SIAM J. MATH. ANAL.
Vol. 32, No. 5, pp. 1094–1116

STABILITY CONDITIONS FOR PATTERNS OF NONINTERACTING LARGE SHOCK WAVES*

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Abstract. In this paper we study different conditions whose presence is required for

A. the admissibility and stability of large shocks present in solutions of a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension

$$u_t + f(u)_x = 0,$$

B. the solvability and L^1 well posedness of the Cauchy problem for the above equation, near solutions containing large and stable, but noninteracting shock waves.

We compare the corresponding conditions of type A and B appearing in the literature; in particular, we show that the finiteness and stability conditions used in our most recent works generalize and/or unify these conditions in appropriate ways.

Key words. conservation laws, shock waves, stability conditions, large BV data

AMS subject classifications. 35L65, 35L45

PII. S0036141000367503

1. Introduction.

Consider the Cauchy problem for an $n \times n$ system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0, \tag{1.1}$$

$$u(0, \cdot) = \bar{u}. \tag{1.2}$$

In the study of local existence and stability of solutions to (1.1), (1.2), due to the finite speed of propagation one is led to consider the special case where the initial data $\bar{u}$ is a small perturbation of a Riemann data:

$$\bar{u}(x) = \begin{cases} u^-, & x < 0, \\ u^+, & x > 0. \end{cases} \tag{1.3}$$

In this case, several results in the literature have shown that existence and stability of solutions can be obtained under a suitable linearized stability condition for the solutions of (1.1), (1.2), (1.3). (For a general theory of conservation laws in one space dimension, cf. [B], [D], [Sm].)

The main purpose of this paper is to compare the various assumptions of this kind and to prove their equivalence. We shall restrict ourselves to the case where the solution of (1.1), (1.2), (1.3) consists of $m+1$ constant states, $m \in \{2, \ldots, n\}$, separated by (possibly large) admissible shocks, say, in the characteristic families

*Received by the editors February 7, 2000; accepted for publication (in revised form) October 17, 2000; published electronically February 21, 2001. This research was partially supported by the European TMR Network on Hyperbolic Conservation Laws ERBFMRXCT960033.

http://www.siam.org/journals/sima/32-5/36750.html

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$i_1 < \cdots < i_m$. Calling these intermediate states $u_0^0 = u^-, u_0^1, u_0^2, \ldots, u_0^m = u^+$, and Λ^q the speed of the i_q shock, the linearized system has the form

$$v_t + Df(u_0^q) \cdot v_x = 0, \qquad x/t \in (\Lambda^q, \Lambda^{q+1}). \tag{1.4}$$

Along shock lines we have the boundary equations obtained by linearizing the Rankine–Hugoniot equations that yield the linear dependence of the strengths of the outgoing waves on the components of the incoming wave vector interacting with the i_q large shock under consideration:

$$\epsilon_k^{out} = \sum_{\substack{s:1\ldots n\\ incoming}} W_q^{k,s} \cdot \epsilon_s^{in} \tag{1.5}$$

(see Figure 1.1).

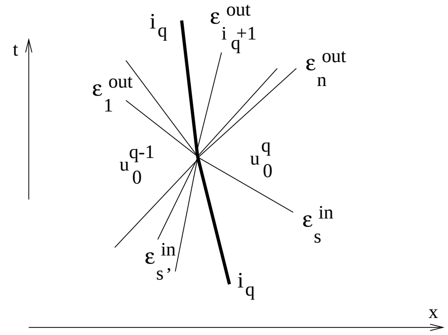

Fig. 1.1.

As we have mentioned, under some classical assumptions on the flux f in (1.1), which are recalled below, a variety of results concerning the (global) existence and uniqueness of admissible solutions to (1.1), (1.2) and their L^1 stability have been recently established [BC], [BM], [Le], [LT], [Scho], [W].

In all of these works, the conditions of two different natures are necessarily introduced:

A. conditions yielding the admissibility and stability of each of the large shocks in the reference solution of (1.1), (1.2), (1.3),
B. conditions guaranteeing the BV *stability* of the linearized system (1.4) [Scho], [BM], [Le], [W]. In [Scho], it is proved that they imply the local existence of solutions to the Cauchy problem, for data $\bar{u}$ suitably close to (1.3), and conditions providing the L^1 *stability* of the system (1.4) [BM], [BC], [Le]. It was proved that these in turn imply the L^1 stability of the nonlinear system (1.1), on a domain $\mathcal{D}$ of small BV perturbations of the data (1.3).

Our paper is organized as follows. In section 2 we focus on the conditions of type A, in particular, the well-known Majda stability condition [M].

Section 3 discusses different conditions of type B. In [Le], [LT], the stability conditions are formulated in terms of the existence of a suitable family of weights $w_s^q > 0$ such that the corresponding BV or L^1 norm of any solution of the linearized system (1.4) is nonincreasing in time. The main result of section 3 (Theorem 3.2) will show that the Schochet BV stability assumptions [Scho] are equivalent to BV stability assumptions in [Le]. Also, the L^1 stability condition in [BM], [Le], will appear to imply the mentioned BV stability (Theorem 3.1).

In the last section we treat the case of systems of $n = 2$ equations, with the presence of $m = 2$ large shocks and deal with the corresponding conditions introduced in [BC], [W], [LT].

We end this section by recalling the setting of the Cauchy problem (1.1), (1.2) (compare [Le]). In the n-dimensional state space $m+1$ distinct states $\{u_0^q\}_{q=0}^m$ are fixed, with their corresponding open disjoint neighborhoods $\{\Omega^q\}_{q=0}^m$ such that

- $f : \Omega \longrightarrow \mathbf{R}^n$ is smooth and defined on $\Omega = \bigcup_{q=0}^{m} \Omega^q \subset \mathbf{R}^n$.
- f is strictly hyperbolic in Ω, that is, at each point $u \in \Omega$, the matrix $Df(u)$ has n real and simple eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$.
- Each characteristic field of (1.1) is either linearly degenerate or genuinely nonlinear, that is, with a basis $\{r_k(u)\}_{k=1}^n$ of corresponding right eigenvectors of $Df(u)$, $Df(u)r_k(u) = \lambda_k(u)r_k(u)$, each of the n directional derivatives $r_k\nabla\lambda_k$ vanishes either identically or nowhere.

The solution to (1.1), (1.2) with the initial data

$$\bar{u}(x) = \begin{cases} u_0^0, & x < 0, \\ u_0^m, & x > 0 \end{cases} \tag{1.6}$$

is given by m shocks (u_0^{q-1}, u_0^q), $q : 1 \ldots m$, belonging to respective characteristic families i_q and travelling with respective speeds Λ^q:

$$u(t,x) = \begin{cases} u_0^0, & x < \Lambda^1 t, \\ u_0^q, & \Lambda^q t < x < \Lambda^{q+1} t, \quad q : 1 \ldots m-1, \\ u_0^m, & x > \Lambda^m t, \end{cases} \tag{1.7}$$

as in Figure 1.2.

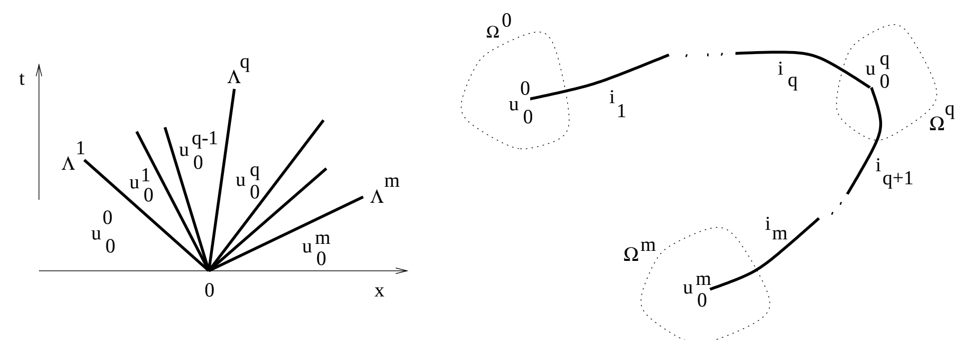

FIG. 1.2.

2. Stability of large shocks revisited. In this section we discuss the conditions of type A. Since every shock (u_0^{q-1}, u_0^q) has to be treated separately, it is not restrictive to assume that $m = 1$ and simplify the notation $u_0^0 = u^-, u_0^1 = u^+, \Omega^0 = \Omega^-, \Omega^1 = \Omega^m = \Omega^+, i_1 = i, \Lambda^1 = \Lambda$.

In this setting, for (1.7) to be a distributional solution of (1.1), (1.2), (1.3), the Rankine–Hugoniot conditions must be satisfied:

$$f(u^-) - f(u^+) = \Lambda(u^- - u^+). \tag{2.1}$$

Second, our i-shock is assumed to be compressive in the sense of Lax [L], that is,

$$\lambda_i(u^-) > \Lambda > \lambda_i(u^+). \tag{2.2}$$

Finally, in order to treat the Cauchy problem (1.1), (1.2), with $\bar{u}$ in (1.2) being a perturbation of (1.3), one must guarantee the so-called stability of the shock (u^-, u^+). This condition, introduced and justified in [LT], [Le], [BC] is the following:

(2.3) There exists a smooth function $\Psi : \Omega^- \times \Omega^+ \longrightarrow \mathbf{R}^{n-1}$ such that

(i) $\Psi(u^0, u^1) = 0$ iff the states u^0 and u^1 can be connected by a (large) shock of the ith characteristic family, with the speed $\Lambda(u^0, u^1)$. The Rankine–Hugoniot condition holds: $f(u^0) - f(u^1) = \Lambda(u^0, u^1)(u^0 - u^1)$. In particular, $\Psi(u^-, u^+) = 0$ and $\Lambda(u^-, u^+) = \Lambda$.

(ii)

$$\operatorname{rank} \frac{\partial \Psi}{\partial u^0}(u^-, u^+) = \operatorname{rank} \frac{\partial \Psi}{\partial u^1}(u^-, u^+) = n - 1.$$

(iii) The $n - 1$ vectors

$$\Big\{ \frac{\partial \Psi}{\partial u^0}(u^-, u^+) \cdot r_k(u^-) \Big\}_{k=1}^{i-1} \cup \Big\{ \frac{\partial \Psi}{\partial u^1}(u^-, u^+) \cdot r_k(u^+) \Big\}_{k=i+1}^{n}$$

are linearly independent.

Under these hypotheses one can see that if only the sets Ω^-, Ω^+ are small enough, then any Riemann problem $(u^0, u^1) \in \Omega^- \times \Omega^+$ for (1.1) has a unique self-similar solution composed of n shocks or rarefaction waves. The ith wave is a large i compressive Lax shock, connecting some states in the domains Ω^- and Ω^+.

In [Scho], the stability of the large shock (u^-, u^+) satisfying (2.1), (2.2) is understood in the classical sense of Majda:

(2.4) The n vectors

$$r_1(u^-), \dots, r_{i-1}(u^-), u^- - u^+, r_{i+1}(u^+), \dots, r_n(u^+)$$

are linearly independent.

Obviously for weak shocks (2.4) is always satisfied, and equivalent to (2.3)(iii). The main result of this section discusses this same situation in the general case.

Theorem 2.1. *Let (u^-, u^+) be a Rankine–Hugoniot shock such that its speed Λ in* (2.1) *is not an eigenvalue of $Df(u^-)$ nor of $Df(u^+)$. Then the conditions* (2.3) *and* (2.4) *are equivalent.*

The proof of Theorem 2.1 relies on the construction of a particular function Ψ_0, whose zero level set consists of those pairs of states $(u^0, u^1) \in \Omega^- \times \Omega^+$ that can be connected by an admissible i-shock as in (2.3)(i).

We define Ψ_0 as follows:

$$\Psi_0(u^0, u^1) = \Big\{ \Big\langle f(u^1) - f(u^0),\ V_k(u^1 - u^0) \Big\rangle \Big\}_{k=1}^{n-1}, \tag{2.5}$$

where V_k are any smooth functions defined on a neighborhood of the vector $u_0 = u^+ - u^- \neq 0$ with values in $\mathbf{R}^n$, and such that for every u the space

$$\mathrm{span}\{V_1(u), \dots, V_{n-1}(u)\}$$

is the orthogonal complement of the vector u.

LEMMA 2.2. *$\{V_k\}_{k=1}^{n-1}$ can be taken so that*

$$V_k(u_0) = -[DV_k(u_0)]^T \cdot u_0 \quad \forall k : 1 \dots n-1. \tag{2.6}$$

Proof. By $e_1, \dots, e_n$ we denote the standard Euclidean base of $\mathbf{R}^n$.

For u close to e_n define the vectors $\{\widetilde{V}_k(u)\}_{k=1}^{n-1}$ applying the Gramm–Schmidt orthogonalization process to n linearly independent vectors $u, e_1, \dots, e_{n-1}$. Namely, set

$$\begin{aligned} &\widetilde{V}_1(u) = e_1 - \langle e_1, u\rangle \cdot \tfrac{u}{|u|^2}, \\ &\widetilde{V}_k(u) = e_k - \left[\langle e_k, u\rangle \cdot \tfrac{u}{|u|^2} + \sum_{s=1}^{k-1} \langle e_k, \widetilde{V}_s(u)\rangle \cdot \widetilde{V}_s(u)\right] \quad \forall k : 2 \dots n-1. \end{aligned} \tag{2.7}$$

Note that

$$\widetilde{V}_k(e_n) = e_n \quad \forall k : 1 \dots n-1 \tag{2.8}$$

and

- $\langle \widetilde{V}_k(u), u\rangle = 0 \quad \forall k : 1 \dots n-1,$
- $\{\widetilde{V}_k\}_{k=1}^{n-1}$ are smooth functions of u.

Thus, $\mathrm{span}\{\widetilde{V}_1(u), \dots, \widetilde{V}_{n-1}(u)\}$ always complements orthogonally the vector u.

Moreover, using (2.8) and the fact that $\widetilde{V}_k \in \mathrm{span}(e_1, \dots, e_k, u)$, by the explicit formulas (2.7) one proves inductively that

$$D\widetilde{V}_k(e_n) = [d_{sl}]_{s,l:1\dots n}, \quad d_{sl} = \begin{cases} -1 & \text{for } (s,l) = (n,k), \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Now for u close to u_0 define

$$V_k(u) = A^{-1} \cdot \widetilde{V}_k(Au), \tag{2.10}$$

where A is an orthogonal transformation composed with an appropriate dilation such that $Au_0 = e_n$. Consequently

$$A^{-1} = |u_0|^2 A^T. \tag{2.11}$$

Obviously $\{V_k\}_{k=1}^{n-1}$ are smooth functions, and by the corresponding property of $\{\widetilde{V}_k\}_{k=1}^{n-1}$ they span the orthogonal complement of its argument vector.

By (2.10), (2.11), (2.9), and (2.8) we get

$$\begin{aligned} [DV_k(u_0)]^T \cdot u_0 =& A^T \cdot [D\widetilde{V}_k(e_n)]^T \cdot (A^T)^{-1} \cdot u_0 = A^{-1} \cdot [D\widetilde{V}_k(e_n)]^T \cdot Au_0 \\ =& -A^{-1} e_k = -A^{-1} \cdot \widetilde{V}_k(Au_0) = -V_k(u_0), \end{aligned}$$

which proves (2.6). ∎

Using the above lemma one finds a convenient formula for the derivatives of Ψ_0:

$$\frac{\partial \Psi_0}{\partial u^0}(u^-, u^+) = -V \cdot [Df(u^-) - \Lambda Id], \tag{2.12}$$

$$\frac{\partial \Psi_0}{\partial u^1}(u^-, u^+) = V \cdot [Df(u^+) - \Lambda Id], \tag{2.13}$$

where V is the $(n-1) \times n$ matrix, whose rows are the vectors $V_1(u_0), \dots, V_{n-1}(u_0)$. Note that since rank $V = n-1$, then Λ is neither an eigenvalue of $Df(u^-)$ nor $Df(u^+)$, which in view of (2.12), (2.13) implies

$$\text{rank}\frac{\partial \Psi_0}{\partial u^0}(u^-, u^+) = \text{rank}\frac{\partial \Psi_0}{\partial u^1}(u^-, u^+) = n-1. \tag{2.14}$$

Proof of Theorem 2.1.

Step 1. By (2.12), (2.13) we get

$$\frac{\partial \Psi_0}{\partial u^0}(u^-, u^+) \cdot r_k(u^-) = -(\lambda_k(u^-) - \Lambda) \cdot V \cdot r_k(u^-) \quad \forall k : 1 \dots i-1,$$

$$\frac{\partial \Psi_0}{\partial u^1}(u^-, u^+) \cdot r_k(u^+) = (\lambda_k(u^+) - \Lambda) \cdot V \cdot r_k(u^+) \quad \forall k : i+1 \dots n.$$

Since $\Lambda \notin \{\lambda_k(u^-)\}_{k=1}^{i-1} \cup \{\lambda_k(u^+)\}_{k=i+1}^{n}$ we see that the condition (2.3)(iii) for our function Ψ_0 is satisfied iff the vectors $\{V \cdot r_k(u^-)\}_{k=1}^{i-1} \cup \{V \cdot r_k(u^+)\}_{k=i+1}^{n}$ are linearly independent, which is in turn equivalent to Majda's condition (2.4), as ker $V = \text{span}(u_0)$. We have thus shown that (2.4) is equivalent to (2.3)(iii) for the function Ψ_0.

Recalling (2.14), one sees this way that (2.4) implies (2.3).

Step 2. Now we turn toward proving the converse implication. Let Ψ be any function satisfying (2.3). In particular, by (2.3)(ii), rank $D\Psi(u^-, u^+)$ is maximal and equal to $n-1$. The same is true for $D\Psi_0(u^-, u^+)$, by (2.14), so

$$\text{rank}\ \ D\Psi(u^-, u^+) = \text{rank}\ \ D\Psi_0(u^-, u^+). \tag{2.15}$$

Another important remark is that

$$\ker\ \ D\Psi(u^-, u^+) = \ker\ \ D\Psi_0(u^-, u^+). \tag{2.16}$$

The spaces in (2.16) both coincide with the tangent space of the manifold $(\Psi_0)^{-1}(0)$ at point (u^-, u^+).

The following simple fact of linear algebra will be used in what follows.

Lemma 2.3. *Let $A, B : \mathbf{R}^n \longrightarrow \mathbf{R}^s$ be two linear operators, $s < n$. Assume that* rank $A =$ rank $B = s$ *and* ker $A =$ ker B. *Then for any s vectors $v_1, \dots, v_s \in \mathbf{R}^n$ it holds that the vectors $\{Av_k\}_{k=1}^s$ are linearly independent iff $\{Bv_k\}_{k=1}^s$ are linearly independent.*

In view of (2.15), (2.16), we can apply Lemma 2.2 to the linear operators

$$D\Psi(u^-, u^+),\ D\Psi_0(u^-, u^+) : \mathbf{R}^{2n} \longrightarrow \mathbf{R}^{n-1}$$

and the following set of $n-1$ test vectors in $\mathbf{R}^{2n}$:

$$\{[r_k(u^-)^T, 0 \dots 0]^T\}_{k=1}^{i-1} \cup \{[0 \dots 0, r_k(u^+)^T]^T\}_{k=i+1}^{n}.$$

By (2.3)(iii) we receive that the same condition is satisfied by our function Ψ_0. This in turn, is equivalent to (2.4), as shown in Step 1. ∎

The proof of Theorem 2.1 shows that if the function Ψ as in (2.3) exists, then it can be replaced by the function Ψ_0, in this case necessarily enjoying the properties (2.3)(i)–(2.3)(iii).

3. *BV* and L^1 stability conditions compared. In this and the next sections we discuss different stability conditions of type B, used in [BC], [W], [Scho], and [Le]. Recall that these conditions guarantee the well posedness of the problem (1.1), (1.2) and the existence of the Lipschitz continuous semigroup of solutions, whose domain contains all the small $L^1 \cap BV$ perturbations of the initial data $\bar{u}$ in (1.6) (compare [Le]).

We show the equivalence of the Schochet BV stability condition (called in [Scho] the finiteness condition) with the BV stability condition used in [Le], as well as with the Wang BV stability condition [W], and the equivalence of L^1 stability condition from [BM], [Le] with the one introduced in [BC] for 2×2 systems.

Also (see Remark 3.8), we position our work to some of the results found in [LY].

We start by recalling the mentioned conditions.

3.1. *BV* stability condition. There exist positive weights $w_1^q, \dots, w_n^q$ (for every $q : 0 \dots m$) such that the following holds. Consider a small wave of a family $k \le i_q$, hitting from the right the large initial i_q-shock (u_0^{q-1}, u_0^q), as in Figure 3.1. Then

$$\sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| < 1 \tag{3.1}$$

at $\epsilon_1^{in} = \cdots = \epsilon_k^{in} = \cdots = \epsilon_n^{in} = 0$.

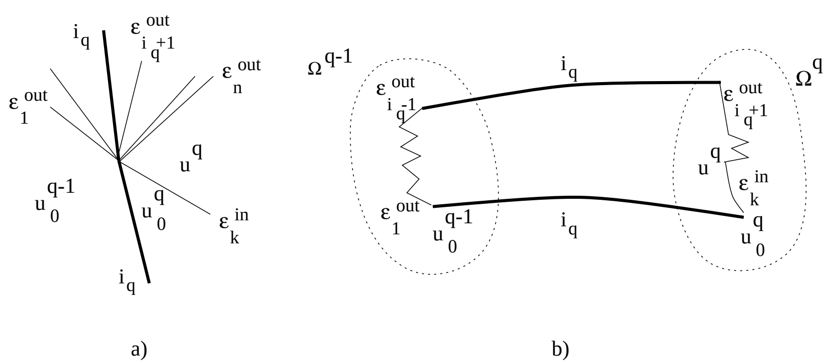

Fig. 3.1.

Symmetrically, in the case when a small k-wave with $k \ge i_q$ hits the shock (u_0^{q-1}, u_0^q) from the left (compare Figure 3.2), there holds

$$\sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| < 1 \tag{3.2}$$

at $\epsilon_1^{in} = \cdots = \epsilon_k^{in} = \cdots = \epsilon_n^{in} = 0$.

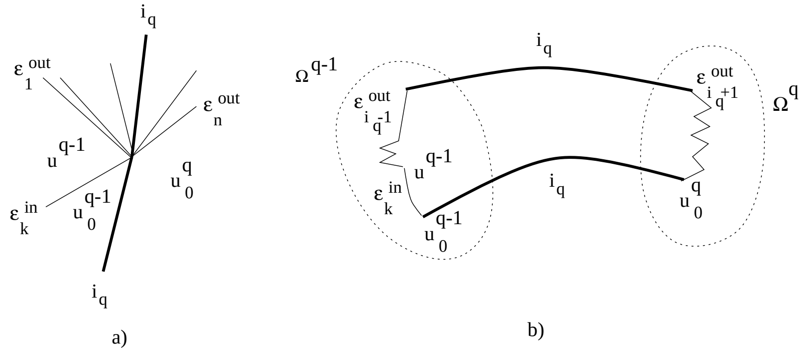

FIG. 3.2.

Regarding w_s^q as the weight given to an s-wave located in the region between the $q-1$ and the qth large shock, conditions (3.1), (3.2) simply say that every time a small wave hits a large shock, the total weighted strength of the outgoing small waves is smaller than the weighted strength of the incoming wave.

3.2. L^1 stability condition [Le], [BM]. There exist positive weights $w_1^q, \dots, w_n^q$ (for every $q : 0 \dots m$) such that in the setting of Figure 3.1

$$\sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right) \right| + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right) \right| < 1 \tag{3.3}$$

at $\epsilon_1^{in} = \cdots = \epsilon_k^{in} = \cdots = \epsilon_n^{in} = 0$, while in the setting of Figure 3.2

$$\sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right) \right| + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right) \right| < 1 \tag{3.4}$$

at $\epsilon_1^{in} = \cdots = \epsilon_k^{in} = \cdots = \epsilon_n^{in} = 0$.

Note that since the weights $\{w_i^0\}_{i=1}^n$ and $\{w_i^m\}_{i=1}^n$ appear only in one inequality (3.1) or (3.2), then the corresponding BV stability estimates for the leftmost large shock (u_0^0, u_0^1) and the rightmost (u_0^{m-1}, u_0^m) may take the following, simplified form:

$$\sum_{s=i_2}^{n} \frac{w_s^1}{w_k^1} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \; \epsilon_s^{out} \right| < 1 \tag{3.1a}$$

for all small waves of families $k \le i_1$, hitting the first shock i_1 from the right, and

$$\sum_{s=1}^{i_{m-1}} \frac{w_s^{m-1}}{w_k^{m-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \; \epsilon_s^{out} \right| < 1 \tag{3.2a}$$

for all small waves of families $k \ge i_m$, hitting the last shock i_m from the left.

The analogous simplifications may be easily done for the L^1 stability estimates (3.3) and (3.4).

Also, for $q \notin \{1, m\}$, (3.1) and (3.2) can be rewritten as follows:

$$\sum_{s=1}^{i_{q-1}} \frac{w_s^{q-1}}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| + \sum_{s=i_{q+1}}^{n} \frac{w_s^q}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| < 1, \tag{3.1a}$$

$$\sum_{s=1}^{i_{q-1}} \frac{w_s^{q-1}}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| + \sum_{s=i_{q+1}}^{n} \frac{w_s^q}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| < 1. \tag{3.2a}$$

Analogously, the L^1 stability condition (3.3) and (3.4) for $q \notin \{1, m\}$ may be formulated with the correspondingly changed summation ranges.

THEOREM 3.1. *The L^1 stability condition* (3.3), (3.4) *implies the BV stability condition* (3.1), (3.2).

Proof. In view of the preceding remarks, assume that the L^1 stability condition (3.3a) and (3.4a) holds, with weights $\{w_s^q\}$. For $q : 1 \ldots m-1$ and $s : 1 \ldots n$ define

$$\widetilde{w}_s^q = \left| \lambda_s(u_0^q) - \Lambda^{q+1} \right| \cdot w_s^q.$$

We will show that the BV stability condition (3.1), (3.2) is satisfied for all $q : 1 \ldots m$.

Indeed, to prove (3.1), compute

$$\begin{aligned} &\sum_{s=1}^{i_{q-1}} \frac{\widetilde{w}_s^{q-1}}{\widetilde{w}_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| + \sum_{s=i_{q+1}}^{n} \frac{\widetilde{w}_s^q}{\widetilde{w}_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| \\ &\qquad = \sum_{s=1}^{i_{q-1}} \frac{w_s^{q-1}}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| \cdot \frac{|\lambda_s(u_0^{q-1}) - \Lambda^q|}{|\lambda_k(u_0^q) - \Lambda^{q+1}|} \\ &\qquad\quad + \sum_{s=i_{q+1}}^{n} \frac{w_s^q}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| \cdot \frac{|\lambda_s(u_0^q) - \Lambda^{q+1}|}{|\lambda_k(u_0^q) - \Lambda^{q+1}|} < 1 \end{aligned}$$

by (3.3a) and the following easily received inequalities:

$$\begin{aligned} \left| \lambda_k(u_0^q) - \Lambda^{q+1} \right| &> |\lambda_k(u_0^q) - \Lambda^q| \quad \forall k \le i_q, \\ \left| \lambda_k(u_0^q) - \Lambda^{q+1} \right| &< |\lambda_k(u_0^q) - \Lambda^q| \quad \forall k \ge i_{q+1}. \end{aligned}$$

The estimate (3.2) is justified in a similar way. □

3.3. The Schochet BV stability condition [Scho]. In connection with (3.1) and (3.2), for every $q : 1 \ldots m$ define four nonnegative matrices, expressing the strengths of outgoing waves in terms of the strengths of the incoming small waves, interacting with the large initial i_q-shock:

(3.5)

- interaction from the right, waves outgoing to the right

$$M_q^{rr} = [a_{sk}^q], \quad s : i_{q+1} \dots n, \; k : 1 \dots i_q,$$

- interaction from the right, waves outgoing to the left

$$M_q^{rl} = [a_{sk}^q], \quad s : 1 \dots i_{q-1}, \; k : 1 \dots i_q,$$

- interaction from the left, waves outgoing to the right

$$M_q^{lr} = [a_{sk}^q], \quad s : i_{q+1} \dots n, \; k : i_q \dots n,$$

- interaction from the left, waves outgoing to the left

$$M_q^{ll} = [a_{sk}^q], \quad s : 1 \dots i_{q-1}, \; k : i_q \dots n.$$

In all of the above definitions

$$a_{sk}^q = \left| \frac{\partial \epsilon_s^{out}}{\partial \epsilon_k^{in}} \right|$$

at $\epsilon_1^{in} = \cdots = \epsilon_k^{in} = \cdots = \epsilon_n^{in} = 0$.

Note that in (3.5) the range of s (indexing the outgoing small waves) depends on the neighboring large shock (of the family i_{q-1} or i_{q+1}). Indeed, it is relevant to keep track of only these newborn waves that in the future may possibly interact with large shocks, thus changing the global wave pattern.

Keeping the above comment in mind, we also remark that the notation for the matrices $M_1^{rl}, M_1^{lr}, M_1^{ll}, M_m^{rr}, M_m^{lr}, M_m^{rl}$ is ambiguous, however, in view of what we have said, the precise form of these matrices is irrelevant in the following analysis.

Consider the first pair of large shocks (u_0^0, u_0^1) and (u_0^1, u_0^2) and a tuple $\gamma = [\gamma_k]_{k:i_2 \dots n}$ of small waves travelling in the region between these shocks, and approaching the second one. By interaction of γ with (u_0^1, u_0^2), then, interaction of the newborn "reflected" waves with (u_0^0, u_0^1) and so on, further waves travelling in the region between the two large shocks are produced. Call

$$R^1 = M_1^{rr}. \tag{3.6}$$

The total strength of such waves, belonging to the characteristic families $k \geq i_2$, is then seen to be

$$\left[Id + R^1 M_2^{ll} + \left(R^1 M_2^{ll} \right)^2 + \cdots \right] |\gamma| = \left(Id - R^1 M_2^{ll} \right)^{-1} |\gamma| \doteq P^{1-2} |\gamma|$$

(where $|\gamma| = [|\gamma_k|]_{k:i_2 \dots n}$), provided that the first finiteness requirement

$$\text{all eigenvalues of } R^1 \cdot M_2^{ll} \text{ are } < 1 \text{ in absolute value} \tag{3.7}$$

is satisfied.

Now, view the pair of the first two large shocks as a single entity. The reflection matrix R^{1-2}, expressing the strengths of the outgoing small waves of families $k \geq i_3$, exiting the region between the first and the second large waves to the right of the latter one, in terms of the incoming waves of the families $k \leq i_2$, possibly interacting with the $(i_1 - i_2)$ couple of large shocks from the right, has the form

$$R^{1-2} = M_2^{rr} + M_2^{lr} P^{1-2} R^1 M_2^{rl}.$$

The natural finiteness requirement for the triple $(i_1 - i_2 - i_3)$ of large shocks, analogous to (3.7) is then

$$\text{all eigenvalues of } R^{1-2} \cdot M_3^{ll} \text{ are } < 1 \text{ in absolute value.}$$

Proceeding in the same manner and viewing any fixed combination $(i_1 - \cdots - i_q)$ of consecutive large shocks as a single entity, influencing its succeeding large wave i_{q+1}, we obtain the following $(m-1)$ assertions that constitute the announced Schochet BV stability condition:

$$\begin{aligned} &\text{spRad}(F^{1-2}) < 1, \\ &\text{spRad}(F^{1-2-3}) < 1, \\ &\quad\vdots \\ &\text{spRad}(F^{1-\cdots-m}) < 1 \end{aligned} \tag{3.8}$$

(spRad stands here for the spectral radius of the reference matrix). The finiteness matrices F are defined inductively, together with the corresponding reflection and production matrices R, P, by recalling (3.6) and setting

$$F^{1-\cdots-q} \doteq R^{1-\cdots-(q-1)} \cdot M_q^{ll} \qquad \text{for } q : 2 \ldots m, \tag{3.9}$$

$$P^{1-\cdots-q} \doteq \left(Id - F^{1-\cdots-q}\right)^{-1} \qquad \text{for } q : 2 \ldots m, \tag{3.10}$$

$$R^{1-\cdots-q} \doteq M_q^{rr} + M_q^{lr} P^{1-\cdots-q} R^{1-\cdots-(q-1)} M_q^{rl} \qquad \text{for } q : 2 \ldots m-1. \tag{3.11}$$

3.4. *BV* stability condition. The main theorem of this subsection is the following.

THEOREM 3.2. *The BV stability condition* (3.1), (3.2) *is equivalent to the Schochet BV stability condition* (3.8).

To prove Theorem 3.2, we need two abstract results on matrix theory.

LEMMA 3.3. *Let $Q = [q_{sk}]_{s,k:1\ldots n}$ be an $n \times n$ matrix with nonnegative entries: $q_{sk} \geq 0$. The following conditions are equivalent:*

(i) $\text{spRad}(Q) < 1$.

(ii) *There exists a diagonal matrix $W = \text{diag}(w_1, \ldots, w_n)$ with positive diagonal entries $w_s > 0$ such that $\| WQW^{-1} \|_1 < 1$.*
Here the norm of an $n \times n$ matrix $P = [p_{sk}]_{s,k:1\ldots n}$ is defined by

$$\| P \|_1 = \max_{k:1\ldots n} \sum_{s=1}^{n} |p_{sk}|.$$

The above lemma, which came up independently in the investigations leading to this paper, follows also from the results in [LY, Theorem 1 in Appendix 1]; thus for brevity we omit its proof.

LEMMA 3.4. *Let A, B be two $n \times n$ matrices with nonnegative entries:*

$$A = [a_{sk}]_{s,k:1\ldots n}, \quad B = [b_{sk}]_{s,k:1\ldots n}.$$

Assume that there exist two sets of indices $col, ver \subset \{1 \dots n\}$ *with the properties*

- $col \cap ver = \emptyset$,
- $\forall k \notin col \ \forall s : 1 \dots n, \quad a_{sk} = b_{ks} = 0$,
- $\forall s \notin ver \ \forall k : 1 \dots n, \quad a_{sk} = b_{ks} = 0$.

Then the following two statements are equivalent:

(i) *There exists* $W = \mathrm{diag}(w_1, \dots, w_n)$ *with all* $w_k > 0$ *such that* $\| WAW^{-1} \|_1 < 1$ *and* $\| WBW^{-1} \|_1 < 1$.

(ii) *There exists* $W = \mathrm{diag}(w_1, \dots, w_n)$ *with all* $w_k > 0$ *such that* $\| WABW^{-1} \|_1 < 1$.

The matrix norm $\| \cdot \|_1$ *is defined as in Lemma* 3.3.

Proof. (i) $\Rightarrow$ (ii). This implication is an obvious consequence of the fact that $\| \cdot \|_1$ is a matrix norm.

(ii) $\Rightarrow$ (i). Since $WABW^{-1} = (WAW^{-1})(WBW^{-1})$, we may without loss of generality assume that $\| AB \|_1 < 1$ and prove the existence of a diagonal matrix W satisfying (i). By (ii) we have

$$\sum_{s \in col} \left[b_{sk} \cdot \sum_{r \in ver} a_{rs} \right] < 1 \quad \forall k \in ver.$$

For a fixed $\epsilon > 0$ define

$$w_k = \begin{cases} \sum_{s \in ver} a_{sk} + \epsilon & \text{for } k \in col, \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{s \in ver} w_s a_{sk} = \sum_{s \in ver} a_{sk} < w_k \quad \forall k \in col,$$

$$\sum_{s \in col} w_s b_{sk} = \sum_{s \in col} \left(\sum_{r \in ver} a_{rs} \right) b_{sk} + \sum_{s \in col} \epsilon b_{sk} < 1 = w_k \quad \forall k \in ver,$$

provided that ϵ is small enough.

We have thus proved that $\| WAW^{-1} \|_1 < 1$ and $\| WBW^{-1} \|_1 < 1$. ◻

For every matrix $M_q^{xy}, \ x, y \in \{l, r\}$, define the corresponding square $n \times n$ matrix $\widetilde{M}_q^{xy}$ by completing all the "missing" entries with zeros. For example, in view of (3.5)

$$\widetilde{M}_1^{rr} = [\widetilde{a}_{sk}]_{s,k:1\dots n}, \qquad \widetilde{a}_{sk} = \begin{cases} a_{sk} & \text{for } s : i_2 \dots n, \ k : 1 \dots i_1, \\ 0 & \text{otherwise.} \end{cases}$$

The next lemma shows some possible reformulations of our BV stability condition (3.1), (3.2).

Lemma 3.5. *The following conditions are equivalent to the BV stability condition* (3.1), (3.2) :

(i) *There exist* $m-1$ *diagonal matrices* $\{W^q\}_{q=1}^{m-1}$ *with positive diagonal entries such that*

(3.12)
$$\| W^1 \widetilde{M}_1^{rr} (W^1)^{-1} \|_1 < 1,$$

(3.13)
$$\begin{array}{l} \| W^{q-1} \widetilde{M}_q^{ll} (W^{q-1})^{-1} + W^q \widetilde{M}_q^{lr} (W^{q-1})^{-1} \|_1 < 1 \\ \| W^q \widetilde{M}_q^{rr} (W^q)^{-1} + W^{q-1} \widetilde{M}_q^{rl} (W^q)^{-1} \|_1 < 1 \end{array} \qquad \forall q : 2 \dots m-1,$$

(3.14)
$$\| W^{m-1} \widetilde{M}_m^{ll} (W^{m-1})^{-1} \|_1 < 1.$$

(ii) *Define two block square matrices of the dimension* $(m-1) \cdot n$*:*

$$Odd_m = \begin{bmatrix} \widetilde{M}_1^{rr} & 0 & \dots & \dots & 0 \\ 0 & \widetilde{M}_3^{ll} & \widetilde{M}_3^{rl} & 0 & \vdots \\ \vdots & \widetilde{M}_3^{lr} & \widetilde{M}_3^{rr} & 0 & \\ \vdots & 0 & 0 & \widetilde{M}_5^{ll} & \\ 0 & \dots & & & \ddots \end{bmatrix},$$

$$Even_m = \begin{bmatrix} \widetilde{M}_2^{ll} & \widetilde{M}_2^{rl} & 0 & \dots & 0 \\ \widetilde{M}_2^{lr} & \widetilde{M}_2^{rr} & 0 & \dots & \\ 0 & 0 & \widetilde{M}_4^{ll} & \widetilde{M}_4^{rl} & \\ \vdots & \vdots & \widetilde{M}_4^{lr} & \widetilde{M}_4^{rr} & \\ 0 & & & & \ddots \end{bmatrix}.$$

Then

$$\text{spRad}(Odd_m \cdot Even_m) < 1. \tag{3.15}$$

Proof. The condition (i) is obviously equivalent to (3.1), (3.2) if we define $W^q = \text{diag}(w_1^q, \dots, w_n^q)$ for all $q : 1 \dots m-1$.

Note that (3.12), (3.13), (3.14) are equivalent to

$$\| W \cdot Odd_m \cdot W^{-1} \| < 1, \qquad \| W \cdot Even_m \cdot W^{-1} \| < 1, \tag{3.16}$$

where W is the block diagonal matrix of the dimension $(m-1) \cdot n$ given by

$$W = \text{diag}(W^1, \dots, W^{m-1}).$$

By Lemma 3.3 and Lemma 3.4, (3.16) is in turn equivalent to (3.15), which proves (ii). ∎

Before we give the proof of Theorem 3.2, we need one more result of a technical nature.

LEMMA 3.6. *Let A, B be two $n \times n$ matrices with nonnegative entries such that* $\| A + B \|_1 < 1$. *Then* $\| B \cdot (Id - A)^{-1} \|_1 < 1$.

Proof. Note first that since $\| A \|_1 < 1$, then the matrix $Id - A$ is invertible and its inverse

$$(Id - A)^{-1} = Id + A + A^2 + \cdots$$

has nonnegative entries. From the assumption it follows moreover that

$$\sum_{i=1}^{n} [B]_{ik} < 1 - \sum_{i=1}^{n} [A]_{ik} = \sum_{i=1}^{n} [Id - A]_{ik},$$

for every $k : 1 \ldots n$, and thus

$$\begin{aligned} \sum_{i=1}^{n} [B \cdot (Id - A)^{-1}]_{ik} &= \sum_{s=1}^{n} \left(\sum_{i=1}^{n} [B]_{is} \right) \cdot [(Id - A)^{-1}]_{sk} \\ &< \sum_{s=1}^{n} \left(\sum_{i=1}^{n} [Id - A]_{is} \right) \cdot [(Id - A)^{-1}]_{sk} \\ &= \sum_{i=1}^{n} [(Id - A) \cdot (Id - A)^{-1}]_{ik} = 1, \end{aligned}$$

for every $k : 1 \ldots n$, which proves our lemma. $\square$

Now we are ready to give the following proof.

Proof of Theorem 3.2.

Step 1. (3.1), (3.2) $\Rightarrow$ (3.8). We use the equivalent form of the BV stability condition (3.1), (3.2) given in Lemma 3.5(i).

We first show that

$$\forall q : 1 \ldots m - 1 \quad \| W^q \cdot \widetilde{R}^{1-\cdots-q} \cdot (W^q)^{-1} \|_1 < 1. \tag{3.17}$$

We proceed by induction on q. For $q = 1$, (3.17) is equivalent to (3.12) in view of (3.6). For $q : 2 \ldots m - 1$, by (3.11) we have

$$\begin{aligned} W^q \cdot \widetilde{R}^{1-\cdots-q} \cdot (W^q)^{-1} = {} & W^q \widetilde{M}_q^{rr} (W^q)^{-1} \\ & + \left[W^q \widetilde{M}_q^{lr} \widetilde{P}^{1-\cdots-q} \widetilde{R}^{1-\cdots-(q-1)} (W^{q-1})^{-1} \right] \\ & \cdot \left[W^{q-1} \widetilde{M}_q^{rl} (W^q)^{-1} \right]. \end{aligned}$$

The desired conclusion (3.17) will thus follow from the second inequality in (3.13) provided that

$$\| W^q \widetilde{M}_q^{lr} \widetilde{P}^{1-\cdots-q} \widetilde{R}^{1-\cdots-(q-1)} (W^{q-1})^{-1} \|_1 < 1. \tag{3.18}$$

Note that

$$
\begin{aligned}
&W^q \widetilde{M}_q^{lr} \widetilde{P}^{1-\cdots-q} \widetilde{R}^{1-\cdots-(q-1)} (W^{q-1})^{-1} \\
&\quad = W^q \widetilde{M}_q^{lr} \cdot \left(Id - \widetilde{R}^{1-\cdots-(q-1)} \widetilde{M}_q^{ll} \right)^{-1} \cdot \widetilde{R}^{1-\cdots-(q-1)} (W^{q-1})^{-1} \\
&\quad = \left[W^q \widetilde{M}_q^{lr} (W^{q-1})^{-1} \right] \\
&\qquad \cdot \left\{ Id - \left[W^{q-1} \widetilde{R}^{1-\cdots-(q-1)} (W^{q-1})^{-1} \right] \cdot \left[W^{q-1} \widetilde{M}_q^{ll} (W^{q-1})^{-1} \right] \right\}^{-1} \\
&\qquad \cdot \left[W^{q-1} \widetilde{R}^{1-\cdots-(q-1)} (W^{q-1})^{-1} \right].
\end{aligned} \tag{3.19}
$$

Setting

$$A = W^{q-1} \widetilde{M}_q^{ll} (W^{q-1})^{-1}, \qquad B = W^q \widetilde{M}_q^{lr} (W^{q-1})^{-1}$$

and combining Lemma 3.6 with the inductive assumption

$$\| W^{q-1} \cdot \widetilde{R}^{1-\cdots-(q-1)} \cdot (W^{q-1})^{-1} \|_1 < 1,$$

we get (3.18) by (3.19) and thus complete the proof of (3.17).

We now prove inductively that the BV stability condition (3.1), (3.2) implies (3.8). For $m = 2$, the conditions (3.12) and (3.14) are by Lemmas 3.3 and 3.4 equivalent to

$$\text{all eigenvalues of } \widetilde{M}_1^{rr} \cdot \widetilde{M}_2^{ll} \text{ are } < 1 \text{ in absolute value.} \tag{3.20}$$

However,

$$\text{Spec } M_1^{rr} M_2^{ll} \subset \text{Spec } \widetilde{M}_1^{rr} \widetilde{M}_2^{ll} \subset \left(\text{Spec } M_1^{rr} M_2^{ll} \right) \cup \{0\},$$

thus (3.20) is equivalent to

$$\text{spRad}(F^{1-2}) < 1,$$

which is in turn precisely the condition (3.8).

Note that we proved above even more than we need to at this point—we proved the equivalence of (3.1), (3.2), and (3.8) in case $m = 2$ of only two large shocks present.

Let now $m > 2$. Since (3.13) for $q = m - 1$ implies

$$\| W^{q-2} \widetilde{M}_{q-1}^{ll} (W^{q-1})^{-1} \|_1 < 1,$$

by the inductive assumption we get

$$\text{spRad}(F^{1-\cdots-q}) < 1 \quad \forall q : 2 \ldots m-1.$$

However, by (3.14) and (3.17) for $q = m - 1$, in view of Lemma 3.4 and definition (3.9)

$$\| W^{m-1} \widetilde{F}^{1-\cdots-m} (W^{m-1})^{-1} \|_1 < 1,$$

which by Lemma 3.3 implies finally

$$\operatorname{spRad}(F^{1-\cdots-m}) < 1.$$

This finishes the proof of (3.1), (3.2) $\Rightarrow$ (3.8). $\square$

Step 2. (3.8) $\Rightarrow$ (3.1), (3.2). We use the equivalent form of the BV stability condition (3.1), (3.2) given in Lemma 3.5(ii).

We proceed by induction on m. For $m = 2$ the assertion has already been established in Step 1. Let $m > 2$ and fix $\lambda \geq 1$. We will show that

$$\det(Odd_m \cdot Even_m - \lambda Id) \neq 0, \tag{3.21}$$

which by the property of nonnegative matrices mentioned in the proof of Lemma 3.3 will prove the theorem.

Assume first that m is an odd number. By known formulae on the determinant of block matrices (see [G]) and a few easy computations one gets

$$\begin{aligned} &\det(Odd_m \cdot Even_m - \lambda Id) \\ &\quad = \det(Odd_{m-1} \cdot Even_{m-1} - \lambda Id) \\ &\qquad \cdot \det\Big(\widetilde{M}^{ll}_m \widetilde{M}^{rr}_{m-1} + \widetilde{M}^{ll}_m \cdot A_m \cdot (\lambda Id - Odd_{m-1} \cdot Even_{m-1})^{-1} \\ &\qquad\qquad \cdot B_m \cdot \widetilde{M}^{rl}_{m-1} - \lambda Id\Big), \end{aligned} \tag{3.22}$$

where A_m is an $n \times ((m-2)\cdot n)$ block matrix of the form

$$A_m = \begin{bmatrix} 0 & \ldots & \ldots & 0 & \widetilde{M}^{lr}_{m-1} \end{bmatrix},$$

and B_m is an $((m-2)\cdot n) \times n$ block matrix

$$B_m = \begin{bmatrix} 0 & \ldots & 0 & \widetilde{M}^{rl}_{m-2} & \widetilde{M}^{rr}_{m-2} \end{bmatrix}^T,$$

while Odd_{m-1} and $Even_{m-1}$ are defined analogously to Odd_m and $Even_m$ as in Lemma 3.5(ii).

Note that the Schochet condition (3.8) implies (by the inductive assumption)

$$\det(Odd_{m-1} \cdot Even_{m-1} - \lambda Id) \neq 0, \tag{3.23}$$

$$\operatorname{spRad}(F^{1-\cdots-m}) < 1. \tag{3.24}$$

By the definitions (3.9)–(3.11)

$$F^{1-\cdots-m} = M^{ll}_m \cdot \Big[M^{rr}_{m-1} + M^{lr}_{m-1}\big(Id - F^{1-\cdots-(m-1)}\big)^{-1} \cdot R^{1-\cdots-(m-2)} M^{rl}_{m-1}\Big].$$

Thus, in view of (3.23) and (3.24), the needed (3.21) will follow from (3.22) provided that

$$\begin{aligned} &A_m \cdot (Id - Odd_{m-1} \cdot Even_{m-1})^{-1} \cdot B_m \\ &\qquad = \widetilde{M}^{lr}_{m-1} \cdot \big(Id - \widetilde{F}^{1-\cdots-(m-1)}\big)^{-1} \cdot \widetilde{R}^{1-\cdots-(m-2)}. \end{aligned} \tag{3.25}$$

By the same kind of reasoning it is possible to prove that for m even, (3.21) is a consequence of the formula

$$\begin{aligned} C_m \cdot (Id - Odd_{m-1} \cdot Even_{m-1})^{-1} \cdot D_m \\ = \big(Id - \widetilde{F}^{1-\dots-(m-1)}\big)^{-1} \cdot \widetilde{R}^{1-\dots-(m-2)} \cdot \widetilde{M}^{rl}_{m-1}, \end{aligned} \tag{3.26}$$

where C_m is an $n \times ((m-2)\cdot n)$ block matrix of the form

$$C_m = \begin{bmatrix} 0 & \dots & 0 & \widetilde{M}^{lr}_{m-2} & \widetilde{M}^{rr}_{m-2} \end{bmatrix},$$

and D_m is an $((m-2)\cdot n) \times n$ block matrix

$$D_m = \begin{bmatrix} 0 & \dots & \dots & 0 & \widetilde{M}^{rl}_{m-1} \end{bmatrix}^T.$$

In the remaining part of the proof we will concentrate on showing that (3.25) holds for every odd number m. The proof of (3.26) is entirely the same, so we leave it to the careful reader.

We are going to prove (3.25) by induction on odd numbers m. For $m = 3$, the left-hand side of (3.25) reduces to

$$\widetilde{M}^{lr}_2 \cdot (Id - \widetilde{M}^{rr}_1 \cdot \widetilde{M}^{ll}_2)^{-1} \cdot \widetilde{M}^{rr}_1,$$

which is precisely equal to $\widetilde{M}^{lr}_2 \cdot \big(Id - \widetilde{F}^{1-2}\big)^{-1} \cdot \widetilde{R}^1$ by (3.6) and (3.9).

For $m > 3$ and odd, computing $(Id - Odd_{m-1} \cdot Even_{m-1})^{-1}$ in terms of the matrices $Odd_{m-3}, Even_{m-3}$, and the basic block-interaction matrices M^{xy}_q, we receive the equivalent form of the left-hand side of the formula (3.25):

$$\begin{aligned} & A_m \cdot (Id - Odd_{m-1} \cdot Even_{m-1})^{-1} \cdot B_m \\ &= \begin{bmatrix} 0 & \widetilde{M}^{lr}_{m-1} \end{bmatrix} \\ &\quad \cdot \Bigg\{ Id - \begin{bmatrix} \widetilde{M}^{ll}_{m-2} & \widetilde{M}^{rl}_{m-2} \\ \widetilde{M}^{lr}_{m-2} & \widetilde{M}^{rr}_{m-2} \end{bmatrix} \cdot \begin{bmatrix} \widetilde{M}^{rr}_{m-3} & 0 \\ 0 & \widetilde{M}^{ll}_{m-1} \end{bmatrix} \\ &\quad - \begin{bmatrix} \widetilde{M}^{ll}_{m-2} \\ \widetilde{M}^{lr}_{m-2} \end{bmatrix} \cdot A_{m-2} \cdot (Id - Odd_{m-3} \cdot Even_{m-3})^{-1} \\ &\quad \cdot B_{m-2} \cdot \begin{bmatrix} \widetilde{M}^{rl}_{m-3} & 0 \end{bmatrix} \Bigg\}^{-1} \cdot \begin{bmatrix} \widetilde{M}^{rl}_{m-2} \\ \widetilde{M}^{rr}_{m-2} \end{bmatrix}. \end{aligned} \tag{3.27}$$

Using the inductive assumption and the definition (3.11) we reformulate the right-hand side of (3.27):

$$
\begin{aligned}
&A_m \cdot (Id - Odd_{m-1} \cdot Even_{m-1})^{-1} \cdot B_m \\
&\quad = \left[\begin{array}{cc} 0 & \widetilde{M}^{lr}_{m-1} \end{array}\right] \\
&\qquad \cdot \left\{ Id - \left[\begin{array}{cc} \widetilde{M}^{ll}_{m-2} & \widetilde{M}^{rl}_{m-2} \\ \widetilde{M}^{lr}_{m-2} & \widetilde{M}^{rr}_{m-2} \end{array}\right] \cdot \left[\begin{array}{cc} \widetilde{M}^{rr}_{m-3} & 0 \\ 0 & \widetilde{M}^{ll}_{m-1} \end{array}\right] \right. \\
&\qquad\qquad - \left[\begin{array}{c} \widetilde{M}^{ll}_{m-2} \\ \widetilde{M}^{lr}_{m-2} \end{array}\right] \cdot \widetilde{M}^{lr}_{m-3} \cdot (Id - \widetilde{F}^{1-\cdots-(m-3)})^{-1} \\
&\qquad\qquad\qquad \left. \cdot \widetilde{R}^{1-\cdots-(m-4)} \left[\begin{array}{cc} \widetilde{M}^{rl}_{m-3} & 0 \end{array}\right] \right\}^{-1} \cdot \left[\begin{array}{c} \widetilde{M}^{rl}_{m-2} \\ \widetilde{M}^{rr}_{m-2} \end{array}\right] \\
&\quad = \left[\begin{array}{cc} 0 & \widetilde{M}^{lr}_{m-1} \end{array}\right] \cdot \left\{ Id - \left[\begin{array}{cc} \widetilde{M}^{ll}_{m-2} & \widetilde{M}^{rl}_{m-2} \\ \widetilde{M}^{lr}_{m-2} & \widetilde{M}^{rr}_{m-2} \end{array}\right] \right. \\
&\qquad \left. \cdot \left[\begin{array}{cc} \begin{array}{l} \widetilde{M}^{rr}_{m-3} + \\ \quad \widetilde{M}^{lr}_{m-3}(Id - \widetilde{F}^{1-\cdots-(m-3)})^{-1} \cdot \\ \qquad \cdot \widetilde{R}^{1-\cdots-(m-4)} \widetilde{M}^{rl}_{m-3} \end{array} & 0 \\ 0 & \widetilde{M}^{ll}_{m-1} \end{array}\right] \right\}^{-1} \cdot \left[\begin{array}{c} \widetilde{M}^{rl}_{m-2} \\ \widetilde{M}^{rr}_{m-2} \end{array}\right] \\
&\quad = \left[\begin{array}{cc} 0 & \widetilde{M}^{lr}_{m-1} \end{array}\right] \cdot \left\{ Id - \left[\begin{array}{cc} \widetilde{M}^{ll}_{m-2} & \widetilde{M}^{rl}_{m-2} \\ \widetilde{M}^{lr}_{m-2} & \widetilde{M}^{rr}_{m-2} \end{array}\right] \right. \\
&\qquad\qquad \left. \cdot \left[\begin{array}{cc} \widetilde{R}^{1-\cdots-(m-3)} & 0 \\ 0 & \widetilde{M}^{ll}_{m-1} \end{array}\right] \right\}^{-1} \cdot \left[\begin{array}{c} \widetilde{M}^{rl}_{m-2} \\ \widetilde{M}^{rr}_{m-2} \end{array}\right].
\end{aligned}
\tag{3.28}
$$

Calling

$$
\begin{aligned}
X &= Id - \widetilde{M}^{ll}_{m-2} \widetilde{R}^{1-\cdots-(m-3)}, \\
Y &= -\widetilde{M}^{rl}_{m-2} \widetilde{M}^{ll}_{m-1}, \\
Z &= -\widetilde{M}^{lr}_{m-2} \widetilde{R}^{1-\cdots-(m-3)}, \\
W &= Id - \widetilde{M}^{lr}_{m-2} \widetilde{M}^{ll}_{m-1},
\end{aligned}
$$

we rewrite the right-hand side of (3.28):

$$
\begin{aligned}
&\left[\begin{array}{cc} 0 & \widetilde{M}^{lr}_{m-1} \end{array}\right] \cdot \left[\begin{array}{cc} X & Y \\ Z & W \end{array}\right]^{-1} \cdot \left[\begin{array}{c} \widetilde{M}^{rl}_{m-2} \\ \widetilde{M}^{rr}_{m-2} \end{array}\right] \\
&\qquad = \widetilde{M}^{lr}_{m-1} \cdot \Big(-(W - ZX^{-1}Y)^{-1} Z X^{-1} \cdot \widetilde{M}^{rl}_{m-2} \\
&\qquad\qquad\qquad + (W - ZX^{-1}Y)^{-1} \cdot \widetilde{M}^{rr}_{m-2} \Big) \\
&\qquad = \widetilde{M}^{lr}_{m-1} \cdot (W - ZX^{-1}Y)^{-1} \cdot \big(\widetilde{M}^{rr}_{m-2} - ZX^{-1} \cdot \widetilde{M}^{rl}_{m-2} \big) \\
&\qquad = \widetilde{M}^{lr}_{m-1} \cdot (Id - \widetilde{R}^{1-\cdots-(m-2)} \widetilde{M}^{ll}_{m-1})^{-1} \cdot \widetilde{R}^{1-\cdots-(m-2)},
\end{aligned}
\tag{3.29}
$$

because, by definitions (3.9)–(3.11)

$$W - ZX^{-1}Y = Id - \widetilde{R}^{1-\cdots-(m-2)}\widetilde{M}^{ll}_{m-1},$$
$$\widetilde{M}^{rr}_{m-2} - ZX^{-1} \cdot \widetilde{M}^{rl}_{m-2} = \widetilde{R}^{1-\cdots-(m-2)}.$$

The equality (3.29) together with (3.28) prove (3.25). The proof of Step 2 and thus also the proof of Theorem 3.2 is complete. □

3.5. L^1 stability condition. In connection with (3.3) and (3.4), we define the matrices $N^{rr}_q, N^{rl}_q, N^{lr}_q$, and N^{ll}_q $(q : 1 \dots m)$, having the same dimensions as their corresponding matrices M^{xy}_q in (3.5), and with their (nonnegative) entries given by

$$b_{sk} = a_{sk} \cdot \left| \frac{\lambda_s(u^q_0) - \Lambda^q}{\lambda_k(u^q_0) - \Lambda^q} \right| \quad \text{in } N^{rr}_q,$$
$$b_{sk} = a_{sk} \cdot \left| \frac{\lambda_s(u^{q-1}_0) - \Lambda^q}{\lambda_k(u^q_0) - \Lambda^q} \right| \quad \text{in } N^{rl}_q,$$
$$b_{sk} = a_{sk} \cdot \left| \frac{\lambda_s(u^q_0) - \Lambda^q}{\lambda_k(u^{q-1}_0) - \Lambda^q} \right| \quad \text{in } N^{lr}_q,$$
$$b_{sk} = a_{sk} \cdot \left| \frac{\lambda_s(u^{q-1}_0) - \Lambda^q}{\lambda_k(u^{q-1}_0) - \Lambda^q} \right| \quad \text{in } N^{ll}_q.$$

Using the analysis of the previous subsection, we can now state the following.

Proposition 3.7. *The L^1 stability condition* (3.3), (3.4) *is equivalent to the condition* (3.8)*, where the matrices $F^{1-\cdots-q}$ are defined as in* (3.9)–(3.11)*, with every matrix M^{xy}_q replaced by the corresponding one N^{xy}_q. In particular, for $m = 2$,* (3.8) *reduces to*

$$\text{the spectral radius of an } n \times n \text{ matrix}$$
$$|\mathcal{S} - \Lambda^1 Id| \cdot \widetilde{M}^{rr}_1 \cdot |\mathcal{S} - \Lambda^1 Id|^{-1} \cdot |\mathcal{S} - \Lambda^2 Id| \cdot \widetilde{M}^{ll}_2 \cdot |\mathcal{S} - \Lambda^2 Id|^{-1} \tag{3.30}$$
$$\text{is smaller than } 1,$$

where

$$|\mathcal{S} - \Lambda Id| = \mathrm{diag}(|\lambda_1(u^1_0) - \Lambda|, \dots, |\lambda_n(u^1_0) - \Lambda|).$$

Remark 3.8. It has recently been brought to our attention that conditions similar to our BV and L^1 stability conditions, though expressed in the language of matrix analysis, can be found in the book [LY].

The authors investigate the (short time) existence and regularity of classical solutions to the so-called typical boundary value problems on fan-shaped domains for quasi-linear hyperbolic systems with smooth coefficients. In particular, they show the existence of a unique C^1 solution to this problem, provided that the so-called minimal characterizing number of the characterizing matrix for the typical boundary value problem is smaller than 1 (Theorem 1.1 in Chapter 4). If the same holds for the second characterizing matrix (see paragraph 4 in Chapter 7), then the corresponding solution is C^2 regular (Theorem 1.1 in Chapter 7).

These results can well be applied to the quasi-linear system (1.4) with the boundary conditions (1.5) along the boundaries of the angular domains given by the large shocks in the solution of (1.1), (1.2), (1.3). The boundary conditions (1.5) appear already in the solvable form (see Lemma 5.10 in Chapter 2), that is, some of the components of u at the vertex $x = 0, t = 0$ (namely, the components corresponding to the outgoing modes) are explicitly expressed as functions of the others (corresponding to the incoming modes). It is not hard to notice that the characterizing matrix of this problem is made up of the quantities $\{\frac{\partial}{\partial \epsilon_k^{in}}\ \epsilon_s^{out}\}$ in such a way that its minimal characterizing number is smaller than 1 iff our BV stability condition holds. In a similar manner, the mentioned solvability condition for the second characterizing matrix, containing the numbers $\{\frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right)\}$, is equivalent to our L^1 stability condition.

The results in [LY] thus imply the local in time existence of the piecewise C^1 (respectively, C^2) solution to the problem (1.1), (1.2) with $\bar{u}$ smooth except at the point $x = 0$, where it induces the Riemann problem "close" to (u^-, u^+).

4. Systems of two equations. In the particular case $n = m = 2,\ i_1 = 1,\ i_2 = 2$, the matrices M_1^{rr} and M_2^{ll} reduce to single numbers, and the L^1 stability condition (3.30) appears in a simple form:

$$\left| \frac{\partial \epsilon_2^{out}}{\partial \epsilon_1^{in}}\ _{|\epsilon_1^{in} = 0} \right| \cdot \left| \frac{\partial \epsilon_1^{out}}{\partial \epsilon_2^{in}}\ _{|\epsilon_2^{in} = 0} \right| \cdot \frac{\lambda_1(u_0^1) - \Lambda^2}{\lambda_1(u_0^1) - \Lambda^1} \cdot \frac{\lambda_2(u_0^1) - \Lambda^1}{\lambda_2(u_0^1) - \Lambda^2} < 1. \tag{4.1}$$

Similarily, the BV stability condition (3.1), (3.2) is equivalent to

$$\left| \frac{\partial \epsilon_2^{out}}{\partial \epsilon_1^{in}}\ _{|\epsilon_1^{in} = 0} \right| \cdot \left| \frac{\partial \epsilon_1^{out}}{\partial \epsilon_2^{in}}\ _{|\epsilon_2^{in} = 0} \right| < 1. \tag{4.2}$$

In both (4.1) and (4.2) the first derivative corresponds to the right interaction with the large shock of the first family, while the second derivative corresponds to the left interaction with the large shock of the second characteristic family.

In what follows we show that (4.1) and (4.2) are equivalent, respectively, to the appropriate conditions providing stability results in [BC] and [W].

4.1. The Bressan–Colombo L^1 stability condition [BC]. In the setting of [BC],

$$\kappa_1 = \frac{\partial \epsilon_2^{out}}{\partial \epsilon_1^{in}}\ _{|\epsilon_1^{in} = 0} = - \frac{\left\langle \frac{\partial \Psi^2(u_0^0, u_0^1)}{\partial u^1}, r_1(u_0^1) \right\rangle}{\left\langle \frac{\partial \Psi^2(u_0^0, u_0^1)}{\partial u^1}, r_2(u_0^1) \right\rangle}$$

and

$$\kappa_2 = \frac{\partial \epsilon_1^{out}}{\partial \epsilon_2^{in}}\ _{|\epsilon_2^{in} = 0} = - \frac{\left\langle \frac{\partial \Psi^1(u_0^1, u_0^2)}{\partial u^1}, r_2(u_0^1) \right\rangle}{\left\langle \frac{\partial \Psi^1(u_0^1, u_0^2)}{\partial u^1}, r_1(u_0^1) \right\rangle},$$

where

$$\Psi^1(u^1, u^2) = \langle l_1(u^1, u^2), u^1 - u^2 \rangle,$$

$$\Psi^2(u^0, u^1) = \langle l_2(u^0, u^1), u^0 - u^1 \rangle,$$

l_1 and l_2 being the left eigenvectors of the averaged flux gradient matrix between the reference points u.

One sees that the Bressan–Colombo stability condition

$$\left|\kappa_1 \cdot \frac{\lambda_1(u_0^1) - \Lambda^2}{\lambda_1(u_0^1) - \Lambda^1}\right| \cdot \left|\kappa_2 \cdot \frac{\lambda_2(u_0^1) - \Lambda^1}{\lambda_2(u_0^1) - \Lambda^2}\right| < 1$$

is precisely (4.1).

4.2. The Wang *BV* stability condition [W]. In [W], (1.1), (1.7) is assumed to satisfy the following finiteness condition:

Let

$$\begin{aligned} &\left(\Lambda^1 Id - Df(u_0^1)\right)^{-1}(u_0^1 - u_0^0) = \alpha r_1(u_0^1) + \beta r_2(u_0^1), \\ &\left(Df(u_0^1) - \Lambda^2 Id\right)^{-1}(u_0^2 - u_0^1) = \gamma r_1(u_0^1) + \delta r_2(u_0^1). \end{aligned} \tag{4.3}$$

Then

$$|\beta\gamma| < |\alpha\delta|. \tag{4.4}$$

The above condition is a reduction of a multidimensional BV stability condition (to be found in [Me]) to the case of one space dimension.

THEOREM 4.1. *Assume that both shocks in the reference solution* (1.7) *(recall that $m = 2$) are Majda stable and Lax admissible. Then the condition* (4.4) *is equivalent to the BV stability condition* (4.2).

Proof. It is enough to show that in the context of (4.3), (4.4), (4.2), there hold

$$\left|\frac{\beta}{\alpha}\right| = \left|\frac{\partial \epsilon_2^{out}}{\partial \epsilon_1^{in}}_{|\epsilon_1^{in} = 0}\right|, \tag{4.5}$$

$$\left|\frac{\gamma}{\delta}\right| = \left|\frac{\partial \epsilon_1^{out}}{\partial \epsilon_2^{in}}_{|\epsilon_2^{in} = 0}\right|. \tag{4.6}$$

We focus on (4.5) and thus the case when the large shock (u_0^0, u_0^1) is hit from the right by a small wave of the first characteristic family and strength ϵ_1^{in}. The proof of (4.6) is entirely similar, so we omit it.

Let $F : \Omega^0 \times \Omega^1 \times I \longrightarrow \mathbf{R}$ be defined as follows (I is here a small neighborhood of $0 \in \mathbf{R}$):

$$F(u^-, u^+, \epsilon) = \Psi_0(u^-, \widetilde{\Phi}_2(u^+, \epsilon)),$$

where Ψ_0 is as in (2.5), (2.6) (its existence is implied by the proof of Theorem 2.1, in view of the Majda stability of the first large shock). The functions $\widetilde{\Phi}_i : \Omega^1 \times I \longrightarrow \Omega^1$ for $i = 1, 2$ are such that

$$\widetilde{\Phi}_i(u^+, \epsilon) = u^- \quad \text{iff} \quad \Phi_i(u^-, \epsilon) = u^+,$$

where $\Phi_i : \Omega^1 \times I \longrightarrow \Omega^1$ for a fixed u^- coincides with the ith rarefaction curve in the positive part of I, and for $\epsilon \in I$ negative follows the ith shock curve through the argument point u (compare [L]). It is not hard to notice that $\frac{\partial}{\partial \epsilon}\widetilde{\Phi}_i(u, 0) = -r_i(u)$.

The fundamental equation relating the strengths ϵ_1^{in} and ϵ_2^{out} in (4.5) has by (2.6) the form

$$F(u_0^0, \Phi_1(u_0^1, \epsilon_1^{in}), \epsilon_2^{out}) = 0. \tag{4.7}$$

Differentiating (4.7) with respect to ϵ_1^{in} at $\epsilon_1^{in} = 0$ and using (2.13), we receive

$$\begin{aligned} 0 &= \frac{\partial \Phi_0}{\partial u^1}(u_0^0, u_0^1) \cdot r_1(u_0^1) - \frac{\partial \Phi_0}{\partial u^1}(u_0^0, u_0^1) \cdot r_2(u_0^1) \cdot \frac{\partial \epsilon_2^{out}}{\partial \epsilon_1^{in}}_{|\epsilon_1^{in} = 0} \\ &= V_1(u_0^1 - u_0^0)^T \cdot \left[Df(u_0^1) - \Lambda^1 Id\right] \cdot \left(r_1(u_0^1) - r_2(u_0^1) \cdot \frac{\partial \epsilon_2^{out}}{\partial \epsilon_1^{in}}_{|\epsilon_1^{in} = 0} \right). \end{aligned} \tag{4.8}$$

Since $V_1(u_0^1 - u_0^0)$ is orthogonal to $u_0^1 - u_0^0$, (4.8) is equivalent to

$$\left[Df(u_0^1) - \Lambda^1 Id\right] \cdot \left(r_1(u_0^1) - r_2(u_0^1) \cdot \frac{\partial \epsilon_2^{out}}{\partial \epsilon_1^{in}}_{|\epsilon_1^{in} = 0} \right) = s \cdot (u_0^1 - u_0^0), \tag{4.9}$$

with some $s \neq 0$, as Λ^1 is not an eigenvalue of $Df(u_0^1)$. The first formula in (4.3) is equivalent to

$$\left[Df(u_0^1) - \Lambda^1 Id\right] \cdot \left(-\alpha r_1(u_0^1) - \beta r_2(u_0^1)\right) = (u_0^1 - u_0^0),$$

and thus by (4.9) we get (4.5). ◻

Acknowledgment. We wish to thank an anonymous referee for bringing to our attention the book [LY].

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