

# On the $L^1$ stability of multi-shock solutions to the Riemann problem

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## 1. Introduction

In this article we present a summary of some recent results concerning the  $L^1$  stability of non-linear large shock waves, that arise in the study of strictly hyperbolic systems of conservation laws in one space dimension. For the detailed discussion and the proofs we refer to papers [6] [7] [8] [9].

The system we consider has the following general form:

$$u_t + f(u)_x = 0 \tag{1.1}$$

with the flux function  $f$  satisfying:

- $f : \Omega \rightarrow \mathbf{R}^n$  is smooth, defined on some open set  $\Omega \subset \mathbf{R}^n$ .
- (1.1) is strictly hyperbolic in  $\Omega$ , that is: at every point  $u \in \Omega$  the matrix  $Df(u)$  has  $n$  real and simple eigenvalues  $\lambda_1(u) < \dots < \lambda_n(u)$ .
- Each characteristic field of (1.1) is either linearly degenerate or genuinely nonlinear, that is: with a basis  $\{r_k(u)\}_{k=1}^n$  of corresponding right eigenvectors of  $Df(u)$ ,  $Df(u)r_k(u) = \lambda_k(u)r_k(u)$ , each of the  $n$  directional derivatives  $r_k \nabla \lambda_k$  vanishes either identically or nowhere.

In this setting, it has been shown in [2] [3] that the Cauchy problem (1.1) (1.2) with:

$$u(0, x) = \bar{u}(x) \tag{1.2}$$

is wellposed in  $L^1(\mathbf{R}, \mathbf{R}^n)$ , within the class of initial data  $\bar{u} \in L^1 \cap BV(\mathbf{R}, \mathbf{R}^n)$  having suitably small total variation. Namely, the entropy solutions of (1.1) (1.2) constitute a semigroup which is Lipschitz continuous with respect to time and initial data. A major question which remains open is whether existence and uniqueness of solutions also holds for arbitrarily large initial data. As a first step towards this analysis we study the wellposedness of (1.1) (1.2) with the initial data  $\bar{u}$  being a small perturbation of fixed Riemann data  $(u_0^0, u_0^M)$ . The solution of the latter consists of  $M \in \{2 \dots n\}$  (large) waves of different characteristic families; the discussion we present below concerns a particular case when all these large waves are shocks.

More precisely,  $M + 1$  distinct states  $\{u_0^q\}_{q=0}^M$  are fixed and their respective sufficiently small neighbourhoods  $\Omega^q$  are chosen, with  $\Omega = \bigcup_{q=0}^M \Omega^q$ . We assume that the problem (1.1) (1.2) with

$$\bar{u}(x) = \begin{cases} u_0^0 & x < 0 \\ u_0^M & x > 0 \end{cases} \tag{1.3}$$

has an  $M$ -shock solution:

$$u(t, x) = \begin{cases} u_0^0 & x < \Lambda^1 t \\ u_0^q & \Lambda^q t < x < \Lambda^{q+1} t, \quad q : 1 \dots M - 1 \\ u_0^M & x > \Lambda^M t, \end{cases} \tag{1.4}$$

in which the states  $u_0^q$  are joined by  $M$  (large) shocks  $(u_0^{q-1}, u_0^q)$ ,  $q : 1 \dots M$ , travelling with respective speeds  $\Lambda^q$  (see Figure 1.1).

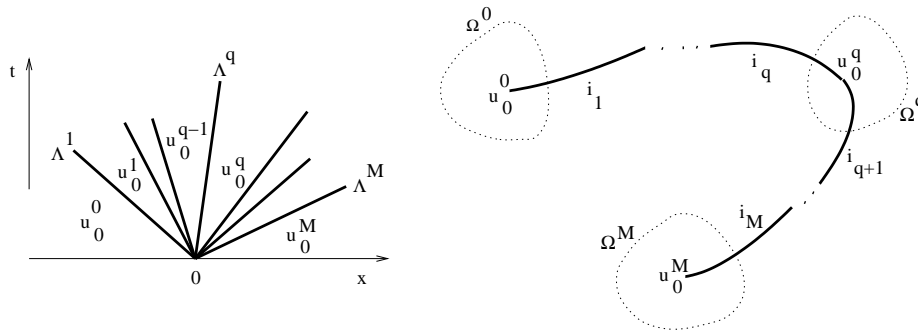


FIGURE 1.1

The following standard conditions on the nature of the large shocks are assumed. For (1.4) to be a distributional solution of (1.1) (1.2) (1.3), we need that for every shock  $q : 1 \dots M$  the Rankine-Hugoniot conditions are satisfied:

$$f(u_0^{q-1}) - f(u_0^q) = \Lambda^q (u_0^{q-1} - u_0^q). \tag{1.5}$$

Moreover, the shocks  $(u_0^{q-1}, u_0^q)$  are said to belong to the corresponding  $i_q$ -characteristic families ( $1 \leq i_1 < i_2 < \dots < i_M \leq n$ ) and assumed to be compressive in the sense of Lax [5]:

$$\lambda_{i_q}(u_0^{q-1}) > \Lambda^q > \lambda_{i_q}(u_0^q). \tag{1.6}$$

Finally, we require that all large shocks are stable in the sense of Majda [11], that is; for every  $q : 1 \dots M$ :

The  $n$  vectors

$$r_1(u_0^{q-1}), \dots, r_{i_q-1}(u_0^{q-1}), u_0^q - u_0^{q-1}, r_{i_q+1}(u_0^q), \dots, r_n(u_0^q) \tag{1.7}$$

are linearly independent.

The following questions arise naturally:

- A. Do we have the (global in time and space) existence of an 'admissible' solution  $u$  to (1.1) (1.2) when  $\bar{u}$  stays 'close' to the Riemann data (1.3)?
- B. In case the answer to A is positive, is the solution  $u$  stable under small perturbations of its initial data?

Unlike in the case of small initial data, the assumptions introduced so far are not sufficient to ensure the positive answer to any of the above questions. The more, even the solvability of Riemann problems  $(u^-, u^+)$  with  $u^-, u^+ \in \Omega$  is not just a simple consequence of the existence of the solution (1.4), but requires an additional hypothesis. This and other stability conditions implying positive answers to questions A and B will be introduced and discussed in the next section.

### 2. Stability conditions

Consider a small wave of a family  $k \leq i_q$ , travelling with speed  $\lambda_k^{in}$ , and hitting from the right the large initial  $i_q$ -shock  $(u_0^{q-1}, u_0^q)$ , as in Figure 2.1. Condition (1.7) guarantees that the Riemann problem  $(u_0^{q-1}, u_0^q)$  can be uniquely solved; let  $M_q^r$  be the  $(n - 1) \times i_q$  matrix expressing the strengths of the small outgoing waves (travelling with respective speeds  $\lambda_s^{out}$ ) in terms of the strength incoming in the considered interaction, while another matrix  $N_q^r$  encloses also the information on the shift ratios after/before the interaction:

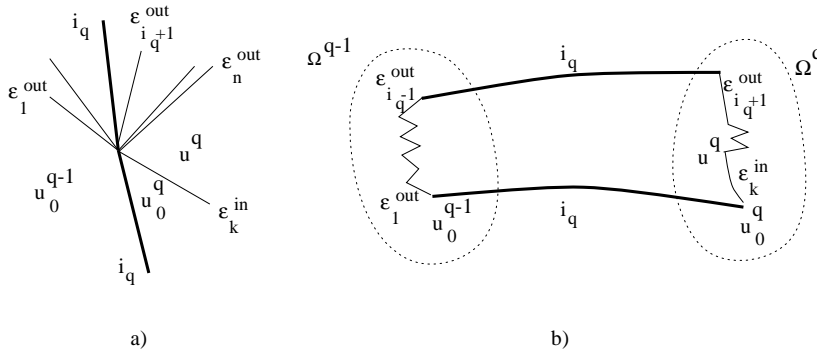


FIGURE 2.1

$$\begin{aligned}
 M_q^r &= [a_{sk}^q]_{\substack{s:1\dots n, s \neq i_q, \\ k:1\dots i_q}}, & a_{sk}^q &= \frac{\partial \epsilon_s^{out}}{\partial \epsilon_k^{in}} \Big|_{\epsilon_k^{in}=0} \\
 N_q^r &= [b_{sk}^q]_{\substack{s:1\dots n, s \neq i_q, \\ k:1\dots i_q}}, & b_{sk}^q &= \frac{\partial}{\partial \epsilon_k^{in}} \Big|_{\epsilon_k^{in}=0} \left( \epsilon_s^{out} \cdot \frac{\lambda_s^{out} - \Lambda^q}{\lambda_k^{in} - \Lambda^q} \right).
 \end{aligned}
 \tag{2.1}$$

Analogously, we define the matrices  $M_q^l, N_q^l$ , describing outgoing/incoming strengths or strengths and shifts ratios in wave patterns when the interacting small wave approaches from the left; in this case the index  $k$  changes from  $i_q$  to  $n$  (see Figure 2.2).

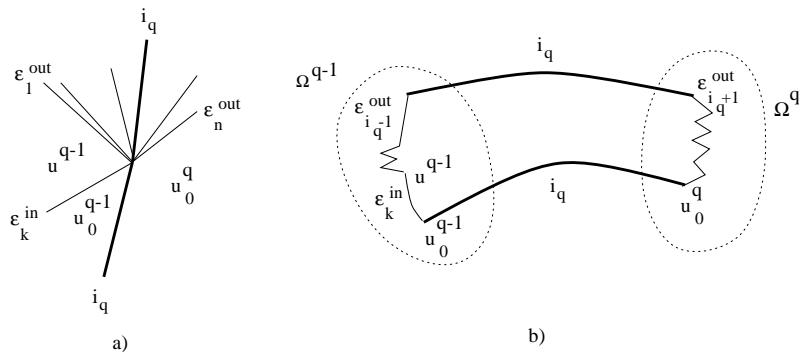


FIGURE 2.2

Define now the square  $M \cdot (n - 1)$  dimensional matrix  $\mathbf{W}_0$ :

$$\mathbf{W}_0 = \begin{bmatrix} [\Theta] & M_1^r & & & \\ M_2^l & [\Theta] & M_2^r & & \\ & M_3^l & [\Theta] & M_3^r & \\ & & \ddots & \ddots & \\ & & & M_M^l & [\Theta] \end{bmatrix}$$

(here  $[\Theta]$  stands for the  $(n-1) \times (n-1)$  zero matrix), and another matrix  $\widetilde{\mathbf{W}}_0$ , in the same manner as  $\mathbf{W}_0$ , but with the submatrices  $M_q$  replaced by the corresponding  $N_q$ . Finally, define the nonnegative matrices  $\mathbf{W}_1 := |\mathbf{W}_0|$  and  $\mathbf{W}_2 := |\widetilde{\mathbf{W}}_0|$ , that consists of the absolute values of the entries of the corresponding matrices.

Now we are ready to formulate our first set of stability conditions:

$$\text{FINITENESS CONDITION :} \quad 1 \notin \text{spec } \mathbf{W}_0, \quad (2.2)$$

$$\text{BV STABILITY CONDITION :} \quad \text{specRad } \mathbf{W}_1 < 1, \quad (2.3)$$

$$L^1 \text{ STABILITY CONDITION :} \quad \text{specRad } \mathbf{W}_2 < 1. \quad (2.4)$$

Above, 'spec' stands for the spectrum and 'specRad' for the spectral radius of a given matrix.

**Proposition 2.1.** *The condition (2.2) is weaker than (2.3), which is in turn implied by (2.4).*

The conditions (2.3) and (2.4), easy to check for the concrete wave patterns, are however less convenient when one needs to control the change in the total variation or the  $L^1$  norm of the profile of the solution  $u(t, \cdot)$  across the interaction time. We thus reformulate our conditions in the following way.

WEIGHTED BV STABILITY CONDITION

There exist positive weights  $w_1^q, \dots, w_n^q$  (for every  $q : 0 \dots M$ ) such that in the setting of Figure 2.1 and Figure 2.2, respectively:

$$\begin{aligned} \sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^q} \cdot |a_{sk}^q| + \sum_{s=i_q+1}^n \frac{w_s^q}{w_k^q} \cdot |a_{sk}^q| < 1, \\ \sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^{q-1}} \cdot |a_{sk}^q| + \sum_{s=i_q+1}^n \frac{w_s^q}{w_k^{q-1}} \cdot |a_{sk}^q| < 1. \end{aligned} \tag{2.5}$$

**Remark.** Regarding  $w_s^q$  as the weight given to an  $s$ -wave located in the region between the  $q-1$  and the  $q$ -th large shock, conditions (2.5) simply say that, every time a small wave hits a large shock, the total weighted strength of the outgoing small waves is smaller than the weighted strength of the incoming wave.

WEIGHTED  $L^1$  STABILITY CONDITION

In the setting of Figure 2.1 and Figure 2.2, respectively:

$$\begin{aligned} \sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^q} \cdot |b_{sk}^q| + \sum_{s=i_q+1}^n \frac{w_s^q}{w_k^q} \cdot |b_{sk}^q| < 1, \\ \sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^{q-1}} \cdot |b_{sk}^q| + \sum_{s=i_q+1}^n \frac{w_s^q}{w_k^{q-1}} \cdot |b_{sk}^q| < 1. \end{aligned} \tag{2.6}$$

**Proposition 2.2.** *The conditions (2.3) and (2.5) are equivalent. The conditions (2.4) and (2.6) are equivalent.*

**3. Examples**

Before we present our main results concerning the wellposedness of the system (1.1), we first examine the introduced stability conditions for two well known hyperbolic conservative systems: the  $p$ -system and the full  $\gamma$ -law gas dynamics.

**Proposition 3.1.** *For the  $p$ -system:*

$$u_t - v_x = 0, \quad v_t + p(u)_x = 0,$$

where  $p > 0$ ,  $p' < 0$ ,  $p'' > 0$  and  $u > 0$ , the condition (2.4) (and thus also (2.3)) is satisfied for any initial shock pattern (1.4).

**Proposition 3.2.** *For the  $\gamma$ -gas-law Euler equations:*

$$\rho_t + (\rho v)_x = 0, \quad (\rho v)_t + (\rho v^2 + P)_x = 0,$$

$$\left(\frac{\gamma-1}{2}\rho v^2 + P\right)_t + \left(\frac{\gamma-1}{2}\rho v^3 + \gamma P v\right)_x = 0,$$

there exist two critical adiabatic exponents  $\gamma_2 > \gamma_1 > 1$  such that the following is true. For  $\gamma > \gamma_2$  the condition (2.4) is always satisfied, and for  $\gamma > \gamma_1$  the condition (2.3) holds. On the other hand, for every  $\gamma \in (1, \gamma_1)$  there indeed exist Riemann problems for which (2.3) fails; similarly for every  $\gamma \in (\gamma_1, \gamma_2)$  there exist a pattern (1.4) such that (2.4) fails (although (2.3) holds).

### 4. Main results

In this section we explain how the stability conditions (2.2) (2.3) (2.4) imply (in different extents) the wellposedness of (1.1). Our first result concerns the solvability of Riemann problems with initial states in  $\Omega$ .

**Proposition 4.1.** *Let the Finiteness Condition (2.2) hold. With any Riemann data  $(u^-, u^+)$ ,  $u^- \in \Omega^i, u^+ \in \Omega^j, 0 \leq i \leq j \leq M$ , (1.1) has a unique self-similar solution, attaining  $n + 1$  states, consecutively connected by:*

- weak waves of the corresponding families (if both left and right states of a pair under consideration belong to the same set  $\Omega^q, i \leq q \leq j$ ),
- $j - i$  large shocks, joining the states belonging to different sets  $\Omega^q$ ,

as in Figure 4.1.

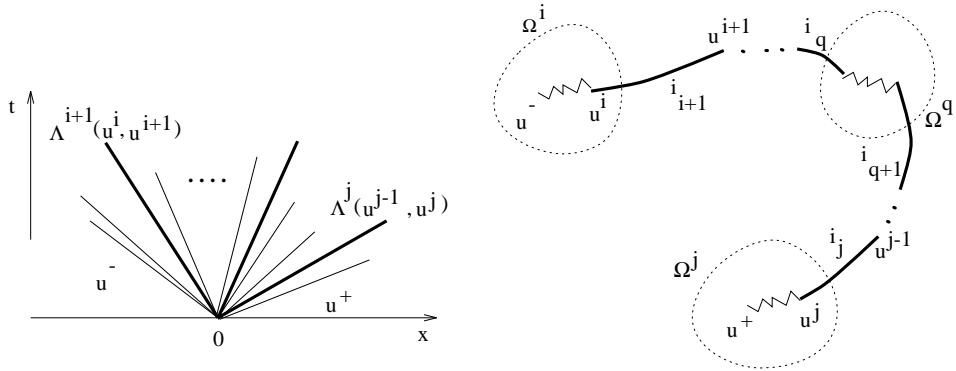


FIGURE 4.1

Now we turn to the main point of this article. Define the domain  $\tilde{\mathcal{D}}_{\delta_0}$  by:

$$\tilde{\mathcal{D}}_{\delta_0} = \text{cl} \left\{ u : \mathbf{R} \rightarrow \mathbf{R}^n; \text{ there exist points } x^1 < x^2 < \dots < x^M \text{ in } \mathbf{R} \right.$$

$$\left. \begin{aligned} \text{such that calling } \tilde{u}(x) = & \begin{cases} u_0^0 & x < x^1 \\ u_0^q & x^q < x < x^{q+1}, \quad q : 1 \dots M-1 \\ u_0^M & x > x^M \end{cases} \quad (4.1) \\ \text{we have: } u - \tilde{u} \in L^1(\mathbf{R}, \mathbf{R}^n) \text{ and T.V.}(u - \tilde{u}) \leq \delta_0 \end{aligned} \right\},$$

with the closure taken in  $L^1_{\text{loc}}(\mathbf{R}, \mathbf{R}^n)$ .

Our main results are the following:

**Theorem A** *If the Stability Condition (2.5) is satisfied then there exists  $\delta_0 > 0$  such that for every  $\bar{u} \in \tilde{\mathcal{D}}_{\delta_0}$  (1.1) (1.2) has a solution (defined for all times  $t \geq 0$ ).*

**Theorem B** *If the Stability Condition (2.6) is satisfied then there exist  $\delta_0 > 0$ ,  $L > 0$ , a closed domain  $\mathcal{D}_{\delta_0} \subset L^1_{\text{loc}}(\mathbf{R}, \mathbf{R}^n)$  containing  $\tilde{\mathcal{D}}_{\delta_0}$ , and a continuous semigroup  $S : [0, \infty) \times \mathcal{D}_{\delta_0} \rightarrow \mathcal{D}_{\delta_0}$  such that:*

- (i)  $S(0, \bar{u}) = \bar{u}$ ,  
 $S(t + s, \bar{u}) = S(t, S(s, \bar{u})) \quad \forall t, s \geq 0 \quad \forall \bar{u} \in \mathcal{D}_{\delta_0}$ .
- (ii)  $\|S(t, \bar{u}) - S(s, \bar{w})\|_{L^1} \leq L \cdot (|t - s| + \|\bar{u} - \bar{w}\|_{L^1}) \quad \forall t, s \geq 0 \quad \forall \bar{u}, \bar{w} \in \mathcal{D}_{\delta_0}$ .
- (iii) *Each trajectory  $t \mapsto S(t, \bar{u})$  is a solution of (1.1) (1.2).*

Towards the proof of Theorem A, one explicitly defines the Glimm potential, measuring the total strength of all small waves in the approximate wave front tracking solutions of (1.1), and the possible amount of interaction between themselves or with the large shocks. Thanks to the condition (2.5), this potential is nonincreasing in time, and grants us the *BV* stability estimates yielding the global existence of the solution, as in Theorem A.

To prove Theorem B, the Lyapunov functional, measuring the  $L^1$  distance between the profiles of the wave front tracking approximate solutions, is introduced. Our functional is motivated by the similar one in [3]; the difference is that it contains some extra terms accounting for the interactions and coupling of the small waves against the large shocks. The key point in the analysis is to prove that this functional 'almost' decreases in time along any pair of approximate wave front tracking solutions to (1.1) - this property allows us to conclude that the solutions obtained in Theorem A constitute a Lipschitz continuous semigroup  $S$ .

## 5. Relations to previous works

We now comment on the relation of the results in this article to other papers. In [12], Schochet was the first to introduce a  $BV$  stability condition, giving positive answer to question A. This condition is formulated inductively with respect to the number of large shocks  $M$  and uses the language of matrix analysis, in the spirit of our condition (2.3). As shown and accompanied by a more detailed discussion in [7], the Schochet condition and our conditions (2.3) and (2.5) are equivalent. In [7], one can also find the proofs of Propositions 2.1 and 2.2.

In [1], Bressan and Colombo consider the general Riemann problem for systems of two equations and assuming the corresponding  $L^1$  stability condition, answer question B positively. More recently, the paper [9] proves Theorems A and B (for systems of  $n \geq 2$  equations) in the presence of only two large shocks, of characteristic families  $i$  and  $j > i$ ; indeed in the case  $M = 2$ ,  $i_1 = i$ ,  $i_2 = j$ , the conditions (2.5) and (2.6) reduce to the corresponding conditions of [9].

Theorems A and B as stated in this article are proved in [6]. Substantial differences between  $M = 2$  and  $M > 2$  occur in particular in the proof of Theorem B. Namely, the straightforward generalization of the Lyapunov functional introduced in [9] does not provide a functional decreasing along the wave front tracking solutions, for  $M > 2$ . On the other hand, our new functional defined in [6], reduces when  $M = 2$  to a Lyapunov functional that can be seen as a simplification of the one from [9].

The Stability Condition (2.6), which came up naturally in the investigations leading to [6], was earlier introduced in [4] (formulae (3.42) and (3.43)), to guarantee the wellposedness of associated linearized variational systems.

Proposition 4.1 and the Finiteness Condition (2.2) are corollaries of the results in [8]. Recently it was brought to our attention that conditions similar to (2.2) – (2.4) could be found in [10], where the authors address the existence of smooth solutions to (1.1).

Proposition 3.1 follow from the discussion in [1], the existence of  $\gamma_1$  in Proposition 3.2 is clear from [12] in view of our Proposition 2.2. The part of Proposition 3.2 concerning the critical exponent  $\gamma_2$  appears here for the first time.

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