\ ,\ v \in T_pM\big\}\ .$$

In what follows, the symbol $C^1_T(M)$ will denote the metric subspace of the Banach space $\big(C^1_T(\mathbf{R}^k)\ ,\ \|\cdot\|_1\big)$, where $\|\cdot\|_1$ is the usual C^1 norm, of all the T-periodic, C^1 functions $x : \mathbf{R} \longrightarrow M$ and, analogously, by $C_T(TM)$ we mean the metric space of T-periodic, continuous functions $x : \mathbf{R} \longrightarrow TM$, with the metric inherited from the Banach space $C_T(\mathbf{R}^k \times \mathbf{R}^k)$ (the "sup" norm).

Given a subspace A of a topological space S and a subset B of A, we denote by $\mathrm{Fr}_A(B)$ and $\overline{B}^A$ respectively, the boundary and the closure of B relative to A.

As in [4], we tacitly assume some natural identifications; for example we identify a point $p \in M$ with the constant function $t \mapsto p$ in $C^1_T(M)$, or a function $x \in C^1_T(M)$ with $(x,\dot{x}) \in C_T(TM)$. Also, we regard each of the above spaces as the zero-slices of the space obtained as the Cartesian product of $[0,\infty)$ and the space under consideration. In this manner, M becomes a subset of $[0,\infty) \times C^1_T(M)$ and of $[0,\infty) \times C_T(TM)$ as well, and so on.

In the same spirit, by $h_{|M} : M \longrightarrow \mathbf{R}^k$ we understand the function given by $h_{|M}(p) = h(p,0)$.

Finally, by $\mathcal{E}$ we denote the topological vector space of all functions $f : \mathbf{R} \times TM \longrightarrow \mathbf{R}^k$ having the properties (P1) – (P3), endowed with the topology given by the following fundamental system of neighbourhoods of 0:

$$\big\{U_{K,\varepsilon} : K \text{ is a compact subset of } TM\ ,\ \varepsilon > 0\big\},$$

where

$$U_{K,\varepsilon} = \big\{f \in \mathcal{E} :\ \text{for a.a. } t \in [0,T],\ \text{for all } (p,v) \in K,\quad |f(t,p,v)| < \varepsilon\big\}.$$

Remark 2.1. *If we restrict our attention to the space of continuous perturbing functions $\mathcal{E} \cap C^0\big([0,T] \times TM\ ,\ \mathbf{R}^k\big)$, the above topology on $\mathcal{E}$ induces the compact-open topology.*

Let us recall an important definition from [2]. We will say that (λ, x) is a solution pair of (1) if:

- $\lambda \geq 0$,
- $x \in C^1_T(M)$ and $\dot{x}$ is absolutely continuous,
- for a.a. $t \in \mathbf{R}$, $\Pi_{T_{x(t)}M}(\ddot{x}(t)) = h(x(t),\dot{x}(t)) + \lambda f(t,x(t),\dot{x}(t))$,

where, for a fixed subspace $E \subset \mathbf{R}^k$, $\Pi_E : \mathbf{R}^k \longrightarrow E$ is the orthogonal projection of $\mathbf{R}^k$ onto E. From now on, X will denote the subset of $[0,\infty) \times C^1_T(M)$ of all the solution pairs of (1).

We quote an important known result (see e.g. [1]):

Theorem 2.2. *Let $T^2M = \{(p,v,u) \in \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{R}^k : p \in M;\ v,u \in T_pM\}$. There exists exactly one smooth function $\nu : T^2M \longrightarrow \mathbf{R}^k$, such that:*

(i) *for any $(p,v,u) \in T^2M$, $\nu(p,v,u) \in (T_pM)^\perp$,*
(ii) *for any $p \in M$, $\nu(p,\cdot,\cdot) : T_pM \times T_pM \longrightarrow (T_pM)^\perp$ is bilinear and symmetric,*
(iii) *$(u,w) \in T_{(p,v)}TM$ if and only if $u \in T_pM$ and $\Pi_{(T_pM)^\perp}(w) = \nu(p,v,u)$.*

Define

$$\widehat{h} : TM \longrightarrow \mathbf{R}^k \times \mathbf{R}^k \ ;\ \widehat{h}(p,v) = (v, r(p,v) + h(p,v)),$$

$$\bar{f} : \mathbf{R} \times TM \longrightarrow \mathbf{R}^k \times \mathbf{R}^k \ ;\ \bar{f}(t,p,v) = \big(0, f(t,p,v)\big),$$

where $r : TM \longrightarrow \mathbf{R}^k$ is given by: $r(p,v) = \nu(p,v,v)$. Since, by Theorem 2.2, $\widehat{h}$ and $\bar{f}$ are tangent to TM, we may consider a first order ODE on TM

$$\dot{\xi} = \widehat{h}(\xi) + \lambda \bar{f}(t,\xi)\ ,\ \lambda \geq 0, \tag{3}$$

where we put $\xi(t) = (\xi_1(t)\ ,\ \xi_2(t))$, with $\xi_1(t) \in M$ and $\xi_2(t) \in T_{\xi_1(t)}M$.
We say that (λ,ξ) is a solution pair of (3), if:

- $\lambda \geq 0$,
- $\xi \in C_T(TM)$ and ξ is absolutely continuous,
- for a.a. $t \in \mathbf{R}$, $\dot{\xi}(t) = \widehat{h}(\xi(t)) + \lambda \bar{f}(t,\xi(t))$.

We denote the set of all solution pairs of (3) by $\widehat{X}$. Notice that $(\lambda,x) \in X$ if and only if $(\lambda,(x,\dot{x})) \in \widehat{X}$.

Let U be an open subset of a smooth, boundaryless manifold $N \subset \mathbf{R}^l$, and $v : N \longrightarrow \mathbf{R}^l$ be a continuous tangent vector field which is admissible on U, i.e. such that the set $v^{-1}(0) \cap U$ is compact. Then, one can associate to the pair (v,U) an integer number, called the degree of the vector field v in U, and denoted by $\deg(v,U)$ which, roughly speaking, counts (algebraically) the number of zeros of v in U (see e.g. [8], and references therein). Given an isolated zero p of v, it is convenient to introduce the index $\mathrm{i}(v,p)$ of v at p as follows: $\mathrm{i}(v,p) = \deg(v,U)$, where U is any open neighborhood of p such that $v^{-1}(0) \cap U = \{p\}$.

In the flat case, namely if U is an open subset of $\mathbf{R}^k$, $\deg(v,U)$ is just the Brouwer degree (with respect to zero) of v in any bounded open set V containing $v^{-1}(0)$ and such that $\overline{V} \subset U$. One can see that all the standard properties of the Brouwer degree on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, etc. are still valid in the more general context of differentiable manifolds.

Now we recall the notion of T-resonancy for first order ODE's on manifolds, introduced in [3]. Let $N \subset \mathbf{R}^l$ be as above, and $g : N \longrightarrow \mathbf{R}^l$ be a continuous tangent vector field. We say that a point $p \in g^{-1}(0)$ is T-resonant for g if:

- g is C^1 in a neighbourhood of p,

- the linear equation on T_pN (note that $g'(p) \in \mathrm{End}(T_pN)$)

$$\dot{x} = g'(p)x$$

admits nontrivial (i.e. nonzero) T-periodic solutions.

Correspondingly, $p \in g^{-1}(0)$ is non-T-resonant for g, if g is C^1 in a neighbourhood of p and the only T-periodic solution of the above equation is the trivial one. Notice that p is non-T-resonant for g if and only if the spectrum $\mathrm{spec}\big(g'(p)\big)$ of $g'(p)$ contains no eigenvalues of the form $\frac{2\pi n i}{T}$, with $n \in \mathbf{Z}$. Thus, in particular, if p is non-T-resonant for g, then p is an isolated zero of g, with index ± 1.

3. Branches of solution pairs

We quote two results, which will be usefull in the sequel.

Theorem 3.1 ([10]). *Let $N \subset \mathbf{R}^l$ be a boundaryless, smooth manifold, $g : N \longrightarrow \mathbf{R}^l$ be a continuous tangent vector field, and $\Phi : \mathbf{R} \times N \longrightarrow \mathbf{R}^l$ be such that the mapping $\Phi_1 : \mathbf{R} \times TN \longrightarrow \mathbf{R}^l$ given by $\Phi_1(t,p,v) = \Phi(t,p)$ satisfies (P1)–(P3) (with M replaced by N). Denote by Y the set of solution pairs of the following first order ODE on N:*

$$\dot{x} = g(x) + \lambda\Phi(t,x)\ ,\ \lambda \geq 0 \tag{4}$$

(defined as the set $\widehat{X}$, with N instead of TM) and let Ω be an open subset of $[0,\infty) \times C_T(N)$ such that the degree $\deg(g, \Omega \cap N)$ is well defined and nonzero (note that $\Omega \cap N$ makes sense by the identifications introduced in the preceding section). Then there exists a set $\Gamma \subset \Omega$, satisfying:

(i) $\Gamma \subset Y \setminus g^{-1}(0)$ *(i.e. Γ contains only "nontrivial" solution pairs of (4)),*
(ii) Γ *is connected,*
(iii) $\overline{\Gamma}^{[0,\infty)\times C_T(N)} \cap \big(\Omega \cap g^{-1}(0)\big) \neq \emptyset$,
(iv) Γ *is not contained in any compact subset of* Ω.

If, additionally, N is closed in $\mathbf{R}^l$ and $\Omega = [0,\infty) \times C_T(N)$, then Γ is unbounded.

Theorem 3.2 ([4]). *Let U be an open subset of TM. Then $h_{|M}$ is admissible (for the topological degree) on $U \cap M$ if and only if $\widehat{h}$ is admissible on U, and in the case of admissibility we have*

$$\deg(\widehat{h}, U) = \deg(-h_{|M}, U \cap M).$$

Now we adapt Theorem 3.1 to the case of second order ODE's on M.

Theorem 3.3. *(M need not be compact.) Let Ω be an open subset of $[0,\infty) \times C^1_T(M)$ such that $\deg(h_{|M}, \Omega \cap M)$ is well defined and nonzero. Then there exists a set $\Gamma \subset \Omega$, satisfying:*

(i) $\Gamma \subset X \setminus (h_{|M})^{-1}(0)$ *(i.e. Γ contains only "nontrivial" solution pairs of (1)),*
(ii) Γ *is connected,*
(iii) $\overline{\Gamma}^{[0,\infty)\times C^1_T(M)} \cap \big(\Omega \cap (h_{|M})^{-1}(0)\big) \neq \emptyset$,
(iv) Γ *is not contained in any compact subset of* Ω.

If, additionally, M is closed in $\mathbf{R}^k$ and $\Omega = [0,\infty) \times C^1_T(M)$, then Γ is unbounded.

Proof. (Compare the proof of Theorem 4.2 in [4].) Let $\widehat{\Omega}$ be an open subset of $[0,\infty) \times C_T(TM)$ such that $\widehat{\Omega} \cap ([0,\infty) \times C^1_T(M)) = \Omega$. Since $\widehat{\Omega} \cap M = \Omega \cap M$, by Theorem 3.2 we have

$$\deg(\widehat{h}, \widehat{\Omega} \cap TM) = \deg(-h_{|M}, \Omega \cap M) = (-1)^m \deg(h_{|M}, \Omega \cap M) \neq 0$$

(here m is the dimension of the manifold M).

Thus, by Theorem 3.1, there exists a connected set $\mathcal{G}$, contained in $\widehat{\Omega} \cap \big(\widehat{X} \setminus \widehat{h}^{-1}(0)\big)$ such that $\overline{\mathcal{G}}^{[0,\infty)\times C_T(TM)} \cap \big(\Omega \cap \widehat{h}^{-1}(0)\big) \neq \emptyset$ and $\mathcal{G}$ is not contained in any compact subset of $\widehat{\Omega}$. The proof is completed defining Γ equal to the above set $\mathcal{G}$ regarded as a subset of $[0,\infty) \times C^1_T(M)$. ∎

Corollary 3.4 (A continuation principle)**.** *(M need not be compact, only closed in $\mathbf{R}^k$.) Let Ω_0 be an open bounded subset of $C^1_T(M)$ such that:*

(i) $\deg(h_{|M}, \Omega_0 \cap M)$ *is well defined and nonzero,*

(ii) $\Big([0,1] \times \mathrm{Fr}_{C^1_T(M)} \Omega_0\Big) \cap X = \emptyset$.

Then $(\{1\} \times \Omega_0) \cap X \neq \emptyset$ (in other words, there exists a T-periodic solution of (1) for $\lambda = 1$, which is an element of Ω_0).

Proof. Define $\Omega = [0,\infty) \times \Omega_0$. Since

$$\deg(h_{|M}, \Omega \cap M) = \deg(h_{|M}, \Omega_0 \cap M) \neq 0,$$

there exists a set $\Gamma \subset \Omega \cap \big(X \setminus (h_{|M})^{-1}(0)\big)$ as in Theorem 3.3. The connected set $\overline{\Gamma} = \overline{\Gamma}^{[0,\infty)\times C^1_T(M)}$ is contained in X (as X is closed in $[0,\infty) \times C^1_T(M)$), so $\overline{\Gamma} \cap ([0,\bar{\lambda}] \times \mathrm{Fr}_{C^1_T(M)} \Omega_0) = \emptyset$. On the other hand, Ascoli's theorem together with the closedness of M implies that bounded and closed subset of X are actually compact. Then, using the connectedness of $\overline{\Gamma}$, we have $\overline{\Gamma} \cap \big(\{1\} \times \Omega_0\big) \neq \emptyset$. ∎

4. T-RESONANCY REVISITED

We will say that a point $p \in (h_{|M})^{-1}(0)$ is second order T-resonant for h, if $(p,0) \in TM$ is T-resonant for $\widehat{h}$.

Assuming h to be C^1 in a neighbourhood of $(p,0)$ in TM, we see that $p \in (h_{|M})^{-1}(0)$ is second order non-T-resonant, if and only if $(\widehat{h})'(p,0)$ does not have eigenvalues of the form $\frac{2\pi n i}{T}$ with $n \in \mathbf{Z}$.

Since

$$T_{(p,0)}TM = T_pM \times T_pM,$$

the linear operator $(\widehat{h})'(p,0) : T_{(p,0)}TM \longrightarrow T_{(p,0)}TM$ is represented by the block matrix:

$$\begin{pmatrix} 0 & I \\ D_1h(p,0) & D_2h(p,0) \end{pmatrix},$$

where I is the identity on T_pM. This immediately gives a description of the second order non-T-resonancy of p, namely the unique T-periodic solution of the linear equation (on T_pM)

$$\ddot{y} = D_2h(p,0)\dot{y} + (h_{|M})'(p)y$$

is the trivial one.

By the Schur formulas (see e.g. [5])

$$\begin{aligned}\det((\widehat{h})'(p,0)-\mu\bar{I}) &= \det\begin{pmatrix} -\mu I & I \\ D_1h(p,0) & D_2h(p,0)-\mu I \end{pmatrix} \\ &= \det\Big(-D_1h(p,0)-\mu D_2h(p,0)+\mu^2 I\Big),\end{aligned}$$

where $\bar{I}$ is the identity on $T_{(p,0)}TM$. Hence $\frac{2\pi ni}{T} \in \operatorname{spec}((\widehat{h})'(p,0))$, for some $n \in \mathbf{Z}$, if and only if

$$\det\left(D_1h(p,0)+\frac{2\pi ni}{T}D_2h(p,0)+\left(\frac{2\pi n}{T}\right)^2 I\right)=0. \tag{5}$$

In the particular case when $D_2h(p,0)=0$, the formula (5) implies that p is second order non-T-resonant if and only if

$$-\left(\frac{2n\pi}{T}\right)^2 \notin \operatorname{spec}((h_{|M})'(p)), \tag{6}$$

for any $n \in \mathbf{Z}$.

Another interesting particular case is when h is the sum of a positional vector field and a friction, that is h is of the form: $h(p,v)=g(p)-\mu v$ with $\mu>0$. In this case, formula (5) yields that $p \in g^{-1}(0)$ is second order non-T-resonant if and only if

$$-\left(\frac{2\pi n}{T}\right)^2+\mu\frac{2\pi ni}{T} \notin \operatorname{spec}(g'(p)), \tag{7}$$

for any $n \in \mathbf{Z}$.

We need a sharpened version of Lemma 3.3 from [3]

Lemma 4.1. *Let N, g be as in Theorem 3.1. Let $p \in g^{-1}(0)$ be non-T-resonant for g. Fix a compact neighbourhood $K \subset N$ of p and a number $\varepsilon_K > 0$. Then for any sufficiently small neighbourhood V of p in $C_T(N)$ there exists a real number $\delta_V > 0$ such that the set $[0,\delta_V] \times \operatorname{Fr}_{C_T(N)}V$ does not contain any solution pair of (4), whenever Φ is as in Theorem 3.1 and*

$$|\Phi(t,x)| < \varepsilon_K \quad \text{for a.a. } t \in [0,T], \text{ for all } x \in K. \tag{8}$$

Proof. Let W be an open neighbourhood of p in $C_T(N)$ such that $\{0\} \times \overline{W}^{C_T(N)}$ does not contain any solution pair of (4) different from $(0,p)$. The existence of such a set was shown in the proof of Lemma 3.3 in [3]. Define

$$W_1 = \{x \in W : \ x(t) \in \operatorname{int}_N K \text{ for all } t \in \mathbf{R}\}.$$

W_1 is an open neighbourhood of p in $C_T(N)$.

Take an open subset V of W_1, containing p. Assume by contradiction that there is no number δ_V as in the assertion. Then, there exist three sequences $\{\lambda_n\} \subset [0,\infty)$, $\{x_n\} \subset \operatorname{Fr}_{C_T(N)}V$ and $\{\Phi_n\}$, with Φ_n as in Theorem 3.1, such that:

(i) $\{\lambda_n\}$ converges to 0,
(ii) each x_n is absolutely continuous,
(iii) each Φ_n satisfies (8),
(iv) $\dot{x}_n(t) = g\big(x_n(t)\big) + \lambda_n\Phi_n\big(t,x_n(t)\big)$, for a.a. $t \in \mathbf{R}$ and all n.

By Ascoli's theorem, we may without loss of generality assume that $\{x_n\}$ converges in $C_T(N)$ to some $x_0 \in \mathrm{Fr}_{C_T(N)} V$, and $\dot{x}_0(t) = g\big(x_0(t)\big)$ for a.a. $t \in \mathbf{R}$. Since $x_0 \in \overline{W}^{C_T(N)}$ and $x_0 \neq p$, we obtain the desired contradiction. ∎

Now we specialize the above lemma for the second order case.

Corollary 4.2. *Let $p \in h^{-1}_{|M}(0)$ be second order non-T-resonant for h. Fix a compact neighbourhood K of $(p,0)$ in TM and $\varepsilon_K > 0$. Then for any sufficiently small neighbourhood V of p in $C^1_T(M)$ there exists a real number $\delta_V > 0$ such that the set $[0,\delta_V] \times \mathrm{Fr}_{C^1_T(M)} V$ does not contain any solution pair of (1), whenever $f \in U_{K,\varepsilon_K}$.*

Proof. By the definition of second order non-T-resonancy, we know that the point $(p,0) \in TM$ is non-T-resonant for $\widehat{h}$. For a fixed set V, as in the assertion, let $\widehat{V}$ be an open subset of $C_T(TM)$ such that $V = \widehat{V} \cap C^1_T(M)$. Note that if V is small enough, $\widehat{V}$ can be chosen as small as required in Lemma 4.1, so there exists the corresponding number $\delta_{\widehat{V}} > 0$. Since $\mathrm{Fr}_{C^1_T(M)} V \subset \mathrm{Fr}_{C_T(TM)} \widehat{V}$, one can define $\delta_V = \delta_{\widehat{V}}$. ∎

We are now in a position to give a multiplicity result for the following second order differential equation:

$$\ddot{x}_\pi = h(x,\dot{x}) + f(t,x,\dot{x})\ , \tag{9}$$

where h and f are as in (1).

Theorem 4.3. *(M need not be compact.) Let $p_1,\dots,p_n \in (h_{|M})^{-1}(0)$ be second order non-T-resonant for h. Then there exists an open neighbourhood $U \subset \mathcal{E}$ of 0 such that for every $f \in U$ equation (9) has at least n geometrically distinct (more precisely, with pairwise disjoint images) T-periodic solutions.*

Proof. We claim that if p is a second order non-T-resonant zero of $h_{|M}$ then, given a sufficiently small compact neighborhood C of p in M, there exists an open neighborhood U_p of 0 in $\mathcal{E}$ such that for every $f \in U_p$ equation (9) has a T-periodic solution whose image is contained in C.

To see this, take a compact subset K of TM such that $K \cap M = C$ and let V be an open subset of $C_T(TM)$ containing $(p,0)$. Shrinking V if necessary, we can assume that:

(i) the image of any $x \in V \cap C^1_T(M)$ is contained in C,
(ii) $(\widehat{h}_{|V\cap TM})^{-1}(0) = \{(p,0)\}$,
(iii) there exists $\delta > 0$ such that $[0,\delta] \times \mathrm{Fr}_{C^1_T(M)}(V \cap C^1_T(M))$ does not contain any solution pair of (1) for any $f \in U_{K,1}$.

Since

$$(V \cap C^1_T(M)) \cap M = (V \cap TM) \cap M = V \cap M\ ,$$

by (ii) and Theorem 3.2 we get

$$\begin{aligned} \deg\Big(h_{|M}, \big(V \cap C^1_T(M)\big) \cap M\Big) &= \deg\Big(h_{|M}, \big(V \cap TM\big) \cap M\Big) \\ &= (-1)^m \deg\Big(\widehat{h}, V \cap TM\Big) = \pm 1\ . \end{aligned}$$

Thus, by Corollary 3.4, one can see that $U_p = U_{K,\delta}$ fulfils our claim.

To complete the proof, for $i = 1, \dots, n$, choose pairwise disjoint compact neighborhoods C_i of p_i as above. By the first part of the proof, to each of the C_i it corresponds an open neighborhood U_{p_i} of 0 in $\mathcal{E}$ such that for every $f \in U_{p_i}$ equation (9) has a T-periodic solution whose image is contained in C_i. Hence $U = \bigcap_{i=1}^n U_{p_i}$ is an open neighborhood of 0 in $\mathcal{E}$ which has the desired properties. ∎

In order to illustrate Theorem 4.3, let us consider the second order differential equation

$$\ddot{x} = (\operatorname{grad} G)(x) - \mu\dot{x} + f(t, x, \dot{x}), \tag{10}$$

where $G : M \longrightarrow \mathbf{R}$ is a C^2 function with nondegenerate critical points (i.e. a *Morse function*), and $\mu > 0$. By standard computations, using the fact that the frictional coefficient is nonzero, one can show that the critical points of G are second order non-T-resonant zeros of $h(p, v) = (\operatorname{grad} G)(p) - \mu v$. Furthermore, from the weak Morse inequality (see e.g. [7]) it follows that $h_{|M}$ has at least

$$b(M) = \sum_{i=1}^{m} b_i(M)$$

critical points, where $b_i(M)$ denotes the i-th Betti number of M. Hence Theorem 4.3 implies that (10) has at least $b(M)$ geometrically independent T-periodic solutions, for every f belonging to a suitable neighbourhood of 0 in $\mathcal{E}$.

5. Genericity and multiplicity

In this section we will consider a particular case of equation (1),

$$\ddot{x}_\pi = g(x) + \lambda f(t, x, \dot{x}) \ , \ \lambda \geq 0, \tag{11}$$

where $g : M \longrightarrow \mathbf{R}^k$ is a C^r $(r \geq 1)$, tangent vector field. An easy but important result is the following:

Lemma 5.1. *Let $g : M \longrightarrow \mathbf{R}^k$ be a C^1 tangent vector field on M. Let $p_1, \dots, p_n$ be nondegenerate zeros of g. There exists a C_c^∞ function $\nu : M \longrightarrow \mathbf{R}$ such that $p_1, \dots, p_n$ are second order non-T-resonant zeros of $g + \rho \operatorname{grad}\nu$ for any $\rho \in (0, 1]$.*

Proof. Let us introduce the notation: $\mathcal{S} = \{-(2n\pi/T)^2 : \ n \in \mathbf{Z}\}$.

For a given nondegenerate zero $p \in M$ of g we choose $\delta_p > 0$ such that $\operatorname{spec}(g'(p) + \rho\delta_p I) \cap \mathcal{S} = \emptyset$ for every $\rho \in (0, 1]$ (here I stands for the identity on T_pM); for instance we may take $\delta_p = \min\{2\pi^2/T^2 \, , \, d/2\}$, where

$$d = \min\big\{ \, |s - e| : s \in \mathcal{S} \, , \, e \in \operatorname{spec}\big(g'(p)\big) \setminus \mathcal{S}\big\} > 0.$$

Let $\{v_1, \dots, v_k\}$ be an orthonormal basis of $\mathbf{R}^k$ such that $\{v_1, \dots, v_m\}$ spans T_pM. Define a C^∞ function $E : \mathbf{R}^k \longrightarrow \mathbf{R}$ by

$$E(q) = \frac{\delta_p}{2} \sum_{i=1}^{m} \langle q - p \, , \, v_i \rangle^2 \ , \quad q \in \mathbf{R}^k.$$

Fix an open, relatively compact neighbourhood U of p in M and a C^∞ function $\sigma : M \longrightarrow [0, 1]$ such that $\operatorname{supp}(\sigma) \subset U$ and $p \in \operatorname{int}_M(\sigma^{-1}(1))$. If we define $w : M \longrightarrow \mathbf{R}$ by $w(q) = \sigma(q) \, E(q)$, we get: $\operatorname{supp}(w) \subset U$ and, since $(\operatorname{grad} w)(p) = 0$, $(g + \rho \operatorname{grad} w)'(p) = g'(p) + \rho\delta_p I$. Hence $\operatorname{spec}\big((g + \rho \operatorname{grad} w)'(p)\big) \cap \mathcal{S} = \emptyset$ for every $\rho \in (0, 1]$. Thus, by (6), p is second order non-T-resonant for $g + \operatorname{grad} w$.

Using the procedure described above, we construct C^∞ functions $w_i : M \longrightarrow \mathbf{R}$, for each point p_i. Without loss of generality we may have $\mathrm{supp}(w_i) \cap \mathrm{supp}(w_j) = \emptyset$ for $i \neq j$. The function $\nu = \sum_{i=1}^n w_i$, fulfils the requirements. ■

Denote by $\mathfrak{X}^r(M)$, $r \geq 0$, the subspace of the Banach space $C^r(M, \mathbf{R}^k)$ made up of the C^r tangent vector fields on M. Let us consider the set $\mathfrak{X}^r_T(M)$ of all functions $g \in \mathfrak{X}^r(M)$ having the property that there exists an open set $U \subset \mathcal{E}$, containing 0, such that the equation (2) has at least $|\chi(M)|$ geometrically distinct T-periodic solutions (i.e. functions $x \in C^1_T(M)$ with $(1, x)$ beeing a solution pair for (11)), for every $f \in U$. As a consequence, whenever $g \in \mathfrak{X}^r_T(M)$, for every $f \in \mathcal{E}$ there exists a positive number $\bar\lambda$ such that for every $\lambda \in [0, \bar\lambda)$ the equation (11) has at least $|\chi(M)|$ geometrically distinct T-periodic solutions.

We will show that $\mathfrak{X}^r_T(M)$ is "generic" in $\mathfrak{X}^r(M)$.

Theorem 5.2. *The set $\mathfrak{X}^r_T(M)$, $r \geq 0$, is open and dense in $\mathfrak{X}^r(M)$.*

Proof. Let us prove first the openess of $\mathfrak{X}^r_T(M)$ in $\mathfrak{X}^r(M)$. Take $g \in \mathfrak{X}^r_T(M)$ and an open neighbourhood U of 0 in $\mathcal{E}$, such that (2) has at least $|\chi(M)|$ geometrically distinct solutions. Without loss of generality, we may assume $U = U_{K,\delta}$ for some $\delta > 0$ and a compact subset K of TM. Let $B^r_{\delta/2}(0)$ denote the ball centered at 0, of radius $\delta/2$ in the space $\mathfrak{X}^r(M)$. Then $g + B^r_{\delta/2}(0)$ is an open neighbourhood of g in $\mathfrak{X}^r(M)$ such that $g + B^r_{\delta/2}(0) \subset \mathfrak{X}^r_T(M)$.

We now prove the density. Consider first the case $r \geq 1$. It is a well known consequence of the Thom transversality theorem that the set of the C^r tangent vector fields on M whose zeros are nondegenerate is dense in $\mathfrak{X}^r(M)$ (see e.g. [9]). By the Poincaré-Hopf theorem, such vector fields have at least $|\chi(M)|$ zeros, hence, by Lemma 5.1, also the set A^r_T of the vector fields in $\mathfrak{X}^r(M)$ which have at least $|\chi(M)|$ second order non-T-resonant zeros is dense in $\mathfrak{X}^r(M)$. Hence $\mathfrak{X}^r_T(M) \supset A^r_T$ is, in turn, dense.

Let us now take $r = 0$. Since the set A^1_T is dense in $\mathfrak{X}^1(M)$, it is also dense in $\mathfrak{X}^0(M)$, so $\mathfrak{X}^0_T(M)$ is dense in $\mathfrak{X}^0(M)$ and the proof is complete. ■

Remark 5.3. *We have already observed that, given a Morse function G, equation (10) admits at least $b(M)$ geometrically distinct T-periodic solutions provided that $\mu > 0$. In the case when* $\mathrm{grad}G$ *is replaced by a function g with only nondegenerate zeros, the situation becomes slightly more complicated. As in the proof of Theorem 5.2, we have $\#\{g^{-1}(0)\} \geq |\chi(M)|$. Let $p_1, \ldots, p_{|\chi(M)|}$ be nondegenerate zeros of $g^{-1}(0)$. By (7), in order p_j to be second order non-T-resonant for $h(p, v) = g(p) - \mu v$, we must have*

$$-\left(\frac{2n\pi}{T}\right)^2 + \mu\frac{2n\pi i}{T} \notin \mathrm{spec}\big(g'(p_j)\big),$$

for any $n \in \mathbf{Z}$. Obviously this condition is fulfiled for any but a finite number of values of $\mu > 0$. Hence, given g with only nondegenerate zeros, for all but a finite number of values $\mu > 0$ there exists $U \subset \mathcal{E}$ such that the equation

$$\ddot{x}_\pi = g(x) - \mu\dot{x} + f(t, x, \dot{x}) \tag{12}$$

admits at least $|\chi(M)|$ geometrically distinct T-periodic solutions whenever $f \in U$.

Remark 5.4. *Let us fix $\mu > 0$ in (12) and consider the set $\mathfrak{X}^r_{T,\mu}(M)$ of all the functions $g \in \mathfrak{X}^r(M)$ with the property that there exists an open neighbourhood*

$U \subset \mathcal{E}$ of 0, such that (12) has at least $|\chi(M)|$ geometrically distinct T-periodic solutions for any $f \in U$. With the same argument used in the proof of Lemma 5.1 and Theorem 5.2, it can be shown that $\mathfrak{X}^r_{T,\mu}(M)$ is open and dense in $\mathfrak{X}^r(M)$.

If we restrict our attention to a less general class of vector fields, we are able to give a result sharper than Theorem 5.2. Consider the following equation:

$$\ddot{x}_\pi = (\operatorname{grad} G)(x) + f(t, x, \dot{x}), \tag{13}$$

where $G : M \longrightarrow \mathbf{R}$ is of class C^r $(r \geq 1)$.

Denote by $\mathcal{G}^r_T(M)$ the subspace of $C^r(M, \mathbf{R})$ of all the C^r functions G having the property that there exists an open set $U \subset \mathcal{E}$, containing 0, such that (13) has at least $b(M)$ geometrically distinct T-periodic solutions. We will show that $\mathcal{G}^r_T(M)$ is "generic" in $C^r(M, \mathbf{R})$.

Theorem 5.5. *The set $\mathcal{G}^r_T(M)$, $r \geq 1$, is open and dense in $C^r(M, \mathbf{R})$.*

Proof. For the openess of $\mathcal{G}^r_T(M)$ in $C^r(M, \mathbf{R})$ we proceed as in the first part of the proof of Theorem 5.2.

Let us prove the density. Assume first $r \geq 2$. It is well known (see e.g. [6]) that Morse functions of class C^r constitute a dense, open subset of $C^r(M, \mathbf{R})$. Let G be a Morse function. By the weak Morse inequality (see e.g. [7]):

$$b(M) \leq \# \left\{ (\operatorname{grad} G)^{-1}(0) \right\} .$$

Lemma 5.1 yields the density in $C^r(M, \mathbf{R})$ of the set D_T of the functions $G \in C^r(M, \mathbf{R})$ with $b(M)$ non-T-resonant zeros of grad$G$.

In the case $r = 1$, we proceed as in the last part of the proof of Theorem 5.2, showing that $\mathcal{G}^1_T(M)$ is open and dense in $C^1(M, \mathbf{R})$. ■

Let us compare, by an example, the kind of information carried by Theorem 5.2 and Theorem 5.5. For instance, in the case when M is the two-dimensional sphere S^2, we have $\left|\chi(S^2)\right| = 2 = b(S^2)$ hence the two theorems assert, respectively, that equations (2) and (13) have "generically" two T-periodic solutions for small T-periodic perturbations. On the other hand, when the manifold is the two-dimensional torus $\mathbf{T}^2$, we have $\left|\chi(\mathbf{T}^2)\right| = 0$ while $b(\mathbf{T}^2) = 4$. Hence, in this case, no useful information follows from the former theorem while the latter guarantees that on $\mathbf{T}^2$ the equation (13) has "generically" four T-periodic solutions for any T-periodic perturbations, small enough.

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