ON THE OPTIMAL CONSTANTS IN KORN'S AND GEOMETRIC RIGIDITY ESTIMATES, IN BOUNDED AND UNBOUNDED DOMAINS, UNDER NEUMANN BOUNDARY CONDITIONS

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Abstract. We are concerned with the optimal constants: in the Korn inequality under tangential boundary conditions on bounded sets $\Omega \subset \mathbb{R}^n$, and in the geometric rigidity estimate on the whole $\mathbb{R}^2$. We prove that the latter constant equals $\sqrt{2}$, and we discuss the relation of the former constants with the optimal Korn's constants under Dirichlet boundary conditions, and in the whole $\mathbb{R}^n$, which are well known to equal $\sqrt{2}$. We also discuss the attainability of these constants and the structure of deformations/displacement fields in the optimal sets.

1. Introduction and the main results

In this paper we are concerned with the optimal constants in the Korn inequality [10, 11] and in the Friesecke-James-Müller geometric rigidity estimate [7, 8].

Let Ω be an open, bounded, and connected subset of $\mathbb{R}^n$ with Lipschitz continuous boundary. The Korn inequality [10, 11, 13] states that there exists a constant $C(\Omega)$ depending only on Ω, such that for all $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ there holds:

$$\min_{A \in so(n)} \|\nabla u - A\|_{L^2(\Omega)} \leq C(\Omega) \|D(u)\|_{L^2(\Omega)}, \tag{1.1}$$

where by $D(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$ we mean the symmetric part of ∇u.

Let now $\vec{n}$ denote the outward unit normal on $\partial\Omega$. Given (1.1), it is not hard to deduce (see Lemma 2.1) the following version of Korn's inequality subject to tangential boundary conditions. Namely, there exists a constant $\kappa(\Omega)$, depending only on Ω, such that for all $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ satisfying $u \cdot \vec{n} = 0$ on $\partial\Omega$ there holds:

$$\min_{A \in L_\Omega} \|\nabla u - A\|_{L^2(\Omega)} \leq \kappa(\Omega) \|D(u)\|_{L^2(\Omega)}, \tag{1.2}$$

where by L_Ω above we denote the linear space of skew-symmetric matrices that are gradients of affine maps tangential on the boundary of Ω:

$$L_\Omega = \left\{A \in so(n);\ \exists a \in \mathbb{R}^n \quad \forall x \in \partial\Omega \quad (Ax + a) \cdot \vec{n}(x) = 0\right\}.$$

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The optimal constant in (1.2) is given by:

$$\kappa(\Omega) = \sup\Big\{ \min_{A\in L_\Omega} \|\nabla u - A\|_{L^2(\Omega)};\ u\in W^{1,2}(\Omega,\mathbb{R}^n),\ u\cdot\vec n = 0 \text{ on } \partial\Omega \text{ and } \|D(u)\|_{L^2(\Omega)} = 1\Big\}, \tag{1.3}$$

and we aim to study its relation to Korn's constant in the whole $\mathbb{R}^n$, which is $\sqrt{2}$ (see Lemma 2.2):

$$\kappa(\mathbb{R}^n) = \sup\Big\{\|\nabla u\|_{L^2(\mathbb{R}^n)};\ u\in W^{1,2}(\mathbb{R}^n,\mathbb{R}^n),\ \|D(u)\|_{L^2(\mathbb{R}^n)} = 1\Big\} = \sqrt{2}. \tag{1.4}$$

In this setting, our first set of main results is:

Theorem 1.1. *For any open, bounded, Lipschitz, connected $\Omega\subset\mathbb{R}^n$:*

$$\kappa(\Omega) \geq \kappa(\mathbb{R}^n) = \sqrt{2}. \tag{1.5}$$

In fact, $\kappa(\Omega)$ may be arbitrarily large. In Example 3.3 we will recall our construction in [12] which implies that for a sequence of thin shells around a sphere, the Korn constants go to ∞ as the thickness goes to 0. On the other hand, as we show in Example 3.2, there is: $\kappa([0,1]^2) = \sqrt{2}$. [1] We however have:

Theorem 1.2. *Assume that there exists a sequence $\{u_k\}_{k=1}^\infty$, $u_k\in W^{1,2}(\Omega,\mathbb{R}^n)$, $u_k\cdot\vec n = 0$ on $\partial\Omega$, with the following properties:*

(i) *u_k converges to 0 weakly in $W^{1,2}(\Omega,\mathbb{R}^n)$,*
(ii) *$\|D(u_k)\|_{L^2(\Omega)} = 1$,*
(iii) *$\lim_{k\to\infty}\|\nabla u_k\|_{L^2(\Omega)} = \kappa(\Omega)$.*

Then $\kappa(\Omega) = \sqrt{2}$.

Theorem 1.3. *If $\kappa(\Omega) > \sqrt{2}$ then the supremum in the definition (1.3) is attained. More precisely, for every $A_0\in L_\Omega$ there exists $u\in W^{1,2}(\Omega,\mathbb{R}^n)$, such that $u\cdot\vec n = 0$ on $\partial\Omega$, $D(u)\not\equiv 0$, and:*

$$\min_{A\in so(n)}\|\nabla u - A\|_{L^2(\Omega)} = \|\nabla u - A_0\|_{L^2(\Omega)} = \kappa(\Omega)\|D(u)\|_{L^2(\Omega)}. \tag{1.6}$$

Theorem 1.4. *The vector fields u for which Korn's constant $\kappa(\Omega)$ is attained:*

$$\Big\{u\in W^{1,2}(\Omega,\mathbb{R}^n);\ u\cdot\vec n = 0 \text{ on } \partial\Omega,\ u \text{ satisfies (1.6) for some } A_0\in L_\Omega\Big\}; \tag{1.7}$$

form a closed linear subspace of $W^{1,2}(\Omega,\mathbb{R}^n)$. Moreover, if $\kappa(\Omega) > \sqrt{2}$ then this space is of finite dimension.

In the second part of this paper we concentrate on the nonlinear version of Korn's inequality, namely the Friesecke-James-Muller geometric rigidity estimate [7, 8]. It states that for an open, bounded, smooth and connected domain $\Omega\subset\mathbb{R}^n$, there exists a constant $\kappa_{nl}(\Omega)$ depending only on Ω, such that for every $u\in W^{1,2}(\Omega,\mathbb{R}^n)$ there holds:

$$\min_{R\in SO(n)}\|\nabla u - R\|_{L^2(\Omega)} \leq \kappa_{nl}(\Omega)\|\mathrm{dist}(\nabla u, SO(n))\|_{L^2(\Omega)}. \tag{1.8}$$

[1] We are able to prove that for smooth domains there always holds: $\kappa(\Omega) > \sqrt{2}$. The proof of this fact will appear elsewhere.

Define:

$$\kappa_{nl}(\mathbb{R}^n) = \sup\left\{ \min_{R\in SO(n)} \frac{\|\nabla u - R\|_{L^2(\mathbb{R}^n)}}{\|\mathrm{dist}(\nabla u, SO(n))\|_{L^2(\mathbb{R}^n)}};\ u\in W^{1,2}_{loc}(\mathbb{R}^n,\mathbb{R}^n),\right.$$
$$\left. \mathrm{dist}(\nabla u, SO(n)) \in L^2(\mathbb{R}^n)\setminus\{0\}\right\}. \tag{1.9}$$

Our results in this context are restricted to dimension 2:

Theorem 1.5. *We have:* $\kappa_{nl}(\mathbb{R}^2) = \sqrt{2}$. *In particular:*

$$\forall u \in W^{1,2}_{loc}(\mathbb{R}^2,\mathbb{R}^2) \quad \mathrm{dist}(\nabla u, SO(2)) \in L^2(\mathbb{R}^2) \implies$$
$$\min_{R\in SO(n)} \|\nabla u - R\|_{L^2(\mathbb{R}^2)} \le \sqrt{2}\|\mathrm{dist}(\nabla u, SO(2))\|_{L^2(\mathbb{R}^2)}.$$

Theorem 1.6. *For every rotation* $R_0 \in SO(2)$ *there exists* $u \in W^{1,2}_{loc}(\mathbb{R}^2,\mathbb{R}^2)$ *with* $\mathrm{dist}(\nabla u, SO(2)) \in L^2(\mathbb{R}^2)\setminus\{0\}$ *such that:*

$$\min_{R\in SO(2)} \|\nabla u - R\|_{L^2(\mathbb{R}^2)} = \|\nabla u - R_0\|_{L^2(\mathbb{R}^2)}$$
$$= \sqrt{2}\|\mathrm{dist}(\nabla u(x), SO(2))\|_{L^2(\mathbb{R}^2)}. \tag{1.10}$$

Theorem 1.7. *The vector fields for which the nonlinear Korn constant in (1.9) is attained, namely:*

$$\Big\{u \in W^{1,2}_{loc}(\mathbb{R}^2,\mathbb{R}^2);\ \mathrm{dist}(\nabla u, SO(2)) \in L^2(\mathbb{R}^2),$$
$$u \textit{ satisfies } (1.10) \textit{ for some } R_0\in SO(2)\Big\}$$

have the defining property that their gradients are of the form:

$$\nabla u(x) = R_0 R(\alpha(x)) + \begin{bmatrix} a(x) & b(x) \\ b(x) & -a(x)\end{bmatrix} \qquad \textit{with } R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix}, \tag{1.11}$$

for some $\alpha, a, b \in L^2(\mathbb{R}^2)$. *Conversely, for every* $\alpha\in L^2(\mathbb{R}^2)$ *there exists* $a, b \in L^2(\mathbb{R}^2)$ *and* $u \in W^{1,2}_{loc}(\mathbb{R}^2,\mathbb{R}^2)$ *such that (1.10) and (1.11) hold.*

The proofs of the three Theorems above are independent from the proof of (1.8) in [7]. They rely on the conformal-anticonformal decomposition of 2×2 matrices, and it is not clear how this construction and methods could be extended to yield a result in higher dimensions $n > 2$.

There is an extensive literature relating to Korn's inequality and its applications, notably in linear elasticity [2, 3, 10, 13]. On the other hand, the nonlinear estimate (1.8) plays crucial role in models in nonlinear elasticity [8, 7]. Indeed, the relation between these two estimates is clear if we recall that the tangent space to $SO(n)$ at Id is $so(n)$. The blow-up rate and properties of $\kappa(\Omega)$ for thin spherical-like domains around a given surface were studied in [12]. The relations of $\kappa(\Omega)$ with the measure of axisymmetry of Ω have been discussed in [5]. An interesting extension of both Korn's and the geometric rigidity estimates under mixed growth conditions has been recently established in [4].

2. Preliminaries

Recall that the linear space of skew-symmetric matrices is:

$$so(n) = \{A \in \mathbb{R}^{n\times n};\ A = -A^T\}$$

while $SO(n)$ stands for the group of proper rotations:

$$SO(n) = \{R \in \mathbb{R}^{n\times n}; R^T = R^{-1} \text{ and } \det R = 1\}.$$

The scalar product and the (Frobenius) norm in the space of $n \times n$ (real) matrices $\mathbb{R}^{n\times n}$ are given by:

$$A : B = \mathrm{tr}(A^T B) \qquad |A|^2 = A : A.$$

We first notice the following characterization of the minimiser in (1.2):

Lemma 2.1. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^n)$, $u \cdot \vec{n} = 0$ on $\partial\Omega$. Then the minimum in the left hand side of (1.2) is attained, uniquely, at:*

$$A_0 = \mathbb{P}_{L_\Omega} \fint_\Omega \nabla u,$$

where $\mathbb{P}_{L_\Omega}$ denotes the orthogonal projection of $\mathbb{R}^{n\times n}$ on L_Ω.

Proof. Let $A_0 \in L_\Omega$ be a minimiser of $\|\nabla u - A\|^2_{L^2(\Omega)}$ over L_Ω. Taking the derivative in the direction of $A \in L_\Omega$, one obtains:

$$\forall A \in L_\Omega \qquad \int_\Omega (\nabla u - A_0) : A = 0.$$

Equivalently, there holds:

$$\left(\fint_\Omega \nabla u - A_0\right) \in L_\Omega^\perp,$$

which implies the lemma. ■

For convenience of the reader, we now sketch the proof of (1.4).

Lemma 2.2. *For every open, Lipschitz, connected $\Omega \subset \mathbb{R}^n$, the Korn constant under Dirichlet boundary conditions equals $\sqrt{2}$:*

$$\kappa_0(\Omega) = \sup\Big\{\|\nabla u\|_{L^2(\Omega)};\ u \in W_0^{1,2}(\Omega, \mathbb{R}^n),\ \|D(u)\|_{L^2(\Omega)} = 1\Big\} = \sqrt{2}. \tag{2.1}$$

Proof. For every $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ we have:

$$\begin{aligned} 2\int_\Omega |D(u)|^2 &= \int |\nabla u|^2 + \int \nabla u : (\nabla u)^T = \int |\nabla u|^2 + \int \mathrm{tr}(\nabla u)^2 \\ &= \int |\nabla u|^2 + \int |\mathrm{div}\ u|^2 + \int \left(\mathrm{tr}(\nabla u)^2 - (\mathrm{tr}\nabla u)^2\right). \end{aligned} \tag{2.2}$$

When, additionally, Ω is bounded and $u \in W_0^{1,2}(\Omega, \mathbb{R}^n)$, this implies that:

$$2\int_\Omega |D(u)|^2 = \int |\nabla u|^2 + \int |\mathrm{div}\ u|^2,$$

because $\left(\mathrm{tr}(\nabla u)^2 - (\mathrm{tr}\nabla u)^2\right)$ is a null-Lagrangean, i.e. its integral depends only on the boundary value of u on $\partial\Omega$. We therefore conclude that, in this case: $\|\nabla u\|_{L^2(\Omega)} \leq \sqrt{2}\|D(u)\|_{L^2(\Omega)}$. The same inequality is also true on unbounded domains, because of the density of $\mathcal{C}_c^\infty(\Omega, \mathbb{R}^n)$ in $W_0^{1,2}(\Omega, \mathbb{R}^n)$.

To prove that $\sqrt{2}$ is optimal and that it is attained, it is enough to take $u \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^n)$ with $\operatorname{div} u = 0$ (when $n = 3$, take $u = \operatorname{curl} v$ for any compactly supported v). This achieves the proof. ∎

We now recall the Poincaré inequality for tangential vector fields. The proof, which can be found in [1], is deduced through a standard argument by contradiction.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, Lipschitz set. For every $u \in W^{1,2}(\Omega, \mathbb{R}^n)$, $u \cdot \vec{n} = 0$ on $\partial\Omega$, there holds:*

$$\|u\|_{L^2(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^2(\Omega)}, \tag{2.3}$$

where the constant $C(\Omega)$ depends only on Ω (it is independent of u).

3. The optimal Korn constant $\kappa(\Omega)$: a proof of Theorem 1.1 and two examples

In the course of proof of Theorem 1.1, we will use the following observation:

Proposition 3.1. *For any $f \in L^2(\mathbb{R}^n)$ there holds:*

$$\lim_{R\to\infty} R^{-n/2}\|f\|_{L^1(B_R)} = 0,$$

on the ball $B_R = \{x \in \mathbb{R}^n,\ |x| \leq R\}$.

Proof. Fix $\epsilon > 0$. For m sufficiently large, one has $\|f\|_{L^2(\mathbb{R}^n\setminus B_m)} < \epsilon$. Denote by ω_n the volume of the unit ball B_1 in $\mathbb{R}^n$. Take any $R > m$ so that: $\left(\frac{m}{R}\right)^{n/2} \|f\|_{L^2(\mathbb{R}^n)} \leq \epsilon$. Then:

$$\begin{aligned} R^{-n/2}\|f\|_{L^1(B_R)} &= R^{-n/2}\left(\int_{B_R\setminus B_m} |f| + \int_{B_m} |f|\right) \\ &\leq R^{-n/2}|B_R|^{1/2}\epsilon + R^{-n/2}|B_m|^{1/2}\|f\|_{L^2(\mathbb{R}^n)} \\ &\leq \omega_n^{1/2}\epsilon + \left(\frac{m}{R}\right)^{n/2}\omega_n^{1/2}\|f\|_{L^2(\mathbb{R}^n)} \leq 2\omega_n^{1/2}\epsilon, \end{aligned}$$

which achieves the proof. ∎

Proof of Theorem 1.1

1. Without loss of generality we may assume that $0 \in \Omega$. Let $u \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^n)$ with $\|D(u)\|_{L^2(\mathbb{R}^n)} = 1$. Define the sequence $u_k \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^n)$ by: $u_k(x) = k^{n/2-1}u(kx)$. One has:

$$\|\nabla u_k\|_{L^2(\mathbb{R}^n)} = \|\nabla u\|_{L^2(\mathbb{R}^n)}, \qquad \|D(u_k)\|_{L^2(\mathbb{R}^n)} = \|D(u)\|_{L^2(\mathbb{R}^n)} = 1.$$

Let now $\phi \in \mathcal{C}_c^\infty(\Omega)$ be a nonnegative function, equal identically to 1 in a neighborhood of 0, and define: $v_k = \phi u_k$. Clearly $v_k \in W_0^{1,2}(\Omega, \mathbb{R}^n)$ and:

$$\nabla v_k = \phi\nabla u_k + u_k \otimes \nabla\phi.$$

We claim that:

$$\lim_{k\to\infty} \|\nabla v_k\|_{L^1(\Omega)} = 0, \tag{3.1}$$

$$\lim_{k\to\infty} \|\nabla v_k\|_{L^2(\Omega)} = \|\nabla u\|_{L^2(\mathbb{R}^n)}, \tag{3.2}$$

$$\lim_{k\to\infty} \|D(v_k)\|_{L^2(\Omega)} = \|D(u)\|_{L^2(\mathbb{R}^n)} = 1. \tag{3.3}$$

To prove the claim, notice first that:

$$\lim_{k\to\infty} \|u_k \otimes \nabla\phi\|_{L^2(\Omega)} \leq \lim_{k\to\infty} \|\nabla\phi\|_{L^\infty} k^{-1} \|u\|_{L^2(\mathbb{R}^n)} = 0.$$

On the other hand, for all $i, j : 1 \ldots n$:

$$\lim_{k\to\infty} \left\| \phi \frac{\partial}{\partial x_i} u_k^j \right\|^2_{L^2(\Omega)} = \lim_{k\to\infty} \int_{\mathbb{R}^n} \left| \phi(x/k) \frac{\partial}{\partial x_i} u^j(x) \right|^2 \,\mathrm{d}x = \left\| \frac{\partial}{\partial x_i} u^j \right\|^2_{L^2(\Omega)}.$$

Thus we obtain (3.2) and (3.3). Similarly:

$$\lim_{k\to\infty} \|\phi \nabla u_k\|_{L^1(\Omega)} \leq \lim_{k\to\infty} \|\phi\|_{L^\infty} k^{-n/2} \|\nabla u\|_{L^1(k\Omega)} = 0,$$

where the last equality follows by Proposition 3.1. Hence we conclude (3.1) as well.

2. Notice that by Lemma 2.1:

$$\begin{aligned} \min_{A\in L_\Omega} \|\nabla v_k - A\|_{L^2(\Omega)} &= \left\| \nabla v_k - \mathbb{P}_{L_\Omega} ⨍_\Omega \nabla v_k \right\|_{L^2(\Omega)} \\ &\geq \|\nabla v_k\|_{L^2(\Omega)} - \left\| \mathbb{P}_{L_\Omega} ⨍_\Omega \nabla v_k \right\|_{L^2(\Omega)} \geq \|\nabla v_k\|_{L^2(\Omega)} - |\Omega|^{-1/2} \|\nabla v_k\|_{L^1(\Omega)}. \end{aligned}$$

Now, by (3.1) and (3.2), the right hand side of the above inequality converges to $\|\nabla u\|_{L^2(\mathbb{R}^n)}$ as $k \to \infty$. On the other hand, by (1.2) and (1.3), the left hand side is bounded by $\kappa(\Omega)\|D(v_k)\|_{L^2(\Omega)}$. Therefore, passing to the limit and using (3.3), we obtain:

$$\|\nabla u\|_{L^2(\mathbb{R}^n)} \leq \kappa(\Omega) \|D(u)\|_{L^2(\mathbb{R}^n)} = \kappa(\Omega).$$

Recalling the definition (1.4) the theorem follows. ■

Example 3.2. We now show that $\kappa(Q) = \sqrt{2}$ for $Q = [0,1]^2 \subset \mathbb{R}^2$.

Firstly, observe that (see Theorem 9.4 [12]) $L_\Omega \neq \{0\}$ if and only if Ω has a rotational symmetry. When this is not the case, then:

$$\kappa(\Omega) = \sup\left\{ \frac{\|\nabla u\|_{L^2(\Omega)}}{\|D(u)\|_{L^2(\Omega)}};\ u \in W^{1,2}(\Omega, \mathbb{R}^n),\ u \cdot \vec{n} = 0 \text{ on } \partial\Omega \right\}. \tag{3.4}$$

In view of (3.4) and Theorem 1.1, it is hence enough to prove that for every $u \in W^{1,2}(Q, \mathbb{R}^2)$ satisfying $u^1(0, x_2) = u^1(1, x_2) = 0$ and $u^2(x_2, 0) = u^2(x_1, 0) = 0$ for all $x_1, x_2 \in [0,1]$, there holds:

$$\int_Q |\nabla u|^2 \leq 2 \int_Q |D(u)|^2. \tag{3.5}$$

Consider first a regular vector field $u \in \mathcal{C}^2(\bar{Q}, \mathbb{R}^2)$. As in (2.2), we obtain:

$$\int_Q |D(u)|^2 = \frac{1}{2}\int_Q |\nabla u|^2 + \frac{1}{2}\int_Q |\mathrm{div}\ u|^2 + \int_Q \left(\partial_1 u^2 \partial_2 u^1 - \partial_1 u^1 \partial_2 u^2 \right). \tag{3.6}$$

Note that:

$$\int_Q \left(\partial_1 u^2 \partial_2 u^1 - \partial_1 u^1 \partial_2 u^2 \right) = \int_Q \partial_1 (u^2 \partial_2 u^1) - \int_Q \partial_2 (u^2 \partial_1 u^1),$$

and that both terms in the right hand side of the above equality integrate to 0 on Q, because of the assumed boundary condition. Thus, (3.6) yields (3.5) for $u \in \mathcal{C}^2$.

It now suffices to check that every $u \in W^{1,2}(Q, \mathbb{R}^2)$ with $u \cdot \vec{n} = 0$ on ∂Q, can be approximated by a sequence of $\mathcal{C}^2(\bar{Q}, \mathbb{R}^2)$ vector fields satisfying the same boundary condition. To this end, define the extension $\bar{u}^1 \in W^{1,2}([0,1]\times[-1,2],\mathbb{R})$ of the component $u^1 \in W^{1,2}(Q,\mathbb{R})$, by:

$$\forall x_1 \in [0,1] \quad \forall x_2 \in [-1,2] \qquad \bar{u}^1(x) = \begin{cases} u^1(x) & \text{if } \ x_2 \in [0,1] \\ u^1(x_1,-x_2) & \text{if } \ x_2 \in [-1,0] \\ u^1(x_1, 2-x_2) & \text{if } \ x_2 \in [1,2]. \end{cases}$$

Let $\phi : (-1,2) \to \mathbb{R}$ be a nonnegative, smooth and compactly supported function, equal to 1 on $[0,1]$. Then $\phi\bar{u}^1 \in W_0^{1,2}([0,1]\times[-1,2],\mathbb{R})$, and thus $\phi\bar{u}^1$ can be approximated in $W^{1,2}$ by a sequence $u_k^1 \in \mathcal{C}_c^\infty([0,1]\times[-1,2],\mathbb{R})$. Clearly, u_k^1 converges to u^1 on Q, and each $u_k^1(x) = 0$ whenever $x_1 \in \{0,1\}$.

In a similar manner, we construct smooth approximating sequence $\{u_k^2\}_{k\geq 1}$. Writing $u_k = (u_k^1, u_k^2) \in \mathcal{C}^\infty(\bar{Q},\mathbb{R}^2)$, we obtain the desired approximations of u. ■

Example 3.3. We now recall the construction [12] of a family of domains $\Omega^h \subset \mathbb{R}^n$ parametrised by $0 < h \ll 1$, with the property that:

$$\kappa(\Omega^h) \to \infty \quad \text{ as } h \to 0.$$

Let S denote the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ and let $g : S \to (0, \frac{1}{3})$ be a smooth function on S. Define:

$$\Omega^h = \Big\{(1+t)x;\ x \in S,\ t \in \big(hg(x) - h, hg(x)\big)\Big\}.$$

Clearly, we may request from function g to be such that no Ω^h has any rotational symmetry, and hence $L_{\Omega^h} = \{0\}$ implies (3.4) for all h.

Let now $v : S \to \mathbb{R}^n$ be a tangent vector field given by a rotation: $v(x) = a \times x$, for some $a \in \mathbb{R}^n$. Define $u^h \in W^{1,2}(\Omega^h,\mathbb{R}^n)$:

$$u^h(x+tx) = \Big((1+t)\mathrm{Id} + hx \otimes \nabla g(x)\Big)v(x) = (1+t)(a\times x) + \langle a, x \times \nabla g(x)\rangle x.$$

One can check that u^h is tangent at $\partial\Omega^h$ and that:

$$\|\nabla u^h\|_{L^2(\Omega^h)} \geq Ch^{1/2}, \qquad \|D(u^h)\|_{L^2(\Omega^h)} \leq Ch^{3/2}.$$

Hence we conclude the blow-up of Korn's constant: $\kappa(\Omega^h) \geq Ch^{-1}$ in the vanishing thickness $h \to 0$. ■

4. The optimal Korn constant $\kappa(\Omega)$: proofs of Theorems 1.2, 1.3 and 1.4

Proof of Theorem 1.2

1. From (ii) and (iii) we see that the sequences:

$$\{|\nabla u_k|^2 \chi_\Omega \ \mathrm{d}x\}_{k=1}^\infty \quad \text{and} \quad \{|D(u_k)|^2 \chi_\Omega \ \mathrm{d}x\}_{k=1}^\infty$$

are bounded in the space of Radon measures $\mathcal{M}(\mathbb{R}^n)$. Therefore (possibly passing to subsequences), they converge weakly in $\mathcal{M}(\mathbb{R}^n)$ to some μ, ν, concentrated on $\bar{\Omega}$. That is:

$$\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \qquad \lim_{k\to\infty} \int_\Omega \phi^2 |\nabla u_k|^2 \ \mathrm{d}x = \int_{\mathbb{R}^n} \phi^2 \ \mathrm{d}\mu, \qquad \lim_{k\to\infty} \int_\Omega \phi^2 |D(u_k)|^2 \ \mathrm{d}x = \int_{\mathbb{R}^n} \phi^2 \ \mathrm{d}\nu. \tag{4.1}$$

In particular, one has:

$$\mu(\bar\Omega) = \kappa(\Omega)^2, \qquad \nu(\bar\Omega) = 1. \tag{4.2}$$

We now assume that:

$$\kappa(\Omega) > \kappa(\mathbb{R}^n), \tag{4.3}$$

and derive a contradiction. We will distinguish two cases: when $\mu(\Omega) > 0$ and $\mu(\Omega) = 0$.

2. First, notice that:

$$\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \qquad \int_{\mathbb{R}^n} \phi^2 \,\mathrm{d}\mu \leq \kappa(\Omega)^2 \int_{\mathbb{R}^n} \phi^2 \,\mathrm{d}\nu. \tag{4.4}$$

Indeed, for a given ϕ as above consider the sequence $v_k = \phi u_k \in W^{1,2}(\Omega, \mathbb{R}^n)$. Clearly $v_k \cdot \vec n = 0$ on $\partial\Omega$ and by Lemma 2.1 we have:

$$\left\| \nabla v_k - \mathbb{P}_{L_\Omega} \fint_\Omega \nabla v_k \right\|_{L^2(\Omega)} \leq \kappa(\Omega) \|D(v_k)\|_{L^2(\Omega)}. \tag{4.5}$$

Since $\nabla v_k = \phi \nabla u_k + u_k \otimes \nabla\phi$, the sequence $\fint_\Omega \nabla v_k$ converges to 0 in $\mathbb{R}^{n\times n}$ by (i). The same convergence must be true for the respective sequence of projections. Similarly, $\lim_{k\to\infty} \|u_k \otimes \nabla\phi\|_{L^2(\Omega)} = 0$ by (i). Hence (4.5), after passing to the limit with $k \to \infty$ yields:

$$\lim_{k\to\infty} \|\phi \nabla u_k\|_{L^2(\Omega)} \leq \kappa(\Omega) \lim_{k\to\infty} \|\phi D(u_k)\|_{L^2(\Omega)},$$

which in view of (4.1) proves (4.4).

Assume now that $\mu(\Omega) > 0$. In this case we are ready to derive a contradiction. Let B be an open ball, compactly contained in Ω, with $\mu(B) > 0$. By (4.4):

$$\mu(\bar\Omega \setminus B) \leq \kappa(\Omega)^2 \nu(\bar\Omega \setminus B). \tag{4.6}$$

On the other hand, recalling the definition (1.4) and reasoning exactly as in the proof of (4.4), we get:

$$\forall \phi \in \mathcal{C}_c^\infty(B) \qquad \int_B \phi^2 \,\mathrm{d}\mu \leq \kappa(\mathbb{R}^n)^2 \int_B \phi^2 \,\mathrm{d}\nu,$$

which implies:

$$\mu(B) \leq \kappa(\mathbb{R}^n)^2 \nu(B). \tag{4.7}$$

Now, both sides of (4.7) are positive, so by (4.3): $\mu(B) < \kappa(\Omega)^2 \nu(B)$. Together with (4.6) this yields:

$$\mu(\bar\Omega) < \kappa(\Omega)^2 \nu(\bar\Omega),$$

contradicting (4.2).

3. It remains to consider the case $\mu(\Omega) = 0$, when the measure μ concentrates on $\partial\Omega$, due to the lack of the equiintegrability of the sequence $\{|\nabla u_k|^2\}_{k=1}^\infty$ close to $\partial\Omega$. We will prove that:

$$\mu(\partial\Omega) \leq \kappa(\mathbb{R}^n)^2 \nu(\partial\Omega). \tag{4.8}$$

Both sides of (4.8) are positive, and so (4.3) in view of the assumption $\mu(\Omega) = 0$ implies:

$$\mu(\bar\Omega) = \mu(\partial\Omega) < \kappa(\Omega)^2 \nu(\partial\Omega) \leq \kappa(\Omega)^2 \nu(\bar\Omega),$$

contradicting (4.2). This will end the proof of the theorem.

Towards establishing (4.8), let $\theta : [0,\infty) \longrightarrow [0,1]$ be a smooth, non-increasing function such that:

$$\theta(t) = 1 \text{ for } t \in [0,1], \qquad \theta(t) = 0 \text{ for } t \geq 2.$$

Define: $\phi_k(x) = \theta(k\mathrm{dist}(x,\partial\Omega))$. For large k we have $\phi_k \in \mathcal{C}_c^\infty((\partial\Omega)_\epsilon)$ on a small open neighborhood $(\partial\Omega)_\epsilon$ of $\partial\Omega$. By (4.1), for some increasing sequence $\{n_k\}_{k=1}^\infty$:

$$\mu(\partial\Omega) = \lim_{k\to\infty} \|\phi_k \nabla u_{n_k}\|^2_{L^2(\Omega)}, \qquad \nu(\partial\Omega) = \lim_{k\to\infty} \|\phi_k D(u_{n_k})\|^2_{L^2(\Omega)}. \tag{4.9}$$

To simplify the notation, we will pass to subsequences and write $n_k = k$.

Define the extension of u_k on $(\partial\Omega)_\epsilon$ by reflecting the normal components oddly and tangential components evenly, across $\partial\Omega$. That is, denoting by $\pi : (\partial\Omega)_\epsilon \longrightarrow \partial\Omega$ the projection onto $\partial\Omega$ along the normal vectors $\vec{n}$, so that:

$$\big(x - \pi(x)\big) \parallel \vec{n}(\pi(x)) \qquad \forall x \in (\partial\Omega)_\epsilon,$$

let, for all $x \in (\partial\Omega)_\epsilon \setminus \Omega$:

$$\begin{aligned} u_k(x)\cdot\vec{n}(\pi(x)) &= -u_k(2\pi(x)-x)\cdot\vec{n}(\pi(x)),\\ u_k(x)\cdot\tau &= u_k(2\pi(x)-x)\cdot\tau \qquad \forall \tau \in T_{\pi(x)}\partial\Omega. \end{aligned} \tag{4.10}$$

Since $u_k\cdot\vec{n} = 0$ on $\partial\Omega$, the above defined extension u_k is still $W^{1,2}$ regular. By (1.4) there holds:

$$\|\nabla(\phi_k u_k)\|_{L^2(\mathbb{R}^n)} \leq \kappa(\mathbb{R}^n)\|D(\phi_k u_k)\|_{L^2(\mathbb{R}^n)}.$$

Again, by taking $\{n_k\}$ in (4.9) converging to ∞ sufficiently fast, we may without loss of generality assume that $\|u_k\|_{L^2(\Omega)} \leq 1/k^2$. Therefore:

$$\lim_{k\to\infty} \|\phi_k\nabla u_k\|_{L^2(\mathbb{R}^n)} \leq \kappa(\mathbb{R}^n) \lim_{k\to\infty} \|\phi_k D(u_k)\|_{L^2(\mathbb{R}^n)}. \tag{4.11}$$

Consider the quantity:

$$I = \lim_{k\to\infty}\bigg\{ \int_{\mathbb{R}^n\setminus\Omega} |\phi_k\nabla u_k|^2 - \int_\Omega |\phi_k\nabla u_k|^2 \bigg\}.$$

After changing the variables in the first integral and noting that:

$$\det\nabla(2\pi(x)-x) = \det\,(2\nabla\pi(x) - \mathrm{Id}) = -1 + \mathcal{O}(1)|x-\pi(x)|,$$

we obtain:

$$\begin{aligned} I &= \lim_{k\to\infty}\int_\Omega \Big\{|\phi_k(x)\nabla u_k(2\pi(x)-x)|^2 - |\phi_k(x)\nabla u_k(x)|^2\Big\}\,\mathrm{d}x\\ &= \lim_{k\to\infty}\int_{\Omega\cap\{\mathrm{dist}(x,\partial\Omega)<1/k\}} \Big\{|\nabla u_k(2\pi(x)-x)|^2 - |\nabla u_k(x)|^2\Big\}\,\mathrm{d}x. \end{aligned} \tag{4.12}$$

The definition of extension (4.10) yields now the following identities, for each $x \in (\partial\Omega)_\epsilon$ and each $\tau,\eta \in T_{\pi(x)}\partial\Omega$:

$$\begin{aligned} &\partial_\tau(u_k\cdot\eta)(2\pi(x)-x) = \Big(1 + \mathcal{O}(1)|x-\pi(x)|\Big)\partial_\tau(u_k\cdot\eta)(x),\\ &\partial_{\vec{n}(\pi(x))}(u_k\cdot\eta)(2\pi(x)-x) = -\partial_{\vec{n}(\pi(x))}(u_k\cdot\eta)(x),\\ &\partial_\tau(u_k\cdot\vec{n}(\pi(x))(2\pi(x)-x) = \Big(-1 + \mathcal{O}(1)|x-\pi(x)|\Big)\partial_\tau(u_k\cdot\vec{n}(\pi(x))(x),\\ &\partial_{\vec{n}(\pi(x))}(u_k\cdot\vec{n}(\pi(x))(2\pi(x)-x) = \partial_{\vec{n}(\pi(x))}(u_k\cdot\vec{n}(\pi(x))(x). \end{aligned} \tag{4.13}$$

Since $\eta\partial_\tau v_k = \partial_\tau(v_k\eta) - v_k\partial_\tau\eta$, we see that equating the contribution of all components in (4.12) and recalling (iii) we have:

$$I = 0.$$

In the same manner, (4.13) implies that $|D(u_k)(2\pi(x)-x)|^2$ equals to $|D(u_k)(x)|^2$ plus lower order terms whose integrals on $\Omega\cap\{\mathrm{dist}(x,\partial\Omega)<1/k\}$ vanish, as $k\longrightarrow\infty$. Hence also:

$$II = \lim_{k\to\infty}\left\{\int_{\mathbb{R}^n\setminus\Omega}|\phi_k D(u_k)|^2 - \int_\Omega |\phi_k D(u_k)|^2\right\} = 0.$$

Therefore:

$$\begin{aligned}\lim_{k\to\infty}\|\phi_k\nabla u_k\|_{L^2(\mathbb{R}^n)} &= 2\lim_{k\to\infty}\|\phi_k\nabla u_k\|_{L^2(\Omega)},\\ \lim_{k\to\infty}\|\phi_k D(u_k)\|_{L^2(\mathbb{R}^n)} &= 2\lim_{k\to\infty}\|\phi_k D(u_k)\|_{L^2(\Omega)}.\end{aligned}\tag{4.14}$$

Combining (4.14), (4.11) with (4.9) proves (4.8). ∎

Proof of Theorem 1.3

It is enough to assume that $A_0 = 0$. Let $\{u_k\}_{k=1}^\infty$ be a maximizing sequence of (1.3), that is: $u_k\in W^{1,2}(\Omega,\mathbb{R}^n)$, $u_k\cdot\vec{n}=0$ on $\partial\Omega$, $\|D(u_k)\|_{L^2(\Omega)}=1$ and $\lim_{k\to\infty}\left\|\nabla u_k - \mathbb{P}_{L_\Omega}\fint_\Omega\nabla u_k\right\|_{L^2(\Omega)} = \kappa(\Omega)$.

By modifying u_k we may, without loss of generality, assume that:

$$\mathbb{P}_{L_\Omega}\fint\nabla u_k = 0, \qquad \lim_{k\to\infty}\|\nabla u_k\|_{L^2(\Omega)} = \kappa(\Omega).\tag{4.15}$$

Using Lemma 2.3 (after possibly passing to a subsequence), we have:

$$u_k \rightharpoonup u \quad \text{weakly in } W^{1,2}(\Omega,\mathbb{R}^n),\tag{4.16}$$

for some u satisfying $u\cdot\vec{n}=0$ on $\partial\Omega$.

We now show that (1.6) holds with $A_0=0$. First of all, by applying Theorem 1.2 to the sequence $\{u_k\}$, we see that $u\neq 0$. Further, (4.16) implies that $\mathbb{P}_{L_\Omega}\fint\nabla u = \lim_{k\to\infty}\mathbb{P}_{L_\Omega}\fint\nabla u_k = 0$, so:

$$\|\nabla u\|_{L^2(\Omega)} \leq \kappa(\Omega)\|D(u)\|_{L^2(\Omega)}.\tag{4.17}$$

Since $\mathbb{P}_{L_\Omega}\fint\nabla(u_k-u)=0$, there also holds:

$$\|\nabla(u_k-u)\|_{L^2(\Omega)} \leq \kappa(\Omega)\|D(u_k-u)\|_{L^2(\Omega)}.$$

Squaring both sides of the above inequality, passing to the limit with $k\to\infty$ and recalling (4.15) and (4.16), we obtain:

$$\kappa(\Omega)^2 - \|\nabla u\|^2_{L^2(\Omega)} \leq \kappa(\Omega)^2\left(1-\|D(u)\|^2_{L^2(\Omega)}\right).$$

Together with (4.17) this proves:

$$\|\nabla u\|_{L^2(\Omega)} = \kappa(\Omega)\|D(u)\|_{L^2(\Omega)},$$

yielding the result. ∎

Proof of Theorem 1.4

1. Let E be the set in (1.7). It is clear that $u\in E$ implies $\lambda u\in E$, for all $\lambda\in\mathbb{R}$. If $u_1,u_2\in E$, then by Lemma 2.1:

$$\left\|\nabla u_i - \mathbb{P}_{L_\Omega}\fint_\Omega\nabla u_i\right\|_{L^2(\Omega)} = \kappa(\Omega)\|D(u_i)\|_{L^2(\Omega)} \qquad \forall i=1,2.\tag{4.18}$$

On the other hand, by the linearity of the operator $\mathbb{P}_{L_\Omega}$ and by (1.2), (1.3):
(4.19)
$$\left\|\nabla(u_1 \pm u_2) - \left(\mathbb{P}_{L_\Omega} \fint_\Omega \nabla u_1 \pm \mathbb{P}_{L_\Omega} \fint_\Omega \nabla u_2\right)\right\|_{L^2(\Omega)} \leq \kappa(\Omega)\|D(u_1 \pm u_2))\|_{L^2(\Omega)}.$$

Squaring the two inequalities in (4.19) and equating the terms from (4.18) we obtain:

$$\left\langle \nabla u_1 - \mathbb{P}_{L_\Omega} \fint_\Omega \nabla u_1,\ \nabla u_2 - \mathbb{P}_{L_\Omega} \fint_\Omega \nabla u_2 \right\rangle_{L^2(\Omega)} = \kappa(\Omega)^2 \left\langle D(u_1), D(u_2)\right\rangle_{L^2(\Omega)}.$$

Therefore, (4.19) is actually true as the equality. We hence conclude that $u_1 + u_2 \in E$, proving that E is a linear space.

The closedness of E follows by noting that if a sequence u_k converges to u in $W^{1,2}(\Omega, \mathbb{R}^n)$ then the minimizing matrices $\mathbb{P}_{L_\Omega} \fint \nabla u_k$ converge to $\mathbb{P}_{L_\Omega} \fint \nabla u$.

2. To prove the second claim, we argue by contradiction. Assume that the space E is of infinite dimension. Then it admits a Hilbertian (orthonormal in $W^{1,2}(\Omega, \mathbb{R}^n)$) basis $\{u_k\}_{k=1}^\infty$. It is easy to see that there must be:

$$u_k \rightharpoonup 0 \quad \text{weakly in } W^{1,2}(\Omega, \mathbb{R}^n). \tag{4.20}$$

We now notice that:

$$\liminf_{k\to\infty} \|D(u_k)\|_{L^2(\Omega)} > 0. \tag{4.21}$$

Because otherwise, by Korn's inequality (1.2) there would be:

$$\liminf_{k\to\infty} \left\|\nabla u_k - \mathbb{P}_{L_\Omega} \fint_\Omega \nabla u_k\right\|_{L^2(\Omega)} = 0,$$

and since by (4.20) $\lim_{k\to\infty} \fint \nabla u_k = 0$, there follows that $\liminf_{k\to\infty} \|\nabla u_k\|_{L^2(\Omega)} = 0$. In view of the Poincaré inequality (2.3), we hence obtain $\liminf_{k\to\infty} \|u_k\|_{W^{1,2}(\Omega)} = 0$, in contradiction with the orthonormality of the sequence $\{u_k\}_{k=1}^\infty$.

Define: $v_k = u_k/\|D(u_k)\|_{L^2(\Omega)}$. Clearly, there holds:

$$\|D(v_k)\|_{L^2(\Omega)} = 1, \qquad \|\nabla v_k\|_{L^2(\Omega)} = \kappa(\Omega),$$

and because of (4.21) we also have: $v_k \rightharpoonup 0$ weakly in $W^{1,2}(\Omega, \mathbb{R}^n)$. By Theorem 1.2 there follows $\kappa(\Omega) = \kappa(\mathbb{R}^n) = \sqrt{2}$, which is a desired contradiction. ■

5. The optimal geometric rigidity constant in $\mathbb{R}^2$

To prove Theorems 1.5, 1.6, 1.7 we need some preliminary discussion.

Lemma 5.1. *Assume that $w \in L^2_{loc}(\mathbb{R}^n)$ and $\Delta w = 0$. If $w = f + g$ with $f \in L^2(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$, then $w \equiv const$.*

Proof. Fix $x_0, y_0 \in \mathbb{R}^n$. For any $r > 0$ we have:

$$\begin{aligned}|w(x_0) - w(y_0)| &= \left| \fint_{B_r(x_0)} w - \fint_{B_r(y_0)} w \right| = \frac{1}{|B_r|} \left| \int_{B_r(x_0)\Delta B_r(y_0)} w \right| \\ &\leq \left(\frac{1}{|B_r|} \int_{B_r(x_0)\Delta B_r(y_0)} |f| \right) + \left(\frac{1}{|B_r|} \int_{B_r(x_0)\Delta B_r(y_0)} |g| \right) \\ &\leq \frac{|B_r(x_0)\Delta B_r(y_0)|^{1/2}}{|B_r|} \|f\|_{L^2} + \frac{|B_r(x_0)\Delta B_r(y_0)|}{|B_r|} \|g\|_{L^\infty} \\ &\leq \left(\frac{1}{|B_r|} + \frac{|B_r(x_0)\Delta B_r(y_0)|}{|B_r|} \right) \left(\|f\|_{L^2} + \|g\|_{L^\infty} \right),\end{aligned}$$

where by Δ we denote the symmetric difference of two sets: $B_1 \Delta B_2 = (B_1 \setminus B_2) \cup (B_2 \setminus B_1)$. The quantity in the first parentheses above clearly converges to 0 as $r \to \infty$. Therefore $w(x_0) = w(y_0)$, which achieves the proof. ■

Lemma 5.2. *Let $f \in L^2(\mathbb{R}^2, \mathbb{R}^{2\times 2})$ and let $u \in W^{1,2}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ satisfy:*

$$-\Delta u = \mathrm{div}\ f \qquad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{5.1}$$

Then u can be decoupled as:

$$u = v + w; \qquad v, w \in L^2_{loc},\ \nabla v \in L^2,\ \nabla w \in L^2_{loc},\ -\Delta w = 0 \ \ \text{in } \mathbb{R}^2. \tag{5.2}$$

Moreover:

$$\nabla v = \lim_{m\to\infty} \nabla v_m \qquad \text{strongly in } L^2(\mathbb{R}^n),\ \text{for some} \ \ v_m \in \mathcal{C}^\infty_c(\mathbb{R}^2, \mathbb{R}^2). \tag{5.3}$$

Proof. For each $m \in \mathbb{N}$, let v_m be the solution to:

$$\begin{cases} v_m \in W^{1,2}_0(B_m) \\ -\displaystyle\int_{\mathbb{R}^2} \nabla v_m : \nabla \phi = \int_{\mathbb{R}^2} f : \nabla \phi \qquad \forall \phi \in W^{1,2}_0(B_m), \end{cases} \tag{5.4}$$

whose existence and uniqueness follow from the Lax-Milgram theorem, together with:

$$\|\nabla v_m\|_{L^2} \leq \|f\|_{L^2(B_m)} \leq \|f\|_{L^2(\mathbb{R}^2)}.$$

Therefore, passing to a subsequence:

$$\nabla v_m \rightharpoonup z \quad \text{ weakly in } L^2(\mathbb{R}^2) \tag{5.5}$$

and also :

$$\mathrm{curl}\ z = 0 \ \ \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{5.6}$$

Condition (5.6) is now equivalent to: $z = \nabla v$. This can be seen, for example, via Helmholtz decomposition [6]:

$$z = z_0 + \nabla v; \qquad v \in L^2_{loc},\ \nabla v, z_0 \in L^2,\ \mathrm{div}\ z_0 = 0 \text{ in } \mathcal{D}'.$$

Since from (5.6) also curl $z_0 = 0$, hence the components of z_0 satisfy the Cauchy-Riemann equations, and therefore $\Delta z_0 = 0$. Recalling that $z_0 \in L^2(\mathbb{R}^2)$ it follows by Lemma 5.1 that $z_0 = 0$. Consequently, by (5.5):

$$\nabla v_m \rightharpoonup \nabla v \quad \text{ weakly in } L^2(\mathbb{R}^2). \tag{5.7}$$

Passing to the limit in (5.4), we obtain: $-\Delta v = \mathrm{div}\ f$ in $\mathcal{D}'$, hence $-\Delta w = 0$, for $w = u - v$ and (5.2) is proven.

Finally, by Mazur's theorem and (5.7), ∇v is the strong L^2-limit of $\nabla \tilde{v}_m$ which are gradients of some finite (in fact, convex) linear combinations $\tilde{v}_m$ of v_m. Clearly, each $\tilde{v}_m \in W_0^{1,2}(B_{r_m})$ and the result in (5.3) follows by density of $\mathcal{C}_c^\infty(B_{r_m})$ in $W_0^{1,2}(B_{r_m})$. ■

Remark 5.3. Note that one can directly show that ∇v_m in Lemma 5.2 converges strongly in $L^2(\mathbb{R}^2)$. Let $k > m$. Extending v_m by zero to $\mathbb{R}^2$, so that $v_m \in W_0^{1,2}(B_k)$, and taking $\phi = v_m$ in the equation (5.4) written for v_k, we get:

$$\int_{B_k} \nabla v_k : \nabla v_m = -\int_{B_k} f : \nabla v_m = -\int_{B_m} f : \nabla v_m = \int_{B_m} |\nabla v_m|^2.$$

The last equality above follows by taking $\phi = v_m$ in the equation (5.4) written for v_m. Now, passing $m \to \infty$ implies, by the weak convergence in (5.5):

$$\lim_{m\to\infty} \int_{\mathbb{R}^2} |\nabla v_m|^2 = \int_{\mathbb{R}^2} \nabla v_k : \nabla v.$$

Finally, passing $k \to \infty$ yields:

$$\lim_{m\to\infty} \int_{\mathbb{R}^2} |\nabla v_m|^2 = \int_{\mathbb{R}^2} |\nabla v|^2.$$

The claim (5.3) now follows, since convergence of norms in presence of the weak convergence implies strong convergence in $L^2(\mathbb{R}^2)$.

Lemma 5.4. *Let $u \in W_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ and $\nabla u \in L^2(\mathbb{R}^2)$. Then:*

$$\int_{\mathbb{R}^2} \det \nabla u = 0. \tag{5.8}$$

Proof. Since $\Delta u = \operatorname{div} \nabla u$ in $\mathcal{D}'(\mathbb{R}^2)$ we may apply Lemma 5.2 to $f = \nabla u \in L^2(\mathbb{R}^2)$ and write $u = v + w$ satisfying (5.2). Since $\Delta w = 0$ and $\nabla w = \nabla u - \nabla v \in L^2(\mathbb{R}^2)$, it follows from Lemma 5.1 that $\nabla w = 0$ and hence by (5.3):

$$\nabla u = \nabla v = \lim_{m\to\infty} \nabla v_m \quad \text{ strongly in } L^2(\mathbb{R}^2), \text{ for some } v_m \in \mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{R}^2).$$

It remains to prove (5.8) for $u \in \mathcal{C}_c^\infty$, which is a standard argument. Let $\operatorname{supp} u \subset B_r$. We have:

$$\begin{aligned}\int_{\mathbb{R}^2} \det \nabla u &= \int_{B_r} (\partial_1 u^1 \partial_2 u^2 - \partial_1 u^2 \partial_2 u^1) \\ &= \int_{B_r} (\partial_1(u^1 \partial_2 u^2) - \partial_2(u^1 \partial_1 u^2)) = \int_{\partial B_r} (u^1 \partial_2 u^2, u^1 \partial_1 u^2)\vec{n} = 0,\end{aligned}$$

where we used integration by parts and the divergence theorem. ■

We finally need to recall the conformal–anticonformal decomposition of 2×2 matrices. Let $\mathbb{R}_c^{2\times 2}$ and $\mathbb{R}_a^{2\times 2}$ denote, respectively, the spaces of conformal and anticonformal matrices:

$$\mathbb{R}_c^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix};\ a, b \in \mathbb{R} \right\}, \qquad \mathbb{R}_a^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix};\ a, b \in \mathbb{R} \right\}.$$

It is easy to see that $\mathbb{R}^{2\times 2} = \mathbb{R}_c^{2\times 2} \oplus \mathbb{R}_a^{2\times 2}$ because both spaces have dimension 2 and they are mutually orthogonal: $A : B = 0$ for all $A \in \mathbb{R}_c^{2\times 2}$ and $B \in \mathbb{R}_a^{2\times 2}$.

For $F = [F_{ij}]_{i,j:1,2} \in \mathbb{R}^{2\times 2}$, its projections F^c on $\mathbb{R}^{2\times 2}_c$, and F^a on $\mathbb{R}^{2\times 2}_a$ are:

$$F^c = \frac{1}{2}\left[\begin{array}{cc} F_{11}+F_{22} & F_{12}-F_{21} \\ F_{21}-F_{12} & F_{11}+F_{22} \end{array}\right], \qquad F^a = \frac{1}{2}\left[\begin{array}{cc} F_{11}-F_{22} & F_{12}+F_{21} \\ F_{12}+F_{21} & F_{22}-F_{11} \end{array}\right].$$

It follows that:

$$F = F^c + F^a \quad \text{and} \quad |F|^2 = |F^c|^2 + |F^a|^2 \tag{5.9}$$

and, by a direct calculation:

$$\det F = 2(|F^c|^2 - |F^a|^2). \tag{5.10}$$

Since $SO(2) = \left\{\left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right],\ \theta \in [0, 2\pi)\right\} \subset \mathbb{R}^{2\times 2}_c$, it also follows that:

$$\text{dist}(F, SO(2)) \geq \text{dist}(F, \mathbb{R}^{2\times 2}_c) = |F^a| \tag{5.11}$$

which implies:

$$|\text{cof } F - F| = |-2F^a| \leq 2\text{dist}(F, SO(2)). \tag{5.12}$$

Finally, recall that the cofactor matrix in dimension 2 is given by:

$$\text{cof } F = \left[\begin{array}{cc} F_{22} & -F_{21} \\ -F_{12} & F_{11} \end{array}\right].$$

We now state the following first result towards proving Theorem 1.5.

Lemma 5.5. *Let* $u \in W^{1,2}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ *and assume that* $\text{dist}(\nabla u, SO(2)) \in L^2(\mathbb{R}^2, \mathbb{R})$. *Then there exists* $R_0 \in SO(2)$ *such that:*

$$\int_{\mathbb{R}^2} |\nabla u(x) - R_0|^2 \ \mathrm{d}x \leq 2 \int_{\mathbb{R}^2} \text{dist}^2(\nabla u(x), SO(2)) \ \mathrm{d}x.$$

Proof. From the assumption $\text{dist}(\nabla u, SO(2)) \in L^2(\mathbb{R}^2)$ and (5.12) we deduce:

$$f := \text{cof } \nabla u - \nabla u \in L^2(\mathbb{R}^2).$$

Taking divergence of f and recalling that div cof $\nabla u = 0$ we obtain that $-\Delta u =$ div f. In view of Lemma 5.2 we now write:

$$u = v + w \tag{5.13}$$

where v and w satisfy (5.2). We now prove that:

$$\nabla w \equiv R_0 \in SO(2). \tag{5.14}$$

For $\epsilon > 0$ sufficiently small, define:

$$g(x) = \left\{ \begin{array}{ll} \mathbb{P}_{SO(2)} \nabla u(x) & \text{if } \text{dist}(\nabla u(x), SO(2)) < \epsilon \\ \text{Id} & \text{otherwise} \end{array} \right.$$

Then:

$$\nabla w = g + h; \quad g \in L^\infty(\mathbb{R}^2) \text{ and } h \in L^2(\mathbb{R}^2). \tag{5.15}$$

The assertion $h = \nabla w - g = \nabla u - g + \nabla v \in L^2(\mathbb{R}^2)$ follows from the assumption $\text{dist}(\nabla u, SO(2)) \in L^2(\mathbb{R}^2)$ as follows. We already know that $\nabla v \in L^2(\mathbb{R}^2)$ by (5.13).

For $h_1 = \nabla u - g$ note that $|h_1(x)| = \mathrm{dist}(\nabla u, SO(2))$ when $\mathrm{dist}(\nabla u(x), SO(2)) < \epsilon$, while when $\mathrm{dist}(\nabla u(x), SO(2)) \geq \epsilon$, we have:

$$\begin{aligned} |h_1(x)| = |\nabla u(x) - \mathrm{Id}| &\leq \mathrm{dist}(\nabla u(x), SO(2)) + \mathrm{diam}\big(SO(2)\big) \\ &\leq \left(1 + \frac{\mathrm{diam}\big(SO(2)\big)}{\epsilon}\right) \mathrm{dist}(\nabla u(x), SO(2)). \end{aligned}$$

Since ∇w is harmonic in $\mathbb{R}^2$, (5.15) implies that $\nabla w \equiv R_0$ is constant by Lemma 5.1. But $\mathrm{dist}(R_0, SO(2)) \leq \mathrm{dist}(\nabla u, SO(2)) + |\nabla v| \in L^2(\mathbb{R}^2)$, so $R_0 \in SO(2)$ and (5.14) is now established.

By (5.14) and (5.13) we have:

$$\nabla u = \nabla v + R_0. \tag{5.16}$$

Since $\int \det \nabla v = 0$ by Lemma 5.4, we obtain by (5.10):

$$\int_{\mathbb{R}^2} |(\nabla v)^c|^2 = \int_{\mathbb{R}^2} |(\nabla v)^a|^2. \tag{5.17}$$

Consequently:

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla u - R_0|^2 = \int |\nabla v)|^2 = \int |(\nabla v)^c|^2 + \int |(\nabla v)^a|^2 = 2 \int |(\nabla v)^a|^2 \\ = 2 \int |(\nabla v + R_0)^a|^2 \leq 2 \int \mathrm{dist}^2(\nabla v + R_0, SO(2)) \\ = 2 \int_{\mathbb{R}^2} \mathrm{dist}^2(\nabla u, SO(2)), \end{aligned}$$

where we used (5.16), (5.9), (5.17), (5.11) and the fact that $(R_0)^a = 0$. This achieves the proof. ■

Proof of Theorem 1.6

1. Without loss of generality we may assume that $R_0 = \mathrm{Id}$. We shall look for a function $u \in W^{1,2}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ such that:

$$\nabla u(x) = R(\alpha(x)) + \begin{bmatrix} a(x) & b(x) \\ b(x) & -a(x) \end{bmatrix}, \quad \text{with } R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \tag{5.18}$$

and:

$$\nabla u(x) - \mathrm{Id} \in L^2(\mathbb{R}^2) \tag{5.19}$$

Indeed, note that by Lemma 5.4, (5.19) and (5.10):

$$\int_{\mathbb{R}^2} |(\nabla u)^c - \mathrm{Id}|^2 = \int_{\mathbb{R}^2} |(\nabla u)^a|^2.$$

Hence, by (5.9):

$$\int_{\mathbb{R}^2} |\nabla u - \mathrm{Id}|^2 = 2 \int_{\mathbb{R}^2} |(\nabla u)^a|^2 = 2 \int_{\mathbb{R}^2} \mathrm{dist}^2(\nabla u, SO(2)).$$

because $(\nabla u)^c = R(\alpha) \in SO(2)$. Since $(\nabla u)^a = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ it also follows that $\int_{\mathbb{R}^2} \mathrm{dist}^2(\nabla u, SO(2)) = 2 \int_{\mathbb{R}^2} (a^2 + b^2)$.

On the other hand, there is always the unique rotation R which makes the quantity in the left hand side of (1.10) finite:

$$\int_{\mathbb{R}^2} |\nabla u - R|^2 \geq \frac{1}{2}\int_{\mathbb{R}^2} |R - \mathrm{Id}|^2 - \int_{\mathbb{R}^2} |\nabla u - \mathrm{Id}|^2.$$

This proves the theorem, provided (5.18) and (5.19) hold.

2. We shall show that for any $\alpha \in L^2(\mathbb{R}^2, \mathbb{R})$ there exists a vector field $g = (a,b)^T \in L^2(\mathbb{R}^2, \mathbb{R}^2)$ satisfying (5.18). Then (5.19) will follow automatically, as:

$$\int |R(\alpha) - \mathrm{Id}|^2 = 2\int (\cos\alpha - 1)^2 + (\sin\alpha)^2 = 2\int (2 - 2\cos\alpha) \leq 2\int |\alpha|^2.$$

The last inequality above follows by noting that the function $\alpha \mapsto \alpha^2 + 2\cos\alpha - 2$ attains its minimum value 0 at $\alpha = 0$, since $(\alpha^2 + 2\cos\alpha - 2)' = 2(\alpha - \sin\alpha)$ is positive for $\alpha > 0$ and negative for $\alpha < 0$.

Fix $\alpha \in L^2(\mathbb{R}^2)$. The map $u \in W^{1,2}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ with ∇u of the form (1.11) exists if and only if the right hand side in (1.11) is curl-free, i.e.:

$$\left\{ \begin{array}{ll} \mathrm{curl}\ g = \mathrm{div}\ f & \quad \text{in } \mathcal{D}'(\mathbb{R}^2) \\ \mathrm{div}\ g = \mathrm{curl}\ f & \end{array} \right. \tag{5.20}$$

where:

$$f = (\sin\alpha,\ \cos\alpha - 1)^T \in L^2(\mathbb{R}^2, \mathbb{R}^2).$$

The system (5.20) can be solved by Fourier transform, namely:

$$g = \mathcal{F}^{-1}(h), \qquad h(x) = -\left\langle \frac{x^\perp}{|x|}, \mathcal{F}(f)(x) \right\rangle \frac{x}{|x|} + \left\langle \frac{x}{|x|}, \mathcal{F}(f)(x) \right\rangle \frac{x^\perp}{|x|}, \tag{5.21}$$

where $x^\perp = (-x_2, x_1)$. Here $\mathcal{F}$ stands for the Fourier transform of $L^2(\mathbb{R}^2, \mathbb{C})$ and we identify the complex variable functions with the $\mathbb{R}^2$-valued vector fields.

Note that from (5.21) it follows that:

$$\begin{array}{ll} \forall x \in \mathbb{R}^2 & \langle \mathcal{F}(g)(x), x^\perp \rangle = \langle \mathcal{F}(f)(x), x \rangle \\ & \langle \mathcal{F}(g)(x), x \rangle = -\langle \mathcal{F}(f)(x), x^\perp \rangle \end{array} \tag{5.22}$$

which precisely implies (5.20). Therefore, for every $f \in L^2(\mathbb{R}^2)$ there exists a unique $g \in L^2(\mathbb{R}^2)$ solving (5.20). This achieves the proof of the theorem. Moreover:

$$\|g\|_{L^2} = \|h\|_{L^2} = \|\mathcal{F}(f)\|_{L^2} = \|f\|_{L^2},$$

by Plancherel identity and by inspecting (5.21). ■

This concludes the proof of Theorem 1.5 as well. In view of the argument in the above proof, Theorem 1.7 will follow in view of:

Lemma 5.6. *If $\int |\nabla u - R_0|^2 = 2\int \mathrm{dist}^2(\nabla u, SO(2)) < \infty$ then ∇u must be of the form (5.18) with $R(\alpha) - R_0 \in L^2(\mathbb{R}^2)$.*

Proof. Note that by (5.9): $|\nabla u - R|^2 = |(\nabla u)^c - R|^2 + |(\nabla u)^a|^2$ for any $R \in SO(2)$. Hence taking infimum over all rotations, we get:

$$\mathrm{dist}^2(\nabla u, SO(2)) = \mathrm{dist}^2((\nabla u)^c, SO(2)) + |(\nabla u)^a|^2. \tag{5.23}$$

In particular:

$$(\nabla u)^a \in L^2(\mathbb{R}^2).$$

Further, by (5.10) and Lemma 5.4:

$$\int |(\nabla u)^c - R_0|^2 = \int |(\nabla u)^a|^2.$$

Therefore, by (5.9) and (5.23):

$$\begin{aligned}\int |(\nabla u)^c - R_0|^2 &= \frac{1}{2}\int |\nabla u - R_0|^2 = \int \mathrm{dist}^2(\nabla u, SO(2)) \\ &= \int \mathrm{dist}^2((\nabla u)^c, SO(2)) + \int |(\nabla u)^a|^2 \\ &= \int \mathrm{dist}^2((\nabla u)^c, SO(2)) + \int |(\nabla u)^c - R_0|^2,\end{aligned}$$

which implies that $\int \mathrm{dist}^2((\nabla u)^c, SO(2)) = 0$ and hence: $(\nabla u(x))^c \in SO(2)$ for a.e. x. Consequently, ∇u has the form in (5.18) and:

$$R(\alpha) - R_0 = \nabla u - R_0 - (\nabla u)^a \in L^2(\mathbb{R}^2)$$

by (5.23). ■

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