BRANCHES OF FORCED OSCILLATIONS IN DEGENERATE SYSTEMS OF SECOND ORDER ODEs

MARTA LEWICKA AND MARCO SPADINI

1. Introduction

This paper is devoted to studying the set of oscillations of a mass point, constrained to a smooth manifold, and forced by an autonomous vector field G with a periodic perturbation F. We focus on a class of systems where G is "degenerate": its set of zeros being a noncompact submanifold of the constraint. There seem to be no results in the literature for this general case while the "extreme" cases (i.e., when $G \equiv 0$ or $G^{-1}(0)$ is compact) are well understood. For instance, in [2] there are studied branches of T-periodic solutions to second order differential equations of the form

$$\ddot{\xi}_\pi = \lambda F(t, \xi, \dot{\xi}) \qquad \lambda \geq 0, \tag{E1}$$

where F is tangent to a given differentiable manifold X and is T-periodic in t, under the assumption that the averaged vector field

$$p \mapsto ⨍_0^T F(t,p,0)\,\mathrm{d}t := \frac{1}{T}\int_0^T F(t,p,0)\,\mathrm{d}t$$

is admissible for the degree (that is, the set of its zeros is compact). In [4], T-periodic solutions to equations of the form

$$\ddot{\xi}_\pi = G(\xi, \dot{\xi}) + \lambda F(t, \xi, \dot{\xi}) \qquad \lambda \geq 0, \tag{E2}$$

are studied under the assumption that G is admissible for the degree. In this case, the average of F plays no role. As we said, little is known about the case when $G(\cdot, 0)^{-1}(0)$ is noncompact.

In this paper we wish to address, at least partially, this problem. We examine the case when X is the Cartesian product of two manifolds and G is constantly zero on one of them. In particular, this approach allows us to recover known results about (E1) and (E2).

Let M and N be two smooth manifolds in $\mathbb{R}^k$. Consider the following system of two coupled second order ODEs:

$$\begin{cases} \ddot{x}_{\pi_M} = \lambda f(t, x, \dot{x}, y, \dot{y}), \\ \ddot{y}_{\pi_N} = g(x, \dot{x}, y, \dot{y}) + \lambda h(t, x, \dot{x}, y, \dot{y}), \end{cases} \tag{1.1}$$

under the following assumptions on vector fields f, h, g:

(A1)
- (i) $f : \mathbb{R} \times TM \times TN \longrightarrow \mathbb{R}^k$ is continuous, T-periodic in t and tangent to M, that is: $f(t, p, v, q, w) \in T_pM$ for all $t \in \mathbb{R}$, $p \in M$, $v \in T_pM$, $q \in N$, $w \in T_qN$,
- (ii) $h : \mathbb{R} \times TM \times TN \longrightarrow \mathbb{R}^k$ is continuous, T-periodic in t and tangent to N,
- (iii) $g : TM \times TN \longrightarrow \mathbb{R}^k$ is continuous and tangent to N.

2000 *Mathematics Subject Classification.* 34C25, 34C40.

Key words and phrases. Coupled differential equations, branches of periodic solutions, fixed point index.

In (1.1) λ is a nonnegative parameter and the subscripts π_M, π_N denote the projections on the tangent spaces to M and N, respectively. That is, for example, $\ddot{x}_{\pi_M}(t)$ denotes the orthogonal projection of the acceleration $\ddot{x}(t) \in \mathbb{R}^k$ onto $T_{x(t)}M$.

In studying (1.1), the following vector field, tangent to $M \times N$, is of importance:

$$\nu : M \times N \longrightarrow \mathbb{R}^k \times \mathbb{R}^k, \quad \nu(p,q) = \left(\fint_0^T f(t,p,0,q,0)\ \mathrm{d}t,\ g(p,0,q,0) \right).$$

Given a manifold $X \subset \mathbb{R}^s$, by $\mathcal{C}^1_T(X)$ we denote the space of T-periodic $\mathcal{C}^1$ functions from $\mathbb{R}$ to X, with the topology inherited from the Banach space $C^1\big([0,T],\mathbb{R}^s\big)$. We will also identify points on X with constant functions from $\mathbb{R}$ to X. Thus, if S is a subset of $\mathcal{C}^1_T(X)$, by $S \cap X$ we mean the set of those points of X, that regarded as constant maps belong to S.

Our main result is the following:

Theorem 1.1. *Assume (A1) and let Ω be an open subset of $[0,\infty) \times \mathcal{C}^1_T(M \times N)$ such that*

$$\deg(\nu, \Omega \cap (M \times N))$$

is well defined and nonzero. Then there exists a connected set $\Gamma \subset \Omega$ enjoying the properties:

- (i) *every triple $(\lambda, x, y) \in \Gamma$ is a solution to (1.1),*
- (ii) *if $(\lambda, x, y) \in \Gamma$ then the parameter $\lambda > 0$ or $(x,y) \notin M \times N$ (that is, (x,y) is not constant),*
- (iii) *$\bar{\Gamma} \cap (\{0\} \times \nu^{-1}(0)) \cap \Omega \neq \emptyset$, where $\bar{\Gamma}$ stands for the closure of Γ in $[0,\infty) \times \mathcal{C}^1_T(M \times N)$,*
- (iv) *$\bar{\Gamma} \cap \Omega$ is not contained in any compact subset of Ω.*

In particular, if $M \times N$ is closed in $\mathbb{R}^{2k}$ and $\Omega = [0,\infty) \times \mathcal{C}^1_T(M \times N)$ then Γ is unbounded.

When either N or M is a singleton, our result reduces to Theorem 2.2 of [2] and Theorem 4.2 of [4], respectively.

The structure of this short paper is as follows. In Section 2 we compute the fixed point index of the T-translation operator associated to the reduced first order system, which is a version of (1.1) on the tangent bundle $T(M \times N)$. Section 3 contains the proof of Theorem 1.1 and an example illustrating the theory.

The results presented here are in the spirit of [5] where the first order case is discussed. The techniques we use are close to those of, e.g. [2, 4], therefore we describe only the main new ingredients and refer to those papers for a more detailed exposition.

2. Reduction to a first order system.

Towards a proof of Theorem 1.1, we conveniently express the system (1.1) in the first order form. Given a manifold M, one can prove (see, e.g. [1]) that there exists a unique smooth map $r_M : TM \longrightarrow \mathbb{R}^k$ such that for any $\mathcal{C}^2$ curve $x : \mathbb{R} \longrightarrow M$, $r_M(x(t), \dot{x}(t))$ is the orthogonal projection of $\ddot{x}(t)$ onto $T_{x(t)}(M)^\perp$. The map r_M satisfies, in particular, $r_M(p,v) \in (T_pM)^\perp$ and:

$$|r_M(x(t), \dot{x}(t))| = \kappa_M(x(t), \dot{x}(t)) \cdot |\dot{x}(t)|^2, \tag{2.1}$$

where $\kappa_M(p,v)$ is the normal curvature of M at p in the direction of v. Hence (1.1) can be equivalently written as a first order system on $TM \times TN$:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = r_M(x_1,x_2) + \lambda f(t,x_1,x_2,y_1,y_2), \\ \dot{y}_1 = y_2, \\ \dot{y}_2 = r_N(y_1,y_2) + g(x_1,x_2,y_1,y_2) + \lambda h(t,x_1,x_2,y_1,y_2). \end{cases} \tag{2.2}$$

For $t \geq 0$, denote by $P_t^\lambda(p,v,q,w)$ the value at time t (when defined) of the solution to (2.2) which takes as initial values:

$$x_1(0) = p,\ \ x_2(0) = v,\ \ y_1(0) = q,\ \ y_2(0) = w. \tag{2.3}$$

Lemma 2.1. *Let f, g, h be $\mathcal{C}^1$ vector fields satisfying (A1). Assume that for some relatively compact open subset U of $TM \times TN$ we have that:*

(i) *P_T^0 is well defined on $\bar{U}$,*
(ii) *every fixed point of P_T^0 on ∂U corresponds to a constant solution of (1.1), $(x_1,y_1) = (p,q)$,*
(iii) *ν has no zeros on the boundary (in $M \times N$) of the set $U \cap (M \times N)$.*

Then for $\lambda > 0$ sufficiently small:

$$\operatorname{ind}(P_T^\lambda, U) = \deg\big(\nu, U \cap (M \times N)\big).$$

Proof. For a given $\lambda \geq 0$ and $\mu \in [0,1]$, let $H(\lambda,p,v,q,w,\mu) \in TM \times TN$ be the value at time T of the solution to:

$$\begin{cases} \dot{x}_1 = & x_2, \\ \dot{x}_2 = & r_M(x_1,x_2) \\ & \quad +\lambda\Big(\mu f(t,x_1,x_2,y_1,y_2) + (1-\mu) ⨍_0^T f(t,x_1,x_2,y_1,y_2)\ \mathrm{d}t\Big), \\ \dot{y}_1 = & y_2, \\ \dot{y}_2 = & r_N(y_1,y_2) + g(x_1,x_2,y_1,y_2) + \lambda\mu h(t,x_1,x_2,y_1,y_2), \end{cases} \tag{2.4}$$

satisfying (2.3).

1. We first claim that for every small λ, the mapping $H(\lambda,\cdot) : \bar{U} \times [0,1] \longrightarrow TM \times TN$ is an admissible homotopy for the fixed point index. We argue by contradiction and assume that there are sequences $\lambda_i \to 0$, $\mu_i \to \mu_0 \in [0,1]$, $(p_i,v_i,q_i,w_i) \to (p_0,v_0,q_0,w_0) \in \partial U$ such that the corresponding solutions $(x_1^i,x_2^i,y_1^i,y_2^i)$ of (2.4) satisfy $x_1^i(T) = p_i$, $x_2^i(T) = v_i$, $y_1^i(T) = q_i$, $y_2^i(T) = w_i$. Clearly the sequence $(x_1^i,x_2^i,y_i^i,y_2^i)$ converges uniformly on $[0,T]$ to a T-periodic solution of (2.2) with $\lambda = 0$. In view of (ii), there must be $v_0 = 0$, $w_0 = 0$ and

$$g(p_0,0,q_0,0) = 0. \tag{2.5}$$

We will now show that also $\nu(p_0,q_0) = 0$ and hence obtain a contradiction with (iii). By (2.1) and in view of the periodicity of (x_1^i,x_2^i) we have:

$$\begin{aligned} \int_0^T |r_M(x_1^i(t),x_2^i(t))|\ \mathrm{d}t &\leq C_1 \int_0^T |x_2^i(t)|^2\ \mathrm{d}t \leq C_2 \int_0^T \left(\int_0^T |\dot{x}_2^i(s)|\mathrm{d}s\right)^2 \mathrm{d}t \\ &\leq C_2T^2 \int_0^T |\dot{x}_2^i(t)|^2\ \mathrm{d}t \leq C_3 \int_0^T |r_M(x_1^i(t),x_2^i(t))|^2\ \mathrm{d}t + C_3\lambda_i^2, \end{aligned} \tag{2.6}$$

where C_1, C_2 and C_3 are positive constants which may depend on T, f and the geometry of M but are independent of i. To see the second inequality in (2.6), notice that x_2^i is the derivative of a periodic function x_1^i, and thus any component of x_2^i must have a zero in $[0,T]$.

The last inequality in (2.6) follows from (2.4) and the following simple calculation:

$$\int_0^T \left| \mu f(t,x_1^i,x_2^i,y_1^i,y_2^i) + (1-\mu) \fint_0^T f(t,x_1^i,x_2^i,y_1^i,y_2^i)\ \mathrm{d}t \right|^2 \ \mathrm{d}t$$
$$= \mu^2 \int_0^T |f|^2 + (1-\mu^2) T \cdot | \fint_0^T f|^2 \leq \int_0^T |f(t,x_1^i,x_2^i,y_1^i,y_2^i)|^2\ \mathrm{d}t$$

The last quantity above is clearly bounded, independently of i, because all trajectories $(x_1^i, x_2^i, y_1^i, y_2^i)$ are contained in a compact region of $TM \times TN$.

Using (2.1) again and since x_2^i converges to 0, we obtain for sufficiently large i:

$$C_3 \int_0^T \left| r_M(x_1^i(t), x_2^i(t))|^2 \ \mathrm{d}t \leq \frac{1}{2} \int_0^T |r_M(x_1^i(t), x_2^i(t)) \right| \ \mathrm{d}t,$$

Thus, by (2.6):

$$\int_0^T |r_M(x_1^i(t), x_2^i(t))| \ \mathrm{d}t \leq 2C_3 \lambda_i^2.$$

Integrating on $[0,T]$ the second equation in (2.4) we get:

$$\left| \int_0^T f(t,x_1^i,x_2^i,y_1^i,y_2^i)\ \mathrm{d}t \right| = \frac{1}{\lambda_i} \left| \int_0^T r_M(x_1^i(t), x_2^i(t))\ \mathrm{d}t \right| \leq 2C_3 \lambda_i,$$

which after passing to the limit implies: $0 = \int_0^T f(t,p_0,0,q_0,0)\ \mathrm{d}t$. Hence by (2.5) we obtain $\nu(p_0,q_0) = 0$.

2. By the homotopy invariance of the fixed point index, we conclude that for every small $\lambda > 0$ there holds:

$$\mathrm{ind}\ (P_T^\lambda, U) = \mathrm{ind}\ (H(\lambda, \cdot, \mu = 0), U).$$

The last index above is by Theorem 2.1 [3] equal to deg $(-\nu_\lambda, U)$, where

$$\nu_\lambda(p,v,q,w) = \left(v, r_M(p,v) + \lambda \fint_0^T f(t,p,v,q,w)\ \mathrm{d}t, w, r_N(q,w) + g(p,v,q,w) \right).$$

Further, Lemma 3.2 [4] implies that:

$$\deg\ (-\nu_\lambda, U) = \deg\ (\tilde{\nu}_\lambda, U \cap (M \times N))$$

where $\tilde{\nu}_\lambda(p,q) = (\lambda \fint_0^T f(t,p,0,q,0)\ \mathrm{d}t, g(p,0,q,0))$. On the other hand, clearly:

$$\deg\ (\tilde{\nu}_\lambda, U \cap (M \times N)) = \deg\ (\nu, U \cap (M \times N))$$

which ends the proof of the Lemma. ■

3. A proof of Theorem 1.1 and an example

We will use the following abstract result from [2]:

Lemma 3.1. *Let Y be a locally compact metric space and let K be a nonempty, compact subset of it. Assume that any compact subset of Y containing K has nonempty boundary. Then $Y \setminus K$ contains a connected set whose closure intersects K and is not compact.*

Proof of Theorem 1.1. We prove the result under the additional assumption that f, g, h are $\mathcal{C}^1$. The extension to nonsmooth vector fields follows in a straightforward manner, as in [5].

Let W be the subset of $[0,\infty) \times TM \times TN$ given by:

$$W = \{(\lambda, x_1(0), x_2(0), y_1(0), y_2(0));\ (\lambda, x_1, x_2, y_1, y_2) \in \Omega\},$$

and set

$$S = W \cap \{(\lambda, x_1(0), x_2(0), y_1(0), y_2(0));\ (\lambda, x_1, x_2, y_1, y_2) \text{ solves } (2.2)\},$$

$$K = S \cap (\{0\} \times \nu^{-1}(0)).$$

We will prove that the set $S \setminus K$ has a connected subset which meets K and whose closure is not compact. This will be done by checking the assumptions of Lemma 3.1 for the pair (Y, K) with:

$$Y = S \setminus \{(0, p, 0, q, 0);\ g(p, 0, q, 0) = 0 \text{ and } \nu(p, q) \neq 0\}.$$

In the sequel, given any set $A \subset [0, \infty) \times TM \times TN$ and $\lambda \geq 0$, we will denote $A_\lambda = \{(p, v, q, w);\ (\lambda, p, v, q, w) \in A\}$.

Firstly, since by assumption we have deg $(\nu, W_0) \neq 0$, we conclude that K must be nonempty. Because of the regularity of g, arguing as in the first part of the proof of Lemma 2.1 one can show that any sequence $(\lambda_i, p_i, v_i, q_i, w_i) \in S$ with $\lambda_i \to 0^+$ converges to a point in Y, and conclude that Y is locally compact.

Assume now, by contradiction, that Y has a compact subset C, containing K and with empty boundary in Y. Choose an open set $A \subset W$ so that $A \cap Y = C$ and $\partial A \cap S = \emptyset$. In particular $\partial A_0 \cap K = \emptyset$. Now, by Lemma 2.1 we see that for a sufficiently small $\lambda > 0$:

$$\begin{aligned} \text{ind}\,(P_T^\lambda, A_\lambda) &= \text{deg}\,(\nu, A_\lambda \cap (M \times N)) \\ &= \text{deg}\,(\nu, A_0 \cap (M \times N)) = \text{deg}\,(\nu, W_0) \neq 0. \end{aligned} \tag{3.1}$$

On the other hand, the map $\delta \mapsto \text{ind}\,(P_T^\delta, A_\delta)$ is constant in view of the generalized homotopy invariance of the fixed point index. Recalling the compactness of C, its value must equal 0 for some $\delta > 0$, when P_T^δ has no fixed points in A_δ. This, however, contradicts (3.1) and ends the proof of the theorem. ■

Observe that the connected set Γ in Theorem 1.1 might be contained in the slice $\{0\} \times \mathcal{C}_T^1(M \times N)$, as in the system:

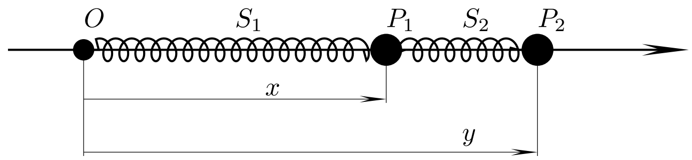
$$\begin{cases} \ddot{x} = \lambda f(t, x, y), \\ \ddot{y} = -y + \lambda \sin t, \end{cases}$$

where we put $M = N = \mathbb{R}$, $T = 2\pi$.

Example. Let $n \in \mathbb{N}$ be an odd number and consider the two coupled ODEs:

$$\begin{cases} \ddot{x} = -x - \alpha \dot{x} + \mu (y - x)^n, \\ \ddot{y} = \mu (x - y)^n + f(t), \end{cases} \tag{3.2}$$

describing the mechanical system as in the figure below.



There are two equal masses P_1 and P_2 and a fixed point O confined to a linear rail and connected by two springs: a nonlinear spring S_2 (whose elastic force is proportional to the n-th power of the displacement) and a linear spring S_1. Moreover, P_1 is subject to friction and P_2 to a T-periodic force f with nonzero average. In (3.2) $\alpha > 0$ is the friction coefficient and $\mu > 0$ is a parameter used to control the stiffness of S_2.

We apply Theorem 1.1 to show that for small $\mu > 0$, (3.2) admits a T-periodic solution. With the change of variable $\lambda = \mu^n$, $\xi = \lambda x$, $\eta = \lambda y$ the system becomes:

$$\begin{cases} \ddot{\xi} = -\xi - \alpha\dot{\xi} + \lambda(\eta - \xi)^n, \\ \ddot{\eta} = \lambda\Big((\xi - \eta)^n + f(t)\Big). \end{cases} \tag{3.3}$$

Take $\Omega = [0, \infty) \times C^1_T(\mathbb{R}^2)$, and notice that the degree of the vector field

$$\nu(p, q) = \left((p - q)^n + \fint_0^T f(t)\ \mathrm{d}t, -q\right)$$

relative to $\Omega \cap \mathbb{R}^2$ is nonzero. By Theorem (1.1), (3.3) has an unbounded connected set of of T-periodic solutions that branches from $\left(0, -\left(\fint_0^T f(t)\ \mathrm{d}t\right)^{1/n}, 0\right)$. This proves the claim.

References

[1] Furi M. *Second order differential equations on manifolds and forced oscillations*, proc. conference on "Topological methods in differential equations and inclusions", Université de Montreal, 1994, NATO-ASI series C, Kluwer Acad. Publ., Dordrecht NL.

[2] Furi M., Pera M.P., *A continuation principle for periodic solutions of forced motion equations on manifolds and applications to bifurcation theory*, Pacific J. Math. **160** (1993), 219–244.

[3] Furi M. and Spadini M., *On the fixed point index of the flow and applications to periodic solutions of differential equations on manifolds,* Boll. Un. Mat. Ital. (7) 10-A (1996), 333-346.

[4] Furi M., Spadini M., *Branches of forced oscillations for periodically perturbed second order autonomous ODE's on manifolds*, Journal of differential equations **154** (1999), pp.96-106.

[5] Spadini M., *Branches of harmonic solutions to periodically perturbed coupled differential equations on manifolds.* Discrete and Continuous Dyn. Syst. 2006.

University of Minnesota, 206 Church St. SE, Minneapolis, MN 55455, USA
E-mail address: lewicka@math.umn.edu

Dipartimento di Matematica Applicata, Via S. Marta 3, I-50139 Firenze, Italy
E-mail address: marco.spadini@unifi.it