TEMPORAL ASYMPTOTICS FOR THE P'TH POWER NEWTONIAN FLUID IN ONE SPACE DIMENSION

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Abstract. The balance laws of mass, momentum and energy are considered for a p'th power Newtonian fluid undergoing one dimensional longitudinal motions. For initial-boundary value problems involving fixed endpoints held at a prescribed temperature or insulated, we prove exponential convergence of solutions to equilibria for generic initial data. The estimates for different boundary conditions are presented in a unified manner by utilising the thermodynamic concept of availability.

1. Introduction.

This article is concerned with initial-boundary value problems for a p'th power Newtonian fluid (see [D]) undergoing longitudinal one-dimensional motions. The governing equations, which express the balance of mass, momentum and energy in Lagrangian form, are as follows

$$\begin{aligned} \xi_t &= \nu_x, \\ \nu_t &= \left(-\frac{\theta}{\xi^p} + \mu\frac{\nu_x}{\xi}\right)_x, \\ c_v\theta_t &= \left(-\frac{\theta}{\xi^p} + \mu\frac{\nu_x}{\xi}\right)\nu_x + \left(\kappa\frac{\theta_x}{\xi}\right)_x, \end{aligned} \tag{G}$$

where ξ (specific volume), ν (velocity), θ (absolute temperature) are unknown functions of $(x,t) \in [0,1] \times [0,\infty)$, and $\mu, \kappa, c_v > 0$ are given constants. For boundary conditions, we take zero velocity endpoints

$$\nu(0,t) = 0 = \nu(1,t), \tag{V}$$

along with either the Dirichlet condition

$$\theta(0,t) = \Theta = \theta(1,t), \tag{D}$$

where $\Theta > 0$ is a prescribed constant, or the Neumann condition

$$\theta_x(0,t) = 0 = \theta_x(1,t), \tag{N}$$

while the initial conditions are given by

$$\xi(x,0) = \xi_0(x), \qquad \nu(x,0) = \nu_0(x), \qquad \theta(x,0) = \theta_0(x). \tag{I}$$

Finally, the specific volume ξ and the absolute temperature θ are subject to the physical constraints

$$\xi > 0 \quad \text{and} \quad \theta > 0. \tag{C}$$

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The two initial-boundary value problems given by (G), (V), (I), (C), along with the Dirichlet condition (D) or Neuman condition (N), will be referred to as $(IBVP)_D$ and $(IBVP)_N$ respectively.

The p'th power Newtonian fluid is a linearly viscous, Fourier heat-conducting compressible gas, whose pressure $\mathcal{P}$ and internal energy e are given by

$$\mathcal{P} = \frac{\theta}{\xi^p}, \qquad e = c_v\theta,$$

with the pressure exponent $p \geq 1$ and constant specific heat $c_v > 0$. The stress S and the heat flux q then appear in Lagrangian form as follows

$$S = -\frac{\theta}{\xi^p} + \mu\frac{\nu_x}{\xi}, \qquad q = -\kappa\frac{\theta_x}{\xi},$$

where the constants $\mu, \kappa > 0$ are the viscosity and heat conductivity of the fluid. We may now express the momentum and energy equations in (G) in the concise balance law form

$$\nu_t = S_x, \tag{M}$$

$$\left(e + \frac{1}{2}\nu^2\right)_t = (S\nu - q)_x\,. \tag{E}$$

Initial-boundary value problems for the compressible Navier-Stokes ($p = 1$) equation in one space dimension have been extensively studied following the seminal paper [KS], where the existence and uniqueness of global classical solutions to the corresponding $(IBVP)_N$ was established. The ideas of this work have been extended to a variety of other physically natural boundary conditions (e.g. stress free) and more general pressure laws, along with weak existence theorems in Sobolev and BV spaces (see [CHT], [H], [J], [Ka], [KN], [L], [N1], [S]). The temporal asymptotics of solutions have also been studied, and exponential rates of convergence established (see [HL] and [N2]); while the treatment of the Dirichlet temperature condition (D) can be found in [N0].

In this paper we prove the exponential convergence of the specific volume, velocity and temperature to their respective equilibrium values (Theorem II). The main step in the argument is the derivation of the pointwise uniform bounds on the specific volume (see Theorem I (i)). A central difficulty here, that is associated with the pinned endpoint boundary condition, is the presence of an a priori unknown impulse $\int_0^t S(1,\tau)\mathrm{d}\tau$, arising at the boundary. We obtain the requisite bound, namely

$$0 < \underline{\xi} < \xi(x,t) < \overline{\xi}, \quad \forall (x,t) \in [0,1] \times [0,\infty),$$

through an analysis of the momentum balance, in combination with estimates following from the entropy identity (3.2) and convexity arguments.

The next step in the proof of convergence involves establishing a global L^2 estimate on the temperature gradient:

$$\int_0^\infty \int_0^1 \theta_x^2(x,t)\mathrm{d}x\mathrm{d}t < \infty$$

(see Theorem I (ii)). Here, the main difficluty arises from the Dirichlet temperature condition, due to the a priori unknown energy flux $\int_0^t [q(0,\tau) - q(1,\tau)]\mathrm{d}\tau$, through the boundary. This is circumvented by identifying a thermodynamic potential, the availablity [1], which is adapted to either temperature boundary condition (D) or (N), and then serves as a Lyapunov function for solutions.

The layout of this paper is as follows. In Section 2 we state the main theorems, while in Section 3 we gather some preliminary global estimates relevant for their proofs. The key pointwise estimate on the specific volume is obtained in Section 4, followed by the proof of the global L^2 temperature gradient estimate in Section 5. The final two Sections are devoted to establishing the exponential convergence of solutions to equilibrium states.

2. Main results.

The existence theory for the initial-boundary value problems may be conveniently formulated in terms of the spaces of Hölder continuous functions $C^{2+\alpha}[0,1]$, and $C^{\alpha,\alpha/2}\left([0,1]\times[0,\infty)\right)$ which arise naturally in the theory of parabolic partial differential equations, see [K]. Throughout the paper we adopt the convention that any constant that appears will depend at most on the $C^{2+\alpha}$ norms of the initial data, $\min_{x\in[0,1]}\xi_0(x)$ and $\min_{x\in[0,1]}\theta_0(x)$. Also, we denote such generic "small" constants by λ, and "large" constants by Λ.

For convenience and without loss of generality we may assume $\int_0^1 \xi_0(x)\mathrm{d}x = 1$. Then from conservation of mass and condition (V) we have

$$\int_0^1 \xi(x,t)\mathrm{d}x = 1. \tag{2.1}$$

We first state an existence and uniqueness result which follows from [W].

Theorem . *Consider the initial-boundary value problems given by $(IBVP)_D$ or $(IBVP)_N$. Set $\alpha\in(0,1)$ and let the initial data $\xi_0,\nu_0,\theta_0\in C^{2+\alpha}[0,1]$ satisfy the physical constraints $\xi_0,\theta_0>0$ and be compatible with the relevant boundary conditions. Then there exists a unique classical solution (ξ,ν,θ) on $[0,1]\times[0,\infty)$ with $\xi\in C^{1+\alpha,1+\alpha/2}\left([0,1]\times[0,\infty)\right)$, $\nu,\theta\in C^{2+\alpha,1+\alpha/2}\left([0,1]\times[0,\infty)\right)$.*

Our first main result concerns uniform pointwise bounds on the specific volume and global L^2 bounds on the temperature gradient.

Theorem I. *Let (ξ,ν,θ) be as in the previous Theorem. There exist constants $\underline{\xi},\bar{\xi}>0$, such that*

$$\begin{aligned} &\text{(i)} \quad \underline{\xi} \le \xi(x,t) \le \overline{\xi}, \\ &\text{(ii)} \quad \int_0^\infty\int_0^1 \theta_x^2 \mathrm{d}x\mathrm{d}t < \infty. \end{aligned}$$

Based on the above estimates we are able to establish the exponential convergence of solutions to equilibrium states. Here, a distinction arises between the $(IBVP)_D$ and $(IBVP)_N$ with respect to the limiting temperature. More precisely, seting

$$\bar{\Theta} = \begin{cases} \Theta & \text{for } (IBVP)_D, \\ \dfrac{1}{c_v}\displaystyle\int_0^1\left(c_v\theta_0 + \frac{1}{2}\nu_0^2\right)\mathrm{d}x & \text{for } (IBVP)_N, \end{cases}$$

[1] Ericksen refers to this as ballistic free energy [E].

we have the following convergence result:

Theorem II. *Let (ξ, ν, θ) be as before. Then*

$$\text{(i)} \quad \int_0^1 \left(\xi_x^2 + \nu_x^2 + \theta_x^2\right)(x,t)\mathrm{d}x \le \Lambda e^{-\lambda t},$$

$$\text{(ii)} \quad \max_{x\in[0,1]} \left(|\xi(x,t) - 1| + |\nu(x,t)| + |\theta(x,t) - \bar{\Theta}|\right) \le \Lambda e^{-\lambda t}.$$

The established convergence rate is a reflection of the underlying parabolic structure of the governing equations (G) induced by the presence of viscosity and heat conduction.

3. Energy and entropy estimates.

In this Section we identify relevant thermodynamic quantities and establish global bounds thereon. First, we note that the entropy η of a p'th power Newtonian fluid is a concave function

$$\eta(\theta, \xi) = c_v \ln\theta + h(\xi), \qquad \text{where } h(\xi) = \begin{cases} \ln\xi & \text{for } p = 1 \\ \frac{1}{p-1}\left(1 - \xi^{1-p}\right) & \text{for } p > 1 \end{cases}, \tag{3.1}$$

which evaluated along a solution to (M) and (E) satisfies the following standard entropy identity:

$$\eta_t = \mu\frac{\nu_x^2}{\xi\theta} + \kappa\frac{\theta_x^2}{\xi\theta^2} - \left(\frac{q}{\theta}\right)_x. \tag{3.2}$$

For the Neumann problem, the entropy flux, $-q/\theta$, is zero at the boundary. But, for the Dirichlet problem this is generally not true due to the entropy exchange with the heat bath. However, for either boundary condition we can track the global entropy change (fluid and heat bath) through the quantity

$$A := e + \frac{1}{2}\nu^2 - \bar{\Theta}\eta = c_v(\theta - \bar{\Theta}\ln\theta) - \bar{\Theta}h(\xi) + \frac{1}{2}\nu^2. \tag{3.3}$$

By combining (E) and (3.2) we obtain the availability identity,

$$\left(e + \frac{1}{2}\nu^2 - \bar{\Theta}\eta\right)_t = \left(S\nu - \left(1 - \frac{\bar{\Theta}}{\theta}\right)q\right)_x - \bar{\Theta}\left(\mu\frac{\nu_x^2}{\xi\theta} + \kappa\frac{\theta_x^2}{\xi\theta^2}\right), \tag{3.4}$$

which reveals $\int_0^1 A\mathrm{d}x$ as a Lyapunov functional for both $(IBVP)_D$ and $(IBVP)_N$. This is due to the presence of the entropy production term $\mu\nu_x^2/(\xi\theta) + \kappa\theta_x^2/(\xi\theta^2)$ induced by the dissapative mechanisms of viscosity $\mu > 0$ and heat conduction $\kappa > 0$; see formula (7.1).

All subsequent results, unless otherwise indicated, apply to both boundary value problem $(IBVP)_D$ and $(IBVP)_N$.

Lemma 3.1.
(i) $\displaystyle\int_0^1 \nu^2(x,t)\mathrm{d}x \le \Lambda$,
(ii) $\displaystyle\lambda \le \int_0^1 \theta(x,t)\mathrm{d}x \le \Lambda,$
(iii) $\displaystyle\int_0^t \int_0^1 \left(\mu\frac{\nu_x^2}{\xi\theta} + \kappa\frac{\theta_x^2}{\xi\theta^2}\right)\mathrm{d}x\mathrm{d}\tau \le \Lambda.$

Proof. Integrating (3.1) in space and then using Jensen's inequality and incorporating (2.1) we get

$$\int_0^1 \eta \mathrm{d}x \le c_v \ln\left(\int_0^1 \theta \mathrm{d}x\right) + h\left(\int_0^1 \xi \mathrm{d}x\right) = c_v \ln\left(\int_0^1 \theta\mathrm{d}x\right). \tag{3.5}$$

Integrating (3.4) over $[0,1]\times[0,t]$, and noting boundary conditions (V) and either (D) or (N), we arrive at:

$$\int_0^1 \left(c_v\theta + \frac{1}{2}\nu^2 - \bar{\Theta}\eta\right)\mathrm{d}x + \bar{\Theta}\int_0^t\int_0^1 \left(\mu\frac{\nu_x^2}{\xi\theta} + \kappa\frac{\theta_x^2}{\xi\theta^2}\right)\mathrm{d}x\mathrm{d}\tau \le \Lambda.$$

Hence, in view of (3.5):

$$\begin{aligned}\int_0^1 \left(c_v\theta + \frac{1}{2}\nu^2\right)\mathrm{d}x + \bar{\Theta}\int_0^t\int_0^1 \left(\mu\frac{\nu_x^2}{\xi\theta} + \kappa\frac{\theta_x^2}{\xi\theta^2}\right)\mathrm{d}x\mathrm{d}\tau \\ \le \Lambda + c_v\bar{\Theta}\ln\left(\int_0^1 \theta\mathrm{d}x\right).\end{aligned} \tag{3.6}$$

In particular, $\int_0^1 \theta\mathrm{d}x \le \Lambda + \bar{\Theta}\ln\left(\int_0^1 \theta\mathrm{d}x\right)$, which yields (ii). Using (ii) in (3.6) we establish (i) and (iii). ∎

4. Pointwise bounds on the specific volume ξ.

This Section is devoted to proving Theorem I (i). The central difficulty here stems from the presence of an a priori unknown impulse $I = \int S(1,s)\mathrm{d}s$ arising at the boundary, due to the pinned endpoint condition (V).

The main step is the derivation of the uniform upper bound on ξ, and is motivated by ideas first appearing in [KS]. Time dependent lower bounds on ξ then follow directly, while the uniform bound from below requires a more detailed study of the impulse I. The presence of viscosity and heat conduction, as well as the thermodynamic structure of the problem, play a crucial role in the argument.

For convenience we introduce the notation

$$\theta_m(t) = \max_{x\in[0,1]}\theta(x,t), \quad \xi_m(t) = \max_{x\in[0,1]}\xi(x,t), \quad \nu_m(t) = \max_{x\in[0,1]}\nu(x,t).$$

The following preliminary lemma will be of later use.

Lemma 4.1.
(i) $\displaystyle\theta_m(t) \le \Lambda\left(1 + \xi_m(t)\int_0^1 \frac{\theta_x^2}{\xi\theta^2}\mathrm{d}x\right),$
(ii) $\displaystyle\theta(x,t) \ge \lambda - \Lambda\int_0^1 \frac{\theta_x^2}{\theta^2}\mathrm{d}x.$

Proof. To prove (i), calculate:

$$\begin{aligned}\theta(x,t) &\leq \left[\theta^{1/2}(y,t)+\frac{1}{2}\int_0^1 \frac{\mid\theta_x\mid}{\theta^{1/2}}\mathrm{d}x\right]^2 \leq 2\left[\theta(y,t)+\frac{1}{4}\left(\int_0^1 \frac{|\theta_x|}{\theta^{1/2}}\mathrm{d}x\right)^2\right]\\ &\leq 2\left[\theta(y,t)+\frac{1}{4}\left(\int_0^1 \xi\theta\mathrm{d}x\right)\left(\int_0^1 \frac{\theta_x^2}{\xi\theta^2}\mathrm{d}x\right)\right].\end{aligned} \tag{4.1}$$

Integrating (4.1) with respect to y over $[0,1]$, by Lemma 3.1 (ii) we get (i).

In a similar manner we receive

$$\begin{aligned}\theta^{1/2}(x,t) &\geq \theta^{1/2}(y,t)-\frac{1}{2}\int_0^1 \frac{\mid\theta_x\mid}{\theta^{1/2}}\mathrm{d}x\\ &\geq \theta^{1/2}(y,t)-\frac{1}{2}\left(\int_0^1 \theta\mathrm{d}x\right)^{1/2}\left(\int_0^1 \frac{\theta_x^2}{\theta^2}\mathrm{d}x\right)^{1/2}\end{aligned}$$

which upon squaring and recalling Lemma 3.1 (ii), yields

$$\theta(x,t) \geq \frac{1}{2}\theta(y,t)-\Lambda\int_0^1 \frac{\theta_x^2}{\theta^2}\mathrm{d}x.$$

Now integrating in y over $[0,1]$ and using Lemma 3.1 (ii) we obtain (ii). ∎

We also need a technical lemma, proved in [N2].

Lemma 4.2. *Let $\alpha(t)$ and $\omega(t)$ be continuous functions, α nonnegative and ω satisfying:*

$$\lambda e^{\lambda(t-\tau)} \leq \exp\left\{\int_\tau^t \omega(s)\mathrm{d}s\right\} \leq \Lambda e^{\Lambda(t-\tau)}$$

for every pair of nonnegative $\tau \leq t$. Then

$$\limsup_{t\to+\infty}\int_0^t \exp\left\{-\int_\tau^t \omega(s)\mathrm{d}s\right\}\alpha(\tau)\mathrm{d}\tau \leq \Lambda\limsup_{t\to+\infty}\int_t^{t+1}\alpha(s)\mathrm{d}s.$$

We are now in a position to present:

Proof of Theorem I (i). For the convenience of the reader, we divide the proof into three steps.

STEP 1. Integrating the momentum balance (M) over $[x,1]\times[\tau,t]$ we get

$$\begin{aligned}\int_1^x [\nu(r,t)-\nu(r,\tau)]\,\mathrm{d}r &= \int_\tau^t S(x,s)\mathrm{d}s - \int_\tau^t S(1,s)\mathrm{d}s\\ &= -\int_\tau^t S(1,s)\mathrm{d}s - \int_\tau^t \frac{\theta}{\xi^p}(x,s)\mathrm{d}s + \mu\ln\xi(x,s)|_{s=\tau}^{s=t}.\end{aligned}$$

Setting $\mathcal{M}(x,\tau,t) := \int_1^x[\nu(r,t)-\nu(r,\tau)]\mathrm{d}r$, and the impulse

$$\mathcal{I}(\tau,t) := \int_\tau^t S(1,s)\mathrm{d}s,$$

we rewrite the above equation in the form

$$\int_\tau^t \frac{\theta}{\xi^p}(x,s)\mathrm{d}s = \mu\ln\xi(x,t)-\mu\ln\xi(x,\tau)-\mathcal{I}(\tau,t)-\mathcal{M}(x,\tau,t). \tag{4.2}$$

Multiplying (4.2) by p/μ and taking exponentials allows one to check readily that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\xi^p\cdot\exp\left\{-\frac{p}{\mu}\mathcal{M}-\frac{p}{\mu}\mathcal{I}\right\}\right]=\frac{p}{\mu}\;\theta\exp\left\{-\frac{p}{\mu}\mathcal{M}-\frac{p}{\mu}\mathcal{I}\right\}.$$

Hence,

$$\begin{aligned}\xi^p(x,t)\cdot\exp&\left\{-\frac{p}{\mu}\mathcal{M}(x,\tau,t)-\frac{p}{\mu}\mathcal{I}(\tau,t)\right\}\\ &=\xi^p(x,\tau)+\int_\tau^t\frac{p}{\mu}\;\theta(x,s)\cdot\exp\left\{-\frac{p}{\mu}\mathcal{M}(x,\tau,s)-\frac{p}{\mu}\mathcal{I}(\tau,s)\right\}\mathrm{d}s.\end{aligned}\tag{4.3}$$

We check at once that $\mathcal{M}$ is a bounded quantity. Indeed, using Jensen's inequality and Lemma 3.1 (i) we have

$$|\mathcal{M}(x,\tau,t)|^2\le 2\int_0^1\left[\nu^2(x,t)+\nu^2(x,\tau)\right]\mathrm{d}x\le\Lambda.\tag{4.4}$$

Introducing (4.4) in (4.3) we arrive at

$$\begin{aligned}\lambda\left[\xi^p(x,\tau)+\int_\tau^t\theta(x,s)e^{-\frac{p}{\mu}\mathcal{I}(\tau,s)}\mathrm{d}s\right]&\le\xi^p(x,t)\cdot e^{-\frac{p}{\mu}\mathcal{I}(\tau,t)}\\ &\le\Lambda\left[\xi^p(x,\tau)+\int_\tau^t\theta(x,s)e^{-\frac{p}{\mu}\mathcal{I}(\tau,s)}\mathrm{d}s\right].\end{aligned}\tag{4.5}$$

STEP 2. In this step we prove the uniform upper bound on ξ. First, from (2.1) it follows that

$$\int_0^1\xi^p(x,t)\mathrm{d}x\le\xi_m^{p-1}(t)\int_0^1\xi(x,t)\mathrm{d}x=\xi_m^{p-1}(t).\tag{4.6}$$

On the other hand, by Jensen's inequality

$$\int_0^1\xi^p(x,\tau)\mathrm{d}x\ge\left(\int_0^1\xi(x,\tau)\mathrm{d}x\right)^p=1.\tag{4.7}$$

¿From the right inequality in (4.5) with $\tau=0$, Lemma 4.1 (i) shows that:

$$\begin{aligned}\xi_m^p(t)\cdot e^{-\frac{p}{\mu}\mathcal{I}(0,t)}\le\Lambda\Bigg[&1+\int_0^t e^{-\frac{p}{\mu}\mathcal{I}(0,s)}\mathrm{d}s\\ &+\int_0^t\xi_m(s)\cdot e^{-\frac{p}{\mu}\mathcal{I}(0,s)}\left(\int_0^1\frac{\theta_x^2}{\xi\theta^2}\mathrm{d}x\right)(s)\mathrm{d}s\Bigg].\end{aligned}\tag{4.8}$$

Since by (4.7) $\xi_m^p\ge 1$, it is clear that $\xi_m(t)\le\xi_m^p(t)$, and thus from (4.8), by means of the Gronwall inequality, we obtain

$$\xi_m^p(t)\cdot e^{-\frac{p}{\mu}\mathcal{I}(0,t)}\le\Lambda\left(1+\int_0^t e^{-\frac{p}{\mu}\mathcal{I}(0,s)}\mathrm{d}s\right)\cdot\exp\left\{\Lambda\int_0^t\int_0^1\frac{\theta_x^2}{\xi\theta^2}\mathrm{d}x\mathrm{d}s\right\}.$$

By Lemma 3.1 (iii) it follows that

$$\xi_m^p(t)\le\Lambda e^{\frac{p}{\mu}\mathcal{I}(0,t)}\left(1+\int_0^t e^{-\frac{p}{\mu}\mathcal{I}(0,s)}\mathrm{d}s\right).\tag{4.9}$$

On the other hand, setting $\tau=0$ in the left inequality (4.5), then integrating over the spatial interval $[0,1]$, and utilizing the estimates (4.6), (4.7) and Lemma 3.1

(ii), we receive the following bound:

$$\xi_m^{p-1}(t) \geq \lambda e^{\frac{p}{\mu}\mathcal{I}(0,t)} \left(1 + \int_0^t e^{-\frac{p}{\mu}\mathcal{I}(0,s)} \mathrm{d}s\right). \tag{4.10}$$

Now, (4.9) and (4.10) give

$$\xi_m^p(t) \leq \Lambda \xi_m^{p-1}(t),$$

from which we conclude the existence of $\overline{\xi}$ such that

$$\xi(x,t) \leq \overline{\xi} \tag{4.11}$$

STEP 3. Our next concern will be the lower bound on ξ. Integrating (4.5) in x over $[0,1]$ and recalling (4.7), (4.11) and Lemma 3.1 (ii) we see that

$$\lambda \leq e^{\frac{p}{\mu}\mathcal{I}(\tau,t)} \left(1 + \int_\tau^t e^{-\frac{p}{\mu}\mathcal{I}(\tau,s)} \mathrm{d}s\right) \leq \Lambda. \tag{4.12}$$

Setting $\tau = 0$ in the left inequality in (4.5) while utilizing Lemma 4.1 (ii) and (4.12), we have

$$\begin{aligned} \xi^p(x,t) &\geq \lambda e^{\frac{p}{\mu}\mathcal{I}(0,t)} \cdot \left[\xi_0^p(x) + \int_0^t e^{-\frac{p}{\mu}\mathcal{I}(0,s)} \theta(x,s) \mathrm{d}s\right] \\ &\geq \lambda e^{\frac{p}{\mu}\mathcal{I}(0,t)} \cdot \Bigg[1 + \int_0^t e^{-\frac{p}{\mu}\mathcal{I}(0,s)} \mathrm{d}s \\ &\qquad\qquad - \Lambda \int_0^t e^{-\frac{p}{\mu}\mathcal{I}(0,s)} \left(\int_0^1 \frac{\theta_x^2}{\theta^2} \mathrm{d}x\right)(s) \mathrm{d}s\Bigg] \\ &\geq \lambda - \Lambda \int_0^t e^{\frac{p}{\mu}\mathcal{I}(s,t)} \left(\int_0^1 \frac{\theta_x^2}{\theta^2} \mathrm{d}x\right)(s) \mathrm{d}s. \end{aligned} \tag{4.13}$$

Now, by Gronwall's inequality applied to (4.12) we obtain

$$\lambda e^{\lambda(t-\tau)} \leq e^{-\frac{p}{\mu}\mathcal{I}(\tau,t)} \leq \Lambda e^{\Lambda(t-\tau)}.$$

Thus, from Lemma 4.2 with $\omega(t) = -S(1,t)$ and $\alpha(t) = \displaystyle\int_0^1 \frac{\theta_x^2}{\theta^2}(x,t) \mathrm{d}x$, we conclude

$$\limsup_{t\to+\infty} \int_0^t e^{\frac{p}{\mu}\mathcal{I}(s,t)} \left(\int_0^1 \frac{\theta_x^2}{\theta^2} \mathrm{d}x\right)(s) \mathrm{d}s \leq \Lambda \limsup_{t\to+\infty} \int_t^{t+1} \int_0^1 \frac{\theta_x^2}{\theta^2} \mathrm{d}x \mathrm{d}s. \tag{4.14}$$

On the other hand Lemma 3.1 (iii) and (4.11) imply that the function $\int_0^1 \theta_x^2/\theta^2 \mathrm{d}x$ is integrable in $[0,\infty)$, so the right hand side of (4.14) equals zero. Now since the right hand side of the first inequality in (4.13) is a continuous and positive function of t, in view of (4.14) it implies:

$$\xi(x,t) \geq \underline{\xi}.$$

■

5. Global L^2 bound on the temperature gradient θ_x.

In this Section we complete the proof of Theorem I. As in the derivation of the energy and entropy bounds of Section 3, the availability identity (3.4) plays a key role. Estimates associated with this identity and the momentum equation (Lemma 5.2) enable us to deal with the presence of the a priori unknown boundary heat flux $\int_0^t [q(0,\tau) - q(1,\tau)]\,\mathrm{d}\tau$.

In order to avoid excessive repetition, we do not generally refer directly to the previously established Theorem I (i) but use it freely throughout the presented estimates.

Lemma 5.1. (i) $\displaystyle\int_0^t \nu_m^2(\tau)\mathrm{d}\tau \le \Lambda,$

(ii) $\displaystyle\int_0^t\int_0^1 \left(S^2\nu^2 + \nu^2\nu_x^2\right)\mathrm{d}x\mathrm{d}\tau \le \Lambda\left(1 + \int_0^t\int_0^1 \theta^2\nu^2\mathrm{d}x\mathrm{d}\tau\right).$

Proof. Recalling the boundary condition (V) and Lemma 3.1 (ii) we get

$$\nu_m^2(t) \le \left(\int_0^1 |\nu_x(x,t)|\mathrm{d}x\right)^2 \le \left(\int_0^1 \theta\mathrm{d}x\right)\left(\int_0^1 \frac{\nu_x^2}{\theta}\mathrm{d}x\right) \le \Lambda\int_0^1 \frac{\nu_x^2}{\theta}(x,t)\mathrm{d}x.$$

Now, integrating in t and using Lemma 3.1 (iii) we establish (i).

In order to prove (ii) we use (M) and Young's inequality to receive

$$\begin{aligned}\frac{1}{4}\left(\nu^4\right)_t &= S_x\nu^3 = \left(S\nu^3\right)_x - 3S\nu^2\nu_x = \left(S\nu^3\right)_x + 3\frac{\theta}{\xi^p}\nu^2\nu_x - 3\frac{\mu}{\xi}\nu^2\nu_x^2\\ &\le \left(S\nu^3\right)_x - \frac{3}{2}\frac{\mu}{\xi}\nu^2\nu_x^2 + \Lambda\theta^2\nu^2.\end{aligned}$$

Integrating the above inequality over $[0,1]\times[0,t]$, in view of (V) we see that

$$\int_0^1 \nu^4\mathrm{d}x + \int_0^t\int_0^1 \nu^2\nu_x^2\mathrm{d}x\mathrm{d}\tau \le \Lambda\left(1 + \int_0^t\int_0^1 \theta^2\nu^2\mathrm{d}x\mathrm{d}\tau\right).$$

But $S\nu = -\theta\nu/\xi^p + \mu\nu\nu_x/\xi$, and so (ii) follows directly. ∎

Lemma 5.2. *For every $\epsilon > 0$ there exists a constant Λ such that*

$$\int_0^1 \xi_x^2(x,t)\mathrm{d}x + \int_0^t\int_0^1 \xi_x^2\mathrm{d}x\mathrm{d}\tau \le \Lambda + \epsilon\int_0^t\int_0^1 \theta_x^2\mathrm{d}x\mathrm{d}\tau.$$

Proof. The balance of momentum (M) can be rewritten in the form:

$$\left(\mu\frac{\xi_x}{\xi} - \nu\right)_t = \left(\frac{\theta}{\xi^p}\right)_x.$$

Multiplying by $\mu\xi_x/\xi - \nu$ and integrating over $[0,1]$, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\mu\frac{\xi_x}{\xi} - \nu\right)^2\mathrm{d}x = 2\int_0^1 \left(\frac{\theta_x}{\xi^p} - p\frac{\theta\xi_x}{\xi^{p+1}}\right)\left(\mu\frac{\xi_x}{\xi} - \nu\right)\mathrm{d}x, \tag{5.1}$$

which after a simple computation gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\mu\frac{\xi_x}{\xi} - \nu\right)^2\mathrm{d}x &+ \lambda\int_0^1 \theta\left(\mu\frac{\xi_x}{\xi} - \nu\right)^2\mathrm{d}x\\ &\le \Lambda\int_0^1 \left(\mu\frac{\xi_x}{\xi} - \nu\right)\left(\frac{\theta_x}{\xi^p} - \frac{p\theta}{\mu\xi^p}\nu\right)\mathrm{d}x.\end{aligned} \tag{5.2}$$

Noting

$$\left(\mu\frac{\xi_x}{\xi}-\nu\right)^2 \geq \frac{1}{2}\left(\mu\frac{\xi_x}{\xi}\right)^2-\nu^2, \tag{5.3}$$

and recalling Lemma 5.1 (i) and Lemma 3.1 (ii), it follows from integrating (5.2) over $[0,t]$ that

$$\int_0^1 \xi_x^2 \mathrm{d}x + \int_0^t\int_0^1 \theta\xi_x^2 \mathrm{d}x\mathrm{d}\tau \leq \Lambda\left(1+\int_0^t\int_0^1 |\theta_x\nu| + |\theta\xi_x\nu| + |\theta_x\xi_x| \mathrm{d}x\mathrm{d}\tau\right). \tag{5.4}$$

We now proceed to estimate the right hand side of (5.4). Throughout, we let $\delta > 0$ denote a constant which may be chosen as small as one wishes. From Lemma 5.1 (i), Lemma 3.1 (ii) and Young's inequality, we get:

$$\begin{aligned}
&\int_0^t\int_0^1 \left(|\theta_x\nu| + |\theta\xi_x\nu| + |\theta_x\xi_x|\right) \mathrm{d}x\mathrm{d}\tau \\
&\qquad \leq \delta\int_0^t\int_0^1 \left(\theta_x^2+\theta\xi_x^2\right)\mathrm{d}x\mathrm{d}\tau + \Lambda\int_0^t\int_0^1\left(\nu^2+\theta\nu^2+\frac{\theta_x^2}{\theta}\right)\mathrm{d}x\mathrm{d}\tau \\
&\qquad \leq \delta\int_0^t\int_0^1 \left(\theta_x^2+\theta\xi_x^2\right)\mathrm{d}x\mathrm{d}\tau + \Lambda\left(1+\int_0^t\int_0^1 \frac{\theta_x^2}{\theta}\mathrm{d}x\mathrm{d}\tau\right) \\
&\qquad \leq \Lambda + 2\delta\int_0^t\int_0^1\left(\theta_x^2+\theta\xi_x^2\right)\mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{5.5}$$

The last inequality in (5.5) follows by observing that for any $\gamma > 0$ the inequality $1/\theta \leq 1/(\gamma\theta^2) + \gamma/4$ holds uniformly in θ, and then noting Lemma 3.1 (iii).

Now, substituting (5.5) into (5.4), we deduce

$$\int_0^1 \xi_x^2\mathrm{d}x + \int_0^t\int_0^1 \theta\xi_x^2\mathrm{d}x\mathrm{d}\tau \leq \Lambda + \delta\int_0^t\int_0^1 \theta_x^2\mathrm{d}x\mathrm{d}\tau.$$

In view of Lemma 4.1 (ii), this yields:

$$\begin{aligned}
&\int_0^1 \xi_x^2\mathrm{d}x + \lambda\int_0^t\int_0^1 \xi_x^2\mathrm{d}x\mathrm{d}\tau \\
&\qquad \leq \Lambda + \delta\int_0^t\int_0^1 \theta_x^2\mathrm{d}x\mathrm{d}\tau + \Lambda\int_0^t\left(\int_0^1\frac{\theta_x^2}{\theta^2}\mathrm{d}x\right)\left(\int_0^1 \xi_x^2\mathrm{d}x\right)\mathrm{d}\tau.
\end{aligned} \tag{5.6}$$

Hence, by Gronwall's inequality:

$$\int_0^1 \xi_x^2(x,t)\mathrm{d}x \leq \left(\Lambda+\delta\int_0^t\int_0^1\theta_x^2\mathrm{d}x\mathrm{d}\tau\right)\cdot\exp\left\{\Lambda\int_0^t\int_0^1\frac{\theta_x^2}{\theta^2}\mathrm{d}x\mathrm{d}\tau\right\}. \tag{5.7}$$

Note, that in both formulas (5.6) and (5.7), the constant Λ is independent of the choice of δ. Finally, utilizing Lemma 3.1 (iii) and introducing (5.7) into the right hand side of (5.6) concludes our proof. ■

We may now proceed with:

Proof of Theorem I (ii). Recalling the definition of entropy (3.1) we set

$$\omega = c_v\theta + \frac{1}{2}\nu^2 - \bar{\Theta}\eta + \gamma,$$

where $\gamma > 0$ is chosen such that $\omega \geq c_v\theta/2$ (for $\theta \geq 0$ and $\underline{\xi} \leq \xi \leq \overline{\xi}$). Note that by (3.3) we have

$$\omega_x = c_v\left(1 - \frac{\bar{\Theta}}{\theta}\right)\theta_x + \nu\nu_x - \bar{\Theta}h'(\xi)\xi_x.$$

Utilizing the availability identity (3.4), Lemma 5.1 (ii) and the boundary conditions (V) and (D) or (N), we obtain from integration by parts and Young's inequality:

$$\begin{aligned}
\frac{1}{2}\int_0^1 \omega^2(x,t)\mathrm{d}x &\leq \Lambda + \int_0^t\int_0^1 \omega\omega_t \mathrm{d}x\mathrm{d}\tau \\
&\leq \Lambda + \int_0^t\int_0^1 \omega\left(S\nu - \left(1 - \frac{\bar{\Theta}}{\theta}\right)q\right)_x \mathrm{d}x\mathrm{d}\tau \\
&= \Lambda - \int_0^t\int_0^1 \omega_x\left(S\nu + \frac{\kappa}{\xi}\left(1 - \frac{\bar{\Theta}}{\theta}\right)\theta_x\right)\mathrm{d}x\mathrm{d}\tau \qquad (5.8)\\
&\leq \Lambda - \lambda\int_0^t\int_0^1\left(1 - \frac{\bar{\Theta}}{\theta}\right)^2\theta_x^2\mathrm{d}x\mathrm{d}\tau + \Lambda\int_0^t\int_0^1\left(S^2\nu^2 + \nu^2\nu_x^2 + \xi_x^2\right)\mathrm{d}x\mathrm{d}\tau \\
&\leq \Lambda - \lambda\int_0^t\int_0^1\left(1 - \frac{\bar{\Theta}}{\theta}\right)^2\theta_x^2\mathrm{d}x\mathrm{d}\tau + \Lambda\int_0^t\int_0^1\left(\theta^2\nu^2 + \xi_x^2\right)\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

But Lemma 3.1 (iii) gives

$$\begin{aligned}
\int_0^t\int_0^1\left(1 - \frac{\bar{\Theta}}{\theta}\right)^2\theta_x^2\mathrm{d}x\mathrm{d}\tau &\geq \int_0^t\int_0^1\left(\frac{1}{2} - \frac{\bar{\Theta}^2}{\theta^2}\right)\theta_x^2\mathrm{d}x\mathrm{d}\tau \\
&\geq \frac{1}{2}\int_0^t\int_0^1 \theta_x^2\mathrm{d}x\mathrm{d}\tau - \Lambda,
\end{aligned} \qquad (5.9)$$

which, together with (5.8) and recalling that $\omega \geq c_v\theta/2$ implies

$$\begin{aligned}
\int_0^1 \theta^2(x,t)\mathrm{d}x + \int_0^t\int_0^1 \theta_x^2\mathrm{d}x\mathrm{d}\tau &\leq \Lambda + \Lambda\int_0^t\int_0^1\left(\theta^2\nu^2 + \xi_x^2\right)\mathrm{d}x\mathrm{d}\tau \\
&\leq \Lambda + \Lambda\int_0^t \nu_m^2(\tau)\int_0^1 \theta^2(x,\tau)\mathrm{d}x\mathrm{d}\tau + \Lambda\int_0^t\int_0^1 \xi_x^2\mathrm{d}x\mathrm{d}\tau.
\end{aligned} \qquad (5.10)$$

Now from Lemma 5.1 (i) and Gronwall's inequality applied to (5.10) we see that

$$\int_0^1 \theta^2(x,t)\mathrm{d}x \leq \Lambda\left(1 + \int_0^t\int_0^1 \xi_x^2\mathrm{d}x\mathrm{d}\tau\right). \qquad (5.11)$$

Substituting (5.11) into (5.10) and again noting Lemma 5.1 (i), yields

$$\int_0^1 \theta^2\mathrm{d}x + \int_0^t\int_0^1 \theta_x^2\mathrm{d}x\mathrm{d}\tau \leq \Lambda\left(1 + \int_0^t\int_0^1 \xi_x^2\mathrm{d}x\mathrm{d}\tau\right). \qquad (5.12)$$

This estimate when combined with Lemma 5.2 leads to the required result. ∎

Corollary 5.3. $\displaystyle\int_0^1 \xi_x^2(x,t)\mathrm{d}x + \int_0^t\int_0^1\left(\xi_x^2 + \theta^2\xi_x^2\right)\mathrm{d}x\mathrm{d}\tau \leq \Lambda.$

Proof. First, by Lemma 5.2 and Theorem I (ii) we have

$$\int_0^1 \xi_x^2(x,t)\mathrm{d}x + \int_0^t\int_0^1 \xi_x^2\mathrm{d}x\mathrm{d}\tau \leq \Lambda. \qquad (5.13)$$

Recalling Lemma 3.1 (ii) it follows that

$$\theta^2(x,t) \leq \left[\int_0^1 \theta(y,t)\mathrm{d}y + \int_0^1 |\theta_x|\mathrm{d}x\right]^2 \leq \Lambda + \int_0^1 \theta_x^2(x,t)\mathrm{d}x. \tag{5.14}$$

Hence, using (5.13) we arrive at

$$\begin{aligned}\int_0^t\int_0^1 \theta^2\xi_x^2\mathrm{d}x\mathrm{d}\tau &\leq \Lambda\int_0^t\int_0^1 \xi_x^2\mathrm{d}x\mathrm{d}\tau + \int_0^t\left(\int_0^1\theta_x^2\mathrm{d}x\right)\left(\int_0^1\xi_x^2\mathrm{d}x\right)\mathrm{d}\tau\\ &\leq \Lambda\int_0^t\int_0^1\left(\xi_x^2+\theta_x^2\right)\mathrm{d}x\mathrm{d}\tau,\end{aligned}$$

which in view of (5.13) and Theorem I (ii) completes our proof. ■

6. Global bounds on ξ, ν, θ.

This Section is devoted to showing pointwise convergence of ξ, ν and θ (Theorem 6.6), which will justify the linearization step in the proof of Theorem II. We first obtain global bounds on spatial derivatives of ν, θ which follow from the underlying parabolic structure of (M) and (E). Then, from the established estimates and an elementary Sobolev embedding theorem, we deduce the desired convergence.

Lemma 6.1. $\displaystyle\int_0^t\int_0^1 \nu_x^2\mathrm{d}x\mathrm{d}\tau \leq \Lambda.$

Proof. Multiplying the momentum equation (M) by ν, integrating over $[0,1]\times[0,t]$ and using integration by parts, we see that

$$\begin{aligned}\frac{1}{2}\int_0^1 \nu^2(x,t)\mathrm{d}x &\leq \Lambda\left(1+\int_0^t\int_0^1\left(|\theta_x\nu| + |\theta\nu\xi_x|\right)\mathrm{d}x\mathrm{d}\tau\right) + \int_0^t\int_0^1 \nu\left(\mu\frac{\nu_x}{\xi}\right)_x\mathrm{d}x\mathrm{d}\tau\\ &\leq \Lambda\left(1+\int_0^t\int_0^1\left(\theta_x^2+\nu^2+\theta^2\xi_x^2\right)\mathrm{d}x\mathrm{d}\tau\right) - \lambda\int_0^t\int_0^1\nu_x^2\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

In view of Theorem I (ii), Lemma 5.1 (i) and Corollary 5.3 (iii) we obtain the desired result. ■

Lemma 6.2. $\displaystyle\int_0^1 \nu_x^2(x,t)\mathrm{d}x + \int_0^t\int_0^1 \nu_{xx}^2\mathrm{d}x\mathrm{d}\tau \leq \Lambda.$

Proof. Recalling (M), integrating by parts and using Young's inequality gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1\nu_x^2\mathrm{d}x\right) &= 2\int_0^1 \nu_x\nu_{xt}\mathrm{d}x = -2\int_0^1\nu_t\nu_{xx}\mathrm{d}x\\ &\leq \Lambda\int_0^1\left(|\theta_x\nu_{xx}| + |\theta\xi_x\nu_{xx}| + |\xi_x\nu_x\nu_{xx}|\right)\mathrm{d}x - \lambda\int_0^1\nu_{xx}^2\mathrm{d}x \qquad (6.1)\\ &\leq \Lambda\int_0^1\left(\theta_x^2+\theta^2\xi_x^2+\xi_x^2\nu_x^2\right)\mathrm{d}x - \lambda\int_0^1\nu_{xx}^2\mathrm{d}x.\end{aligned}$$

We note the following simple interpolation inequality:

$$\max_{x\in[0,1]}\nu_x^2(x,t) \leq \Lambda\int_0^1\nu_x^2(x,t)\mathrm{d}x + \lambda\int_0^1\nu_{xx}^2(x,t)\mathrm{d}x. \tag{6.2}$$

Hence, in view of Corollary 5.3, the formula (6.2) combined with Lemma 6.1 yields

$$\begin{aligned}\int_0^t\int_0^1 \xi_x^2\nu_x^2 \mathrm{d}x\mathrm{d}\tau &\le \int_0^t \max_{x\in[0,1]}\nu_x^2(x,\tau)\left(\int_0^1 \xi_x^2\mathrm{d}x\right)\mathrm{d}\tau \\ &\le \Lambda+\lambda\int_0^t\int_0^1 \nu_{xx}^2\mathrm{d}x\mathrm{d}\tau.\end{aligned} \tag{6.3}$$

Upon integrating (6.1) over $[0,t]$ and recalling Theorem I (ii), Corollary 5.3 and (6.3), the result follows. ∎

Corollary 6.3. $\displaystyle\int_0^t\int_0^1 \left(\theta^2\nu_x^2+\xi_x^2\nu_x^2+\nu_x^4\right)\mathrm{d}x\mathrm{d}\tau \le \Lambda.$

Proof. Recalling (5.14), Lemma 6.2 and the formula (6.3), we have

$$\begin{aligned}\int_0^t\int_0^1 \left(\theta^2\nu_x^2+\xi_x^2\nu_x^2\right)\mathrm{d}x\mathrm{d}\tau &\le \Lambda\int_0^t\int_0^1 \nu_x^2\mathrm{d}x\mathrm{d}\tau+ \\ \int_0^t\left(\int_0^1\theta_x^2\mathrm{d}x\right)&\left(\int_0^1\nu_x^2\mathrm{d}x\right)\mathrm{d}\tau+\left(\Lambda+\lambda\int_0^t\int_0^1\nu_{xx}^2\mathrm{d}x\mathrm{d}\tau\right) \\ &\le \Lambda\int_0^t\int_0^1\left(\nu_x^2+\theta_x^2+\nu_{xx}^2\right)\mathrm{d}x\mathrm{d}\tau.\end{aligned} \tag{6.4}$$

On the other hand, Lemma 6.1 and (6.2) give

$$\begin{aligned}\int_0^t\int_0^1 \nu_x^4\mathrm{d}x\mathrm{d}\tau &\le \int_0^t \max_{x\in[0,1]}\nu_x^2(x,t)\left(\int_0^1\nu_x^2\mathrm{d}x\right)\mathrm{d}\tau \\ &\le \Lambda\left(1+\int_0^t\int_0^1\nu_{xx}^2\mathrm{d}x\mathrm{d}\tau\right).\end{aligned} \tag{6.5}$$

In view of Theorem I (ii), Lemma 6.1 and Lemma 6.2, the formulas (6.4) and (6.5) imply the result. ∎

Lemma 6.4. $\displaystyle\int_0^1 \theta_x^2(x,t)\mathrm{d}x+\int_0^t\int_0^1\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau\le\Lambda.$

Proof. From (E), integration by parts and Young's inequality, it follows for either boundary conditions (D) or (N) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1\theta_x^2\mathrm{d}x\right) &= 2\int_0^1\theta_x\theta_{xt}\mathrm{d}x=-2\int_0^1\theta_t\theta_{xx}\mathrm{d}x \\ &\le \Lambda\int_0^1\left(|\theta\nu_x\theta_{xx}|+|\nu_x^2\theta_{xx}|+|\theta_x\xi_x\theta_{xx}|\right)\mathrm{d}x-\lambda\int_0^1\theta_{xx}^2\mathrm{d}x \\ &\le \Lambda\int_0^1\left(\theta^2\nu_x^2+\nu_x^4+\theta_x^2\xi_x^2\right)\mathrm{d}x-\lambda\int_0^1\theta_{xx}^2\mathrm{d}x.\end{aligned} \tag{6.6}$$

As in (6.2) we have

$$\max_{x\in[0,1]}\theta_x^2(x,t)\le\Lambda\int_0^1\theta_x^2(x,t)\mathrm{d}x+\lambda\int_0^1\theta_{xx}^2(x,t)\mathrm{d}x. \tag{6.7}$$

Thus, by Corollary 5.3 and Theorem I (ii):

$$\begin{aligned}\int_0^t\int_0^1 \theta_x^2\xi_x^2 \mathrm{d}x\mathrm{d}\tau &\leq \int_0^t \max_{x\in[0,1]} \theta_x^2(x,\tau)\left(\int_0^1 \xi_x^2\mathrm{d}x\right)\mathrm{d}\tau \\ &\leq \Lambda + \lambda\int_0^t\int_0^1 \theta_{xx}^2\mathrm{d}x\mathrm{d}\tau.\end{aligned} \tag{6.8}$$

Now, integrating (6.6) over $[0,t]$ and applying Corollary 6.3 together with (6.8) conclude the proof. ■

The following direct consequence of Lemma 6.4 and (6.8) will be of later use.

Corollary 6.5. $\displaystyle\int_0^t\int_0^1 \theta_x^2\xi_x^2\mathrm{d}x\mathrm{d}\tau \leq \Lambda.$

We are now in a position to prove pointwise convergence of solutions.

Theorem 6.6. *Let $\bar\Theta$ be as in Theorem II. Then:*

(i) $\displaystyle\lim_{t\to+\infty}\int_0^1 \left(\nu_x^2+\theta_x^2+\xi_x^2\right)(x,t)\mathrm{d}x = 0,$

(ii) $\displaystyle\lim_{t\to+\infty}\max_{x\in[0,1]}\left(|\nu(x,t)| + |\theta(x,t)-\bar\Theta| + |\xi(x,t)-1|\right) = 0.$

Proof. First, by (5.1) we observe that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\mu\frac{\xi_x}{\xi}-\nu\right)^2\mathrm{d}x\right| \leq \Lambda\int_0^1\left(\theta_x^2+\xi_x^2+\nu^2+\theta^2\xi_x^2\right)\mathrm{d}x.$$

By Lemmas and Corollaries of Sections 5 and 6, it follows from the above inequality that the derivative of the function $\int_0^1(\mu\xi_x/\xi-\nu)^2\mathrm{d}x$ is integrable in time. Similarly, in view of (6.1) and (6.6), we conclude the time integrability of derivatives of $\int_0^1\nu_x^2(x,t)\mathrm{d}x$ and $\int_0^1\theta_x^2(x,t)\mathrm{d}x$. In addition, by Lemma 6.1, Theorem I (ii), Lemma 5.1 and Corollary 5.3, we see that the functions $\int_0^1\nu_x^2(x,t)\mathrm{d}x, \int_0^1\theta_x^2(x,t)\mathrm{d}x$ and $\int_0^1(\mu\xi_x/\xi-\nu)^2\mathrm{d}x$ are time integrable over $[0,\infty)$. Hence, we receive:

$$\lim_{t\to+\infty}\int_0^1\left(\nu_x^2+\theta_x^2+\left(\mu\frac{\xi_x}{\xi}-\nu\right)^2\right)\mathrm{d}x = 0. \tag{6.9}$$

In particular, since (V) gives

$$|\nu(x,t)| \leq \left(\int_0^1\nu_x^2\mathrm{d}x\right)^{1/2}, \tag{6.10}$$

it follows from (6.9) that

$$\lim_{t\to+\infty}\max_{x\in[0,1]}|\nu(x,t)| = 0. \tag{6.11}$$

Thus, in view of (5.3) and (6.9) we conclude (i).

Now, for the Dirichlet boundary conditions (D) we have

$$|\theta(x,t)-\Theta| \leq \int_0^1|\theta_x|\mathrm{d}x \leq \left(\int_0^1\theta_x^2\mathrm{d}x\right)^{1/2},$$

while for the Neumann boundary conditions (N)

$$\left|\theta(x,t)-\bar{\Theta}\right| \leq \left|\theta(x,t)-\int_0^1 \theta \mathrm{d}x\right| + \frac{1}{2c_v}\int_0^1 \nu^2(t,x)\mathrm{d}x.$$

Utilizing (6.10) and (i) we see that in both cases

$$|\theta(x,t)-\bar{\Theta}|^2 \leq \int_0^1 \left(\theta_x^2+\nu_x^2\right)\mathrm{d}x, \tag{6.12}$$

and thus (6.9) implies

$$\lim_{t\to+\infty}\max_{x\in[0,1]}|\theta(x,t)-\bar{\Theta}|=0. \tag{6.13}$$

Finally, (2.1) yields

$$|\xi(x,t)-1| \leq \left|\xi(x,t)-\int_0^1 \xi \mathrm{d}x\right| \leq \left(\int_0^1 \xi_x^2 \mathrm{d}x\right)^{1/2}, \tag{6.14}$$

and so recalling (i), (6.11) and (6.13) we deduce (ii). ■

7. Exponential convergence results.

In this Section we establish the exponential convergence of solutions to their equilibrium values. Here, the availability identity supplies a natural Lyapunov function $\mathcal{A}$, upon which we build our proof. Noting the pointwise convergence in Theorem 6.6, the result follows through a Taylor expansion associated with $\mathcal{A}$.

More precisely, define the following nonnegative quantities:

$$\mathcal{V}(t) := \int_0^1 \left(\nu_x^2+\theta_x^2\right)(x,t)\mathrm{d}x, \quad \mathcal{D}(t) := \int_0^1 \left(\mu\frac{\xi_x}{\xi}-\nu\right)^2(x,t)\mathrm{d}x,$$

$$\mathcal{A}(t) := \int_0^1 \left(e+\frac{1}{2}\nu^2-\bar{\Theta}\eta+\gamma\right)(x,t)\mathrm{d}x,$$

where $\gamma = c_v\bar{\Theta}(\ln\bar{\Theta}-1)$.

Lemma 7.1. *For some $\epsilon>0$ there holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{A}+\epsilon\mathcal{D}+\epsilon\mathcal{V}\right)+\lambda\left(\mathcal{A}+\epsilon\mathcal{D}+\epsilon\mathcal{V}\right) \leq 0.$$

Proof. First, integrating the availability identity (3.4) over $[0,1]$ and noting conditions (D) or (N) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(t)+\lambda\mathcal{V}(t) \leq 0. \tag{7.1}$$

Observing the boundedness of θ, due to (6.13), it follows from Taylor expansion of ln, that

$$\lambda(\theta-\bar{\Theta})^2 \leq c_v(\theta-\bar{\Theta}\ln\theta)+\gamma \leq \Lambda(\theta-\bar{\Theta})^2. \tag{7.2}$$

Analogously, using (2.1), the boundedness of ξ, and the concavity of h:

$$\lambda\int_0^1(\xi-1)^2\mathrm{d}x \leq -\int_0^1 h(\xi)\mathrm{d}x \leq \Lambda\int_0^1(\xi-1)^2\mathrm{d}x. \tag{7.3}$$

Adding (7.2) and (7.3) yields:

$$\lambda\int_0^1\left((\theta-\bar{\Theta})^2+(\xi-1)^2+\nu^2\right)\mathrm{d}x \leq \mathcal{A} \leq \Lambda\int_0^1\left((\theta-\bar{\Theta})^2+(\xi-1)^2+\nu^2\right)\mathrm{d}x.$$

Hence, by (6.12), (6.14), followed by (5.3) and (6.10) we receive

$$\begin{aligned}\mathcal{A}(t) &\le \Lambda \int_0^1 \left(\theta_x^2 + \xi_x^2 + \nu^2\right) \mathrm{d}x \\ &\le \Lambda \left[\mathcal{D}(t) + \int_0^1 \left(\theta_x^2 + \nu^2\right) \mathrm{d}x\right] \le \Lambda \left(\mathcal{D}(t) + \mathcal{V}(t)\right).\end{aligned} \tag{7.4}$$

In addition, from (5.2), (6.10), (6.13) and Young's inequality, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(t) + \lambda \mathcal{D}(t) \le \Lambda \mathcal{V}(t). \tag{7.5}$$

From (6.1) and (6.6) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(t) + \lambda \int_0^1 \left(\nu_{xx}^2 + \theta_{xx}^2\right) \mathrm{d}x \le \Lambda \left[\mathcal{V}(t) + \int_0^1 \left(\nu_x^4 + \xi_x^2 + \theta_x^2\xi_x^2 + \xi_x^2\nu_x^2\right) \mathrm{d}x\right]. \tag{7.6}$$

Noting the boundedness of $\int_0^1 \left(\xi_x^2 + \nu_x^2\right) \mathrm{d}x$ (by Theorem 6.6), the inequalities (6.2) and (6.7) imply that the integral on the right hand side of (7.6) is estimated by

$$\lambda \int_0^1 \left(\nu_{xx}^2 + \theta_{xx}^2\right) \mathrm{d}x + \Lambda \left(\mathcal{V}(t) + \int_0^1 \xi_x^2 \mathrm{d}x\right).$$

Thus, recalling (5.3) and (6.10):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(t) \le \Lambda \left(\mathcal{D}(t) + \mathcal{V}(t)\right). \tag{7.7}$$

Finally, multiplying (7.5) by a small constant $\epsilon > 0$ and then adding the result to (7.1) we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{A} + \epsilon\mathcal{D}\right) + \lambda \left(\mathcal{D} + \mathcal{V}\right) \le 0.$$

For sufficiently small ϵ we may conclude the proof by (7.4), (7.7) and the above inequality. ■

Proof of Theorem II. By Lemma 7.1 we see immediatelly that

$$\left(\mathcal{A} + \epsilon\mathcal{D} + \epsilon\mathcal{V}\right)(t) \le \Lambda e^{-\lambda t}.$$

Recalling (5.3) and (6.10) we deduce (i). The statement (ii) then follows easily from (6.12), (6.14) and (6.10). ■

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