PLATES WITH INCOMPATIBLE PRESTRAIN

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Abstract. We study effective elastic behavior of the incompatibly prestrained thin plates, where the prestrain is independent of thickness and uniform through the plate's thickness h. We model such plates as three-dimensional elastic bodies with a prescribed pointwise stress-free state characterized by a Riemannian metric G, and seek the limiting behavior as $h \to 0$.

We first establish that when the energy per volume scales as the second power of h, the resulting Γ-limit is a Kirchhoff-type bending theory. We then show the somewhat surprising result that there exist non-immersible metrics G for whom the infimum energy (per volume) scales smaller than h^2. This implies that the minimizing sequence of deformations carries nontrivial residual three-dimensional energy but it has zero bending energy as seen from the limit Kirchhoff theory perspective. Another implication is that other asymptotic scenarios are valid in appropriate smaller scaling regimes of energy.

We characterize the metrics G with the above property, showing that the zero bending energy in the Kirchhoff limit occurs if and only if the Riemann curvatures R_{1213}, R_{1223} and R_{1212} of G vanish identically. We illustrate our findings with examples; of particular interest is an example where $G_{2\times 2}$, the two-dimensional restriction of G, is flat but the plate still exhibits the energy scaling of the Föppl - von Kármán type. Finally, we apply these results to a model of nematic glass, including a characterization of the condition when the metric is immersible, for $G = \mathrm{Id}_3 + \gamma \vec{n} \otimes \vec{n}$ given in terms of the inhomogeneous unit director field distribution $\vec{n} \in \mathbb{R}^3$.

1. Introduction

There are a number of phenomena where thin plates become prestrained in an inhomogeneous and incompatible manner. In such situations, plates develop internal stresses, deform out of plane and assume non-trivial three-dimensional shapes. Growing leaves, gels subjected to differential swelling, electrodes in electrochemical cells, edges of torn plastic sheets, are but a few examples (see [12, 15] and references therein). It has also been suggested that such incompatible prestrains may be exploited as means of actuation of micro-mechanical devices [19, 20].

The Föppl - von Kármán plate theory has been widely used in the literature to study incompatible prestrain-induced bending. It has recently been shown [15] that this theory arises as Gamma-limit of the three-dimensional energies if the prestrain is a smooth perturbation of identity that tends to zero together with the plate's thickness. However, there are situations like those of liquid crystalline solids where the prestrain is not small, and hence a proper formulation of such problems is of interest.

A possible mathematical foundation relies on a model referred to as the "incompatible" theory of elasticity. This model postulates that the three-dimensional elastic body seeks to realize a configuration with a prescribed Riemannian metric G by means of minimizing a potential (or elastic) energy E. The energy indeed measures how far a given deformation u deviates from being an orientation preserving realization of G. Note that the orientation constraint is necessary, since otherwise there always exists a Lipschitz isometric immersion of any G [8]. The infimum of E

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in absence of any forces or boundary conditions is then strictly positive for any non-flat G [14], and this points to the existence of non-zero stress at free equilibria. Of particular interest are situations where the domain occupied by the body is small in one dimension compared to the other two. In such cases, it is of interest to establish asymptotic models.

To be specific, let $\Omega \subset \mathbb{R}^2$ be an open, bounded, smooth and simply connected set. For small $h > 0$ we consider thin plates with mid-plate Ω, given by:

$$\Omega^h = \Omega \times (-\frac{h}{2}, \frac{h}{2}) = \left\{ x = (x', x_3);\ x' \in \Omega,\ |x_3| < \frac{h}{2} \right\}.$$

Let $G : \bar{\Omega} \to \mathbb{R}^{3\times 3}$ be a smooth field of symmetric positive definite matrices, so that it defines a Riemannian metric on Ω^h. In this paper we study the scenario where G is independent of the thickness as well and uniform through the thickness. Specifically, we assume:

$$G(x', x_3) = G(x') = [G_{ij}(x')]_{i,j=1\ldots 3} \qquad \forall (x', x_3) \in \Omega^h. \tag{1.1}$$

We consider the following prestrain energy functional:

$$E^h(u^h) = \frac{1}{h}\int_{\Omega^h} W(\nabla u^h A^{-1})\ \mathrm{d}x \qquad \forall u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3), \tag{1.2}$$

where A is the positive definite symmetric square root of G:

$$A = \sqrt{G},$$

while $W : \mathbb{R}^{3\times 3} \longrightarrow \bar{\mathbb{R}}_+$ is the elastic energy density. In addition to being $\mathcal{C}^2$ regular in a neighborhood of $SO(3)$, the density W is assumed to satisfy the normalization, frame indifference and nondegeneracy conditions as below:

$$\begin{aligned} \exists c > 0 \quad \forall F \in \mathbb{R}^{3\times 3} \quad \forall R \in SO(3) \qquad & W(R) = 0, \quad W(RF) = W(F), \\ & W(F) \geq c\ \mathrm{dist}^2(F, SO(3)). \end{aligned} \tag{1.3}$$

We are interested in understanding the limiting behavior of $\inf E^h$ as $h \to 0$.

Observe that by the classical work of Nash and Kuiper [22, 23], via Gromov's convex integration technique [8], any smooth metric $G_{2\times 2}$ on Ω has a $\mathcal{C}^{1,\alpha}$ (with $\alpha < 1/7$) isometric immersion in $\mathbb{R}^3$ (see also [2, 4]). However, this technique does not provide a control on the second derivatives of the immersions. On the other hand, if an isometric immersion of regularity $W^{2,2}$ exists, then h^2 is the highest possible scaling of the energy E^h (see Corollary 3.2). This scenario takes place e.g. with elliptic or hyperbolic metrics [10]. We thus focus on energy scalings of orders equal or smaller than h^2. To put this in another context, compare with the work of Le Dret and Raoult [13] on membrane theory, where the infimum energy of deformations scales as $\mathcal{O}(1)$.

We explore three issues.

First, in Part A, we establish (Theorems 2.1 and 3.1) that the Γ-limit of $h^{-2}E^h$ consists of a Kirchhoff bending energy functional (2.5) acting on the set of all $W^{2,2}$ isomertic immersions of $G_{2\times 2}$ in $\mathbb{R}^3$. The main tool is the geometric rigidity theorem of Friesecke, James and Müller [7]. In fact, these authors used their theorem to rigorously derive the nonlinear Kirchhoff bending theory as the Γ-limit of the classical nonlinear elasticity ($G = \mathrm{Id}_3$) under the assumption that E^h scales like h^2. Lewicka and Pakzad [14] extended this approach to the setting $G = G^*_{2\times 2} + e_3 \otimes e_3$. Our results further extend this to arbitrary G that satisfies (1.1).

Second, in Part B, we show that even under the seemingly restrictive assumption (1.1) one encounters other non-trivial theories. We first show (Theorems 5.3 and 5.5) that the limit of the infimum of $h^{-2}E^h$ is non-zero if and only if the three Riemann curvatures R_{1212}, R_{1213} and R_{1223} of G do not vanish identically. Therefore, there exist non-immersible metrics G that induce zero bending energy in the Kirchhoff limit. These metrics lead to non-trivial theories at energy scalings smaller than h^2; we provide examples pertaining to the scaling h^4 (Examples 6.1 through 6.5 and Theorem 7.1). Example 6.1 is particularly significant since the metric $G_{2\times 2}$ is then the standard Euclidean metric and also the limiting deformation equals identity! This case can be regarded as a generalized plane stress with Föppl - von Kármán scaling. Further, we show in Example 6.5 that existence of such residual energy is not limited to cases where $G_{2\times 2}$ is flat. In other words, there are situations when one has large incompatible pre-strain, Föppl - von Kármán or milder bending. A complete characterization of these theories remain an ongoing endeavor.

Third, in Part C, we apply our results to liquid crystal glass where metric G is of the form $\mathrm{Id}_3 + \gamma\vec{n}\otimes\vec{n}$ for a unitary director field $\vec{n}\in\mathbb{R}^3$ and a constant parameter γ. It has recently been suggested that such metrics and the resulting deformations be exploited as means of actuation of micro-mechanical devices [19, 20]. We show in Theorem 8.1 that this metric is immersible if and only if $\mathrm{curl}^T\mathrm{curl}\,(\vec{n}\otimes\vec{n}) = 0$. Further, for the general three-dimensional case we show in Theorem 8.5 that the Γ-limit energy essentially conincides with the L^2 norm (squared) of the curvature form: $(\mathrm{Id}_2 - \tilde{\gamma} n\otimes n)F_{2\times 2}(\mathrm{Id}_2 - \tilde{\gamma} n\otimes n)$, where n is the in-plane component of $\vec{n}$ and $\tilde{\gamma}$ is an explicitly given inhomogeneous parameter.

The paper is organized as follows. Part A consists of three sections. We prove the lower bound to $h^{-2}E^h$ in section 2 and the upper bound in section 3. Section 4 specializes the formulas of the bending energy to the isotropic case. Part B also consists of three sections. In section 5, we derive conditions for the scaling $E^h \sim h^2$ to be optimal. We provide examples of the non-trivial (flat or non-flat) limiting configurations in section 6. In section 7 we show a condition for optimality of the scaling $E^h \sim h^4$, in a particular case pertaining to one of the examples. The application to nematic elastomers is in section 8 that is the sole section in Part C.

Throughout the paper, we use the following notation. Given a matrix $F\in\mathbb{R}^{3\times 3}$ we denote its trace by: tr F, its transpose by: F^T, its symmetric part by: sym $F = \frac{1}{2}(F+F^T)$, and its skew part by: skew $F = F - \mathrm{sym}\,F$. We shall use the matrix norm $|F| = (\mathrm{tr}(F^TF))^{1/2}$, which is induced by the inner product $\langle F_1 : F_2\rangle = \mathrm{tr}(F_1^T F_2)$. The $k\times l$ principal minor of a matrix $F\in\mathbb{R}^{3\times 3}$ will be denoted by $F_{k\times l}$. Conversely, for a given $F_{k\times l}\in\mathbb{R}^{k\times l}$, the 3×3 matrix with principal minor equal $F_{k\times l}$ and all other entries equal to 0, will be denoted $F^*_{k\times l}$. All limits are taken as the thickness parameter h vanishes, i.e. when $h\to 0$. Finally, by C we denote any universal constant, independent of h.

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Part A: Bending limit

2. The bending energy: the lower bound

We begin by characterizing sequences of deformations for which $E^h \leq h^2$:

Theorem 2.1. *For a given sequence of deformations $u^h\in W^{1,2}(\Omega^h,\mathbb{R}^3)$ satisfying:*

$$E^h(u^h) \leq Ch^2, \tag{2.1}$$

where C is a uniform constant, there exists a sequence of translations $c^h \in \mathbb{R}^3$, such that the following properties hold for the normalised deformations $y^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$:

$$y^h(x', x_3) = u^h(x', hx_3) - c^h.$$

(i) *There exists $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ such that, up to a subsequence:*

$$y^h \to y \qquad \textit{strongly in } W^{1,2}(\Omega^1, \mathbb{R}^3).$$

(ii) *The deformation y realizes the midplate metric:*

$$(\nabla y)^T \nabla y = G_{2\times 2}. \tag{2.2}$$

Consequently, the unit normal $\vec{N}$ to the surface $y(\Omega)$ and the Cosserat vector $\vec{b}$ below have the regularity $\vec{N}, \vec{b} \in W^{1,2} \cap L^\infty(\Omega, \mathbb{R}^3)$:

$$\vec{N} = \frac{\partial_1 y \times \partial_2 y}{|\partial_1 y \times \partial_2 y|} \qquad \vec{b} = (\nabla y)(G_{2\times 2})^{-1} \left[\begin{array}{c} G_{13} \\ G_{23} \end{array} \right] + \frac{\sqrt{\det G}}{\sqrt{\det G_{2\times 2}}} \vec{N}. \tag{2.3}$$

(iii) *Define the quadratic forms:*

$$\begin{aligned} \mathcal{Q}_3(F) &= D^2 W(\mathrm{Id})(F, F), \\ \mathcal{Q}_2(x', F_{2\times 2}) &= \min\Big\{ \mathcal{Q}_3(\sqrt{G(x')}^{-1} \tilde{F} \sqrt{G(x')}^{-1});\ \tilde{F} \in \mathbb{R}^{3\times 3} \textit{ with } \tilde{F}_{2\times 2} = F_{2\times 2} \Big\}. \end{aligned} \tag{2.4}$$

The form $\mathcal{Q}_3$ is defined for all $F \in \mathbb{R}^{3\times 3}$, while $\mathcal{Q}_2(x', \cdot)$ are defined on $F_{2\times 2} \in \mathbb{R}^{2\times 2}$. Both forms $\mathcal{Q}_3$ and all $\mathcal{Q}_2$ are nonnegative definite and depend only on the symmetric parts of their arguments. We then have the lower bound:

$$\liminf_{h\to 0} \frac{1}{h^2} E^h(u^h) \geq \mathcal{I}_G(y),$$

where:

$$\mathcal{I}_G(y) = \frac{1}{24} \int_\Omega \mathcal{Q}_2\left(x', (\nabla y)^T \nabla \vec{b}\right)\, \mathrm{d}x'. \tag{2.5}$$

Remark 2.2. Consider a particular case, when the metric G in (1.2) has the structure $G = G^*_{2\times 2} + e_3 \otimes e_3$ as in [12, 14]. Then, likewise: $A = \sqrt{G} = A^*_{2\times 2} + e_3 \otimes e_3$, and $A^{-1} e_3 = G^{-1} e_3 = e_3$. From the formula (2.3) it follows that: $\vec{b} = \vec{N}$, and so the asymptotic expansion of approximate minimizers of (1.2) is: $u^h(x', x_3) \approx y(x') + x_3 \vec{N}(x')$. Also, directly from (2.4) we obtain:

$$\begin{aligned} \mathcal{Q}_2(x', F_{2\times 2}) &= \mathcal{Q}_2^0\Big(A^{-1}_{2\times 2}(x') F_{2\times 2} A^{-1}_{2\times 2}(x') \Big) \\ &\text{where } \mathcal{Q}_2^0(F_{2\times 2}) = \min\Big\{ \mathcal{Q}_3(\tilde{F});\ \tilde{F} \in \mathbb{R}^{3\times 3} \text{ with } \tilde{F}_{2\times 2} = F_{2\times 2} \Big\}. \end{aligned}$$

Therefore, the limiting functional has the form:

$$\mathcal{I}_G(y) = \frac{1}{24} \int_\Omega \mathcal{Q}_2^0\Big(A^{-1}_{2\times 2} (\nabla y)^T \nabla \vec{N} A^{-1}_{2\times 2} \Big)\, \mathrm{d}x'$$

and it depends on y only through the second fundamental form $\Pi_y = (\nabla y)^T \nabla \vec{N}$ of the deformed mid-plate $y(\Omega)$. For the isotropic density W (see (4.4)), one gets:

$$\mathcal{Q}_2^0(F_{2\times 2}) = \mathcal{Q}_{2,iso}^0(F_{2\times 2}) = \mu |\mathrm{sym} F_{2\times 2}|^2 + \frac{\lambda\mu}{\lambda + \mu} |\mathrm{tr} F_{2\times 2}|^2. \tag{2.6}$$

We see that we recover the results of [14] exactly.

Before proving Theorem 2.1, we first state the approximation lemma from [14], which is just rephrasing Theorem 10 in [7] in the present non-Euclidean elasticity context.

Lemma 2.3. *Assume (2.1). There exists matrix fields* $Q^h \in W^{1,2}(\Omega, \mathbb{R}^{3\times 3})$ *such that:*

(i) $\displaystyle \frac{1}{h}\int_{\Omega^h} |\nabla u^h(x', x_3) - Q^h(x')|^2 \, \mathrm{d}x \le C\left(h^2 + \frac{1}{h}\int_{\Omega^h} \mathrm{dist}^2(\nabla u^h A^{-1}, SO(3))\, \mathrm{d}x\right),$

(ii) $\displaystyle \int_\Omega |\nabla Q^h(x')|^2 \, \mathrm{d}x' \le C\left(1 + \frac{1}{h^3}\int_{\Omega^h} \mathrm{dist}^2(\nabla u^h A^{-1}, SO(3))\, \mathrm{d}x\right),$

where the constant C *is uniform in* h*, and depends only on the geometry of the mid-plate* Ω *and on the pre-strain tensor* $A = \sqrt{G}$*.*

Proof of Theorem 2.1.

1. By Lemma 2.3 we see that the sequence $\{Q^h\}$ is bounded in L^2, together with its derivatives. Therefore, up to a subsequence:

$$Q^h \rightharpoonup Q \qquad \text{weakly in } W^{1,2}(\Omega, \mathbb{R}^{3\times 3}). \tag{2.7}$$

Consider the rescaled deformations $y^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$ given by:

$$y^h(x', x_3) = u^h(x', hx_3) - \fint_{\Omega^h} u^h.$$

Since:

$$\int_{\Omega^1} |\nabla u^h(x', hx_3) - Q(x')|^2 \le 2\int_{\Omega^1} |\nabla u^h(x', hx_3) - Q^h|^2 + 2\int_\Omega |Q^h - Q|^2,$$

it follows by Lemma 2.3 (i) and (2.7) that:

$$\left[\begin{array}{ccc} \partial_1 y^h & \partial_2 y^h & \dfrac{\partial_3 y^h}{h} \end{array}\right] \to Q \qquad \text{strongly in } L^2(\Omega^1, \mathbb{R}^{3\times 3}). \tag{2.8}$$

In particular, the sequence $\{\nabla y^h\}$ is bounded in L^2. Since $\fint y^h = 0$, by the Poincaré inequality, a subsequence of $\{y^h\}$ converges weakly in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to some limiting field y. On the other hand, $\{\nabla y^h\}$ converge strongly because of (2.8):

$$\nabla_{tan} y^h \to Q_{3\times 2} \ \text{ and } \ \partial_3 y^h \to 0 \qquad \text{strongly in } L^2(\Omega^1).$$

Consequently, the convergence of $\{y^h\}$ is actually strong, and $y = y(x') \in W^{2,2}(\Omega, \mathbb{R}^3)$ with:

$$\nabla y = \nabla_{tan} y = Q_{3\times 2}. \tag{2.9}$$

We have thus proved (i) in Theorem 2.1.

2. Note that by Lemma 2.3 (i):

$$\begin{aligned} &\int_\Omega \mathrm{dist}^2(Q^h A^{-1}, SO(3))\, \mathrm{d}x' \\ &\quad \le \frac{C}{h}\left(\int_{\Omega^h} \mathrm{dist}^2(\nabla u^h A^{-1}, SO(3)) + \int_{\Omega^h} |\nabla u^h(x', x_3) - Q^h(x')|^2 \, \mathrm{d}x\right) \le Ch^2. \end{aligned} \tag{2.10}$$

Therefore, by (2.7):

$$QA^{-1} \in SO(3) \qquad \forall \text{a.e. } x' \in \Omega, \tag{2.11}$$

so, in particular, we obtain (2.2), and automatically:

$$\nabla y \in W^{1,2} \cap L^\infty(\Omega, \mathbb{R}^3). \tag{2.12}$$

Further, by (2.2) and using the formula $a\times(b\times c) = \langle a,c\rangle b - \langle a,b\rangle c$, one gets:

$$\begin{aligned} |\partial_1 y\times\partial_2 y|^2 &= |Ae_1\times Ae_2|^2 = \langle Ae_1, (Ae_1\times Ae_2)\times Ae_1\rangle \\ &= \langle Ae_2, \langle Ae_1, Ae_1\rangle Ae_2 - \langle Ae_1, Ae_1\rangle Ae_1\rangle \\ &= G_{11}G_{22} - G_{12}^2 = \det G_{2\times 2}. \end{aligned} \tag{2.13}$$

Hence, in view of (2.12): $\vec N\in W^{1,2}\cap L^\infty$ and, consequently, the same holds for $\vec b$.

3. We will now prove that, assuming (2.9) and (2.2), condition (2.11) is equivalent to $\vec b = Qe_3$ satisfty (2.3). Indeed, write:

$$\vec b = \alpha_1\partial_1 y + \alpha_2\partial_2 y + \alpha_3\vec N.$$

By (2.13), we obtain:

$$\det Q = \det\left[\begin{array}{ccc} \partial_1 y & \partial_2 y & \alpha_3\vec N\end{array}\right] = \alpha_3|\partial_1 y\times\partial_2 y| = \alpha_3\sqrt{\det G_{2\times 2}}.$$

Now, (2.11) is equivalent to $Q^TQ = G$ and $\det Q>0$, hence (2.11) is further equivalent to:

$$\begin{aligned} G_{13} &= \langle\vec b, \partial_1 y\rangle = \alpha_1 G_{11} + \alpha_2 G_{12} \\ G_{23} &= \langle\vec b, \partial_2 y\rangle = \alpha_1 G_{21} + \alpha_2 G_{22} \\ \sqrt{\det G} &= \det Q = \alpha_3\sqrt{\det G_{2\times 2}}, \end{aligned}$$

which yields:

$$\left[\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right] = (G_{2\times 2})^{-1}\left[\begin{array}{c} G_{13}\\ G_{23}\end{array}\right], \qquad \alpha_3 = \frac{\sqrt{\det G}}{\sqrt{\det G_{2\times 2}}},$$

exactly as claimed in (2.3).

4. We now modify the sequence $\{Q^h\}$ to another sequence $\tilde Q^h\in L^2(\Omega,\mathbb{R}^{3\times 3})$ so that:

$$R^h = \tilde Q^h A^{-1}\in SO(3) \qquad \forall a.e.\ x'\in\Omega,$$

This is done by projecting $\mathbb{P}_{SO(3)}$ onto $SO(3)$ when possible, and setting:

$$\tilde Q^h A^{-1} = \left\{\begin{array}{ll} \mathbb{P}_{SO(3)}(Q^hA^{-1}) & \text{if } Q^hA^{-1}\in\mathcal{O}_\epsilon(SO(3)) \\ \mathrm{Id} & \text{otherwise}\end{array}\right.$$

with a small $\epsilon>0$. Then, by (2.10):

$$\int_\Omega|\tilde Q^h - Q^h|^2 \leq C\int_\Omega|\tilde Q^hA^{-1} - Q^hA^{-1}|^2 \leq C\int_\Omega \mathrm{dist}^2(Q^hA^{-1}, SO(3)) \leq Ch^2. \tag{2.14}$$

In particular, by (2.7):

$$\tilde Q^h \to Q \qquad \text{strongly in } L^2(\Omega,\mathbb{R}^{3\times 3}). \tag{2.15}$$

Define the scaled strains $S^h\in L^2(\Omega^1,\mathbb{R}^{3\times 3})$ by:

$$S^h(x', x_3) = \frac{1}{h}\Big((R^h)^T\nabla u^h(x', hx_3)A^{-1} - \mathrm{Id}\Big).$$

We have, in view of Lemma 2.3 (i) and (2.14):

$$\begin{aligned} \int_{\Omega^1}|S^h|^2 &\leq \frac{C}{h^2}\int_{\Omega^1}|\nabla u^h(x', hx_3) - \tilde Q^h|^2\,\mathrm{d}x \\ &\leq \frac{C}{h^3}\int_{\Omega^h}|\nabla u^h - Q^h|^2 + \frac{C}{h^2}\int_\Omega|Q^h - \tilde Q^h|^2 \leq C, \end{aligned} \tag{2.16}$$

and hence a subsequence of $\{S^h\}$ converges:

$$S^h \rightharpoonup \bar{S} \qquad \text{weakly in } L^2(\Omega^1, \mathbb{R}^{3\times 3}). \tag{2.17}$$

5. We now derive the formula on the limiting strain $\bar{S}$. Consider the difference quotients:

$$f^{s,h}(x', x_3) = \frac{1}{h}\frac{1}{s}(y^h(x', x_3 + s) - y^h(x', x_3)) \in L^2(\Omega^1, \mathbb{R}^{3\times 3}).$$

By (2.8), it follows that:

$$f^{s,h}(x', x_3) = \frac{1}{h}\fint_0^s \partial_3 y^h(x', x_3 + t)\, \mathrm{d}t \to \vec{b}(x') \qquad \text{in } L^2(\Omega^1, \mathbb{R}^3).$$

Similarly:

$$\partial_3 f^{s,h}(x', x_3) = \frac{1}{s}\Big(h^{-1}\partial_3 y^h(x', x_3 + s) - h^{-1}\partial_3 y^h(x', x_3)\Big) \to 0 \qquad \text{strongly in } L^2(\Omega^1, \mathbb{R}^3),$$

while for $i = 1, 2$, by (2.15) and (2.17):

$$\begin{aligned}
\partial_i f^{s,h}(x', x_3) &= \frac{1}{h}\frac{1}{s}\Big(\nabla u^h(x', h(x_3 + s)) - \nabla u^h(x', hx_3)\Big) e_i \\
&= \frac{1}{s} R^h(x')\Big(S^h(x', x_3 + s) - S^h(x', x_3)\Big) Ae_i \\
&\quad \rightharpoonup \frac{1}{s} QA^{-1}\big(\bar{S}(x', x_3 + s) - \bar{S}(x', x_3)\big) Ae_i \qquad \text{weakly in } L^2(\Omega^1, \mathbb{R}^3).
\end{aligned}$$

Concluding:

$$f^{s,h} \rightharpoonup \vec{b} \qquad \text{weakly in } W^{1,2}(\Omega^1, \mathbb{R}^3),$$

and hence:

$$\forall i = 1, 2 \qquad \partial_i \vec{b}(x') = \frac{1}{s} QA^{-1}\big(\bar{S}(x', x_3 + s) - \bar{S}(x', x_3)\big) Ae_i.$$

By (2.11), QA^{-1} may be replaced by $Q^{T,-1}A$, so that:

$$\forall i = 1, 2 \qquad \bar{S}(x', x_3 + s)Ae_i = \bar{S}(x', x_3)Ae_i + sA^{-1}Q^T\partial_i\vec{b},$$

and in view of (2.9) we obtain:

$$\big(A\ \bar{S}(x', x_3 + s)A\big)_{2\times 2} = \big(A\bar{S}(x', x_3)A\big)_{2\times 2} + s(\nabla y)^T\nabla\vec{b}. \tag{2.18}$$

6. We now compute the lower bound on the rescaled energies. Define the 'good' sets:

$$\Omega_h^1 = \{(x', x_3) \in \Omega^1;\ |S^h(x', x_3)|^2 \leq \frac{1}{h}\}.$$

In view of (2.16), it follows the convergence of characteristic functions:

$$\chi_h = \chi_{\Omega_h^1} \to 1 \qquad \text{strongly in } L^1(\Omega^1).$$

and therefore, by (2.17):

$$\chi_h S^h \rightharpoonup \bar{S} \qquad \text{weakly in } L^2(\Omega^1, \mathbb{R}^{3\times 3}). \tag{2.19}$$

For small h, we may Taylor expand W on the 'good' sets, using the definition of S^h:

$$\begin{aligned}
\forall (x', x_3) \in \Omega_h^1 \qquad \frac{1}{h^2} W\Big(\nabla u^h(x', hx_3)A^{-1}\Big) &= \frac{1}{h^2} W(\mathrm{Id} + hS^h(x', x_3)) \\
&= \frac{1}{2}\mathcal{Q}_3(S^h(x', x_3)) + o(|S^h|^2).
\end{aligned}$$

By (2.19), we now obtain:

$$\begin{aligned}\liminf_{h\to 0}\frac{1}{h^2}E^h(u^h) &\geq \liminf_{h\to 0}\frac{1}{h^2}\int_{\Omega_h^1} W\left(\nabla u^h(x',hx_3)A^{-1}\right)\,\mathrm{d}x\\ &= \liminf_{h\to 0}\frac{1}{2}\int_{\Omega^1}\mathcal{Q}_3\left(\chi_h S^h(x',x_3)\right)\geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_3\left(\bar{S}\right).\end{aligned}\tag{2.20}$$

Since the quadratic form $\mathcal{Q}_3$ is nonnegative definite, we obtain:

$$\begin{aligned}\frac{1}{2}\int_{\Omega^1}\mathcal{Q}_3\left(\bar{S}\right) &\geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_2\left((A\;\bar{S}(x',x_3)A)_{2\times 2}\right)\\ &= \frac{1}{2}\int_{\Omega}\mathcal{Q}_2\left((A\;\bar{S}(x',0)A)_{2\times 2}\right)\,\mathrm{d}x' + \frac{1}{2}\left(\int_{-1/2}^{1/2}s^2\,\mathrm{d}s\right)\int_{\Omega}\mathcal{Q}_2\left(\nabla y)^T\nabla\vec{b}\right)\,\mathrm{d}x'\\ &\geq \frac{1}{24}\int_{\Omega}\mathcal{Q}_2\left(\nabla y)^T\nabla\vec{b}\right)\,\mathrm{d}x' = \mathcal{I}_G(y),\end{aligned}$$

where we used (2.18). In view of (2.20), the proof is complete. ■

3. The bending energy: recovery sequence and the upper bound

In this section we prove that the lower bound in Theorem 2.1 is optimal, in the following sense:

Theorem 3.1. *For every isometric immersion $y\in W^{2,2}(\Omega,\mathbb{R}^3)$ of the metric $G_{2\times 2}$ as in (2.2), there exists a sequence of 'recovery deformations' $u^h\in W^{1,2}(\Omega^h,\mathbb{R}^3)$, such that:*

(i) *The rescaled sequence $y^h(x',x_3)=u^h(x',hx_3)$ converges in $W^{1,2}(\Omega^1,\mathbb{R}^3)$ to y.*
(ii) *One has:*

$$\lim_{h\to 0}\frac{1}{h^2}E^h(u^h)=\mathcal{I}_G(y),$$

where the Cosserat vector $\vec{b}$ in the definition (2.5) of the functional $\mathcal{I}_G$ is derived by (2.3).

It immediately follows that:

Corollary 3.2. *Existence of a $W^{2,2}$ regular isometric immersion of the Riemannian metric $G_{2\times 2}$ on Ω in $\mathbb{R}^3$ is equivalent to the upper bound on the energy scaling at minimizers:*

$$\exists C>0\qquad \inf_{u\in W^{1,2}(\Omega^h,\mathbb{R}^3)}E^h(u)\leq Ch^2.$$

Corollary 3.3. *The limiting functional $\mathcal{I}_G$ attains its minimum.*

Proof. Let $\{y_n\}_{n=1}^\infty$ be a minimizing sequence of $\mathcal{I}_G$. By Theorem 3.1, there exists sequences $u_n^h\in W^{1,2}(\Omega^h,\mathbb{R}^3)$ such that: $\lim_{h\to 0}u_n^h(x',hx_3)=y_n$ in $W^{1,2}(\Omega^1,\mathbb{R}^3)$ and $\lim_{h\to 0}\frac{1}{h^2}E^h(u_n^h)=\mathcal{I}_G(y_n)$, for every n. Taking $u^h=u_n^{h(n)}$ for a sequence $h(n)$ converging to 0 as $n\to\infty$ sufficiently fast, we obtain: $E^h(u^h)\leq Ch^2$. Therefore, by Theorem 2.1 there exists a limiting deformation $y\in W^{2,2}(\Omega,\mathbb{R}^3)$ so that:

$$\mathcal{I}_G(y)\leq\liminf_{h\to 0}\frac{1}{h^2}E^h(u^h)=\lim_{n\to\infty}\mathcal{I}_G(y_n)=\inf\mathcal{I}_G,$$

which achieves that y is a minimizer of $\mathcal{I}_G$. ■

Before proving Theorem 3.1, recall that:

$$\forall F_{2\times 2} \in \mathbb{R}^{2\times 2}_{sym} \qquad \mathcal{Q}_2(x', F_{2\times 2}) = \min\Big\{\mathcal{Q}_3(A^{-1}\tilde{F}A^{-1});\ \tilde{F}\in\mathbb{R}^{3\times 3},\ \tilde{F}_{2\times 2} = F_{2\times 2}\Big\} \\ = \min\big\{\mathcal{Q}_3\big(A^{-1}(F^*_{2\times 2} + \mathrm{sym}(c\otimes e_3))A^{-1}\big);\ c\in\mathbb{R}^3\big\}. \tag{3.1}$$

In what follows, by:

$$c(x', F_{2\times 2})$$

we will denote the unique minimizer of the problem in (3.1).

Proof of Theorem 3.1.

1. Let $y \in W^{2,2}(\Omega,\mathbb{R}^3)$ satisfy (2.2). Define the Cosserat vector field $\vec{b} \in W^{1,2}\cap L^\infty(\Omega,\mathbb{R}^3)$ according to (2.3) and let:

$$Q = \Big[\ \partial_1 y \quad \partial_2 y \quad \vec{b}\ \Big] \in W^{1,2}\cap L^\infty(\Omega,\mathbb{R}^{3\times 3}).$$

By Step 2 in the proof of Theorem 2.1, it follows that:

$$QA^{-1} \in SO(3) \qquad \forall \text{a.e. } x'\in\Omega. \tag{3.2}$$

Define the limiting warping field $\vec{d} \in L^2(\Omega,\mathbb{R}^3)$:

$$\vec{d}(x') = Q^{T,-1}\left(c\big(x', (\nabla y)^T\nabla\vec{b}\big) - \frac{1}{2}\nabla|\vec{b}|^2\right). \tag{3.3}$$

Let $\{d^h\}$ be a approximating sequence in $W^{1,\infty}(\Omega,\mathbb{R}^3)$, satisfying:

$$d^h \to \vec{d} \quad \text{strongly in } L^2(\Omega,\mathbb{R}^3), \qquad \text{and} \qquad h\|d^h\|_{W^{1,\infty}} \to 0. \tag{3.4}$$

Note that such sequence can always be derived by reparametrizing (slowing down) a sequence of smooth approximations of $\vec{d}$. Similiarly, consider the approximations $y^h \in W^{2,\infty}(\Omega,\mathbb{R}^3)$ and $\vec{b}^h \in W^{1,\infty}(\Omega,\mathbb{R}^3)$, with the following properties:

$$\begin{aligned} & y^h \to y \quad \text{strongly in } W^{2,2}(\Omega,\mathbb{R}^3), \quad \text{and} \quad \vec{b}^h \to \vec{b} \quad \text{strongly in } W^{1,2}(\Omega,\mathbb{R}^3) \\ & h\left(\|y^h\|_{W^{2,\infty}} + \|\vec{b}^h\|_{W^{1,\infty}}\right) \leq \epsilon \\ & \frac{1}{h^2}|\Omega\setminus\Omega_h| \to 0, \quad \text{where } \Omega_h = \Big\{x'\in\Omega;\ y^h(x') = y(x') \text{ and } \vec{b}^h(x') = \vec{b}(x')\Big\} \end{aligned} \tag{3.5}$$

for some small $\epsilon > 0$. Existence of approximations with the claimed properties follows by partition of unity and truncation arguments, as a special case of the Lusin-type result for Sobolev functions in [18] (see also Proposition 2 in [7]).

We now define $u^h \in W^{1,\infty}(\Omega^h,\mathbb{R}^3)$ by:

$$u^h(x', x_3) = y^h(x') + x_3\vec{b}^h(x') + \frac{x_3^2}{2}d^h(x').$$

Consequently, the rescalings $y^h \in W^{1,\infty}(\Omega^1,\mathbb{R}^3)$ are:

$$y^h(x', x_3) = y^h(x') + hx_3\vec{b}^h(x') + \frac{h^2}{2}x_3^2 d^h(x'),$$

and therefore in view of (3.4) and (3.5), Theorem 3.1 (i) follows.:

2. Define the matrix fields:

$$Q^h(x') = \Big[\ \partial_1 y^h \quad \partial_2 y^h \quad \vec{b}^h\ \Big], \quad B^h(x') = \Big[\ \partial_1\vec{b}^h \quad \partial_2\vec{b}^h \quad d^h\ \Big], \quad D^h(x') = \big[\ \partial_1 d^h \quad \partial_2 d^h \quad 0\ \big],$$

so that:

$$\nabla u^h(x',x_3) = Q^h(x') + x_3 B^h(x') + \frac{x_3^2}{2}D^h(x') \qquad \forall (x',x_3)\in \Omega^h.$$

Since $Q^h = Q$ in the set Ω_h, then by (3.2) and the bound on the Lipschitz constants of y^h and $\vec b^h$ in (3.5), we obtain:

$$\mathrm{dist}(Q^hA^{-1}, SO(3)) \leq \frac{C}{h}\mathrm{dist}(x',\Omega_h) \leq \frac{C}{h}|\Omega\setminus\Omega_h|^{1/2}. \tag{3.6}$$

The last bound above can be easily obtained by noting that if $B_r(x')\subset \Omega\setminus\Omega_h$ then $\pi r^2 \leq |\Omega\setminus\Omega_h|$, which implies $r\leq C|\Omega\setminus\Omega_h|^{1/2}$. For x' close to the boundary of Ω one needs to slightly refine the argument using smoothness of $\partial\Omega$.

Consequently, by (3.6) and (3.5), it follows that for all h sufficiently small:

$$\begin{aligned}\mathrm{dist}\Big(\nabla u^h(x',hx_3)A^{-1}, SO(3)\Big) &\leq \mathrm{dist}(Q^hA^{-1},SO(3)) + h\|B^h\|_{L^\infty} + h^2\|D^h\|_{L^\infty}\\ &\leq \frac{C}{h}|\Omega\setminus\Omega_h|^{1/2} + Ch(\|\nabla\vec b^h\|_{L^\infty} + \|d^h\|_{L^\infty}) + Ch^2\|\nabla d^h\|_{L^\infty} \leq \epsilon_0,\end{aligned}$$

where ϵ_0 is such that the energy density W is bounded and $\mathcal{C}^2$ regular in the neighbourhood $\mathcal{O}_{\epsilon_0}(SO(3))$. Taylor expanding W at the given rotation in (3.2), we compute:

$$\begin{aligned}&\frac{1}{h^2}\int_{\Omega_h\times(-1/2,1/2)} W\Big(\nabla u^h(x',hx_3)A^{-1}\Big)\\ &= \frac{1}{h^2}\int_{\Omega_h\times(-1/2,1/2)} W\left((Q(x') + hx_3B^h(x') + h^2\frac{x_3^2}{2}D^h(x'))A^{-1}\right)\,\mathrm{d}x\\ &= \frac{1}{2}\int_{\Omega_h\times(-1/2,1/2)} D^2W(Q(x')A^{-1})\left((x_3B^h(x') + h\frac{x_3^2}{2}D^h(x'))A^{-1}\right)^{\otimes 2} + \mathcal{O}(h)\,\mathrm{d}x.\end{aligned}$$

Also, by (3.5):

$$\frac{1}{h^2}\int_{(\Omega\setminus\Omega_h)\times(-1/2,1/2)} W\Big(\nabla u^h(x',hx_3)A^{-1}\Big) \leq \frac{C}{h^2}|\Omega\setminus\Omega_h| \quad \to 0.$$

Hence:

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h^2}E^h(u^h) &= \lim_{h\to0}\frac{1}{h^2}\int_{\Omega_h\times(-1/2,1/2)} W\Big(\nabla u^h(x',hx_3)A^{-1}\Big)\\ &= \lim_{h\to0}\frac{1}{2}\int_{\Omega_h\times(-1/2,1/2)} D^2W(QA^{-1})\left((x_3B^h(x') + h\frac{x_3^2}{2}D^h(x'))A^{-1}\right)^{\otimes 2}\,\mathrm{d}x\\ &= \lim_{h\to0}\frac{1}{2}\int_{-1/2}^{1/2}\int_{\Omega_h} x_3^2D^2W(QA^{-1})\Big(B^h(x')A^{-1}\Big)^{\otimes2}\,\mathrm{d}x'\,\mathrm{d}x_3\\ &= \lim_{h\to0}\frac{1}{24}\int_{\Omega_h}\mathcal{Q}_3\left((QA^{-1})^TB^h(x')A^{-1}\right)\\ &= \frac{1}{24}\int_\Omega \mathcal{Q}_3\left(A^{-1}Q^TBA^{-1}\right),\end{aligned}\tag{3.7}$$

where we have used the last convergence in (3.5), the frame invariance of the density function W resulting in: $D^2W(R)(F,F) = D^2W(\mathrm{Id})(R^TF,R^TF) = \mathcal{Q}_3(R^TF)$ valid for all $R\in SO(3)$, and the following convergence:

$$B^h \to B(x') = \left[\begin{array}{ccc} \partial_1 \vec{b} & \partial_2 \vec{b} & \vec{d} \end{array} \right] \qquad \text{strongly in } L^2(\Omega, \mathbb{R}^{3\times 3}).$$

Now, note that by(3.3):

$$\begin{aligned} \mathrm{sym}(Q^T B(x')) &= \mathrm{sym}((\nabla y)^T \nabla \vec{b}) + \mathrm{sym}(e_3 \otimes \left[\begin{array}{c} (\nabla y)^T \vec{d} + \frac{1}{2}\nabla|\vec{b}|^2 \\ \langle \vec{b}, \vec{d}\rangle \end{array} \right]) \\ &= \mathrm{sym}((\nabla y)^T \nabla \vec{b}) + \mathrm{sym}\Big(e_3 \otimes c(x', (\nabla y)^T \nabla \vec{b})\Big). \end{aligned}$$

Therefore, (3.7) becomes:

$$\lim_{h\to 0} \frac{1}{h^2} E^h(u^h) = \frac{1}{24} \int_\Omega \mathcal{Q}_2 \left((\nabla y)^T \nabla \vec{b} \right) \, \mathrm{d}x',$$

achieving the proof of Theorem 3.1. ■

4. The effective density $\mathcal{Q}_2$ and the case of W isotropic

In this section, we further study the 2d functional (2.5) and the inhomogeneous effective energy measure in (3.1). By $L_3 : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ we denote the linear map with the property that:

$$\mathcal{Q}_3(F) = \langle L_3(F) : F\rangle \quad \text{ and } \quad \langle L_3(F) : \tilde{F}\rangle = \langle L_3(\tilde{F}) : F\rangle \qquad \forall F, \tilde{F} \in \mathbb{R}^{3\times 3}.$$

Note that by frame invariace of W in (1.3) one has: $L_3(F) = L_3(\mathrm{sym}F)$ and $\mathrm{skew}(L_3(F)) = 0$.

Lemma 4.1. *Define the matrix field* $M_A : \Omega \to \mathbb{R}^{3\times 3}$ *by:*

$$\forall i : 1 \ldots 3 \qquad M_A e_i = L_3(e_i \otimes A^{-1}e_3)A^{-1}e_3.$$

Then the unique minimizer $c_0 = c(x', F_{2\times 2})$ *in (3.1) is given by:*

$$A^{-1}c_0 = -M_A^{-1} L_3(A^{-1}F^*_{2\times 2}A^{-1})A^{-1}e_3. \tag{4.1}$$

Consequently:

$$\mathcal{Q}_2(F_{2\times 2}) = \mathcal{Q}_3(A^{-1}F^*_{2\times 2}A^{-1}) - \Big\langle M_A^{-1} L_3(A^{-1}F^*_{2\times 2}A^{-1})A^{-1}e_3, L_3(A^{-1}F^*_{2\times 2}A^{-1})A^{-1}e_3\Big\rangle, \tag{4.2}$$

Proof. For $i : 1..3$ we have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}c_i} \mathcal{Q}_3(A^{-1}(F^*_{2\times 2} + c\otimes e_3)A^{-1}) &= 2\Big\langle L_3(A^{-1}(F^*_{2\times 2} + c\otimes e_3)A^{-1}) : A^{-1}e_i \otimes A^{-1}e_3\Big\rangle \\ &= 2\Big\langle A^{-1}L_3\left(A^{-1}(F^*_{2\times 2} + c\otimes e_3)A^{-1}\right)A^{-1} : e_i \otimes e_3\Big\rangle. \end{aligned}$$

Therefore, at the miminizer c_0 we have:

$$\begin{aligned} \nabla_c \mathcal{Q}_3(A^{-1}(F^*_{2\times 2} + c_0\otimes e_3)A^{-1}) &= 2A^{-1}L_3\left(A^{-1}(F^*_{2\times 2} + c_0\otimes e_3)A^{-1}\right)A^{-1}e_3 \\ &= 2A^{-1}L_3(A^{-1}F^*_{2\times 2}A^{-1} + A^{-1}c_0 \otimes A^{-1}e_3)A^{-1}e_3 = 0, \end{aligned}$$

which is equivalent to:

$$-L_3(A^{-1}F^*_{2\times 2}A^{-1})A^{-1}e_3 = L_3(A^{-1}c_0 \otimes A^{-1}e_3)A^{-1}e_3 = M_A A^{-1}c_0,$$

and consequently to (4.1). Then:

$$
\begin{aligned}
\mathcal{Q}_2(F_{2\times 2}) &= \mathcal{Q}_3(A^{-1}F^*_{2\times 2}A^{-1} + A^{-1}c_0 \otimes A^{-1}e_3) \\
&= \Big\langle L_3(A^{-1}F^*_{2\times 2}A^{-1}) + L_3(A^{-1}c_0 \otimes A^{-1}e_3) : A^{-1}F^*_{2\times 2}A^{-1}\Big\rangle \\
&= \Big\langle L_3(A^{-1}F^*_{2\times 2}A^{-1}) : A^{-1}F^*_{2\times 2}A^{-1} + A^{-1}c_0 \otimes A^{-1}e_3\Big\rangle \\
&= \mathcal{Q}_3(A^{-1}F^*_{2\times 2}A^{-1}) - \Big\langle L_3(A^{-1}F^*_{2\times 2}A^{-1}) : M_A^{-1}L_3(A^{-1}F^*_{2\times 2}A^{-1})(A^{-1}e_3 \otimes A^{-1}e_3)\Big\rangle,
\end{aligned} \tag{4.3}
$$

which proves (4.2). ■

We now assume that the energy density W is isotropic, i.e.:

$$\forall F \in \mathbb{R}^{3\times 3} \quad \forall R \in SO(3) \qquad W(RF) = W(F).$$

It is known [9] (see also [7] and Appendix A in [1]) that $\mathcal{Q}_3$ is then given in terms of the Lamé coefficients λ, μ:

$$\mathcal{Q}_3(F) = \mu|\mathrm{sym}F|^2 + \lambda|\mathrm{tr}F|^2, \tag{4.4}$$

and so we also have:

$$L_3(F) = \mu\ \mathrm{sym}F + \lambda(\mathrm{tr}F)\ \mathrm{Id}. \tag{4.5}$$

Lemma 4.2. *Assume that W is isotropic, so that (4.4) holds. Then:*

$$M_A = \frac{\mu}{2}|A^{-1}e_3|^2\mathrm{Id} + (\lambda + \frac{\mu}{2})(A^{-1}e_3 \otimes A^{-1}e_3) \tag{4.6}$$

and, denoting $D = A^{-1}F^*_{2\times 2}A^{-1}$ and $d = A^{-1}e_3$, we have:

$$\forall F_{2\times 2} \in \mathbb{R}^{2\times 2}_{sym} \qquad \mathcal{Q}_2(x', F_{2\times 2}) = \mu\left(|D|^2 - 2\frac{|Dd|^2}{|d|^2} + \frac{\langle Dd, d\rangle^2}{|d|^4}\right) + \frac{\lambda\mu}{\lambda+\mu}\left(\mathrm{tr}D - \frac{\langle Dd, d\rangle}{|d|^2}\right)^2. \tag{4.7}$$

Proof. By (4.5), we obtain:

$$M_A e_i = L_3(e_i \otimes d)d = (\lambda + \frac{\mu}{2})\langle d, e_i\rangle d + \frac{\mu}{2}|d|^2 e_i$$

which gives (4.6). It is easy to check directly the following general formula:

$$(\alpha\ \mathrm{Id} + a \otimes b)^{-1} = \frac{1}{\alpha}\mathrm{Id} - \frac{1}{\alpha(\alpha + \langle a, b\rangle)} a \otimes b.$$

Applying it to $\alpha = \frac{\mu}{2}|d|^2$ and $a = (\lambda + \frac{\mu}{2})d$ and $b = d$, we get:

$$M_A^{-1} = \frac{2}{\mu}\frac{1}{|d|^2}\mathrm{Id} - \frac{2\lambda + \mu}{\mu(\lambda+\mu)}\frac{1}{|d|^4}(d \otimes d).$$

Therefore:

$$\langle M_A^{-1}L_3(D)d, L_3(D)d\rangle = \frac{\lambda^2}{\lambda+\mu}(\mathrm{tr}D)^2 + 2\frac{\lambda\mu}{\lambda+\mu}(\mathrm{tr}D)\frac{\langle Dd, d\rangle}{|d|^2} + 2\mu\frac{|Dd|^2}{|d|^2} - \frac{(2\lambda+\mu)\mu}{\lambda+\mu}\frac{\langle Dd, d\rangle^2}{|d|^4}.$$

Concluding:

$$\mathcal{Q}_2(x', F_{2\times 2}) = \frac{\lambda\mu}{\lambda+\mu}(\mathrm{tr}D)^2 + \mu|D|^2 - 2\frac{\lambda\mu}{\lambda+\mu}(\mathrm{tr}D)\frac{\langle Dd, d\rangle}{|d|^2} - 2\mu\frac{|Dd|^2}{|d|^2} + \frac{(2\lambda+\mu)\mu}{\lambda+\mu}\frac{\langle Dd, d\rangle^2}{|d|^4}$$

which yields (4.7). ■

Theorem 4.3. *Assume that W is isotropic, so that (4.4) holds. Then:*

(4.8)
$$\forall F_{2\times 2}\in\mathbb{R}^{2\times 2}_{sym}\qquad \mathcal{Q}_2(x',F_{2\times 2}) = \mathcal{Q}^0_{2,iso}\left(\sqrt{G_{2\times 2}}^{-1}F_{2\times 2}\sqrt{G_{2\times 2}}^{-1}\right)$$
$$= \mu\left|\sqrt{G_{2\times 2}}^{-1}F_{2\times 2}\sqrt{G_{2\times 2}}^{-1}\right|^2 + \frac{\lambda\mu}{\lambda+\mu}|\mathrm{tr}\left(\sqrt{G_{2\times 2}}^{-1}F_{2\times 2}\sqrt{G_{2\times 2}}^{-1}\right)|^2$$

Proof. Given $v\in\mathbb{R}^3$, we denote $v_{tan}=(v_1,v_2)^T\in\mathbb{R}^2$. As in the proof of Theorem 5.3, given $F\in\mathbb{R}^{3\times 3}$, by $F_{tan}\in\mathbb{R}^{2\times 2}$ we denote the principal 2×2 minor of F, and we let $F_{cross}=(Fe_3)_{tan}=(F_{13},F_{23})^T\in\mathbb{R}^2$. We now use the notation of Lemma 4.2 and identify the terms in (4.7). Call $P=G^{-1}$. Then:

$$\begin{aligned}
|D|^2 &= \left\langle PF^*_{2\times 2}P : F^*_{2\times 2}\right\rangle = \left\langle (PF^*_{2\times 2}P)_{tan} : F_{2\times 2}\right\rangle = \langle P_{tan}F_{2\times 2}P_{tan} : F_{2\times 2}\rangle\\
|Dd|^2 &= \left\langle PF^*_{2\times 2}Pe_3, F^*_{2\times 2}Pe_3\right\rangle = \left\langle (PF^*_{2\times 2}Pe_3)_{tan}, F_{2\times 2}P_{cross}\right\rangle = \langle P_{tan}F_{2\times 2}P_{cross}, F_{2\times 2}P_{cross}\rangle\\
\langle Dd, d\rangle &= \langle PF^*_{2\times 2}Pe_3, e_3\rangle = \langle FP_{cross}, P_{cross}\rangle\\
|d|^2 &= \langle Pe_3, e_3\rangle = P_{33}\\
\mathrm{tr}D &= \mathrm{tr}(PF^*_{2\times 2}) = \mathrm{tr}(P_{tan}F_{2\times 2}).
\end{aligned}$$

Hence, (4.7) becomes:

(4.9)
$$\begin{aligned}
\mathcal{Q}_2(x',F_{2\times 2}) =&\mu\left(\langle P_{tan}F_{2\times 2}P_{tan} : F_{2\times 2}\rangle - 2\frac{\langle P_{tan}F_{2\times 2}P_{cross}, F_{2\times 2}P_{cross}\rangle}{P_{33}} + \frac{\langle FP_{cross},P_{cross}\rangle^2}{(P_{33})^2}\right)\\
&+\frac{\lambda\mu}{\lambda+\mu}\left(\mathrm{tr}(P_{tan}F_{2\times 2}) - \frac{\langle FP_{cross},P_{cross}\rangle}{P_{33}}\right)^2.
\end{aligned}$$

We now identify the terms in the right hand side of (4.8), using the formulas (5.9) and (5.10):

$$\begin{aligned}
|\sqrt{G_{tan}}^{-1}F_{2\times 2}\sqrt{G_{tan}}^{-1}|^2 &= \langle (G_{tan})^{-1}F_{2\times 2}(G_{tan})^{-1} : F_{2\times 2}\rangle\\
&= \langle (P_{tan} - \frac{1}{P_{33}}P_{cross}\otimes P_{cross})F_{2\times 2}(P_{tan} - \frac{1}{P_{33}}P_{cross}\otimes P_{cross}) : F_{2\times 2}\rangle\\
&= \langle P_{tan}F_{2\times 2}P_{tan} : F_{2\times 2}\rangle - \frac{2}{P_{33}}\langle (P_{cross}\otimes P_{cross})FP_{tan} : F_{2\times 2}\rangle\\
&\quad + \frac{1}{(P_{33})^2}\langle (P_{cross}\otimes P_{cross})F(P_{cross}\otimes P_{cross}) : F\rangle\\
&= \langle P_{tan}F_{2\times 2}P_{tan} : F_{2\times 2}\rangle - 2\frac{\langle P_{tan}F_{2\times 2}P_{cross}, F_{2\times 2}P_{cross}\rangle}{P_{33}} + \frac{\langle FP_{cross},P_{cross}\rangle^2}{(P_{33})^2},\\
\mathrm{tr}(\sqrt{G_{tan}}^{-1}F_{2\times 2}\sqrt{G_{tan}}^{-1}) &= \mathrm{tr}\left((G_{tan})^{-1}F_{2\times 2}\right) = \mathrm{tr}\left((P_{tan} - \frac{1}{P_{33}}P_{cross}\otimes P_{cross})F_{2\times 2}\right)\\
&= \mathrm{tr}(P_{tan}F_{2\times 2}) - \frac{1}{P_{33}}\langle FP_{cross},P_{cross}\rangle.
\end{aligned}$$

The equality in (4.8) follows directly by (4.9). ■

Remark 4.4. When $G = G_{2\times 2}^* + e_3 \otimes e_3$ then $d = e_3$ and $Dd = De_3 = 0$, so (4.7) directly becomes:

$$\mathcal{Q}_2(x', F_{2\times 2}) = \mu |D|^2 + \frac{\lambda\mu}{\lambda+\mu}|\mathrm{tr} D|^2, \tag{4.10}$$

which is consistent with (2.6).

Remark 4.5. Call $C(x') = G^{-1}(x')F_{2\times 2}^*$ and note that:

$$\begin{aligned} &\mathrm{tr}\ D = \mathrm{tr}\ C, \quad |D|^2 = \mathrm{tr}\left(C^2\right) \\ &|Dd|^2 = \langle C^2 G^{-1} e_3, e_3\rangle, \quad |d|^2 = \langle G^{-1}e_3, e_3\rangle, \quad \langle Dd, d\rangle = \langle CG^{-1}e_3, e_3\rangle \end{aligned}$$

Consequently, (4.7) can also be equivalently written as:

$$\mathcal{Q}_2(x', F_{2\times 2}) = \mu\left(\mathrm{tr}\left(C^2\right) - 2\frac{\langle C^2G^{-1}e_3, e_3\rangle}{\langle G^{-1}e_3, e_3\rangle} + \frac{\langle CG^{-1}e_3, e_3\rangle^2}{\langle G^{-1}e_3, e_3\rangle^2}\right) + \frac{\lambda\mu}{\lambda+\mu}\left(\mathrm{tr}\ C - \frac{\langle CG^{-1}e_3, e_3\rangle}{\langle G^{-1}e_3, e_3\rangle}\right)^2.$$

Part B: Other limits

5. A characterization of bending: the 3d energy scaling at minimizers

In this section we deduce the following property, complementary to Corollary 3.2:

Theorem 5.1. *The non-vanishing of the three Riemann curvatures of the metric G:*

$$\exists x \in \Omega \qquad \Big(|R_{1212}| + |R_{1213}| + |R_{1223}|\Big)(x) \neq 0 \tag{5.1}$$

is equivalent to the lower bound on the energy scaling at minimizers:

$$\exists c > 0 \qquad \inf_{u\in W^{1,2}(\Omega^h,\mathbb{R}^3)} E^h(u) \geq ch^2. \tag{5.2}$$

Recall that the Riemann curvature tensor $R^{\cdot}_{\cdot\cdot\cdot}$ and its covariant version $R_{\cdot\cdot\cdot\cdot}$ are given by:

$$\begin{aligned} R^s_{ijk} &= \partial_j\Gamma^s_{ik} - \partial_k\Gamma^s_{ij} + \sum_{m=1}^3 \Gamma^s_{jm}\Gamma^m_{ik} - \sum_{m=1}^3 \Gamma^s_{km}\Gamma^m_{ij} \\ R_{sijk} &= \sum_{m=1}^3 G_{sm}R^m_{ijk}, \end{aligned}$$

while the Christoffel symbols are:

$$\Gamma^i_{kl} = \frac{1}{2}\sum_{m=1}^3 G^{im}(\partial_l G_{mk} + \partial_k G_{ml} - \partial_m G_{kl}).$$

Remark 5.2. In [14] we proved for the metric G having a 2d structure $G = (G_{2\times 2})^* + e_3\otimes e_3$, that condition (5.2) is equivalent to the nonimmersability of G, i.e. nonvanishing of its full Riemann curvature tensor R. The reason for this seemingly more restrictive result is that for such G, its flatness is equivalent to the vanishing of the Gaussian curvature of $G_{2\times 2}$, i.e. the 2d flatness of the midplate metric $G_{2\times 2}$. In fact, any isometric immersion of G induces a flat isometric immersion of $G_{2\times 2}$ whose second fundamental form $\Pi = 0$ trivially satisfies the condition (5.5) below. We see that in the general case the curvatures that contribute to the reduced 2d energy $\mathcal{I}_G$ at the scaling h^2 are those listed in (5.1), rather than all the curvatures which naturally contribute towards the residual 3d energy E^h.

The proof of Theorem 5.1 will follow directly from the next two theorems, which we present separately for their independent interest.

Theorem 5.3. *The following conditions are equivalent:*

(i) *The energy functional (2.5) satisfies:*

$$\min \mathcal{I}_G = 0, \tag{5.3}$$

where the minimum is taken over isometric immersions of $G_{2\times 2}$ in $\mathbb{R}^3$ of regularity $W^{2,2}$.

(ii) *There exists a $W^{2,2}$ isometric immersion $y : \Omega \to \mathbb{R}^3$ of $G_{2\times 2}$ such that:*

$$\mathrm{sym}\Big((\nabla y)^T \nabla \vec{b}\Big) = 0 \qquad a.e.\ in\ \Omega. \tag{5.4}$$

where $\vec{b} : \Omega \to \mathbb{R}^3$ is uniquely defined by:

$$\det Q > 0 \quad and \quad Q^T Q = G, \quad where: \ Qe_1 = \partial_1 y,\ Qe_2 = \partial_2 y,\ Qe_3 = \vec{b}.$$

(iii) *There exists a $W^{2,2}$ isometric immersion of $G_{2\times 2}$ in $\mathbb{R}^3$, whose second fundamental form Π is given by the Christoffel symbols of G:*

$$\Pi_{11} = -\frac{1}{\sqrt{G^{33}}}\Gamma_{11}^3, \qquad \Pi_{22} = -\frac{1}{\sqrt{G^{33}}}\Gamma_{22}^3, \qquad \Pi_{12} = -\frac{1}{\sqrt{G^{33}}}\Gamma_{12}^3. \tag{5.5}$$

Corollary 5.4. *The unique (up to rigid motions) minimizing immersion in (5.3) is the immersion y satisfying (5.4) and (5.5). This immersion is automatically smooth (up to the boundary of Ω).*

Proof. Let $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ satisfy (2.2) and $\mathcal{I}_G(y) = 0$. Denoting by $\vec{N} \in W^{1,2}(\Omega, \mathbb{R}^3)$ the unit normal to the surface $y(\Omega)$, we have [10]:

$$\partial_{ij} y = \sum_{m=1}^{2} \gamma_{ij}^m \partial_m y - \Pi_{ij} \vec{N}.$$

Since $G_{2\times 2}$ is smooth, then its Christoffel symbols γ_{ij}^m are smooth, and also the coefficients in Π_{ij} are smooth according to Theorem 5.3. Smoothness of y follows then by a bootstrap argument.

Finally, uniqueness of isometric immersion with a prescribed second fundamental form completes the proof. ■

Theorem 5.5. *Conditions in Theorem 5.3 are further equivalent to the vanishing of the following three Riemann curvatures of G:*

$$R_{112}^3 = R_{221}^3 = R_{1212} = 0, \tag{5.6}$$

which are precisely the Gauss-Codazzi-Meinardi equations for the compatibility of the metric $G_{2\times 2}$ and the second fundamental form Π in (5.5). Condition (5.6) is equivalent to:

$$R_{1212} = R_{1213} = R_{1223} = 0.$$

We now give proofs of Theorem 5.3 and Theorem 5.5.

Proof of Theorem 5.3.

1. Condition (i) holds when there exists $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ satisfying (2.2) and:

$$\mathcal{I}_G(y) = 0. \tag{5.7}$$

The equality (5.7) is clearly equivalent to: $\mathcal{Q}_2(x', (\nabla y)^T\nabla\vec{b}) = 0$ holding for a.e. $x' \in \Omega$. Since $\mathcal{Q}_3(F) = 0$ iff $F \in \mathbb{R}^{3\times 3}$ is skew-symmetric, it follows that (5.7) is further equivalent to (5.4), hence proving (ii).

2. Recall that the matrix field Q in Theorem 2.1 (ii), whose columns are given by vectors $\partial_1 y, \partial_2 y$ and $\vec{b}$, satisfies $Q^TQ = G$. Hence, (5.7) becomes:

$$\begin{aligned} 0 &= \langle \partial_1 y, \partial_1\vec{b}\rangle = \partial_1\langle \partial_1 y, \vec{b}\rangle - \langle \partial_{11}y, \vec{b}\rangle = \partial_1 G_{13} - \langle \partial_{11}y, \vec{b}\rangle \\ 0 &= \langle \partial_2 y, \partial_2\vec{b}\rangle = \partial_2\langle \partial_2 y, \vec{b}\rangle - \langle \partial_{22}y, \vec{b}\rangle = \partial_2 G_{23} - \langle \partial_{22}y, \vec{b}\rangle \\ 0 &= \langle \partial_1 y, \partial_2\vec{b}\rangle + \langle \partial_2 y, \partial_1\vec{b}\rangle = \partial_1 G_{23} + \partial_2 G_{13} - 2\langle \partial_{12}y, \vec{b}\rangle. \end{aligned} \tag{5.8}$$

Given $F \in \mathbb{R}^{3\times 3}$, by $F_{tan} \in \mathbb{R}^{2\times 2}$ let us denote the principal 2×2 minor of F, and we let $F_{cross} = (F_{13}, F_{23})^T \in \mathbb{R}^2$. Call $P = G^{-1}$ Then:

$$G_{tan}P_{tan} + G_{cross}\otimes P_{cross} = \mathrm{Id}_2, \qquad G_{tan}P_{cross} + P_{33}G_{cross} = 0, \tag{5.9}$$

and so consequently:

$$(G_{tan})^{-1} = P_{tan} + (G_{tan})^{-1}G_{cross}\otimes P_{cross} = P_{tan} - \frac{1}{P_{33}}P_{cross}\otimes P_{cross}. \tag{5.10}$$

We therefore obtain:

$$G^{33} = P_{33} = \frac{\det G_{2\times 2}}{\det G}, \qquad (G_{tan})^{-1}G_{cross} = -\frac{1}{P_{33}}P_{cross} = -\frac{1}{G^{33}}(G^{13}, G^{23})^T,$$

where as the standard notation is used: $[G^{ij}]_{i,j:1..3} = G^{-1} = P$. It follows by (2.3) that:

$$\vec{b} = -\frac{1}{G^{33}}\big(G^{13}\partial_1 y + G^{23}\partial_2 y\big) + \frac{1}{\sqrt{G^{33}}}\vec{N},$$

and hence (5.8) becomes:

$$\begin{aligned} \partial_1 G_{13} &= -\frac{1}{G^{33}}\Big(G^{13}\langle \partial_{11}y, \partial_1 y\rangle + G^{23}\langle \partial_{11}y, \partial_2 y\rangle\Big) - \frac{1}{\sqrt{G^{33}}}\Pi_{11} \\ \partial_2 G_{23} &= -\frac{1}{G^{33}}\Big(G^{13}\langle \partial_{22}y, \partial_1 y\rangle + G^{23}\langle \partial_{22}y, \partial_2 y\rangle\Big) - \frac{1}{\sqrt{G^{33}}}\Pi_{22} \\ \frac{1}{2}\big(\partial_1 G_{23} + \partial_2 G_{13}\big) &= -\frac{1}{G^{33}}\Big(G^{13}\langle \partial_{12}y, \partial_1 y\rangle + G^{23}\langle \partial_{12}y, \partial_2 y\rangle\Big) - \frac{1}{\sqrt{G^{33}}}\Pi_{12}, \end{aligned} \tag{5.11}$$

where we used the fact that the coefficients of the second fundamental form Π of the surface $y(\Omega)$ satisfy: $\Pi_{ij} = \langle \partial_i y, \partial_j\vec{N}\rangle = -\langle \partial_{ij}y, \vec{N}\rangle$ for $i,j : 1..2$.

Also, note that $\partial_i G = 2\mathrm{sym}\big((\partial_i Q)^TQ\big)$ for $i = 1,2$, from where we deduce:

$$\begin{aligned} \langle \partial_{11}y, \partial_1 y\rangle &= \frac{1}{2}\partial_1 G_{11}, \qquad \langle \partial_{22}y, \partial_2 y\rangle = \frac{1}{2}\partial_2 G_{22}, \\ \langle \partial_{12}y, \partial_1 y\rangle &= \frac{1}{2}\partial_2 G_{11}, \qquad \langle \partial_{12}y, \partial_2 y\rangle = \frac{1}{2}\partial_1 G_{22}, \\ \langle \partial_{11}y, \partial_2 y\rangle &= \partial_1 G_{12} - \frac{1}{2}\partial_2 G_{11}, \qquad \langle \partial_{22}y, \partial_1 y\rangle = \partial_2 G_{12} - \frac{1}{2}\partial_1 G_{22}. \end{aligned} \tag{5.12}$$

3. We now want to rewrite the equations in (5.11) and the formulas (5.12) using the Christoffel symbols Γ_{ij}^m, $i,j,m = 1..3$ of the metric G. Recall that, since the Levi-Civita connection is metric-compatible, we have:

$$\partial_i G_{jk} = \sum_{m=1}^{3} G_{mk}\Gamma_{ij}^m + \sum_{m=1}^{3} G_{mj}\Gamma_{ik}^m, \tag{5.13}$$

$$\partial_i G^{jk} = -\sum_{m=1}^{3} G^{mk}\Gamma_{mi}^j - \sum_{m=1}^{3} G^{mj}\Gamma_{mi}^k. \tag{5.14}$$

Since $\partial_3 G = 0$, it follows that:

$$\sum_{m=1}^{3} G_{m1}\Gamma_{13}^m = 0, \qquad \sum_{m=1}^{3} G_{m2}\Gamma_{23}^m = 0 \qquad \sum_{m=1}^{3} (G_{m2}\Gamma_{13}^m + G_{m1}\Gamma_{23}^m) = 0.$$

Therefore and in view of (5.13), (5.12) become:

$$\begin{aligned}
\langle \partial_{11}y, \partial_1 y\rangle &= \sum_{m=1}^{3} G_{m1}\Gamma_{11}^m, \qquad & \langle \partial_{22}y, \partial_2 y\rangle &= \sum_{m=1}^{3} G_{m2}\Gamma_{22}^m,\\
\langle \partial_{12}y, \partial_1 y\rangle &= \sum_{m=1}^{3} G_{m1}\Gamma_{12}^m, \qquad & \langle \partial_{12}y, \partial_2 y\rangle &= \sum_{m=1}^{3} G_{m2}\Gamma_{12}^m,\\
\langle \partial_{11}y, \partial_2 y\rangle &= \sum_{m=1}^{3} G_{m2}\Gamma_{11}^m, \qquad & \langle \partial_{22}y, \partial_1 y\rangle &= \sum_{m=1}^{3} G_{m1}\Gamma_{22}^m.
\end{aligned} \tag{5.15}$$

By (5.11), (5.15) and (5.13) we now obtain:

$$\begin{aligned}
\Pi_{11} &= -\frac{1}{\sqrt{G^{33}}}\sum_{m=1}^{3}(G^{13}G_{m1} + G^{23}G_{m2})\Gamma_{11}^m - \sqrt{G^{33}}\sum_{m=1}^{3} G_{m3}\Gamma_{11}^m,\\
\Pi_{22} &= -\frac{1}{\sqrt{G^{33}}}\sum_{m=1}^{3}(G^{13}G_{m1} + G^{23}G_{m2})\Gamma_{22}^m - \sqrt{G^{33}}\sum_{m=1}^{3} G_{m3}\Gamma_{22}^m,\\
\Pi_{12} &= -\frac{1}{\sqrt{G^{33}}}\sum_{m=1}^{3}(G^{13}G_{m1} + G^{23}G_{m2})\Gamma_{12}^m - \sqrt{G^{33}}\sum_{m=1}^{3} G_{m3}\Gamma_{12}^m.
\end{aligned}$$

Since $\sum_{i=1}^{3} G^{i3}G_{mi} = \delta_{m3}$, we conclude (5.5) and note that it is equivalent to (5.7). ■

Proof of Theorem 5.5.

Clearly, Theorem 5.3 (iii) is equivalent (see [10] for details) to the satisfaction of the Codazzi-Mainardi equations for the 2d metric $G_{2\times 2}$ and the second fundamental form Π:

$$\begin{aligned}
\partial_2\left(\frac{1}{\sqrt{G^{33}}}\Gamma_{11}^3\right) - \partial_1\left(\frac{1}{\sqrt{G^{33}}}\Gamma_{12}^3\right) &= \frac{1}{\sqrt{G^{33}}}\left(\sum_{m=1}^{2}\Gamma_{1m}^3\gamma_{12}^m - \sum_{m=1}^{2}\Gamma_{2m}^3\gamma_{11}^m\right)\\
\partial_2\left(\frac{1}{\sqrt{G^{33}}}\Gamma_{12}^3\right) - \partial_1\left(\frac{1}{\sqrt{G^{33}}}\Gamma_{22}^3\right) &= \frac{1}{\sqrt{G^{33}}}\left(\sum_{m=1}^{2}\Gamma_{1m}^3\gamma_{22}^m - \sum_{m=1}^{2}\Gamma_{2m}^3\gamma_{12}^m\right)
\end{aligned} \tag{5.16}$$

together with the Gauss equation:

$$\Gamma_{11}^3\Gamma_{22}^3 - (\Gamma_{12}^3)^2 = G^{33}\kappa \det G_{2\times 2}. \tag{5.17}$$

Above, by γ_{ij}^k we denote the Christoffel symbols of the metric $G_{2\times 2}$, while $\kappa = \kappa(G_{2\times 2})$ is the Gaussian curvature of $G_{2\times 2}$. We now prove that (5.16) and (5.17) are equivalent to (5.6).

1. We first relate the Christoffel symbols γ_{kl}^s with Γ_{kl}^s. By (5.9), the inverse matrix $[g^{ij}]_{i,j=1..2} = [(G_{2\times 2})^{-1}]_{ij}$ is given by: $g^{ij} = G^{ij} - \frac{1}{G^{33}}G^{3i}G^{3j}$. Hence:

$$\begin{aligned}\gamma_{kl}^s &= \frac{1}{2}\sum_{m=1}^{2} g^{sm}(\partial_l G_{mk} + \partial_k G_{ml} - \partial_m G_{kl})\\ &= \frac{1}{2}\sum_{m=1}^{2} G^{sm}(\partial_l G_{mk} + \partial_k G_{ml} - \partial_m G_{kl}) - \frac{1}{2}\frac{G^{3s}}{G^{33}}\sum_{m=1}^{2} G^{3m}\big(\partial_l G_{mk} + \partial_k G_{ml} - \partial_m G_{kl}\big)\\ &= \Gamma_{kl}^s - G^{3s}(\partial_l G_{3k} + \partial_k G_{3l}) - \frac{G^{3s}}{G^{33}}\Big(\Gamma_{kl}^3 - G^{33}(\partial_l G_{3k} + \partial_k G_{3l})\Big)\\ &= \Gamma_{kl}^s - \frac{G^{3s}}{G^{33}}\Gamma_{kl}^3.\end{aligned} \tag{5.18}$$

Also, note that by (5.14), for $i = 1, 2$ we have:

$$\sqrt{G^{33}}\partial_i(\frac{1}{\sqrt{G^{33}}}) = -\frac{1}{2}\frac{\partial_i G^{33}}{G^{33}} = \frac{1}{G^{33}}\sum_{m=1}^{3} G^{m3}\Gamma_{mi}^3. \tag{5.19}$$

2. The first equation in (5.16) now becomes:

$$\begin{aligned}\partial_2\Gamma_{11}^3 - \partial_1\Gamma_{12}^3 - \frac{1}{2}\left(\frac{\partial_2 G^{33}}{G^{33}}\Gamma_{11}^3 - \frac{\partial_1 G^{33}}{G^{33}}\Gamma_{12}^3\right)\\ = \left(\sum_{m=1}^{2}\Gamma_{1m}^3\Gamma_{12}^m - \sum_{m=1}^{2}\Gamma_{2m}^3\Gamma_{11}^m\right) + \frac{G^{32}}{G^{33}}(\Gamma_{11}^3\Gamma_{22}^3 - (\Gamma_{12}^3)^2)\end{aligned}$$

Therefore, in view of (5.19) we obtain:

$$\begin{aligned}R_{121}^3 &= \partial_2\Gamma_{11}^3 - \partial_1\Gamma_{12}^3 + \sum_{m=1}^{3}(\Gamma_{2m}^3\Gamma_{11}^m - \Gamma_{1m}^3\Gamma_{12}^m)\\ &= \frac{1}{G^{33}}\left(G^{33}(\Gamma_{23}^3\Gamma_{11}^3 - \Gamma_{13}^3\Gamma_{12}^3) + \frac{1}{2}(\partial_2 G^{33}\Gamma_{11}^3 - \partial_1 G^{33}\Gamma_{12}^3) + G^{32}(\Gamma_{11}^3\Gamma_{22}^3 - (\Gamma_{12}^3)^2)\right)\\ &= \frac{1}{G^{33}}\left(\sum_{m=1}^{2}(G^{m3}\Gamma_{m1}^3\Gamma_{12}^3 - G^{m3}\Gamma_{m2}^3\Gamma_{11}^3) + G^{32}(\Gamma_{11}^3\Gamma_{22}^3 - (\Gamma_{12}^3)^2)\right),\end{aligned}$$

which gives $R_{121}^3 = 0$ by direct inspection. Similarly, the second equation in (5.16) yields:

$$\begin{aligned}\partial_2\Gamma_{12}^3 - \partial_1\Gamma_{22}^3 - \frac{1}{2}\left(\frac{\partial_2 G^{33}}{G^{33}}\Gamma_{12}^3 - \frac{\partial_1 G^{33}}{G^{33}}\Gamma_{22}^3\right)\\ = \left(\sum_{m=1}^{2}\Gamma_{1m}^3\Gamma_{22}^m - \sum_{m=1}^{2}\Gamma_{2m}^3\Gamma_{12}^m\right) + \frac{G^{31}}{G^{33}}(\Gamma_{11}^3\Gamma_{22}^3 - (\Gamma_{12}^3)^2)\end{aligned}$$

Consequently, using (5.19) as before:

$$\begin{aligned} R^3_{221} &= \partial_2\Gamma^3_{12} - \partial_1\Gamma^3_{22} + \sum_{m=1}^{3}(\Gamma^3_{2m}\Gamma^m_{12} - \Gamma^3_{1m}\Gamma^m_{22}) \\ &= \frac{1}{G^{33}}\left(G^{33}(\Gamma^3_{23}\Gamma^3_{12} - \Gamma^3_{13}\Gamma^3_{22}) + \frac{1}{2}(\partial_2 G^{33}\Gamma^3_{12} - \partial_1 G^{33}\Gamma^3_{22}) - G^{31}(\Gamma^3_{11}\Gamma^3_{22} - (\Gamma^3_{12})^2)\right) \\ &= \frac{1}{G^{33}}\left(\sum_{m=1}^{2}(G^{m3}\Gamma^3_{m1}\Gamma^3_{22} - G^{m3}\Gamma^3_{m2}\Gamma^3_{12}) - G^{31}(\Gamma^3_{11}\Gamma^3_{22} - (\Gamma^3_{12})^2)\right), \end{aligned}$$

which implies $R^3_{221} = 0$.

3. We now turn to proving equivalence of (5.17) with $R_{1212} = 0$. Denoting r^s_{ijk} and r_{sijk} the Riemann curvatures of the metric $G_{2\times 2}$ (where $i,j,k,s = 1..2$) we obtain:

$$\kappa \det G_{2\times 2} = r_{1212} = G_{11}r^1_{212} + G_{12}r^2_{212}$$

Further, for $i = 1,2$ we get by (5.18) and (5.14):

$$\begin{aligned} r^i_{212} &= \partial_1(\Gamma^i_{22} - \frac{G^{3i}}{G^{33}}\Gamma^3_{22}) - \partial_2(\Gamma^i_{12} - \frac{G^{3i}}{G^{33}}\Gamma^3_{12}) \\ &\quad + \sum_{m=1}^{2}(\Gamma^i_{1m} - \frac{G^{3i}}{G^{33}}\Gamma^3_{1m})(\Gamma^m_{22} - \frac{G^{3m}}{G^{33}}\Gamma^3_{22}) - \sum_{m=1}^{2}(\Gamma^i_{2m} - \frac{G^{3i}}{G^{33}}\Gamma^3_{2m})(\Gamma^m_{12} - \frac{G^{3m}}{G^{33}}\Gamma^3_{12}) \\ &= R^i_{212} - \frac{G^{3i}}{G^{33}}(\partial_1\Gamma^3_{22} - \partial_2\Gamma^3_{12}) + \left(\frac{G^{1i}}{G^{33}} - \frac{G^{3i}G^{31}}{(G^{33})^2}\right)(\Gamma^3_{11}\Gamma^3_{22} - (\Gamma^3_{12})^2) \\ &\quad - \frac{G^{3i}}{G^{33}}\sum_{m=1}^{3}(\Gamma^3_{1m}\Gamma^m_{22} - \Gamma^3_{2m}\Gamma^m_{12}). \end{aligned}$$

Consequently, the Gauss equation (5.17) yields:

$$\begin{aligned} R_{1212} &= G_{11}R^1_{212} + G_{12}R^2_{212} + G_{13}R^3_{212} \\ &= \kappa \det G_{2\times 2} + G_{11}(R^1_{212} - r^1_{212}) + G_{12}(R^2_{212} - r^2_{212}) + G_{13}R^3_{212} \\ &= G_{13}R^3_{212} + \frac{1}{G^{33}}(\Gamma^3_{11}\Gamma^3_{22} - (\Gamma^3_{12})^2) - G_{13}(\partial_1\Gamma^3_{22} - \partial_2\Gamma^3_{12}) \\ &\quad - \left(\frac{(1 - G^{13}G_{13})}{G^{33}} + \frac{G^{33}G_{13}G^{31}}{(G^{33})^2}\right)(\Gamma^3_{11}\Gamma^3_{22} - (\Gamma^3_{12})^2) - G_{31}\sum_{m=1}^{3}(\Gamma^3_{1m}\Gamma^m_{22} - \Gamma^3_{2m}\Gamma^m_{12}) \\ &= G_{13}R^3_{212} - G_{13}\left(\partial_1\Gamma^3_{22} - \partial_2\Gamma^3_{12} + \sum_{m=1}^{3}(\Gamma^3_{1m}\Gamma^m_{22} - \Gamma^3_{2m}\Gamma^m_{12})\right) = 0. \end{aligned}$$

Note that we did not use the fact that $R^3_{212} = 0$ in the above calculation.

This completes the proof of (5.6). Theorem 5.5 now follows by the next simple Lemma. ■

Lemma 5.6. *Under condition $R_{1212} = 0$, we have:*

(i) $R^3_{221} = 0$ *if and only if* $R_{1223} = 0$,

(ii) $R^3_{112} = 0$ *if and only if* $R_{1213} = 0$.

Proof. We prove (i), since (ii) follows in exactly the same way. Note that:

$$0 = R_{1221} = G_{1s}R^s_{221}, \quad 0 = R_{2221} = G_{2s}R^s_{221} \quad \text{and} \quad R_{1223} = R_{3221} = G_{3s}R^s_{221}.$$

By invertibility of $G_{2\times 2}$ it hence follows that $R^3_{221} = 0$ is equivalent to: $R^s_{221} = 0$ for all $s = 1, 2, 3$. In view of invertibility of G, this last condition is equivalent to: $R_{1223} = 0$, as claimed. ■

6. Examples

In this section we explore a few examples where the curvature condition of Theorem 5.1 is not satisfied. Consequently, the energy level of the minimizers drops below the h^2 scaling, namely to h^4. We explore various different scenarios where this phenomenon takes place.

Example 6.1. Let $\lambda : \bar\Omega \to \mathbb{R}$ be a smooth positive function and define:

$$G(x', x_3) = G(x') = \operatorname{diag}\big(1, 1, \lambda(x')\big). \tag{6.1}$$

Clearly, the 2d metric $G_{2\times 2} = \mathrm{Id}_2$ has an isometric immersion $y_0(x') = x'$ with the second fundamental form $\Pi = 0$. On the other hand:

$$\forall i, j : 1..2 \qquad \Gamma^3_{ij} = \frac{1}{2\lambda}(\partial_i G_{3j} + \partial_j G_{3i}) = 0,$$

and we see that both conditions (i) and (ii) in Theorem 5.3 are satisfied, so that $\mathcal{I}_G(y_0) = 0$.

We can further check directly that the only possibly non-zero Christoffel symbols are:

$$\forall i : 1..2 \qquad \Gamma^i_{33} = -\frac{1}{2}\partial_i\lambda, \qquad \Gamma^3_{i3} = \frac{1}{2\lambda}\partial_i\lambda.$$

In particular, it easily follows that: $R^3_{121} = R^3_{221} = R_{1212} = 0$, which is consistent with Theorem 5.5. At the same time G is, in general, non-immersible. To see this, recall that the Ricci curvatures are given by: $R_{ij} = \sum_{l=1}^3(\partial_l\Gamma^l_{ij} - \partial_j\Gamma^l_{il}) + \sum_{l,m=1}^3(\Gamma^l_{ij}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{jm})$, for $i, j = 1\ldots 3$. In the present case, we have:

$$\begin{aligned} R_{11} &= \frac{1}{4\lambda^2}\big((\partial_1\lambda)^2 - 2\lambda(\partial_{11}\lambda)\big), \quad R_{12} = \frac{1}{4\lambda^2}\big((\partial_1\lambda)(\partial_2\lambda) - 2\lambda(\partial_{12}\lambda)\big), \quad R_{13} = R_{23} = 0, \\ R_{22} &= \frac{1}{4\lambda^2}\big((\partial_2\lambda)^2 - 2\lambda(\partial_{22}\lambda)\big), \quad R_{33} = -\frac{1}{4\lambda}\big((\partial_1\lambda)^2 - 2\lambda(\partial_{11}\lambda) + (\partial_2\lambda)^2 - 2\lambda(\partial_{22}\lambda)\big). \end{aligned}$$

We hence see that G is immersible if and only if:

$$M := \nabla^2\lambda - \frac{1}{2\lambda}\nabla\lambda \otimes \nabla\lambda \equiv 0 \quad \text{in } \Omega. \tag{6.2}$$

Let us now consider the scaling of the 3d non-Euclidean energy studied in this paper:

$$E^h(u^h) = \frac{1}{h}\int_{\Omega^h} W(\nabla u^h \sqrt{G}^{-1})\,\mathrm{d}x, \qquad \sqrt{G}^{-1} = \operatorname{diag}(1, 1, \frac{1}{\sqrt{\lambda}}),$$

at the following sequence of smooth deformations of Ω^h:

$$u^h(x', x_3) = x' + \Big(-\frac{x_3^2}{4}\partial_1\lambda, -\frac{x_3^2}{4}\partial_2\lambda, \sqrt{\lambda}\,x_3\Big)^T. \tag{6.3}$$

We have: $\big((\nabla u^h)\sqrt{G}^{-1}\big)_{2\times 2} = \mathrm{Id}_2 - \frac{x_3^2}{4}\nabla^2\lambda$, and:

$$\big((\nabla u^h)\sqrt{G}^{-1}\big)e_3 = \Big(-\frac{x_3}{2}\frac{\partial_1\lambda}{\sqrt{\lambda}}, -\frac{x_3}{2}\frac{\partial_2\lambda}{\sqrt{\lambda}}, 1\Big)^T, \qquad \big((\nabla u^h)\sqrt{G}^{-1}\big)^T e_3 = \Big(\frac{x_3}{2}\frac{\partial_1\lambda}{\sqrt{\lambda}}, \frac{x_3}{2}\frac{\partial_2\lambda}{\sqrt{\lambda}}, 1\Big)^T.$$

Recall that for every $F = \mathrm{Id}_3 + \mathcal{A} \in \mathbb{R}^{3\times 3}$ when $\mathcal{A}$ is sufficiently small, we have: $\mathrm{dist}(F, SO(3)) = |\sqrt{F^TF} - \mathrm{Id}| = |\sqrt{\mathrm{Id} + 2\mathrm{sym}\ \mathcal{A} + \mathcal{A}^T\mathcal{A}} - \mathrm{Id}| = \big|\mathrm{sym}\ \mathcal{A} + \frac{1}{2}\mathcal{A}^T\mathcal{A} + o(|\mathrm{sym}\ \mathcal{A} + \frac{1}{2}\mathcal{A}^T\mathcal{A}|)\big|$. Consequently:

$$W\big((\nabla u^h)\sqrt{G}^{-1}\big) \leq C\mathrm{dist}^2\big((\nabla u^h)\sqrt{G}^{-1}, SO(3)\big) \leq Cx_3^4,$$

and therefore:

$$\inf E^h \leq E^h(u^h) \leq \frac{C}{h}\int_{-h/2}^{h/2} x_3^4 \,\mathrm{d}x_3 = Ch^4, \tag{6.4}$$

for any choice of λ in (6.1). In section 7 we will show that this scaling is also optimal, provided that the condition (6.2) does not hold.

Example 6.2. Let $\lambda : \bar{\Omega} \to \mathbb{R}$ be a smooth positive function and consider the metric:

$$G(x', x_3) = G(x') = \lambda(x')\mathrm{Id}_3. \tag{6.5}$$

One checks directly that $\Gamma^i_{kl} = \frac{1}{2\lambda}(\delta_{ik}\partial_l\lambda + \delta_{il}\partial_k\lambda - \delta_{kl}\partial_i\lambda) = \delta_{ik}\partial_l f + \delta_{il}\partial_k f - \delta_{kl}\partial_i f$, where we denote $f = \frac{1}{2}\log\lambda$. We directly compute:

$$R^3_{112} = R^3_{221} = 0 \quad \text{ and } \quad R_{1212} = -\frac{1}{2}\lambda\Delta(\log\lambda) = \lambda^2\kappa(\lambda\mathrm{Id}_2).$$

Therefore, condition (5.6) which is equivalent to $\min \mathcal{I}_G = 0$ according to Theorem 5.5, holds if and only if:

$$\Delta(\log\lambda) = 0, \tag{6.6}$$

or equivalently, when the 2d metric $G_{2\times 2} = \lambda\mathrm{Id}_2$ is flat (immersible in $\mathbb{R}^2$). Note also that since $\Gamma^3_{ij} = 0$ for $i, j : 1..2$ then this is precisely the case when (5.5) of Theorem 5.3 is satisfied.

We now compute the Ricci curvature of G using the conformal rescaling formula:

$$\begin{aligned}\mathrm{Ric}(G) &= \mathrm{Ric}(e^{2f}\mathrm{Id}_3) = -(\nabla^2 f - \nabla f \otimes \nabla f) - \big(\Delta f - |\nabla f|^2\big)\mathrm{Id}_3 \\ &= \Big(-2(\Delta f)\mathrm{Id}_2 + \mathrm{cof}(\nabla^2 f - \nabla f \otimes \nabla f)\Big)^* - (\Delta f + |\nabla f|^2)\ e_3 \otimes e_3.\end{aligned}$$

We observe that G is immersible iff $\mathrm{Ric}(G) = 0$, i.e. when $\nabla f = 0$, which is equivalent to:

$$\lambda \equiv \mathrm{const}. \tag{6.7}$$

Clearly (6.7) implies (6.6), but conversely: there exist non-immersible metrics G for which (6.6) holds i.e. for which the minimum of the residual energy $\mathcal{I}_G$ is 0, and it is attained by the unique (up to rigid motions) smooth isometric immersion $y : \Omega \to \mathbb{R}^2$ of $\lambda\mathrm{Id}_2$.

As in Example 6.1, we now consider scaling of the 3d energies E^h, assuming (6.6). Define:

$$u^h(x', x_3) = y(x')^* + x_3\sqrt{\lambda}e_3 - \frac{x_3^2}{4}\Big((\nabla y)^{-1,T}\nabla\lambda\Big)^*. \tag{6.8}$$

We easily compute that $\big(\nabla u^h(x', x_3)\big)_{2\times 2} = \nabla y + \mathcal{O}(x_3^2)$ and:

$$\begin{aligned}\big(\nabla u^h(x', x_3)\big)e_3 &= \Big(-\frac{x_3}{2}\langle(\nabla y)^{-1,T}\nabla\lambda, e_1\rangle, -\frac{x_3}{2}\langle(\nabla y)^{-1,T}\nabla\lambda, e_2\rangle, \sqrt{\lambda}\Big)^T, \\ \big(\nabla u^h(x', x_3)\big)^T e_3 &= \Big(\frac{x_3}{2\sqrt{\lambda}}\partial_1\lambda, \frac{x_3}{2\sqrt{\lambda}}\partial_2\lambda, \sqrt{\lambda}\Big)^T.\end{aligned}$$

Since $(\nabla y)^T\nabla y = \lambda \mathrm{Id}_2$ it follows that:

$$\begin{aligned} &\Big((\nabla u^h)^T\nabla u^h\Big)_{2\times 2} = \lambda \mathrm{Id}_2 + \mathcal{O}(x_3^2)\\ \forall i=1..2 \quad &\Big((\nabla u^h)^T\nabla u^h\Big)_{3i} = -\frac{x_3}{2}\langle \partial_i y, (\nabla y)^{-1,T}\nabla\lambda\rangle + \frac{x_3}{2}\partial_i\lambda + \mathcal{O}(x_3^2) = \mathcal{O}(x_3^2)\\ &\Big((\nabla u^h)^T\nabla u^h\Big)_{33} = \lambda + \mathcal{O}(x_3^2). \end{aligned}$$

Therefore it follows, by polar decomposition theorem:

$$\begin{aligned} W\big((\nabla u^h)\sqrt{G}^{-1}\big) &\leq C\mathrm{dist}^2\Big(\frac{1}{\sqrt{\lambda}}\nabla u^h, SO(3)\Big) \leq C\mathrm{dist}^2\Big(\sqrt{\frac{1}{\lambda}(\nabla u^h)^T\nabla u^h}, SO(3)\Big)\\ &= C\mathrm{dist}^2\Big(\sqrt{\mathrm{Id}_3 + \mathcal{O}(x_3^2)}, SO(3)\Big) \leq Cx_3^4, \end{aligned}$$

which again yields the scaling h^4, precisely as in (6.4).

Remark 6.3. A more general example of G in the same spirit as above, is: $G(x') = G^*_{2\times 2} + \lambda(x')e_3\otimes e_3$ with $G_{2\times 2}$ immersible in $\mathbb{R}^2$. Since $\Gamma^3_{ij} = 0$ for $i,j = 1..2$, we see that $\min \mathcal{I}_G = 0$, in virtue of Theorem 5.3. On the other hand, one can check directly that taking smooth $y:\Omega\to\mathbb{R}^2$ such that $(\nabla y)^T\nabla y = G_{2\times 2}$ and defining the 3d deformations u^h as in (6.8), it again follows: $W\big((\nabla u^h)\sqrt{G}^{-1}\big) \leq Cx_3^4$. Consequently, the same energy scaling as in (6.4) is valid here as well.

Example 6.4. (i). Let $\lambda_1,\lambda_2,\lambda_3:\bar\Omega\to\mathbb{R}$ be smooth functions such that $\lambda_3 > \lambda_1^2+\lambda_2^2$. Define:

$$G(x',x_3) = G(x') = \begin{bmatrix} 1 & 0 & \lambda_1\\ 0 & 1 & \lambda_2\\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}. \tag{6.9}$$

We directly compute the Christoffel symbols involved in (5.5):

$$\Gamma^3_{11} = \frac{\partial_1\lambda_1}{\lambda_3 - (\lambda_1^2+\lambda_2^2)}, \qquad \Gamma^3_{12} = \frac{\frac{1}{2}(\partial_1\lambda_2 + \partial_2\lambda_1)}{\lambda_3 - (\lambda_1^2+\lambda_2^2)}, \qquad \Gamma^3_{22} = \frac{\partial_2\lambda_2}{\lambda_3 - (\lambda_1^2+\lambda_2^2)}. \tag{6.10}$$

Hence, if $|\partial_1\lambda_1| + |\partial_2\lambda_2| \not\equiv 0$ in Ω, it follows by Theorem 5.3 (iii), that the isometric immersion $y_0(x') = x'$ of $G_{2\times 2} = \mathrm{Id}_2$ is certainly not the immersion for which $\mathcal{I}_G(y_0) = 0$. Of course, this does not preclude the possibility that there exists another immersion $y:\Omega\to\mathbb{R}^3$ of $G_{2\times 2}$ (now necessarily non-flat), for which $\mathcal{I}_G(y) = 0$. As we shall see below, both scenarios are possible.

(ii). Consider a subcase of (6.9), where $\lambda_1 = 0$ and $\lambda_3 = \lambda_2^2 + 1$, so that:

$$G(x',x_3) = G(x') = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & \lambda_2\\ 0 & \lambda_2 & \lambda_2^2+1 \end{bmatrix}.$$

Consequently, by (6.10):

$$\Gamma^3_{11} = 0, \qquad \Gamma^3_{12} = \frac{1}{2}\partial_1\lambda_2, \qquad \Gamma^3_{22} = \partial_2\lambda_2. \tag{6.11}$$

We further compute:

$$\Gamma^1_{11} = \Gamma^2_{11} = 0, \qquad \Gamma^3_{13} = \frac{1}{2}\lambda_2\partial_1\lambda_2, \qquad \Gamma^2_{12} = -\frac{1}{2}\lambda_2\partial_1\lambda_2,$$

and:

$$R^3_{112} = \partial_1\Gamma^3_{12} + (\Gamma^3_{12}\Gamma^2_{12} + \Gamma^3_{13}\Gamma^3_{12}) - (\Gamma^3_{12}\Gamma^1_{11} + \Gamma^3_{22}\Gamma^2_{11}) = \frac{1}{2}\partial_{11}\lambda_2.$$

We see that when $\partial_{11}\lambda_2 \not\equiv 0$ in Ω, then $\min \mathcal{I}_G > 0$, in view of Theorem 5.5. In particular, there is no isometric immersion of $G_{2\times 2}$ satisfying (5.5).

(iii). Consider now a further subcase of (6.9), with $\lambda_1 = 0$, $\lambda_2 = -x_2$ and $\lambda_3 = x_2^2 + 1$, so that:

$$G(x', x_3) = G(x_2) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -x_2 \\ 0 & -x_2 & x_2^2+1 \end{array}\right].$$

By (6.11) we get: $\Gamma^3_{11} = \Gamma^3_{12} = 0$ and $\Gamma^3_{22} = -1$. Let $y(x_1, x_2) = (x_1, \sin x_2, \cos x_2)$ be an isometric immersion of $G_{2\times 2}$ into a cylinder. This immersion has the second fundamental form Π:

$$\Pi = (\nabla y)^T \nabla \vec{N} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right] = -\frac{1}{\sqrt{G^{33}}}\left[\begin{array}{cc} \Gamma^3_{11} & \Gamma^3_{12} \\ \Gamma^3_{12} & \Gamma^3_{22} \end{array}\right].$$

Hence, by Theorem 5.3 (iii) it follows that $\mathcal{I}_G(y) = 0$.

In particular, $R^3_{112} = R^3_{221} = R_{1212} \equiv 0$ in Ω. Metric G is, however, nonimmersible, as a direct calculation of its scalar Ricci curvature shows:

$$S = G^{11}R_{11} + G^{22}R_{22} + G^{33}R_{33} + 2G^{23}R_{23} = -2 + 2x_2^2 \not\equiv 0.$$

Example 6.5. In this example we will have $G_{2\times 2}$ nonimmersible in $\mathbb{R}^2$. Let $\bar{\Omega} \subset \{(x_1, x_2) \in \mathbb{R}^2;\ x_1 > x_2 > 0\}$ and define G:

$$G(x_1, x_2) = \left[\begin{array}{ccc} 1+x_1^2 & x_1x_2 & b_1 + x_1 b_3 \\ x_1x_2 & 1+x_2^2 & b_2 + x_2 b_3 \\ b_1 + x_1 b_3 & b_2 + x_2 b_3 & |\vec{b}|^2 \end{array}\right] \quad \text{where } \vec{b} = \Big(-\frac{1}{3}x_1^3, \frac{1}{3}x_2^3, \frac{1}{2}(x_1^2 - x_2^2)\Big)^T. \tag{6.12}$$

We see that:

$$y(x_1, x_2) = \big(x_1, x_2, \frac{1}{2}(x_1^2 + x_2^2)\big)$$

is an isometric immersion of $G_{2\times 2}$ in $\mathbb{R}^3$. Therefore:

$$\kappa(G_{2\times 2}) = \frac{\partial_{11}y_3\partial_{22}y_3 - (\partial_{12}y_3)^2}{(1 + |\nabla y_3|^2)^2} = \big(1 + x_1^2 + x_2^2\big)^{-2} \neq 0.$$

By Theorem 5.3, we have: $\min \mathcal{I}_G = 0$ iff (5.4) holds. This is equivalent to $\mathrm{sym}(\nabla \vec{b}_{tan} + \nabla b_3 \otimes (x_1, x_2)) = 0$, and further to:

$$\mathrm{sym}(\nabla b_3 \otimes (x_1, x_2)) = -\mathrm{sym}\nabla \vec{b}_{tan}.$$

Given a scalar field b_3, there exists $\vec{b}_{tan}$ so that the above condition is satisfied iff:

$$0 = \mathrm{curl}^T\mathrm{curl}(\nabla b_3 \otimes (x_1, x_2)) = -\Delta b_3.$$

We see that indeed b_3 in (6.12) is harmonic, and that (b_1, b_2) satisfiy: $\mathrm{sym}\nabla\vec{b}_{tan} = \mathrm{diag}(-x_1^2, x_2^2)$, which implies $\mathcal{I}_G(y) = 0$, for the 3d metric $G = Q^TQ$, where $Q = \Big[\partial_1 y, \partial_2 y, \vec{b}\Big]$. Note also that $\det Q > 0$ in $\bar{\Omega}$. Hence, G is given by (6.12). One can check that G is nonimmersible in $\mathbb{R}^3$, by calculating its Ricci curvature (we have used Maple®). In particular, the scalar Ricci curvature of G equals:

$$S = \sum_{i,j=1..3} G^{ij} R_{ij} = \frac{12}{2x_1^2 + 2x_2^2 + 3} \neq 0.$$

In section 8 we will discuss other examples of types of metric G, motivated by the modelling of the nematic glass.

7. Scaling analysis for Example 6.1

In this section we will prove the optimality of the scaling h^4 in (6.4) for every non-immersible metric tensor G of the form in Example 6.1.

Theorem 7.1. *Let $\lambda : \bar{\Omega} \to \mathbb{R}$ be a smooth positive function such that:*

$$M := \nabla^2 \lambda - \frac{1}{2\lambda} \nabla\lambda \otimes \nabla\lambda \not\equiv 0 \qquad in\ \Omega.$$

Define $G(x', x_3) = G(x') = \mathrm{diag}(1, 1, \lambda(x'))$. Then G is non-immersible and there exist $c, C > 0$ such that:

$$ch^4 \leq \inf E^h \leq Ch^4. \tag{7.1}$$

Proof. The equivalence of nonimmersability of G with the condition $M \not\equiv 0$, as well as the upper bound has been established in Example 6.1, see (6.2) and (6.4). Hence it remains to prove the lower bound of (7.1). Recall that the minimizing isometric immersion of $G_{2\times 2}$ here is $y(x') = (x', 0)$, whereas other quantities in the proofs of Theorems 2.1 and 3.1 are:

$$\vec{b} = \sqrt{\lambda} e_3, \qquad Q = A = \mathrm{diag}(1, 1, \sqrt{\lambda})$$

$$\vec{d} = (-\frac{\partial_1 \lambda}{2}, -\frac{\partial_2 \lambda}{2}, 0)^T, \qquad B(x') = \left[\ \partial_1 \vec{b}, \partial_2 \vec{b}, \vec{d}\ \right] = \left[\begin{array}{ccc} 0 & 0 & -\frac{\partial_1 \lambda}{2} \\ 0 & 0 & -\frac{\partial_2 \lambda}{2} \\ \frac{\partial_1 \lambda}{2\sqrt{\lambda}} & \frac{\partial_2 \lambda}{2\sqrt{\lambda}} & 0 \end{array} \right].$$

1. Let $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ be any sequence of deformations, satisfying:

$$E^h(u^h) \leq Ch^4. \tag{7.2}$$

Consider a smooth diffeomorphism $\phi : \Omega^{h_0} \to \mathbb{R}^3$ (for $h_0 > 0$ sufficiently small) given as in (6.3):

$$\phi(x_1, x_2, x_3) = (x_1 - \frac{x_3^2}{4} \partial_1 \lambda,\ x_2 - \frac{x_3^2}{4} \partial_2 \lambda,\ \sqrt{\lambda} x_3)^T.$$

We will write $\mathcal{U}^h = \phi(\Omega^h)$. Since $\nabla\phi = Q + x_3 B - \frac{x_3^2}{4}(\nabla^2 \lambda)^*$, it follows that:

$$(\nabla\phi)A^{-1} = \mathrm{Id}_3 + x_3 BA^{-1} - \frac{x_3^2}{4}(\nabla^2 \lambda)^*.$$

Further, noting that $BA^{-1} \in so(3)$, by polar decomposition theorem we get:

$$\exists R(x) \in SO(3) \qquad (\nabla\phi)A^{-1} = \sqrt{(\nabla\phi)A^{-1}((\nabla\phi)A^{-1})^T} R(x) = (\mathrm{Id}_3 + \mathcal{O}(x_3^2)) R(x).$$

Then:

$$
\begin{aligned}
E^h(u^h) &\geq c\frac{1}{h}\int_{\Omega^h} \mathrm{dist}^2\Big((\nabla u^h)A^{-1}, SO(3)\Big)\ \mathrm{d}x \\
&= c\frac{1}{h}\int_{\mathcal{U}^h} \mathrm{dist}^2\Big(\nabla(u^h\circ\phi^{-1})(\nabla h A^{-1}), SO(3)\Big)\,|\det\nabla h|^{-1}\ \mathrm{d}z \\
&\geq c\frac{1}{h}\int_{\mathcal{U}^h} \mathrm{dist}^2\Big(\nabla(u^h\circ\phi^{-1}), SO(3)\Big)\ \mathrm{d}z + \mathcal{O}(h^4),
\end{aligned}
\tag{7.3}
$$

where we used that:

$$
\mathrm{dist}^2\left(F(\mathrm{Id}_3+\mathcal{O}(x_3^2)), SO(3)\right) \geq c\ \mathrm{dist}^2\left(F, SO(3)(\mathrm{Id}_3+\mathcal{O}(x_3^2))^{-1}\right) \geq c\ \mathrm{dist}^2\left(F, SO(3)\right) + \mathcal{O}(x_3^4).
$$

By the geometric rigidity estimate [7], and since $\det\nabla\phi$ is uniformly bounded away from 0 in Ω, the estimate (7.3) becomes:

$$
\begin{aligned}
E^h(u^h) &\geq c_h\frac{1}{h}\int_{\mathcal{U}^h}\Big|\nabla(u^h\circ\phi^{-1}) - R_h\Big|^2\ \mathrm{d}z + \mathcal{O}(h^4) \geq c_h\frac{1}{h}\int_{\Omega^h}\Big|\nabla u^h - R_h\nabla\phi\Big|^2 + \mathcal{O}(h^4) \\
&\geq c_h\frac{1}{h}\int_{\Omega^h}\Big|\nabla u^h - R_h(Q+x_3B)\Big|^2\ \mathrm{d}x + \mathcal{O}(h^4),
\end{aligned}
\tag{7.4}
$$

for some $R_h\in SO(3)$ and the constants c_h depending on the domains $\mathcal{U}^h$. However, if we replace the integration on Ω^h in (7.3), (7.4), by integration on a small cube $(-\frac{h}{2},\frac{h}{2})^3$, all the subsequent constants, including c_h from the geometric rigidity estimate, will be uniform and independent of ϕ. This leads, by the well known technique of approximation [7, 15, 14], to the following result:

Lemma 7.2. *Assume (7.2). There exists a sequence of rotation fields $R^h\in W^{1,2}(\Omega, SO(3))$, with:*

$$
\frac{1}{h}\int_{\Omega^h}|\nabla u^h - R^h(Q+x_3B)|^2\ \mathrm{d}x \leq Ch^4 \quad \textit{and} \quad \int_\Omega |\nabla R^h|^2 \leq Ch^2.
\tag{7.5}
$$

The proof of Lemma 7.2 reproduces the lines of Theorem 1.6 in [15], hence we omit it. Note that the above is parallel to Lemma 2.3, now with an improved accuracy of the error bound due to the smaller scaling of the energy in (7.2). With the help of (7.5) we now derive the limiting properties of the sequence u^h.

2. Define $\bar R^h = \mathbb{P}_{SO(3)} \fint_{\Omega^h}(\nabla u^h)Q^{-1}\ \mathrm{d}x$. Since: $\mathrm{dist}^2(\fint(\nabla u^h)Q^{-1}, SO(3)) \leq |\fint(\nabla u^h)Q^{-1} - R^h(x')|^2$ for every $x'\in\Omega$, it follows upon integrating and using the Poincarè inequality, together with (7.5), that the projection in $\bar R^h$ is well defined. Indeed:

$$
\begin{aligned}
\mathrm{dist}^2(\fint_{\Omega^h}(\nabla u^h)Q^{-1}, SO(3)) &\leq \fint|\fint(\nabla u^h)Q^{-1}\ \mathrm{d}x - R^h(x')|^2\ \mathrm{d}x' \\
&\leq C\left(\fint|\fint(\nabla u^h)Q^{-1} - \fint R^h|^2 + \fint|R^h - \fint R^h|^2\right) \\
&\leq C\left(\fint_{\Omega^h}|(\nabla u^h)Q^{-1} - R^h|^2 + \fint_{\Omega^h}|\nabla R^h|^2\right) \leq Ch^2.
\end{aligned}
$$

In a similar manner as above and again by (7.5), we observe that:

$$
\begin{aligned}
&\int_\Omega |R^h - \bar R^h|^2 \leq \int_\Omega |R^h - \fint(\nabla u^h)Q^{-1}|^2 \leq Ch^2, \\
&\|(\bar R^h)^T R^h - \mathrm{Id}_3\|^2_{W^{1,2}(\Omega)} \leq Ch^2.
\end{aligned}
\tag{7.6}
$$

In particular, we have the following convergence to some matrix field S on Ω^1, up to a subsequence:

$$\frac{1}{h}\big((\bar{R}^h)^T R^h - \mathrm{Id}_3\big) \rightharpoonup S \qquad \text{weakly in } W^{1,2}(\Omega^1, \mathbb{R}^{3\times 3}), \tag{7.7}$$

3. Define the renormalised deformations $y^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$ by $y^h(x', x_3) = (\bar{R}^h)^T u^h(x', hx_3) - c^h$, where c^h is a constant vector such that by the Poincarè inequality and in view of:

$$\int_{\Omega^1} |(\bar{R}^h)^T \nabla u^h(x', hx_3) - Q|^2 \leq C\frac{1}{h}\int_{\Omega^h} |\nabla u^h - \bar{R}^h Q|^2 \leq Ch^2,$$

there holds:

$$y^h \to y \quad \text{ and } \quad \frac{1}{h}\partial_3 y^h \to \vec{b} \qquad \text{in } W^{1,2}(\Omega^1, \mathbb{R}^3).$$

The above is, naturally, consistent with the results of Theorem 2.1 (i). We now perform the analysis similar to step 5 in the proof of Theorem 2.1. Define the rescaled strains:

$$\mathcal{G}^h(x', x_3) = \frac{1}{h^2}\Big((R^h)^T \nabla u^h(x', hx_3) - (Q + hx_3 B)\Big)A^{-1}.$$

Note that in view of (7.5), $\{\mathcal{G}^h\}$ is uniformly bounded in L^2, so that up to a subsequence:

$$\mathcal{G}^h \rightharpoonup \mathcal{G} \qquad \text{weakly in } L^2(\Omega^1, \mathbb{R}^{3\times 3}). \tag{7.8}$$

Consider the vector fields:

$$\begin{aligned}\frac{1}{h^2}\Big(\partial_3 y^h - h(\vec{b} + hx_3\vec{d})\Big) &= \frac{1}{h}\Big((\bar{R}^h)^T \nabla u^h(x', hx_3) - (Q + hx_3 B)\Big)e_3\\ &= \frac{1}{h}(\bar{R}^h)^T\Big(\nabla u^h(x', hx_3) - R^h(Q + hx_3 B)\Big)e_3 + \frac{1}{h}\Big((\bar{R}^h)^T R^h - \mathrm{Id}\Big)(Q + hx_3 B)e_3.\end{aligned}$$

The first term in the right hand side above converges in $L^2(\Omega^1)$ to 0 by (7.5), while the second term converges to $SQe_3 = S\vec{b}$ by (7.7). Hence we observe the same convergence of the difference quotients:

$$\begin{aligned}f^{s,h}(x', x_3) &= \frac{1}{h^2}\frac{1}{s}\Big(y^h(x', x_3 + s) - y^h(x', x_3) - hs\big(\vec{b} + \big(hx_3 + \frac{hs}{2}\big)\vec{d}\big)\Big)\\ &= \frac{1}{h^2}\fint_0^s \partial_3 y^h(x', x_3 + t) - h\big(\vec{b} + h(x_3 + t)\vec{d}\big)\,\mathrm{d}t,\end{aligned}$$

and of their ∂_3 derivatives, namely:

$$f^{s,h} \to S\vec{b} \quad \text{ and } \quad \partial_3 f^{s,h} \to 0 \qquad \text{in } L^2(\Omega^1, \mathbb{R}^3).$$

Regarding the in-plane derivatives, for $i = 1, 2$ we have:

$$\begin{aligned}&\partial_i f^{s,h}(x', x_3)\\ &\quad = \frac{1}{h^2 s}\Big((\bar{R}^h)^T \partial_i u^h(x', hx_3 + hs) - (\bar{R}^h)^T \partial_i u^h(x', hx_3) - hs\big(\partial_i \vec{b} + h(x_3 + \frac{s}{2})\partial_i \vec{d}\big)\Big)\\ &\quad = \frac{1}{s}\Big((\bar{R}^h)^T R^h \mathcal{G}^h(x', x_3 + s) - (\bar{R}^h)^T R^h \mathcal{G}^h(x', x_3)\Big)e_i\\ &\qquad\qquad + \frac{1}{h}\big((\bar{R}^h)^T R^h - \mathrm{Id}_3\big)Be_i - (x_3 + \frac{s}{2})\partial_i \vec{d}.\end{aligned}$$

Therefore, by (7.8) and (7.7):

$$\partial_i f^{s,h} \rightharpoonup \frac{1}{s}\Big(\mathcal{G}(x', x_3 + s) - \mathcal{G}(x', x_3)\Big)e_i + SBe_i - (x_3 + \frac{s}{2})\partial_i \vec{d} \qquad \text{weakly in } L^2(\Omega^1, \mathbb{R}^3).$$

Concluding, $f^{s,h}$ converges weakly (up to a subsequence) in $W^{1,2}(\Omega^1)$ to $S\vec{b}$, and the limit in the right hand side above must coincide with $\partial_i(S\vec{b})$, yielding:

$$\mathcal{G}(x',x_3)e_i - \mathcal{G}(x',0)e_i = x_3\big(\partial_i(S\vec{b}) - SBe_i\big) + \frac{x_3^2}{2}\partial_i\vec{d} \qquad \text{for } i=1,2. \tag{7.9}$$

4. We now estimate the rescaled energies, as desired. Firstly, observe that for $(x',x_3)\in\Omega^1$:
$W\big((\nabla u^h(x',hx_3))A^{-1}\big) = W\big((R^h(x'))^T(\nabla u^h(x',hx_3))A^{-1}\big) = W\big(\mathrm{Id}_3 + h^2\mathcal{G}^h(x',x_3) + hx_3B(x')A^{-1}\big)$.
Since $BA^{-1}\in so(3)$, it follows by frame invariance that:

$$\begin{aligned} W\big((\nabla u^h(x',hx_3))A^{-1}\big) &= W\Big((e^{-hx_3BA^{-1}})(\mathrm{Id}_3 + h^2\mathcal{G}^h(x',x_3) + hx_3B(x')A^{-1})\Big)\\ &= W\Big(\big(\mathrm{Id} - hx_3BA^{-1} + \frac{h^2x_3^2}{2}(BA^{-1})^2 + \mathcal{O}(h^3)\big)\big(\mathrm{Id}_3 + h^2\mathcal{G}^h(x',x_3) + hx_3B(x')A^{-1}\big)\Big)\\ &= W\Big(\mathrm{Id}_3 + h^2\mathcal{G}^h - \frac{h^2x_3^2}{2}(BA^{-1})^2 + \mathcal{O}(h)h^2\mathcal{G}^h + \mathcal{O}(h^3)\Big). \end{aligned}$$

Define the sets $\Omega_h = \{x\in\Omega^1;\ h|\mathcal{G}^h(x)|\leq 1\}$. Using Taylor's expansion, we now obtain:

$$\begin{aligned} \liminf_{h\to0}\frac{1}{h^4}E^h(u^h) &\geq \liminf_{h\to0}\frac{1}{h^4}\int_{\Omega^1}\chi_{\Omega_h}W\Big(\mathrm{Id}_3 + h^2\mathcal{G}^h - \frac{h^2x_3^2}{2}(BA^{-1})^2 + \mathcal{O}(h^3)\Big)\,\mathrm{d}x\\ &= \liminf_{h\to0}\frac{1}{2}\int_{\Omega^1}\mathcal{Q}_3\Big(\chi_{\Omega_h}(\mathcal{G}^h - \frac{x_3^2}{2}(BA^{-1})^2\Big) \geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_3\Big(\mathcal{G} - \frac{x_3^2}{2}(BA^{-1})^2\Big)\\ &\geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_2\Big(\mathcal{G}(x',x_3)_{2\times2} + \frac{x_3^2}{2}\frac{1}{4\lambda}\nabla\lambda\otimes\nabla\lambda\Big)\,\mathrm{d}x, \end{aligned}$$

where we used the weak sequential lower-semincontinuity of the quadratic form $\mathcal{Q}_3$ in L^2, and since by (7.8) $\chi_{\Omega_h}\to1$ in $L^1(\Omega^1)$, resulting in $\chi_{\Omega_h}\mathrm{sym}\,\mathcal{G}^h \rightharpoonup \mathrm{sym}\,\mathcal{G}$ weakly in $L^2(\Omega^1)$.

By (7.9) we hence arrive at:

$$\begin{aligned} \liminf_{h\to0}\frac{1}{h^4}E^h(u^h) &\geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_2\Big(\mathcal{G}(x',0)_{2\times2} + x_3\big(\nabla(S\vec{b}) - SB\big)_{2\times2} - \frac{x_3^2}{4}M\Big)\,\mathrm{d}x\\ &\geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_2\Big(\mathcal{G}(x',0)_{2\times2} - \frac{x_3^2}{4}M\Big)\,\mathrm{d}x\\ &= \frac{1}{2}\Big(\int_{\Omega^1}\mathcal{Q}_2(\mathcal{G}(x',0)_{2\times2}) - \int_{\Omega^1}\frac{x_3^2}{2}\big\langle L_2\big(\mathcal{G}(x',0)_{2\times2}\big) : M\big\rangle + \int_{\Omega^1}\frac{x_3^4}{16}\mathcal{Q}_2(M)\Big)\\ &= \frac{1}{2}\Big(\int_{\Omega}\mathcal{Q}_2(\mathcal{G}(x',0)_{2\times2}) - \frac{1}{24}\int_{\Omega}\big\langle L_2\big(\mathcal{G}(x',0)_{2\times2}\big) : M\big\rangle + \frac{1}{16}(\frac{1}{144} + \frac{1}{180})\int_{\Omega}\mathcal{Q}_2(M)\,\mathrm{d}x'\Big)\\ &= \frac{1}{2}\Big(\int_{\Omega}\mathcal{Q}_2\big(\mathcal{G}(x',0)_{2\times2} - \frac{1}{48}M\big)\,\mathrm{d}x' + \frac{1}{16}\frac{1}{180}\int_{\Omega}\mathcal{Q}_2(M)\,\mathrm{d}x'\Big)\\ &\geq c\int_{\Omega}\mathcal{Q}_2(M)\,\mathrm{d}x'. \end{aligned}$$

Above, $L_2:\mathbb{R}^{2\times2}\to\mathbb{R}^{2\times2}$ stands for the linear map with the property: $\mathcal{Q}_2(F) = \langle L_2(F):F\rangle$ and $\langle L_2(F):\tilde{F}\rangle = \langle L_2(\tilde{F}):F\rangle$ for all $F,\tilde{F}\in\mathbb{R}^{2\times2}$. The proof of Theorem 7.1 is now complete in view of the nondegeneracy of the quadratic form $\mathcal{Q}_2$ on symmetric matrices. ∎

Part C: Application

8. Application to liquid crystal glass

Nematic elastomers are rubber-like, cross-linked, polymeric solids, which have both positional elasticity (due to the solid response of the polymer chains) and the orientation elasticity (due to the separately deforming director). A nematic glass is a very highly cross-linked nematic elastomer such that the director is effectively constrained to move with the elastomer matrix.

In this section we consider a model of nematic glass [24, 20] whose referential conformation A corresponds to a prolate ellipsoid, elongating the eigenvector $\vec{n}$ by the factor λ, while shrinking the invariant 2d subspace $\vec{n}^\perp = \mathrm{span}(v,w)$ by factor λ^ν:

$$A = \lambda^{-\nu} v\otimes v + \lambda^{-\nu} w\otimes w + \lambda \vec{n}\otimes\vec{n} = \lambda^{-\nu}\big(\mathrm{Id}_3 + (\lambda^{\nu+1}-1)\vec{n}\otimes\vec{n}\big), \qquad \lambda>1, \quad |\vec{n}|=1.$$

In other circumstances, A corresponds to a contraction λ in direction of $\vec{n}$ and an expansion $\lambda^{-\nu}$ in the perpendicular directions, and so λ could also be less than 1 [21]. The coefficient ν is experimentally verified to be in the range $\frac{1}{2}<\nu<2$. Setting $r=\lambda^{\nu+1}$, and writing $\lambda^{-\nu}=r^\delta$ with $\delta=-\frac{\nu}{\nu+1}$, we obtain the metric G and its symmetric square root $A=\sqrt{G}$ given by:

$$G(x',x_3) = G(x') = r^{2\delta}(\mathrm{Id}_3 + (r^2-1)\vec{n}\otimes\vec{n}), \qquad A(x') = r^\delta(\mathrm{Id}_3+(r-1)\vec{n}\otimes\vec{n}). \tag{8.1}$$

We start by the following observation:

Theorem 8.1. *Assume that:*

$$\vec{n}\in S^1 \quad \textit{i.e. } \vec{n}=(n_1,n_2,0)^T\in S^2, \ \textit{with } n=(n_1,n_2)\in S^1. \tag{8.2}$$

Then the following conditions are equivalent:

(i) *the metric G as in (8.1) is immersible, i.e. $G=(\nabla u)^T\nabla u$ for some smooth $u:\Omega^1\to\mathbb{R}^3$,*
(ii) *the Gaussian curvature $\kappa(\mathrm{Id}_2+(r^2-1)n\otimes n)$ vanishes identically in Ω,*
(iii) $\mathrm{curl}^T\mathrm{curl}\ G_{2\times 2}=0$,
(iv) *the following curvatures of G vanish: $R^3_{112}=R^3_{221}=R_{1212}=0$.*

Proof. It is clear that (i) holds iff the Riemann tensor of G vanishes, which is equivalent to:

$$\kappa = \kappa(\mathrm{Id}_2+(r^2-1)n\otimes n)=0.$$

We now calculate the Gaussian curvature κ of the 2d metric $\mathrm{Id}_2+(r^2-1)n\otimes n$ and prove that:

$$\kappa = r^{2\delta}\kappa(G_{2\times 2}) = \frac{1}{2}(r^{-2}-1)\mathrm{curl}^T\mathrm{curl}\ (n\otimes n).$$

This will achieve the lemma, because $R^3_{112}=R^3_{221}=0$ automatically, while $R_{1212}=0$ is equivalent to (ii). We write $r^2-1=\gamma>0$ and compute:

$$\begin{aligned}\big(\det(\mathrm{Id}_2+\gamma n\otimes n)\big)^2\kappa = {}&\det\begin{bmatrix} q & \frac12 e_{,1} & f_{,1}-\frac12 e_{,2}\\ f_{,2}-\frac12 g_{,1} & e & f\\ \frac12 g_{,2} & f & g\end{bmatrix}\\ &-\det\begin{bmatrix} 0 & \frac12 e_{,2} & g_{,1}\\ \frac12 e_{,2}-\frac12 g_{,1} & e & f\\ \frac12 g_{,1} & f & g\end{bmatrix},\end{aligned}$$

where $q=-\frac{1}{2}e_{,22}+f_{,12}-\frac{1}{2}g_{,11}$ and $e=1+\gamma n_1^2$, $f=\gamma n_1 n_2$, $g=1+\gamma n_2^2$. A direct calculation now gives that the right hand side above equals:

$$\begin{aligned}(1+\gamma)q+\gamma^3\cdot 0+\gamma^2\big(&-n_2^2 n_{1,1}n_{2,2}-n_1n_2n_{2,1}n_{2,2}+n_1n_2n_{1,2}n_{2,2}-n_1n_2n_{1,1}n_{1,2}\\ &-n_1^2n_{1,1}n_{2,2}+n_1n_2n_{1,1}n_{2,1}+n_2^2n_{2,1}^2+n_1^2n_{1,2}^2\big)=(1+\gamma)q.\end{aligned}$$

The equality above follows since all the terms in the bracket multiplying γ^2 cancel out. This can be easily seen by substituting $(n_1,n_2)=(\cos\theta,\sin\theta)$ for some angle function $\theta:\Omega\to\mathbb{R}$.

We hence see that $\kappa=0$ iff $q=0$. But note that:

$$q=-\frac{\gamma}{2}\big((n_1^2)_{,22}-2(n_1n_2)_{,12}+(n_2^2)_{,11}\big)=-\frac{\gamma}{2}\mathrm{curl}^T\mathrm{curl}(n\otimes n).$$

Since $\det(\mathrm{Id}_2+(r^2-1)n\otimes n)=1+\gamma$, it follows that $\kappa=-\frac{1}{2}\frac{\gamma}{\gamma+1}\mathrm{curl}^T\mathrm{curl}(n\otimes n)$. ∎

Example 8.2. In accordance with (8.1), the following metric has been put forward in [19] for the description of disclination-mediated thermo-optical response in nematic glass sheets: $G(x',x_3)=\alpha\mathrm{Id}_3+\beta\vec{n}(x')\otimes\vec{n}(x')$, where $\alpha,\beta>0$ are constants, and $\vec{n}$ is as in (8.2) with:

$$n_1=\cos(\theta+\psi),\quad n_2=\sin(\theta+\psi),\qquad \theta=\arctan\frac{x_2}{x_1},\quad \psi\equiv const.$$

Note that θ is the polar angle and so setting the constant $\psi=0$ gives the radial pattern, while $\psi=\pi/2$ gives the azimuthal pattern, and other values of ψ yield spiral patterns. It is easy to check that $\mathrm{curl}^T\mathrm{curl}(n\otimes n)=0$. Therefore, if the simply connected Ω does not contain 0 (since 0 is a singularity for G), then the metrics $G_{2\times 2}$ and G are immersible by Theorem 8.1, and thus $\inf E^h=0$. However if $0\in\Omega$, one may not have a global immersion (implying hence a higher energy scaling; see [20] for a construction with h^2 scaling).

Remark 8.3. Consider any 2d metric $G_{2\times 2}$ with constant eigenvalues $0<\lambda_1\le\lambda_2$:

$$G_{2\times 2}=\lambda_1 v\otimes v+\lambda_2 w\otimes w=\lambda_1(\mathrm{Id}_2-\frac{\lambda_2-\lambda_1}{\lambda_1}w\otimes w).$$

We see that such $G_{2\times 2}$ is flat iff $\mathrm{curl}^T\mathrm{curl}(G_{2\times 2})=0$. Interestingly, $\mathrm{curl}^T\mathrm{curl}$ is the leading order term in the expansion of the Gaussian curvature of a 2d metric at Id_2.

In the 2d case as in (8.2), we directly obtain:

Theorem 8.4. *Assume that G is as in (8.1) with (8.2). Then, Theorems 2.1 and 3.1 hold with the Cosserat vector $\vec{b}$ given by:*

$$\vec{b}=r^\delta\vec{N}$$

and with the limiting functional:

$$\begin{aligned}\mathcal{I}_G(y)=\mathcal{I}_{\vec{n}}(y)&=\frac{1}{24}\int_\Omega\mathcal{Q}_2\left(x',r^\delta(\nabla y)^T\nabla\vec{N}\right)\,\mathrm{d}x'\\ &=\frac{1}{24}\int_\Omega\mathcal{Q}_2^0\left(r^\delta(A_{2\times 2})^{-1}(\nabla y)^T\nabla\vec{N}(A_{2\times 2})^{-1}\right)\,\mathrm{d}x'.\end{aligned}$$

Denote:

$$\alpha=\frac{r-1}{r}.$$

Then:

$$(A_{2\times 2})^{-1}=\frac{1}{r^{1+\delta}}\left(\mathrm{Id}_2+(r-1)n^\perp\otimes n^\perp\right)=\frac{1}{r^\delta}\left(\mathrm{Id}_2-\alpha\ n\otimes n\right)$$

and the quadratic form in the second integrand in (8.4) can be equivalently expressed as:

$$\begin{aligned} & r^{\delta}(A_{2\times 2})^{-1}(\nabla y)^T\nabla \vec N (A_{2\times 2})^{-1} \\ & \qquad = r^{-\delta}\Big((\nabla y)^T\nabla\vec N - 2\alpha \mathrm{sym}\big((n\otimes\partial_n y)\nabla\vec N\big) + \alpha^2\left\langle \partial_n y, \partial_n\vec N\right\rangle n\otimes n\Big). \end{aligned}$$

Moreover, when W is isotropic so that (4.4) holds, we have:

$$\forall F_{2\times 2}\in\mathbb{R}^{2\times 2}_{sym} \qquad \mathcal{Q}_2(x', F_{2\times 2}) = \frac{1}{r^{4\delta}}\mathcal{Q}^0_{2,iso}\Big((\mathrm{Id}_2 - \alpha\ n\otimes n)F_{2\times 2}(\mathrm{Id}_2-\alpha\ n\otimes n)\Big) \tag{8.3}$$

We now turn to the general case of the general 3d director $\vec n$.

Theorem 8.5. *Assume that G is of the form (8.1). Let $n=(n_1,n_2)\in\mathbb{R}^2$ denote the tangential component of the director vector $\vec n$. Denote also:*

$$\gamma = \frac{1}{n_3^2 + |n|^2 r^2}.$$

Then Theorems 2.1 and 3.1 hold with the Cosserat vector $\vec b$ is given by:

$$\vec b = (r^2-1)n_3\gamma(\partial_n y) + r\sqrt{\gamma}(r^\delta \vec N) = \frac{(r^2-1)n_3}{n_3^2+|n|^2r^2}\partial_n y + \frac{r^{1+\delta}}{\sqrt{n_3^2+|n|^2r^2}}\vec N \tag{8.4}$$

and with the limiting functional:

$$\mathcal{I}_G(y) = \mathcal{I}_{\vec n}(y) = \frac{1}{24}\int_\Omega \mathcal{Q}_2\left(x', (\nabla y)^T\nabla\vec b\right)\ \mathrm{d}x'.$$

Moreover, when W is isotropic so that (4.4) holds, we have:

$$\begin{aligned} \forall F_{2\times 2}\in\mathbb{R}^{2\times 2}_{sym} \qquad & \mathcal{Q}_2(x', F_{2\times 2}) = \\ & = \mu\frac{1}{r^{4\delta}}\Big(|F_{2\times 2}|^2 - 2\big((r^2-1)\gamma\big)|F_{2\times 2}n|^2 + \big((r^2-1)\gamma\big)^2\langle F_{2\times 2}n, n\rangle^2\Big) \\ & + \frac{\lambda\mu}{\lambda+\mu}\frac{1}{r^{4\delta}}\Big(\mathrm{tr}F_{2\times 2} - \big((r^2-1)\gamma\big)\langle F_{2\times 2}n,n\rangle\Big)^2. \end{aligned} \tag{8.5}$$

Setting $\tilde\gamma = \frac{1}{|n|^2}\left(1-\sqrt{\gamma}\right)$, the above formula is equivalent to:

$$\mathcal{Q}_2(x', F_{2\times 2}) = \frac{1}{r^{4\delta}}\begin{cases} \mathcal{Q}^0_{2,iso}\Big((\mathrm{Id}_2 - \tilde\gamma\ n\otimes n)F_{2\times 2}(\mathrm{Id}_2 - \tilde\gamma\ n\otimes n)\Big) & \text{if } n_3(x')^2<1, \\ \mathcal{Q}^0_{2,iso}(F_{2\times 2}) & \text{if } n_3(x')^2 = 1. \end{cases} \tag{8.6}$$

Proof. We easily compute:

$$\begin{aligned} & \det G = r^{2+6\delta}, \qquad \det G_{2\times 2} = r^{4\delta}(r^2-(r^2-1)n_3^2) \\ & (G_{2\times 2})^{-1} = \frac{1}{r^{2\delta}}(n_3^2+|n|^2r^2)^{-1}\Big(\mathrm{Id} - (r^2-1)n^\perp\otimes n^\perp\Big), \end{aligned}$$

where $n^\perp = (n_1,n_2)^\perp = (-n_2,n_1)$. Therefore:

$$(G_{2\times 2})^{-1}\begin{bmatrix} G_{13} \\ G_{23}\end{bmatrix} = \frac{(r^2-1)n_3}{n_3^2+|n|^2r^2}n$$

which implies the formula (8.4).

To prove (8.6) in view of Theorem 4.3, it is now enough to check directly that the positive definite matrix $r^{-\delta}\left(\mathrm{Id}_2 - \tilde{\gamma} n \otimes n\right)$ equals $\sqrt{G_{2\times 2}}^{-1}$. Indeed:

$$\begin{aligned}(\mathrm{Id}_2 - \tilde{\gamma} n \otimes n)^2 \big(\mathrm{Id}_2 + (r^2-1) n \otimes n\big) &= \big(\mathrm{Id}_2 + (\tilde{\gamma}|n|^2 - 2\tilde{\gamma}) n \otimes n\big)\big(\mathrm{Id}_2 + (r^2-1) n \otimes n\big) \\ &= \big(\mathrm{Id}_2 - (r^2-1)\gamma n \otimes n\big)\big(\mathrm{Id}_2 + (r^2-1) n \otimes n\big) = \mathrm{Id},\end{aligned}$$

as $\tilde{\gamma}^2|n|^2 - 2\tilde{\gamma} = -(r^2-1)\gamma$. ■

Remark 8.6. The expression in (8.3) is consistent with (8.6), as for $n_3 = 0$ it follows that $\gamma = \frac{r^2-1}{r^2}$ and $\tilde{\gamma} = 1 - 1/r = \alpha$. The expression in (8.3) is also consistent with Remark 4.4, in the following sence. Take $\vec{n} = e_3$. Then $D = r^{-2\delta}\mathrm{diag}(1,1,r^{-1})F^*_{2\times 2}\mathrm{diag}(1,1,r^{-1}) = r^{-2\delta}F^*_{2\times 2}$. Hence, by (4.10):

$$\mathcal{Q}_2(x', F_{2\times 2}) = \frac{1}{r^{4\delta}}\left(\mu|F_{2\times 2}|^2 + \frac{\lambda\mu}{\lambda+\mu}|\mathrm{tr}F_{2\times 2}|^2\right),$$

while (8.5), (8.6) give the same formula directly.

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