A REMARK ON THE GENERICITY OF MULTIPLICITY RESULTS FOR FORCED OSCILLATIONS ON MANIFOLDS

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1. Introduction

Let $M \subset \mathbf{R}^k$ be a boundaryless smooth manifold. In our recent work [6] the genericity of the following property has been proved. If M is compact the perturbed autonomous equation on M

$$\ddot{x}_\pi = g(x) + f(t, x, \dot{x}) \tag{1}$$

has $|\chi(M)|$ geometrically independent T-periodic solutions for any 'small' perturbation f that is T-periodic in t.

In this paper, that can be seen as a continuation of our research in [6], we want to discuss the same property, relatively to the following equation (Theorem 4.3):

$$\ddot{x}_\pi = h(x, \dot{x}) + f(t, x, \dot{x}), \tag{2}$$

where $h : TM \longrightarrow \mathbf{R}^k$ is C^r and tangent to M, and the perturbing function $f : \mathbf{R} \times TM \longrightarrow \mathbf{R}^k$ is T-periodic in t (with $T > 0$ a fixed number), tangent to M and satisfies the usual Carathéodory and admissibility conditions.

In particular, we shall prove that when M is compact, then the set of h such that (2) admits at least $|\chi(M)|$ geometrically independent T-periodic solutions for all functions f small enough, is open and dense in the set of all the C^r tangent vector fields (Corollary 4.4).

The genericity result relative to (2) does not seem to be attainable directly with the methods of [6]. In fact, we proceed in two steps: first, we obtain results in the spirit of [6] but for first order equations in the noncompact case. Secondly, noticing that every second order ODE on M is equivalent to a suitable first order equation on the tangent bundle TM, we get a genericity result for second order equations on (not necessarily compact) manifolds (Theorem 4.3) that reduces to the quoted result when M is compact.

In the sequel, we use the same terminology of [6], and refer to [5, 8] for the notions of differential topology.

2. Preliminaries and notation

Let $N \subset \mathbf{R}^l$ be a boundaryless, n-dimensional, smooth manifold. The general form of the first order ODE on N studied here is the following:

$$\dot{x} = \varphi(x) + \gamma(t, x), \tag{3}$$

where $\varphi : N \longrightarrow \mathbf{R}^l$ is C^r, tangent to N and admissible, i.e. such that $\varphi^{-1}(0)$ is compact. The perturbation $\gamma : \mathbf{R} \times N \longrightarrow \mathbf{R}^l$ is assumed to have the following properties:

(**P1**) (Carathéodory, T-periodicity in t)

- for any $p \in N$, $\gamma(\cdot, p) : \mathbf{R} \longrightarrow \mathbf{R}^l$ is measurable and T-periodic,
- for a.a. $t \in \mathbf{R}$, $\gamma(t, \cdot) : N \longrightarrow \mathbf{R}^l$ is continuous,

(**P2**) (tangency)

- for any $p \in N$ for a.a. $t \in \mathbf{R}$, $\gamma(t, p) \in T_pN$,

(**P3**) (admissibility)

- for any compact $K \subset N$ there exists a function $h_K \in L^1([0,T], \mathbf{R})$ such that for a.a. $t \in [0,T]$, for any $p \in K$,
$$|\gamma(t,p)| < h_K(t).$$

By TM we mean the tangent bundle to the manifold M, that is the subset of $\mathbf{R}^k \times \mathbf{R}^k$ given by
$$TM = \left\{(p, v) \in \mathbf{R}^k \times \mathbf{R}^k : p \in M\ ,\ v \in T_pM\right\}\ .$$
We will say that a continuous map $\varphi : \mathbf{R} \times TM \to \mathbf{R}^k$ such that $\varphi(t, q, v) \in T_qM$ for all $(t, q, v) \in \mathbf{R} \times TM$, tangent to M, though φ is not a tangent vector field on M.

In what follows, the symbol $C^1_T(M)$ will denote the metric subspace of the Banach space $\left(C^1_T(\mathbf{R}^k), \|\cdot\|_1\right)$ of all the T-periodic, C^1 functions $x : \mathbf{R} \longrightarrow M$ with the usual C^1 norm $\|\cdot\|_1$. Analogously, by $C_T(TM)$ we mean the metric space of T-periodic, continuous functions $x : \mathbf{R} \longrightarrow TM$, with the metric inherited from the Banach space $C_T(\mathbf{R}^k \times \mathbf{R}^k)$.

As in [4], we tacitly assume some natural identifications; for example we identify a point $p \in M$ with the constant function $t \mapsto p$ in $C^1_T(M)$, or a function $x \in C^1_T(M)$ with $(x, \dot{x}) \in C_T(TM)$. Also, we regard each of the above spaces as the zero-slices of the space obtained as the Cartesian product of $[0, \infty)$ and the space under consideration. In this manner, M becomes a subset of $[0, \infty) \times C^1_T(M)$ and of $[0, \infty) \times C_T(TM)$ as well, and so on. In the same spirit, by $h_{|M} : M \longrightarrow \mathbf{R}^k$ we understand the function given by $h_{|M}(p) = h(p, 0)$.

Recall that x is a solution of (2) if and only if $\dot{x}$ is absolutely continuous, and for a.a. $t \in \mathbf{R}$
$$\ddot{x}_\pi(t) := \Pi_{T_{x(t)}M}(\ddot{x}(t)) = h(x(t), \dot{x}(t)) + f(t, x(t), \dot{x}(t)),$$
where, for a fixed subspace $E \subset \mathbf{R}^k$, $\Pi_E : \mathbf{R}^k \longrightarrow E$ is the orthogonal projection of $\mathbf{R}^k$ onto E. From now on, X will denote the subset of $C^1_T(M)$ of all the periodic solutions of (2).

Equation (2) is equivalent to the following ODE on TM:
$$\dot{\xi} = \widehat{h}(\xi) + \bar{f}(t, \xi), \tag{4}$$
where:
$$\widehat{h} : TM \longrightarrow \mathbf{R}^k \times \mathbf{R}^k\ ;\ \widehat{h}(p, v) = (v, r(p, v) + h(p, v)),$$
$$\bar{f} : \mathbf{R} \times TM \longrightarrow \mathbf{R}^k \times \mathbf{R}^k\ ;\ \bar{f}(t, p, v) = \big(0, f(t, p, v)\big),$$
and $\xi(t) = (\xi_1(t)\ ,\ \xi_2(t))$, with $\xi_1(t) \in M$ and $\xi_2(t) \in T_{\xi_1(t)}M$.

The map $r : TM \longrightarrow \mathbf{R}^k$ assigns to a fixed $(q, v) \in TM$ the unique vector in $\mathbf{R}^k$ which makes $\big(v, r(q, v)\big)$ tangent to TM at (q, v) (note also that $r(q, v) \in T_qM^\perp$). In this way $\widehat{h}$ (as well as $\bar{f}$) is tangent to TM.

Consider equation (3); we say that a point $p \in \varphi^{-1}(0) \subset N$ is T-resonant for φ if (see e.g. [3])

- φ is C^1 in a neighbourhood of p,
- the linearized equation on T_pN (note that $\varphi'(p) \in \mathrm{End}(T_pN)$)
$$\dot{x} = \varphi'(p)x$$
admits nontrivial (i.e. nonzero) T-periodic solutions.

Note that p is non-T-resonant for φ if and only if the spectrum spec $\big(\varphi'(p)\big)$ of $\varphi'(p)$ contains no eigenvalues of the form $\frac{2\pi n i}{T}$, $n \in \mathbf{Z}$.

Following [6], we say that a point $p \in (h_{|M})^{-1}(0) \subset M$ is second order T-resonant for h, if $(p,0) \in TM$ is T-resonant for $\widehat{h}$. In particular, if h is C^1 in a neighbourhood of $(p,0)$ in TM and $D_2h(p,0) = 0$, the second order T-resonancy is equivalent to

$$-\left(\frac{2n\pi}{T}\right)^2 \in \mathrm{spec}\,(h_{|M})'(p) \text{ for some } n \in \mathbf{Z}.$$

As in [6], we denote by $\mathcal{F}(N)$ the topological vector space of all the functions $\gamma : \mathbf{R} \times N \longrightarrow \mathbf{R}^l$ having the properties (P1) – (P3), endowed with the topology given by the following fundamental system of neighbourhoods of 0:

$$\big\{U_{K,\varepsilon} : K \text{ is a compact subset of } N\ ,\ \varepsilon > 0\big\},$$

where

$$U_{K,\varepsilon} = \big\{\gamma \in \mathcal{F}(N) :\ \text{for a.a. } t \in [0,T],\ \text{for all } p \in K,\quad |\gamma(t,p)| < \varepsilon\big\}.$$

Furthermore, by $\mathcal{E}(M)$ we denote the topological vector space of all the functions $f : \mathbf{R} \times TM \longrightarrow \mathbf{R}^k$ with the properties as in Section 1, and with the topology inherited from $\mathcal{F}(TM) \supset \mathcal{E}(M)$.

3. Genericity of the multiplicity results for first order equations

In this section, that is devoted to first order ODE's on (not necessarily compact) boundaryless manifolds, we show that the set $\mathfrak{X}^{r,s}_T(N)$ of the admissible vector fields φ, tangent to N, and such that:

- $\deg(\varphi, N) = s$,
- the equation (3) has at least s geometrically independent T-periodic solutions for any γ in a suitably 'small' neighbourhood of 0 in $\mathcal{F}(N)$,

is open and dense (relative to the appropriate topology) in the space $\mathfrak{X}^{r,s}(N)$ of all the admissible tangent vector fields having degree equal to s.

Let us denote by $\mathfrak{X}^r(N)$, $r \geq 0$, the vector space of the C^r tangent vector fields to N endowed with the fine (Whitney) topology [5].

For the purpose of future reference, we recall that, given $\varphi \in \mathfrak{X}^r(N)$, the basis of its open neighbourhoods consists of the sets

$$\mathcal{N}^r(\varphi,\Phi,\mathcal{K},E) = \big\{\psi \in \mathfrak{X}^r(N) :\ \| D^k(\varphi\psi_i^{-1})(p) - D^k(\psi\psi_i^{-1})(p) \| < \varepsilon_i \\ \text{for all } p \in \psi_i(K_i),\ k : 0 \ldots r,\ i \in \Lambda\big\},$$

where $\Phi = \{\psi_i, U_i\}_{i\in\Lambda}$ is a locally finite set of charts on N, indexed by the index set Λ, $\mathcal{K} = \{K_i\}_{i\in\Lambda}$ is a family of compact subsets $K_i \subset U_i$ and $E = \{\varepsilon_i\}_{i\in\Lambda}$ a family of positive numbers.

Let $\mathfrak{X}^r_a(N)$ be the subset of $\mathfrak{X}^r(N)$ made up of the C^r admissible vector fields. Observe that $\mathfrak{X}^r_a$ is open, whereas in general, it is not a vector space.

We will say that $s \in \mathbf{N} \cup \{0\}$ is admissible if there exists $\varphi \in \mathfrak{X}^r_a(N)$ such that $|\deg(\varphi, N)| = s$. Given an admissible s, we denote by $\mathfrak{X}^{r,s}(N)$, the set of admissible vector fields $\varphi \in \mathfrak{X}^r_a(N)$ such that $|\deg(\varphi, N)| = s$. Obviously $\mathfrak{X}^{r,n}(N)$ is not a vector space unless N is compact. In fact, as a consequence of the Poincaré-Hopf theorem, when N is compact $s = |\chi(N)|$ is the only possible admissible number.

In the sequel, unless stated differently, s will always denote an admissible integer.

Proposition 3.1. *The set $\mathfrak{X}^{r,s}(N)$, $r \geq 0$, is open in $\mathfrak{X}^r(N)$.*

Proof. Fix a vector field $\varphi \in \mathfrak{X}^{r,s}(N)$. Take $\Phi = \{\Psi_i, U_i\}_{i\in\Lambda}$ a locally finite set of charts on N, and $\mathcal{K} = \{K_i\}_{i\in\Lambda}$ a family of compact subsets $K_i \subset U_i$, such that for some $\sigma \in \mathbf{N}$

$$\bigcup_{i\in\Lambda} K_i = N \quad \text{and} \quad \varphi^{-1}(0) \subset \bigcup_{i=1}^{\sigma} K_i.$$

Take $E = \{\varepsilon_i\}_{i\in\Lambda}$ a family of positive numbers. One sees that if ε_i for $i \notin \{1, \ldots, \sigma\}$ are small enough, then

$$\mathcal{N}^r(\varphi, \Phi, \mathcal{K}, E) \subset \mathfrak{X}^r_a(N).$$

Similarily, ε_i, with $i = 1, \ldots, \sigma$ small, imply by the homotopy property:

$$\mathcal{N}^r(\varphi, \Phi, \mathcal{K}, E) \subset \mathfrak{X}^{r,s}(N).$$

□

Remark. Note that Proposition 3.1 is false if the Whitney topology is replaced by the compact-open (in C^r) one.

Proposition 3.2. *Given a vector field $\varphi \in \mathfrak{X}^{r,s}(N)$, $r \geq 0$, and an open neighbourhood U of φ in $\mathfrak{X}^{r,s}(N)$, there exists $\psi \in U$ such that all the zeros of ψ are nondegenerate. Consequently, by the additivity of the degree, $\#\psi^{-1}(0) \geq s$.*

Proof. We recall that by the Thom transversality theorem, in case $r \geq 1$, the set of the C^r tangent vector fields on N whose zeros are nondegenerate is dense in $\mathfrak{X}^r(N)$ [5, 9]. Since $\mathfrak{X}^{r,s}(N)$ is open in $\mathfrak{X}^r(N)$, U is open in $\mathfrak{X}^r(N)$.

In case $r = 0$ it is enough to note that $\mathfrak{X}^1(N)$ is dense in $\mathfrak{X}^0(N)$ and use the argument above. □

Lemma 3.3. *Assume that $\varphi \in \mathfrak{X}^{r,s}(N)$, $r \geq 1$, has σ nondegenerate zeros $p_1, \ldots, p_\sigma$. Then given a neighbourhood U of φ in $\mathfrak{X}^{r,s}(N)$, there exists $\psi \in U$ such that $p_1, \ldots, p_\sigma$ are non-T-resonant zeros of ψ.*

Proof. Take $\mathcal{S} = \{2\pi ni/T : \ n \in \mathbf{Z}\}$. For $i = 1, \ldots, \sigma$ let $\delta_i > 0$ be such that, for every $\rho \in (0, 1]$

$$\operatorname{spec}(\varphi'(p_i) + \rho\ \delta_i\ \mathrm{Id}_{T_{p_i}N}) \cap \mathcal{S} = \emptyset. \tag{5}$$

For $i = 1, \ldots, \sigma$, let $\{v^i_1, \ldots, v^i_l\}$ be an orthonormal basis of $T_{p_i}N$ and define a smooth function $w_i : N \longrightarrow \mathbf{R}$ by

$$w_i(p) = \frac{\eta_i(p)\ \delta_i}{2}\ \sum_{k=1}^{n} \langle p - p_i\ ,\ v^i_k\rangle^2,$$

where $\eta_i : M \longrightarrow [0,1]$ is smooth with compact support and is equal to 1 in a neighbourhood of p_i. Note that supp $w_i \subset$ supp η_i and

$$(\varphi + \rho \ \operatorname{grad} w_i)'(p_i) = \varphi'(p_i) + \rho \ \delta_i \ \mathrm{Id}_{T_pN}.$$

Thus by (5), p_i is non-T-resonant for $\varphi + \rho \ \operatorname{grad} w_i$ for every $\rho \in (0,1]$.

Without loss of generality, one can assume that supp $\eta_i \cap$ supp $\eta_j = \emptyset$ for $i \neq j$, and $i, j \in \{1, \ldots, \sigma\}$. Define

$$w = \sum_{i=1}^{\sigma} w_i.$$

Taking $\psi = \varphi + \rho \ \operatorname{grad} w$, for $\rho \in (0,1]$ small enough we get the assertion in view of Proposition 3.1. □

Denote by $\mathfrak{X}_T^{r,s}(N)$ the set consisting of those vector fields $\varphi \in \mathfrak{X}^{r,s}(N)$ for which there exists an open neighborhood of U_φ of 0 in $\mathcal{F}(N)$ with the property that equation (3) admits at least s geometrically distinct T-periodic solutions whenever γ is taken in U_φ. Our main result states that such vector fields are generic within $\mathfrak{X}^{r,s}(N)$.

Theorem 3.4. *The set* $\mathfrak{X}_T^{r,s}(N)$*,* $r \geq 0$*, is open in* $\mathfrak{X}^r(N)$ *and dense in* $\mathfrak{X}^{r,s}(N)$*.*

Proof. To prove the first assertion, take $\varphi \in \mathfrak{X}_T^{r,s}(N)$ and let $U_{K,\varepsilon} \subset \mathcal{F}(N)$ be such that (3) admits at least s geometrically distinc T-periodic solutions whenever $\gamma \in U_{K,\varepsilon}$. By Proposition 3.1, take $\mathcal{N}^r(\varphi, \Phi, \mathcal{K}, E) \subset \mathfrak{X}^{r,s}(N)$. Obviously, if $\varepsilon_i < \varepsilon/2$ for all $i \in \Lambda$ such that $K_i \cap K \neq \emptyset$ then also $\mathcal{N}^r(\varphi, \Phi, \mathcal{K}, E) \subset \mathfrak{X}_T^{r,s}(N)$.

We now prove the density. Since $\mathfrak{X}^1(N)$ is dense in $\mathfrak{X}^0(N)$, without loss of generality we can assume that $r \geq 1$.

Fix $\varphi \in \mathfrak{X}_T^{r,s}(N)$ and an open neighbourhood U of φ in $\mathfrak{X}^{r,s}(N)$. By Proposition 3.2 and Lemma 3.3 there exists $\psi \in U$ such that all its zeros $p_1, \ldots, p_\sigma$ $(\sigma \geq s)$ are non-T-resonant. In the remaining part of the proof we show that necessarily $\psi \in \mathfrak{X}_T^{r,s}(N)$.

Indeed, the proof of Theorem 4.1 in [6] shows that for every p_i, $i = 1, \ldots, \sigma$, one can find a sufficiently small compact neighbourhood C_i of p_i in N such that (3) with $\gamma \in U_i$ (U_i a small neighbourhood of 0 in $\mathcal{F}(N)$) has a T-periodic solution whose image is contained in C_i. This finishes the proof. □

As we already remarked, in the case when N is compact the only possible admissible integer is $s = |\chi(N)|$. Indeed, in this case, $\mathfrak{X}^r(N) = \mathfrak{X}^{r,|\chi(N)|}(N)$, and the fine topology coincides with the C^r uniform. Hence we have

Corollary 3.5. *When* N *is compact,* $\mathfrak{X}_T^{r,|\chi(N)|}(N)$*,* $r \geq 0$*, is open and dense in* $\mathfrak{X}^r(N)$ *(with the uniform* C^r *topology).*

We stay with the case N compact and, as in [6], restrict our attention to a particular class of first order systems (3) whose leading term φ is a gradient of some C^r $(r \geq 1)$ function $G : N \longrightarrow \mathbf{R}$, i.e.:

$$\dot{x} = \operatorname{grad} G(x) + \gamma(t, x). \tag{6}$$

Denote by $\mathcal{G}_T^r(N)$ the subspace of $C^r(N, \mathbf{R})$ of all the C^r functions G having the property that there exists an open set $U_G \subset \mathcal{F}(N)$, containing 0, such that (6) has at least

$$b(N) = \sum_{i=1}^{n} b_i(N)$$

geometrically distinct T-periodic solutions. Here $b_i(N)$ denotes the i-th Betti number of N.

In view of the proof of Lemma 3.3 and Theorem 5.5 in [6], one gets

Theorem 3.6. *$\mathcal{G}_T^r(N)$ $(r \geq 1)$ is open and dense in $C^r(N, \mathbf{R})$.*

As remarked in [6], the above theorem gives a stronger result than when applying Theorem 3.4 to equation (6). For instance, if N is the two-dimensional torus $\mathbf{T}^2$, one has $b(\mathbf{T}^2) = 4$ whereas $\chi(\mathbf{T}^2) = 0$.

4. Applications to second order equations

In this section we study the genericity of the multiplicity results for second order ODE's on (not necessarily compact) boundaryless differentiable manifolds. We shall consider the second order analogouses, $\mathfrak{Y}_T^{r,s}(M)$ and $\mathfrak{Y}^{r,s}(M)$, of the spaces $\mathfrak{X}_T^{r,s}$ and $\mathfrak{X}^{r,s}$ defined in the previous section, and show that the former is open and dense in the latter one.

This result will, in particular, yield the claimed generalization of the main result of [6] (Corollary 4.4).

In what follows, we consider $N = TM \subset \mathbf{R}^l$ with $l = 2k$. We will say that $s \in \mathbf{N} \cup \{0\}$ is *second order admissible* if there exists a vector field $h : TM \to \mathbf{R}^k$ tangent to M such that $\left|\deg(h_{|M}, M)\right| = s$. As in the previous section we observe that in the case when M is compact the only possible second order admissible integer is $|\chi(M)|$.

Define the set:

$$\mathfrak{Y}^r(M) = \left\{ \widehat{h} \mid h : TM \longrightarrow \mathbf{R}^k,\ h \in C^r \text{ and tangent to } M \right\}.$$

For s second order admissible, let $\mathfrak{Y}^{r,s}(M) = \mathfrak{Y}^r(M) \cap \mathfrak{X}^{r,s}(TM)$. In other words,

$$\mathfrak{Y}^{r,s}(M) = \left\{ \widehat{h} \mid h : TM \longrightarrow \mathbf{R}^k,\ h \in C^r,\ \text{tangent to } M \text{ and } h_{|M} \in \mathfrak{X}^{r,s}(M) \right\}$$

by Lemma 3.2 in [4]. Since $\mathfrak{Y}^r(M)$ and $\mathfrak{Y}^{r,s}(M)$ are contained in $\mathfrak{X}^r(TM)$, they naturally inherit the topology from $\mathfrak{X}^r(TM)$.

Proposition 4.1. *Let $\varphi \in \mathfrak{Y}^{r,s}(M)$, $r \geq 1$, and let $U \subset \mathfrak{Y}^{r,s}(M)$ be an open set containing φ. Then there exists $\psi \in U$ such that all zeros of ψ are nondegenerate and $\#\psi^{-1}(0) \geq s$.*

Proof. Following the convention introduced above, we can write $\varphi = \widehat{h}_0$, where $h_0 : TM \longrightarrow \mathbf{R}^k$ is C^r, tangent to M and such that $h_{0|M} \in \mathfrak{X}^{r,s}(M)$. Without loss of generality we may assume that $U = \mathcal{N}^r(\varphi, \Phi, \mathcal{K}, E) \cap \mathfrak{Y}^{r,s}(M)$, for some Φ, $\mathcal{K}$ and E. Define $\widetilde{\Phi} = \{\widetilde{\psi}_i, \widetilde{U}_i\}_{i\in\Lambda}$, $\widetilde{\mathcal{K}} = \{\widetilde{K}_i\}_{i\in\Lambda}$, and $\widetilde{E} = \{\widetilde{\varepsilon}_i\}_{i\in\Lambda}$ where $\widetilde{U}_i = U_i \cap M$, $\widetilde{\psi}_i = \psi_{i|\widetilde{U}_i}$, $\widetilde{K}_i = K_i \cap M$, and $\widetilde{\varepsilon}_i = \varepsilon_i/2$. Let $\widetilde{U} = \mathcal{N}^r(h_{0|M}, \widetilde{\Phi}, \widetilde{\mathcal{K}}, \widetilde{E}) \cap \mathfrak{X}^{r,s}(M)$. By Proposition 3.2, there exists $h_1 \in \widetilde{U}$ such that all its zeros are nondegenerate and $\#h_1^{-1}(0) \geq s$.

Let $\sigma : TM \longrightarrow [0,1]$ be a smooth function such that $\sigma_{|M} = 1$. If the support of σ is a small enough neighbourhood of M, then one has that the function $h : TM \longrightarrow \mathbf{R}^k$,

$$h(p,v) = \sigma(p,v)\ h_1(p) + (1-\sigma(p,v))\ h_0(p,v)$$

satisfies

$$\widehat{h} \in \mathcal{N}^r(\widehat{h}_0, \Phi, \mathcal{K}, E).$$

It is easy to check that h is C^r, tangent to M and $h_{|M} = h_1 \in \mathfrak{X}^{r,s}(M)$.

Let $\psi = \widehat{h}$. Then $\psi \in U$ and

$$\psi^{-1}(0) = h_1^{-1}(0).$$

Thus $\#\psi^{-1}(0) \geq s$. Take $p \in \psi^{-1}(0)$. Since

$$T_{(p,0)}TM = T_pM \times T_pM,$$

the linear operator $\psi'(p,0) : T_{(p,0)}TM \longrightarrow T_{(p,0)}TM$ is represented by the block matrix:

$$\begin{pmatrix} 0 & I \\ D_1h(p,0) & D_2h(p,0) \end{pmatrix} = \begin{pmatrix} 0 & I \\ D_1h_1(p,0) & D_2h(p,0) \end{pmatrix}$$

where I is the identity on T_pM. Therefore

$$\det \psi'(p,0) = (-1)^m \det h_1'(p),$$

where m is the dimension of M. Consequently, all zeros of ψ are nondegenerate. □

We now establish a technical lemma that, in the framework of second order differential equations, plays the same role as Lemma 3.3 in the previous section.

Lemma 4.2. *Assume that $\varphi \in \mathfrak{Y}^{r,s}(M)$, $r \geq 1$, has σ nondegenerate zeros $z_1, \ldots, z_\sigma$. Then, given a neighbourhood U of φ in $\mathfrak{Y}^{r,s}(M)$, there exists $\psi \in U$ such that $z_1, \ldots, z_\sigma$ are second order non-T-resonant zeros of ψ.*

Proof. Since φ is in $\mathfrak{Y}^{r,s}(M)$, we have $\varphi = \widehat{h}_0$ for some $h_0 : TM \longrightarrow \mathbf{R}^k$ of C^r class, tangent to M and such that $h_{0|M} \in \mathfrak{X}^{r,s}(M)$ and with the property that the points $p_1, \ldots, p_\sigma$, defined by $(p_i, 0) = z_i$, $i = 1, \ldots, \sigma$ are nondegenerate zeros of $h_{0|M}$.

Exactly as in the proof of Proposition 4.1, but using Lemma 3.3 instead of Proposition 3.2 we get a vector field $\psi = \widehat{h} \in U$ with $p_1, \ldots, p_\sigma$ being (first order) non-T-resonant zeros of $h_{|M}$. Thus $z_1, \ldots, z_\sigma$ are second order non-T-resonant zeros of ψ and the result follows. □

Analogously to the space $\mathfrak{X}_T^{r,s}(N)$ introduced in Section 3, we define the space $\mathfrak{Y}_T^{r,s}(M) \subset \mathfrak{Y}^{r,s}(M)$, containing the second order fields $\widehat{h}$ made out of those h for which the equation (2) admits at least s geometrically distinct solutions whenever f belongs to an appropriate open neighbourhood of 0 in $\mathcal{E}(M)$.

We are now ready to state the second order analogue of Theorem 3.4:

Theorem 4.3. *The set $\mathfrak{Y}_T^{r,s}(M)$, $r \geq 0$, is open and dense in $\mathfrak{Y}^{r,s}(M)$.*

Proof. The proof is done exactly as the proof of Theorem 3.4, in view of Proposition 4.1 and Lemma 4.2. □

The following corollary is the desired generalization of Theorem 5.1 in [6], where the function g in (2) was assumed to depend only on the position p (and not on the speed v).

Corollary 4.4. *When M is compact, $\mathfrak{Y}_T^{r,|\chi(M)|}(M)$, $r \geq 0$, is open and dense in $\mathfrak{Y}^r(M)$.*

The techniques used above allow us also to treat a slightly different problem (compare Remark 5.1 in [6]). When M is compact and h is the sum of a given vector field with only nondegenerate zeros and a friction term, we show that (2) has at least $|\chi(M)|$ geometrically independent T-periodic solutions for almost every friction coefficient and every small enough perturbation f.

More precisely, assume M compact and let $\zeta : TM \longrightarrow \mathbf{R}^k$ be tangent to M. Let μ be a real number. Consider the equation (2) with $h(p,v) = \zeta(p,v) - \mu v$.

Proposition 4.5. *Let M and h be as above, then the following equation:*

$$\ddot{x}_\pi = \zeta(x,\dot{x}) - \mu\dot{x} + f(t,x,\dot{x}),$$

has, for any but at most denumerably many values of μ, at least $|\chi(M)|$ geometrically distinct T-periodic solutions for any f in a suitable neighbourhood of 0 in $\mathcal{E}(M)$.

Proof. Since ζ has only nondegenerate zeros, $\#\zeta_{|M}^{-1}(0) \geq |\chi(M)|$. Therefore, defining $h_\mu(p,v) = \zeta(p,v) - \mu v$, one has $\#(h_\mu)_{|M}^{-1}(0) \geq |\chi(M)|$ for any $\mu \in \mathbf{R}$.

Recall that (compare [6]) a point $(p,0) \in (h_\mu)_{|M}^{-1}(0)$ is non-T-resonant if and only if

$$\det\left(D_1\zeta(p,0) + \frac{2\pi ni}{T} D_2\zeta(p,0) - \mu I + \left(\frac{2\pi n}{T}\right)^2 I\right) \neq 0,$$

for any $n \in \mathbf{Z}$; here I denotes the identity on T_pM. Thus, arguing as in Theorem 4.3 above, the assertion follows. □

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