THE OBSTACLE PROBLEM FOR THE p-LAPLACIAN VIA OPTIMAL STOPPING OF TUG-OF-WAR GAMES

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Abstract. We present a probabilistic approach to the obstacle problem for the p-Laplace operator. The solutions are approximated by running processes determined by tug-of-war games plus noise, and letting the step size go to zero, not unlike the case when Brownian motion is approximated by random walks. Rather than stopping the process when the boundary is reached, the value function is obtained by maximizing over all possible stopping times that are smaller than the exit time of the domain.

1. Introduction

Let $\mathfrak{L}$ be the second order differential operator:

$$\mathfrak{L} = \frac{1}{2}\operatorname{trace}(\sigma\sigma'(x)D_x^2 v),$$

whose matrix coefficient function σ is Lipschitz continuous. Consider the obstacle problem in $\mathbb{R}^N$:

$$\min\left(-\mathfrak{L}v, v - g\right) = 0. \tag{1.1}$$

In order to solve (1.1) probabilistically [18], one first solves the stochastic differential equation:

$$d\mathbf{X}_t = \sigma(\mathbf{X}_t)\, d\mathbf{W}_t, \tag{1.2}$$

starting from x at time $t = 0$. Denoting its solution by $\{\mathbf{X}_t^x, t \geq 0\}$, the value function is defined by taking the supremum over the set $\mathfrak{T}$ of all stopping times valued in $[0, \infty]$:

$$v(x) = \sup_{\tau\in\mathfrak{T}} \mathbb{E}\left[g(\mathbf{X}_\tau^x)\right], \tag{1.3}$$

This value function, under appropriate regularity hypothesis on g, turns out then to be the unique solution to (1.1); for details see Chapter 5 in [18].

The purpose of the present paper is to consider the analog problem when the second order linear differential operator $\mathfrak{L}$ is replaced by the p-Laplacian:

$$-\Delta_p u = -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big), \qquad 2 \leq p < \infty. \tag{1.4}$$

Since the operator $-\Delta_p$ is non-linear, we do not have a suitable variant of the linear stochastic differential equation (1.2) that could be used to write a formula similar to (1.3). Instead, we will show that one can use tug-of-war games with noise as the basic stochastic process. More precisely, we will prove that the solutions to the obstacle problem for the p-Laplacian for $p \in [2, \infty)$, can be interpreted as limits of values of a specific discrete tug-of-war game with noise, when the step-size ϵ, determining the allowed length of move of a token at each step of the game, converges to 0.

To explain our approach, let us first recall a concept of supersolutions suggested by the tug-of-war characterization in [13]. This notion is based on the mean value properties, and it is equivalent to the notion of viscosity supersolution in the class of continuous functions. Namely, let $\Omega \subset \mathbb{R}^N$

Date: November 16, 2015.

Key words and phrases. tug-of-war games, viscosity solutions, p-Laplacian, obstacle problem.

be an open, bounded set with Lipschitz boundary and let $F : \partial\Omega \to \mathbb{R}$ be a Lipschitz continuous boundary data. Choose the parameters α and β as follows:

$$\alpha = \frac{p-2}{N+p}, \qquad \beta = \frac{2+N}{N+p},$$

where $\alpha \geq 0$ since $p \geq 2$, $\beta > 0$, and $\alpha + \beta = 1$.

Supersolutions in the sense of means: A continuous function $v : \Omega \to \mathbb{R} \cup \{\infty\}$ is a supersolution in the sense of means if whenever $\phi \in \mathcal{C}_0^\infty(\Omega)$ is such that $\phi(x) \leq v(x)$ for all $x \in \Omega$, with equality at one point $\phi(x_0) = v(x_0)$ (ϕ touches v from below at x_0), then we have:

$$0 \leq -\phi(x_0) + \frac{\alpha}{2} \sup_{B_\epsilon(x_0)} \phi + \frac{\alpha}{2} \inf_{B_\epsilon(x_0)} \phi + \beta \fint_{B_\epsilon(x_0)} \phi + o(\epsilon^2) \qquad \text{as } \epsilon \to 0^+. \tag{1.5}$$

Above, by $0 \leq h(\epsilon) + o(\epsilon^2)$ as $\epsilon \to 0^+$ we mean that:

$$\lim_{\epsilon \to 0^+} \frac{[h(\epsilon)]^-}{\epsilon^2} = 0.$$

Fixing a scale ϵ, we now consider functions for which (1.5) holds with equality, i.e. without the error term ϵ^2. Let $0 < \epsilon_0 \ll 1$ be a small constant and define the fattened outer boundary set, together with the fattened domain:

$$\Gamma = \{x \in \mathbb{R}^N \setminus \Omega;\ \text{dist}(x, \Omega) < \epsilon_0\}, \qquad X = \Omega \cup \Gamma.$$

ϵ-p-harmonious functions: Let $0 < \epsilon \leq \epsilon_0$. A bounded, Borel function $u : X \to \mathbb{R}$ is ϵ-p-harmonious with boundary values $F : \bar{\Gamma} \to \mathbb{R}$ if:

$$u_\epsilon(x) = \begin{cases} \frac{\alpha}{2} \sup_{B_\epsilon(x)} u_\epsilon + \frac{\alpha}{2} \inf_{B_\epsilon(x)} u_\epsilon + \beta \fint_{B_\epsilon(x)} u_\epsilon & \text{for } x \in \Omega \\ F(x) & \text{for } x \in \Gamma. \end{cases} \tag{1.6}$$

Then, it has been established in [14] that $u = \lim_{\epsilon \to 0} u_\epsilon$ is a solution to the Dirichlet problem:

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ u = F & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Let now $\Psi : \mathbb{R}^n \to \mathbb{R}$ be a bounded, Lipschitz function, which we assume to be compatible with the boundary data: $F(x) \geq \Psi(x)$ for $x \in \partial\Omega$. The function Ψ is interpreted as the obstacle and we consider the problem:

$$\begin{cases} -\Delta_p u \geq 0 & \text{in } \Omega, \\ u \geq \Psi & \text{in } \Omega, \\ -\Delta_p u = 0 & \text{in } \{x \in \Omega;\ u(x) > \Psi(x)\}, \\ u = F & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

That is, we want to find a p-superharmonic function u which takes boundary values F, which is above the obstacle Ψ, and which is actually p-harmonic in the complement of the contact set $\{x \in \overline{\Omega} \colon u(x) = \Psi(x)\}$.

The problem (1.8) has been extensively studied from the variational point of view; see the seminal paper by Lindqvist [10] and the book [6]. In particular, regularity requirements for the domain Ω, the boundary data F and the obstacle Ψ can be vastly generalized. We have, however, focused on the Lipschitz category for technical reasons in our proofs.

Our first result shows how to solve the obstacle problem using ϵ-p-superharmonious functions. The dynamic programing principle (1.9) below is similar to the Wald-Bellman equations of optimal stopping (see Chapter 1 of [21]).

Theorem 1.1. *Let $\alpha \in [0,1)$ and $\beta = 1-\alpha$. Let $F : \bar{\Gamma} \to \mathbb{R}$ and $\Psi : \mathbb{R}^N \to \mathbb{R}$ be two bounded, Borel functions such that $\Psi \leq F$ in $\bar{\Gamma}$. Then there exists a unique bounded Borel function $u : X \to \mathbb{R}$ which satisfies:*

$$u_\epsilon(x) = \begin{cases} \max\left\{\Psi(x), \dfrac{\alpha}{2}\sup_{B_\epsilon(x)} u_\epsilon + \dfrac{\alpha}{2}\inf_{B_\epsilon(x)} u_\epsilon + \beta \fint_{B_\epsilon(x)} u_\epsilon\right\} & \text{for } x \in \Omega \\ F(x) & \text{for } x \in \Gamma. \end{cases} \tag{1.9}$$

After proving Theorem 1.1 in Section §2, we proceed to establishing that $u = \lim_{\epsilon\to 0^+} u_\epsilon$ is the solution to the obstacle problem (1.8). The key step is to show that $\{u_\epsilon\}$ is equicontinuous up to scale ϵ, i.e. the functions u_ϵ may be discontinuous but the discontinuities are of size roughly ϵ. Then, an easy extension of the Arzelá-Ascoli theorem from [14] shows that there are subsequences of $\{u_\epsilon\}$ that converge uniformly to a function u. The standard stability of viscosity solutions yields then that any such limit is a viscosity solution of (1.8). By uniqueness (see Lemma 4.2 and its proof in the Appendix) they all must agree, and we obtain:

Theorem 1.2. *Let $p \in [2,\infty)$ and let $u_\epsilon : \Omega \cup \Gamma \to \mathbb{R}$ be the unique ϵ-p-superharmonious function solving (1.9) with $\alpha = \frac{p-2}{p+N}$ and $\beta = \frac{2+N}{p+N}$. Then u_ϵ converge as $\epsilon \to 0$, uniformly in $\bar{\Omega}$, to a continuous function u which is the unique viscosity solution to the obstacle problem (1.8).*

Towards the proof, the key estimate (in Lemma 4.4) bounds the oscillation of u_ϵ in a uniform way. Namely, given $\eta > 0$ there are small $r_0, \epsilon_0 > 0$ so that whenever $\epsilon < \epsilon_0$, then for all $x_0, y_0 \in \overline{\Omega}$:

$$|x_0 - y_0| < r_0 \implies |u_\epsilon(x_0) - u_\epsilon(y_0)| < \eta. \tag{1.10}$$

To deduce (1.10) we use probabilistic techniques, interpreting u_ϵ as value functions of certain tug-of-war games and finding an appropriate extension of (1.3). In Section 3 we present the details of this construction, involving stochastic processes (tug-of-war games with noise) needed to write down the representation formulas for u_ϵ. For the case of linear equations (that correspond to $p = 2$) with variable coefficients, a similar version of the representation formula (1.11) below is due to Pham [17] and Øksendal-Reikvam [16]. Note that since $\epsilon > 0$ is fixed, we omit it in the statement that follows.

Theorem 1.3. *Let $\alpha, \beta \geq 0$ satisfy $\alpha + \beta = 1$. Let $F : \Gamma \to \mathbb{R}$ and $\Psi : \mathbb{R}^N \to \mathbb{R}$ be two bounded, Borel functions such that $\Psi \leq F$ in Γ. Define:*

$$G : X \to \mathbb{R} \qquad G = \chi_\Gamma F + \chi_\Omega \Psi,$$

where χ_A stands for the characteristic function of a set $A \subset X$. Define the two value functions:

$$u_I(x_0) = \sup_{\tau,\sigma_I}\inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{II}}[G \circ x_\tau], \qquad u_{II}(x_0) = \inf_{\sigma_{II}}\sup_{\tau,\sigma_I} \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{II}}[G \circ x_\tau], \tag{1.11}$$

where sup *and* inf *are taken over all strategies σ_I, σ_{II} and stopping times $\tau \leq \tau_0$ that do not superseed the exit time τ_0 from the set Ω. Then:*

$$u_I = u = u_{II} \quad \text{in } \Omega,$$

where u is a bounded, Borel function satisfying (1.9).

The core of this paper can be found in Section §5, where we use the representation formulas from Theorem 1.3 to establish the oscillation estimate (1.10). We provide full details of the proof for the equicontinuity estimates in Section §4, the proof of Theorem 1.2 in Section §5, and the fact that our games end almost surely in Section §6.

We finish this introduction by discussing other notions of solutions for (1.4) in addition to (1.5):

i) **Weak (or Sobolev) supersolutions:** These are functions $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that:

$$\int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla \phi\rangle \,\mathrm{d}x \geq 0$$

for all test functions $\phi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}_+)$ that are non-negative in Ω.

ii) **Potential theoretic supersolutions or p-superharmonic functions:** A lower-semi-continuous function $v : \Omega \to \mathbb{R}\cup\{\infty\}$ is p-superharmonic if it is not identically ∞ on any connected component of Ω and it satisfies the comparison principle with respect to p-harmonic functions, that is: if $D \Subset \Omega$, and $w \in \mathcal{C}(\bar{D})$ is p-harmonic in D satisfying $w \leq v$ on ∂D, then we must have: $w \leq v$ on D.

iii) **Viscosity supersolutions:** A lower-semicontinuous function $v : \Omega \to \mathbb{R}\cup\{\infty\}$ is a viscosity p-supersolution if it is not identically ∞ on any connected component of Ω, and if whenever $\phi \in \mathcal{C}_0^\infty(\Omega)$ is such that $\phi(x) \leq v(x)$ for all $x \in \Omega$ with equality at one point $\phi(x_0) = v(x_0)$ (ϕ touches v from below at x_0), and $\nabla\phi(x_0) \neq 0$, then we have:

$$-\Delta_p \phi(x_0) \geq 0.$$

The fact that weak supersolutions are potential theoretic and viscosity supersolutions follows from the comparison principle and a regularity argument implying the lower-semicontinuity; see for example Chapter 3 in [6]. The fact that bounded p-superharmonic functions are weak supersolutions was established by Lindqvist in [10].[1] Note that an arbitrary, not necessarily bounded p-superharmonic function v is always the pointwise increasing limit of bounded p-superharmonic functions $v_n = \min\{v, n\}$. The equivalence between viscosity supersolutions and p-superharmonic functions was established in [7]. Therefore, the three notions of supersolution agree on the class of bounded functions; see also the Appendix where for completeness we present the folklore argument stating that viscosity solutions are weak solutions and thus they are unique.

Finally, we recall a classical useful observation. When $u \in \mathcal{C}^2$ and $\nabla u(x) \neq 0$, then one can express the p-Laplacian as a combination of the ordinary Laplacian and the ∞-Laplacian:

$$\Delta_p u(x) = |\nabla u|^{p-2}\big(\Delta u(x) + (p-2)\Delta_\infty u(x)\big), \tag{1.12}$$

where:

$$\Delta_\infty u(x) = \Big\langle \nabla^2 u(x)\frac{\nabla u(x)}{|\nabla u(x)|}, \frac{\nabla u(x)}{|\nabla u(x)|}\Big\rangle. \tag{1.13}$$

The tug-of-war interpretation of the ∞-Laplacian has been developed in the fundamental paper [19]. The obstacle problem for (1.13) has been studied in [15]. A similar treatment as in the present paper, for the double obstacle problem, has been developed in [4].

Acknowledgments. The first author was partially supported by NSF awards DMS-0846996 and DMS-1406730. The second author was partially supported by NSF award DMS-1001179.

[1] In fact, this is also the first reference that we have been able to locate for the classical case $p = 2$.

2. ϵ-p-superharmonious functions: a proof of Theorem 1.1

The proof uses the monotonicity arguments of the Perron method as extended by [11], modified to accommodate the obstacle constraint.

1. The solution to (1.9) will be obtained as the uniform limit of iterations $u_{n+1} = Tu_n$, where, for any bounded Borel function $v : X \to \mathbb{R}$, we define:

$$Tv(x) = \begin{cases} \max\left\{\Psi(x), \dfrac{\alpha}{2}\sup_{B_\epsilon(x)} v + \dfrac{\alpha}{2}\inf_{B_\epsilon(x)} v + \beta \fint_{B_\epsilon(x)} v\right\} & \text{for } x \in \Omega \\ v(x) & \text{for } x \in \Gamma \end{cases} \tag{2.1}$$

and where we put:

$$u_0 = \chi_\Gamma F + \chi_\Omega \left(\inf_X \Psi\right). \tag{2.2}$$

We easily note that $u_1 = Tu_0 \geq u_0$ in X. Consequently: $u_2 = Tu_1 \geq Tu_0 = u_1$ in X and, by induction, the sequence of Borel functions $\{u_n\}$ is nondecreasing in X. Also, $\{u_n\}_{n=1}^\infty$ satisfies:

$$\Psi \leq u_n \leq \max\left\{\sup_\Gamma F, \sup_X \Psi\right\} \quad \text{in } \Omega,$$

and clearly $u_n = F$ on Γ. Therefore, the sequence converges pointwise to a bounded Borel function $u : X \to \mathbb{R}$, satisfying: $u_{|\Gamma} = F$.

2. We now show that the convergence of $\{u_n\}$ to u is uniform in X. This will automatically imply that $u = \lim Tu_n = T(\lim u_n) = Tu$, and hence yield the desired existence result. We argue by contradiction and assume that:

$$M = \lim_{n\to\infty} \sup_{x\in X} (u - u_n)(x) > 0.$$

Fix a small $\delta > 0$ and take $n > 1$ so that:

$$\sup_X\, (u - u_n) < M + \delta \quad \text{and} \quad \forall x \in \Omega \quad \beta \fint_{B_\epsilon(x)} u - u_n \leq \frac{\beta}{|B_\epsilon(x)|}\int_X u - u_n < \delta.$$

The second condition above is justified by the monotone convergence theorem.

Take now $x_0 \in \Omega$ satisfying: $u(x_0) - u_{n+1}(x_0) > M - \delta$. Note that if $u(x_0) = \Psi(x_0)$ then since $u_j(x_0)$ increases to $u(x_0)$ and all $u_j(x_0) \geq \Psi(x_0)$, there would be $u_n(x_0) = \Psi(x_0)$. Therefore $u(x_0) > \Psi(x_0)$ and consequently:

$$\exists m > n \quad u_{m+1}(x_0) - u_{n+1}(x_0) > M - 2\delta \quad \text{and} \quad u_{m+1}(x_0) > \Psi(x_0).$$

We now compute:

$$\begin{aligned} M - 2\delta &< u_{m+1}(x_0) - u_{n+1}(x_0) \\ &= \frac{\alpha}{2}\left(\sup_{B_\epsilon(x_0)} u_m - \sup_{B_\epsilon(x_0)} u_n\right) + \frac{\alpha}{2}\left(\inf_{B_\epsilon(x_0)} u_m - \inf_{B_\epsilon(x_0)} u_n\right) + \beta \fint_{B_\epsilon(x_0)} u_m - u_n \\ &\leq \alpha \sup_{B_\epsilon(x_0)}\, (u_m - u_n) + \beta \fint_{B_\epsilon(x_0)} u_m - u_n \leq \alpha \sup_{B_\epsilon(x_0)}\, (u - u_n) + \beta \fint_{B_\epsilon(x_0)} u - u_n \\ &< \alpha(M + \delta) + \delta = \alpha M + (\alpha + 1)\delta. \end{aligned} \tag{2.3}$$

This implies that $M < \alpha M + (\alpha + 3)\delta$, which clearly is a contradiction for δ sufficiently small, since $\alpha < 1$.

3. We now prove uniqueness of solutions to (1.9). Assume, by contradiction, that u and $\bar{u}$ are two distinct solutions and denote:

$$M = \sup_{\Omega}(u-\bar{u}) > 0.$$

Let $\{x_n\}_{n\geq 1}$ be a sequence of points in Ω such that $\lim_{n\to\infty}(u-\bar{u})(x_n) = M$. Observe that for large n there must be: $u(x_n) > \Psi(x_n)$, since $M > 0$. Therefore, as in (2.3), we get:

$$\begin{aligned}(u-\bar{u})(x_n) &= \frac{\alpha}{2}\left(\sup_{B_\epsilon(x_n)} u - \sup_{B_\epsilon(x_n)} \bar{u}\right) + \frac{\alpha}{2}\left(\inf_{B_\epsilon(x_n)} u - \inf_{B_\epsilon(x_n)} \bar{u}\right) + \beta \fint_{B_\epsilon(x_n)} u - \bar{u} \\ &\leq \alpha \sup_{B_\epsilon(x_n)} (u-\bar{u}) + \beta \fint_{B_\epsilon(x_n)} u - \bar{u} \leq \alpha M + \beta \fint_{B_\epsilon(x_n)} u - \bar{u}.\end{aligned}$$

Passing to the limit with $n\to\infty$, where $\lim x_n = x_0$, we obtain:

$$M \leq \alpha M + \beta \fint_{B_\epsilon(x_0)} u - \bar{u},$$

and hence: $M \leq \fint_{B_\epsilon(x_0)} u - \bar{u}$, since $\beta > 0$. Consequently: $u - \bar{u} = M$ almost everywhere in $B_\epsilon(x_0)$, and hence in particular the following set is nonempty:

$$G = \{x \in X;\ (u-\bar{u})(x) = M\} \neq \emptyset.$$

By the same argument as above, we see that in fact for all $x \in G$, the set $B_\epsilon(x)\setminus G$ has measure 0. We conclude that:

$$u - \bar{u} = M \qquad \text{a.e. in } X$$

which contradicts the fact that $G \cap \Gamma = \emptyset$, and proves the result. ■

We further easily derive the following weak comparison principle:

Corollary 2.1. *Let u and $\bar{u}$ be the unique solutions to (1.9) with the respective boundary data F and $\bar{F}$ and obstacle constraints Ψ and $\bar{\Psi}$, satisfying the assumptions of Theorem 1.1. If $F \leq \bar{F}$ and $\Psi \leq \bar{\Psi}$ then $u \leq \bar{u}$ in Ω.*

Proof. Let $\{u_n\}$ and $\{\bar{u}_n\}$ be the approximating sequences for u and $\bar{u}$, as in the proof of Theorem 1.1. By (2.2): $u_0 \leq \bar{u}_0$, which results in $u_n \leq \bar{u}_n$ for every n, in view of (1.9). Consequently, the limits u and $\bar{u}$ satisfy the same pointwise inequality. ■

3. Game-theoretical interpretation of the ϵ-p-superharmonious functions

We now link the ϵ-p-superharmonious function u solving (1.9) to the probabilistic setting. We define this setting in detail, as this paper is dedicated to analysts rather than probabilists. All the basic concepts can be found in the classical textbook [23].

3.1. **The measure spaces $(X^{\infty,x_0}, \mathcal{F}_n^{x_0})$ and $(X^{\infty,x_0}, \mathcal{F}^{x_0})$.** Fix any $x_0 \in X$ and consider the space of infinite sequences ω (recording positions of token during the game), starting at x_0:

$$X^{\infty,x_0} = \{\omega = (x_0, x_1, x_2 \ldots);\ x_n \in X \text{ for all } n \geq 1\}.$$

For each $n \geq 1$, let $\mathcal{F}_n^{x_0}$ be the σ-algebra of subsets of X^{∞,x_0}, containing sets of the form:

$$A_1 \times \ldots \times A_n := \{\omega \in X^{\infty,x_0};\ x_i \in A_i \text{ for } i : 1 \ldots n\},$$

for all n-tuples of Borel sets $A_1, \ldots, A_n \subset X$. Although the expression in the left hand side above is, formally, a Borel subset of $\mathbb{R}^{Nn}$, we will, with a slight abuse of notation, identify it with the set of infinite histories ω with completely undetermined positions beyond n. Let $\mathcal{F}^{x_0}$

be now defined as the smallest σ-algebra of subsets of X^{∞,x_0}, containing $\bigcup_{n=1}^\infty \mathcal{F}_n^{x_0}$. Clearly, the increasing sequence $\{\mathcal{F}_n^{x_0}\}_{n\geq 1}$ is a filtration of $\mathcal{F}^{x_0}$, and the coordinate projections $x_n(\omega) = x_n$ are $\mathcal{F}^{x_0}$ measurable (random variables) on X^{∞,x_0}.

3.2. **The stopping times τ_0 and τ.** Define the exit time from the set Ω:

$$\tau_0(\omega) = \min\{n \geq 0;\ x_n \in \Gamma\}$$

where we adopt the convention that the minimum over the empty set equals $+\infty$. This way: $\tau_0 : X^{\infty,x_0} \to \mathbb{N} \cup \{+\infty\}$ is $\mathcal{F}^{x_0}$ measurable and, in fact, it is a stopping time with respect to the filtration $\{\mathcal{F}_n^{x_0}\}$, that is:

$$\forall n \geq 0 \qquad \{\omega \in X^{\infty,x_0};\ \tau_0(\omega) \leq n\} \in \mathcal{F}_n^{x_0}. \tag{3.1}$$

Let now $\tau : X^{\infty,x_0} \to \mathbb{N} \cup \{+\infty\}$ be any stopping time (i.e. a random variable satisfying (3.1), where τ_0 is replaced by τ) such that:

$$\tau \leq \tau_0. \tag{3.2}$$

For $n \geq 1$ we define the Borel sets:

$$A_n^\tau = \{(x_0, x_1, \ldots, x_n);\ \exists \omega = (x_0, x_1, \ldots, x_n, x_{n+1}, \ldots) \in X^{\infty,x_0}, \quad \tau(\omega) \leq n\}.$$

By (3.2), it follows that $(x_0, \ldots, x_n) \in A_n^\tau$ whenever $x_n \in \Gamma$.

3.3. **The strategies σ_I, σ_{II}.** For every $n \geq 1$, let $\sigma_I^n, \sigma_{II}^n : X^{n+1} \to X$ be Borel measurable functions with the property that:

$$\sigma_I^n(x_0, x_1, \ldots, x_n),\ \sigma_{II}^n(x_0, x_1, \ldots, x_n) \in B_\epsilon(x_n) \cap X.$$

We call $\sigma_I = \{\sigma_I^n\}_{n\geq 1}$ and $\sigma_{II} = \{\sigma_{II}^n\}_{n\geq 1}$ the strategies of Players I and II, respectively.

3.4. **The probability measure $\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}$.** Fix two parameters $\alpha, \beta \geq 0$, such that: $\alpha + \beta = 1$. Given $\tau, \sigma_I, \sigma_{II}$ as above, we define now a family of probabilistic (Borel) measures on X, parametrised by the finite histories $(x_0, \ldots, x_n)$:

$$\forall n \geq 1 \quad \forall x_1, \ldots, x_n \in X \qquad \gamma_n[x_0, x_1, \ldots, x_n] =$$

$$= \begin{cases} \dfrac{\alpha}{2}\delta_{\sigma_I^n(x_0,x_1,\ldots,x_n)} + \dfrac{\alpha}{2}\delta_{\sigma_{II}^n(x_0,x_1,\ldots,x_n)} + \beta\dfrac{\mathcal{L}_N\lfloor B_\epsilon(x_n)}{|B_\epsilon(x_n)|} & \text{when } (x_0, \ldots, x_n) \notin A_n^\tau \\ \delta_{x_n} & \text{otherwise} \end{cases} \tag{3.3}$$

where δ_y denotes the Dirac delta at a given $y \in X$, while the measure multiplied by β above stands for the N-dimensional Lebesgue measure restricted to the ball $B_\epsilon(x_n)$ and normalized by the volume of this ball. Note that since $\tau \leq \tau_0$, then $\gamma_n[x_0, x_1, \ldots, x_n] = \delta_{x_n}$ whenever $x_n \in \Gamma$.

For every $n \geq 1$ we now define the probability measure $\mathbb{P}^{n,x_0}_{\tau,\sigma_I,\sigma_{II}}$ on $(X^{\infty,x_0}, \mathcal{F}_n^{x_0})$ by setting:

$$\mathbb{P}^{n,x_0}_{\tau,\sigma_I,\sigma_{II}}(A_1 \times \ldots \times A_n) = \int_{A_1} \ldots \int_{A_n} 1 \ \mathrm{d}\gamma_{n-1}[x_0, x_1, \ldots, x_{n-1}] \ldots \mathrm{d}\gamma_0[x_0] \tag{3.4}$$

for every n-tuple of Borel sets $A_1, \ldots, A_n \subset X$. Here, A_1 is interpreted as the set of possible successors x_1 of the initial position x_0, which we integrate $\mathrm{d}\gamma_0[x_0]$, while $x_n \in A_n$ is a possible successor of x_{n-1} which we integrate $\mathrm{d}\gamma_{n-1}[x_0, x_1, \ldots, x_{n-1}]$, etc. The following observation justifies the definition (3.4):

Lemma 3.1. *The family $\{\gamma_n[x_0, x_1, \ldots, x_n]\}$ in (3.3) has the following measurability property. For every $n \geq 1$ and every Borel set $A \subset X$, the function:*

$$X^{n+1} \ni (x_0, x_1, \ldots, x_n) \mapsto \gamma_n[x_0, x_1, \ldots, x_n](A) \in \mathbb{R}$$

is Borel measurable.

It is clear that the family $\{\mathbb{P}^{n,x_0}_{\tau,\sigma_I,\sigma_{II}}\}_{n\geq 1}$ is consistent (see [23]), with the transition probabilities $\gamma_n[x_0, x_1, \ldots, x_n]$. Consequently, in virtue of the Kolmogoroff's Consistency Theorem, it generates uniquely the probability measure $\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}} = \lim_{n\to\infty} \mathbb{P}^{n,x_0}_{\tau,\sigma_I,\sigma_{II}}$ on $(X^{\infty,x_0}, \mathcal{F}_n)$ so that:

$$\forall n \geq 1 \quad \forall A_1 \times \ldots \times A_n \in \mathcal{F}^{x_0}_n \qquad \mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(A_1 \times \ldots \times A_n) = \mathbb{P}^{n,x_0}_{\tau,\sigma_I,\sigma_{II}}(A_1 \times \ldots \times A_n).$$

One can easily prove the following useful observation, which follows by directly checking the definition of conditional expectation:

Lemma 3.2. *Let $v : X \to \mathbb{R}$ be a bounded Borel function. For any $n \geq 1$, the conditional expectation $\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{II}}\{v \circ x_n \mid \mathcal{F}^{x_0}_{n-1}\}$ of the random variable $v \circ x_n$ is a $\mathcal{F}^{x_0}_{n-1}$ measurable function on X^{∞,x_0} (and hence it depends only on the initial n positions in the history $\omega = (x_0, x_1, \ldots) \in X^{\infty,x_0}$), given by:*

$$\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{II}}\{v \circ x_n \mid \mathcal{F}^{x_0}_{n-1}\}(x_0, \ldots, x_{n-1}) = \int_X v \; d\gamma_{n-1}[x_0, \ldots, x_{n-1}].$$

We also have:

Lemma 3.3. *In the above setting, assume that $\beta > 0$. Then each game stops almost surely, i.e.:*

$$\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}\big(\{\tau < \infty\}\big) = 1. \tag{3.5}$$

For convenience of the reader, we give a self-contained proof of this observation in the Appendix.

3.5. **ϵ-p-superharmonious functions and game values.** Before proving Theorem 1.3, we need a lemma on almost optimal selections. This lemma was put forward in [11] and now we present its possible elementary proof.

Lemma 3.4. *Let $u : X \to \mathbb{R}$ be a bounded, Borel function. Fix $\delta, \epsilon > 0$. There exist Borel functions $\sigma_{sup}, \sigma_{inf} : \Omega \to X$ such that:*

$$\forall x \in \Omega \qquad \sigma_{sup}(x), \sigma_{inf}(x) \in B_\epsilon(x) \tag{3.6}$$

and:

$$\forall x \in \Omega \qquad u(\sigma_{sup}(x)) \geq \sup_{B_\epsilon(x)} u - \delta, \qquad u(\sigma_{inf}(x)) \leq \inf_{B_\epsilon(x)} u + \delta. \tag{3.7}$$

Proof. We will prove existence of σ_{sup}, while existence of σ_{inf} follows in a similar manner.

1. Let $u = \chi_A$ for some Borel set $A \subset X$. Without loss of generality $\delta < \frac{1}{3}$. We write $A + B_\delta(0) = \bigcup_{i=1}^\infty B_\delta(x_i)$ as the union of countably many open balls, and define:

$$\forall x \in \Omega \qquad \sigma_{sup}(x) = \begin{cases} x & \text{if } x \notin A + B_\delta(0) \\ x_i & \text{if } x \in B_\delta(x_i) \setminus \bigcup_{j=1}^{i-1} B_\delta(x_j) \end{cases}$$

Clearly, σ_{sup} above is Borel as a pointwise limit of Borel functions.

2. Let $u = \sum_{k=1}^n \alpha_k \chi_{A_k}$ be a simple function, given by disjoint Borel sets $A_k \subset X$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. Without loss of generality $\delta < \min_{k=1\ldots n-1} \frac{\alpha_{k+1}-\alpha_k}{3}$. We now write, as before: $A_k + B_\delta(0) = \bigcup_{i=1}^\infty B_\delta(x_i^k)$, and we subsequently define:

$$\forall x \in \Omega \qquad \sigma_{sup}(x) = \begin{cases} x & \text{if } x \notin (\bigcup_{k=1}^n A_k) + B_\delta(0) \\ x_i^k & \text{if } x \in B_\delta(x_i^k) \setminus \Big(\bigcup_{j=1}^{i-1} B_\delta(x_j^k) \cup \bigcup_{j>k}(A_j + B_\delta(0))\Big) \end{cases}$$

3. In the general case when u is an arbitrary bounded Borel function, consider a simple function v such that $\|u - v\|_{L^\infty(X)} \leq \frac{\delta}{3}$. By the previous construction, there exists $\sigma_{sup} : \Omega \to X$ which is a sup-selection for v, with the error $\frac{\delta}{3}$. Then we have:

$$\forall x\in\Omega \qquad u(\sigma_{sup}(x))\geq v(\sigma_{sup}(x))-\frac{\delta}{3}\geq \sup_{B_\epsilon(x)} v-\frac{2\delta}{3}\geq \sup_{B_\epsilon(x)} u-\delta,$$

and therefore σ_{sup} is also the required sup-selection for the function u. ■

Remark 3.5. If we replace the open balls $B_\epsilon(x)$ in the requirement (3.6) by the closed ones, then the Borel selection satisfying (3.7) may not exist. Take $\epsilon=1, \delta=\frac{1}{3}$ and let $u=\chi_A$ where $A\subset\mathbb{R}^3$ is a bounded Borel set with the property that $A+\bar{B}_1(0)$ is not a Borel set. The existence of A is nontrivial (see [11]) and relies on the existence of a 2d Borel set whose projection on the x_1 axis is not Borel. This result extends the famous construction of Erdos and Stone [5] of a compact (Cantor) set A and a G_δ set B such that $A+B$ is not Borel.

Proof of Theorem 1.3.

1. We first show that:

$$u_{II}\leq u \qquad \text{in } \Omega. \tag{3.8}$$

Fix $\eta>0$ and fix any strategy σ_I and a stopping time $\tau\leq\tau_0$. By Lemma 3.4, there exists a (Markovian) strategy $\sigma_{0,II}$ for Player II, such that $\sigma^n_{0,II}(x_0,\dots x_n)=\sigma^n_{0,II}(x_n)$ and that:

$$\forall n\geq 1 \quad \forall x_n\in X \qquad u(\sigma^n_{0,II}(x_n))\leq \inf_{B_\epsilon(x_n)} u+\frac{\eta}{2^{n+1}} \tag{3.9}$$

Using Lemma 3.2, definition (3.3), suboptimality in (3.9) and the equation (1.9), we compute:

$$\begin{aligned}
\forall(x_0,\dots,x_{n-1})\notin A^\tau_{n-1} \qquad & \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}\{u\circ x_n+\frac{\eta}{2^n}\mid \mathcal{F}^{x_0}_{n-1}\}(x_0,\dots,x_{n-1})\\
&=\int_X u\ \mathrm{d}\gamma_{n-1}[x_0,\dots,x_{n-1}]+\frac{\eta}{2^n}\\
&=\frac{\alpha}{2}u(\sigma_I^{n-1}(x_0,\dots,x_{n-1}))+\frac{\alpha}{2}u(\sigma^{n-1}_{0,II}(x_0,\dots,x_{n-1}))+\beta\fint_{B_\epsilon(x_{n-1})}u+\frac{\eta}{2^n}\\
&\leq \frac{\alpha}{2}\sup_{B_\epsilon(x_{n-1})}u+\frac{\alpha}{2}\inf_{B_\epsilon(x_{n-1})}u+\beta\fint_{B_\epsilon(x_{n-1})}u+\frac{\eta}{2^n}(1+\frac{\alpha}{2})\\
&\leq u(x_{n-1})+\frac{\eta}{2^{n-1}}=\Big(u\circ x_{n-1}+\frac{\eta}{2^{n-1}}\Big)(x_0,\dots,x_{n-1}).
\end{aligned}$$

On the other hand, when $(x_0,\dots,x_{n-1})\in A^\tau_{n-1}$, then $\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}\{u\circ x_n+\frac{\eta}{2^n}\mid\mathcal{F}^{x_0}_{n-1}\}(x_0,\dots,x_{n-1})=u(x_{n-1})+\frac{\eta}{2^n}$ directly from Lemma 3.2 and by (3.3). We therefore obtain that the sequence of random variables $\{u\circ x_n+\frac{\eta}{2^n}\}_{n\geq 0}$ is a supermartingale with respect to the filtration $\{\mathcal{F}^{x_0}_n\}$. It follows that:

$$\begin{aligned}
u_{II}(x_0)&\leq \sup_{\tau,\sigma_I}\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[G\circ x_\tau+\frac{\eta}{2^\tau}]\leq \sup_{\tau,\sigma_I}\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[u\circ x_\tau+\frac{\eta}{2^\tau}]\\
&\leq \sup_{\tau,\sigma_I}\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[u\circ x_0+\frac{\eta}{2^0}]=u(x_0)+\eta,
\end{aligned}$$

where we used the definition of u_{II}, the fact that $G\leq u$, and the Doob's optional stopping theorem in view of the supermartingale property and the uniform boundedness of the random variables $\{u\circ x_{\tau\wedge n}+\frac{\eta}{2^{\tau\wedge n}}\}_{n\geq 0}$. Since $\eta>0$ was arbitrary, (3.8) follows.

2. We now prove that:

$$u\leq u_I \qquad \text{in } \Omega. \tag{3.10}$$

Together with (3.8) and in view of the direct observation from (1.11) that $u_I \leq u_{II}$, (3.10) will imply Theorem 1.3.

Fix $\eta > 0$ and fix any strategy σ_{II}. By Lemma 3.4, there exists a strategy $\sigma_{0,I}$ for Player I, such that $\sigma_{0,I}^n(x_0, \dots x_n) = \sigma_{0,I}^n(x_n)$ and that:

$$\forall n \geq 1 \quad \forall x_n \in X \qquad u(\sigma_{0,I}^n(x_n)) \geq \sup_{B_\epsilon(x_n)} u - \frac{\eta}{2^{n+1}}. \tag{3.11}$$

Define the stopping time $\bar\tau(\omega) = \inf\{n \geq 0;\ u(x_n) = \Psi(x_n)\}$, with the convention that inf over an empty set is $+\infty$. Clearly:

$$(x_0, \dots x_n) \notin A_n^{\bar\tau} \qquad \text{iff} \qquad \forall k = 0 \dots n \quad u(x_k) > \Psi(x_k). \tag{3.12}$$

As before, by Lemma 3.2, the definition (3.3), and the suboptimality in (3.11), we obtain:

$$\begin{aligned}
\forall (x_0, \dots, x_{n-1}) \notin A_{n-1}^{\bar\tau} \qquad & \mathbb{E}^{x_0}_{\tau, \bar\sigma_{0,I}, \sigma_{II}}\{u \circ x_n - \frac{\eta}{2^n} \mid \mathcal{F}^{x_0}_{n-1}\}(x_0, \dots, x_{n-1}) \\
&= \int_X u \ \mathrm{d}\gamma_{n-1}[x_0, \dots, x_{n-1}] - \frac{\eta}{2^n} \\
&= \frac{\alpha}{2} u(\sigma_{0,I}^{n-1}(x_0, \dots, x_{n-1})) + \frac{\alpha}{2} u(\sigma_{II}^{n-1}(x_0, \dots, x_{n-1})) + \beta \fint_{B_\epsilon(x_{n-1})} u - \frac{\eta}{2^n} \\
&\geq \frac{\alpha}{2} \sup_{B_\epsilon(x_{n-1})} u + \frac{\alpha}{2} \inf_{B_\epsilon(x_{n-1})} u + \beta \fint_{B_\epsilon(x_{n-1})} u - \frac{\eta}{2^n}(1 + \frac{\alpha}{2}) \\
&= u(x_{n-1}) - \frac{\eta}{2^n}(1 + \frac{\alpha}{2}) \geq \Big(u \circ x_{n-1} - \frac{\eta}{2^{n-1}}\Big)(x_0, \dots, x_{n-1}),
\end{aligned}$$

where the last equality above follows from (1.9) because of (3.12). For $(x_0, \dots, x_{n-1}) \in A_{n-1}^{\bar\tau}$, we also get, as before: $\mathbb{E}^{x_0}_{\tau, \sigma_{0,I}, \sigma_{II}}\{u \circ x_n - \frac{\eta}{2^n} \mid \mathcal{F}^{x_0}_{n-1}\}(x_0, \dots, x_{n-1}) = u(x_{n-1}) - \frac{\eta}{2^n}$. We now conclude that $\{u \circ x_n - \frac{\eta}{2^n}\}_{n \geq 0}$ is a submartingale with respect to the filtration $\{\mathcal{F}_n^{x_0}\}$, and therefore:

$$\begin{aligned}
u_I(x_0) &\geq \inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\bar\tau, \sigma_{0,I}, \sigma_{II}}[G \circ x_{\bar\tau} - \frac{\eta}{2^{\bar\tau}}] = \inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\bar\tau, \sigma_{0,I}, \sigma_{II}}[u \circ x_{\bar\tau} - \frac{\eta}{2^{\bar\tau}}] \\
&\geq \inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\bar\tau, \sigma_{0,I}, \sigma_{II}}[u \circ x_0 - \frac{\eta}{2^0}] = u(x_0) - \eta,
\end{aligned}$$

where we used the definition of u_I, the fact that $G(x_{\bar\tau}) = u(x_{\bar\tau})$ derived from the definition of $\bar\tau$, and the Doob's optional stopping theorem used to the two stopping times: $\bar\tau$ and 0. Since $\eta > 0$ was arbitrary, we conclude (3.10). ■

4. The main convergence theorem: a proof of Theorem 1.2

First, we recall the definition of viscosity solutions.

Definition 4.1. *We say that a continuous function $u : \bar\Omega \to \mathbb{R}$ is a viscosity solution of the obstacle problem (1.8) if and only if: $u = F$ on $\partial\Omega$ together with $u \geq \Psi$ in Ω, and:*

(i) *for every $x_0 \in \Omega$ and every $\phi \in \mathcal{C}^2(\Omega)$ such that:*

$$\phi(x_0) = u(x_0), \quad \phi < u \ \text{ in } \ \bar\Omega \setminus \{x_0\}, \quad \nabla\phi(x_0) \neq 0, \tag{4.1}$$

there holds: $\Delta_p \phi(x_0) \leq 0$.

(ii) *for every $x_0 \in \Omega$ such that $u(x_0) > \Psi(x_0)$ and every $\phi \in \mathcal{C}^2(\Omega)$ such that:*

$$\phi(x_0) = u(x_0), \quad \phi > u \ \text{ in } \ \bar\Omega \setminus \{x_0\}, \quad \nabla\phi(x_0) \neq 0, \tag{4.2}$$

there holds: $\Delta_p \phi(x_0) \geq 0$.

The fact that variational solutions are viscosity solutions in the sense of Definition 4.1 is due to the equivalence of the local notions of viscosity and weak solutions [7] and the continuity up to the boundary of variational solutions under the regularity hypothesis on F, Ψ, and Ω (see for example [3].) The fact that viscosity solutions are variational solutions is actually equivalent to the following folklore uniqueness result that, for the sake of completeness, we prove in the Appendix:

Lemma 4.2. *Let u and $\bar{u}$ be two viscosity solutions to (1.8) as in Definition 4.1. Then $u = \bar{u}$.*

It is also classical that the unique solution to (1.8) is the pointwise infimum of all p-superharmonic functions that are above the obstacle (see Chapters 5 and 7 in [6]).

Our main approximation result is given in Theorem 4.3 below.

Theorem 4.3. *Let $p \in [2, \infty)$. Let $F : \partial\Omega \to \mathbb{R}$, $\Psi : \bar{\Omega} \to \mathbb{R}$ be two Lipschitz continuous functions, satisfying:*

$$\Psi \leq F \qquad \text{on } \partial\Omega. \tag{4.3}$$

Without loss of generality, we may assume that F, Ψ above are defined on $\bar{\Gamma}$ and X, respectively, and that (4.3) still holds on Γ. Let $u_\epsilon : \Omega \cup \Gamma \to \mathbb{R}$ be the unique ϵ-p-superharmonious function solving (1.9) with $\alpha = \frac{p-2}{p+N}$ and $\beta = \frac{2+N}{p+N}$.

Then u_ϵ converge as $\epsilon \to 0$, uniformly in $\bar{\Omega}$, to a continuous function u which is the unique viscosity solution to the obstacle problem (1.8).

Proof. **1.** We first prove the uniform convergence of u_ϵ, as $\epsilon \to 0$, in $\bar{\Omega}$. This is achieved by verifying the assumptions of the following version of the Ascoli-Arzelá theorem, valid for equibounded (possibly discontinuous) functions with "uniformly vanishing oscillation":

Lemma 4.4. [14] *Let $u_\epsilon : \bar{\Omega} \to \mathbb{R}$ be a set of functions such that:*
(i) $\exists C > 0 \quad \forall \epsilon > 0 \qquad \|u_\epsilon\|_{L^\infty(\bar{\Omega})} \leq C$,
(ii) $\forall \eta > 0 \quad \exists r_0, \epsilon_0 > 0 \quad \forall \epsilon < \epsilon_0 \quad \forall x_0, y_0 \in \bar{\Omega} \qquad |x_0 - y_0| < r_0 \implies |u_\epsilon(x_0) - u_\epsilon(y_0)| < \eta$
Then, a subsequence of u_ϵ converges uniformly in $\bar{\Omega}$, to a continuous function u.

Clearly, solutions u_ϵ to (1.9) as in the statement of Theorem 4.3 are uniformly bounded, by the boundedness of F. The crucial step in the proof of condition (ii) above is achieved by estimating the oscillation of u_ϵ close to the boundary.

Lemma 4.5. *Under the assumptions of Theorem 4.3, let $u_\epsilon : X \to \mathbb{R}$ be the ϵ-p-superharmonious solution to (1.9). Then, for every $\eta > 0$ there exist $r_0, \epsilon_0 > 0$ such that we have:*

$$\forall \epsilon < \epsilon_0 \quad \forall y_0 \in \partial\Omega \quad \forall x_0 \in \bar{\Omega} \quad |x_0 - y_0| < r_0 \implies |u_\epsilon(x_0) - u_\epsilon(y_0)| < \eta. \tag{4.4}$$

We postpone the proof of Lemma 4.5 to Section §5. We now have:

Corollary 4.6. *Under the assumptions of Theorem 4.3, let $\{u_\epsilon\}$ be the sequence of ϵ-p-superharmonious solutions to (1.9). Then $\{u_\epsilon\}$ satisfies condition (ii) in Lemma 4.4.*

Proof. Fix $\eta > 0$ and let r_0, ϵ_0 be as in Lemma 4.5 so that (4.4) holds with $\eta/3$ instead of η. Since Ψ is Lipschitz, we may without loss of generality also assume that:

$$\forall x, y \in X \quad |x - y| < r_0 \implies |\Psi(x) - \Psi(y)| < \eta. \tag{4.5}$$

Note that, consequently, we have:

$$\forall \epsilon < \epsilon_0 \quad \forall x_0, y_0 \in \tilde{\Gamma}_{r_0/3} \quad |x_0 - y_0| < r_0/3 \implies |u_\epsilon(x_0) - u_\epsilon(y_0)| \leq \eta, \tag{4.6}$$

where for any $\delta > 0$ we denote:

$$\tilde{\Gamma}_\delta = \{x \in \bar{\Omega};\ \operatorname{dist}(x, \partial\Omega) \leq \delta\}.$$

In particular, the same implication as in (4.6) holds for $x_0 \in \tilde{\Gamma}_{r_0/6}$ and $y_0 \in \bar{\Omega}$ when $|x_0 - y_0| < r_0/6$.

Let now $x_0, y_0 \in \Omega \setminus \tilde{\Gamma}_{r_0/6}$ and assume that $|x_0 - y_0| < r_0/6$. Define the bounded Borel function $\tilde{F} : \tilde{\Gamma}_{r_0/6} \to \mathbb{R}$ and the Lipschitz obstacle $\tilde{\Psi} : \mathbb{R}^N \to \mathbb{R}$ by:

$$\tilde{F}(z) = u_\epsilon(z - (x_0 - y_0)) + \eta, \qquad \tilde{\Psi}(z) = \Psi(z - (x_0 - y_0)) + \eta.$$

Clearly: $\tilde{F} \geq \tilde{\Psi}$ in $\tilde{\Gamma}_{r_0/6}$, hence by Theorem 1.1 there exists a solution $\tilde{u}_\epsilon : \Omega \to \mathbb{R}$ to (1.9) subject to the boundary data $\tilde{F}$ on $\tilde{\Gamma}_{r_0/6}$, and to the obstacle constraint $\tilde{\Psi}$. Note that by the uniqueness of such solution, there must be:

$$\tilde{u}_\epsilon(z) = u_\epsilon(z - (x_0 - y_0)) + \eta$$

On the other hand: $\tilde{F} \geq u_\epsilon$ in $\tilde{\Gamma}_{r_0/6}$ by (4.6), and also: $\tilde{\Psi} \geq \Psi$ in Ω by (4.5). Corollary 2.1 now implies that $\tilde{u}_\epsilon \geq u_\epsilon$ in $\bar{\Omega}$ and we get:

$$u_\epsilon(x_0) - u_\epsilon(y_0) \leq \tilde{u}_\epsilon(x_0) - u_\epsilon(y_0) = u_\epsilon(y_0) + \eta - u_\epsilon(y_0) = \eta.$$

Exchanging x_0 with y_0, the same argument yields $|u_\epsilon(x_0) - u_\epsilon(y_0)| < \eta$. ∎

2. We now prove that the uniform limit of u_ϵ is a viscosity solution to the obstacle problem (1.8). Clearly, $u = F$ on $\partial\Omega$ and $u \geq \Psi$ in Ω because each ϵ-p-superharmonious function u_ϵ has the same properties. We show that (i) in Definition 4.1 holds.

Let ϕ be a test function as in (4.1). Since x_0 is the minimum of the continuous function $u - \phi$, one can find a sequence of points x_ϵ converging to x_0 as $\epsilon \to 0$, and such that:

$$u_\epsilon(x_\epsilon) - \phi(x_\epsilon) \leq \inf_{\bar{\Omega}} (u_\epsilon - \phi) + \epsilon^3. \tag{4.7}$$

To prove this statement, for every $j \geq 1$ let $a_j = \min_{\bar{\Omega} \setminus B_{1/j}(x_0)} (u - \phi) > 0$ and let $\epsilon_j > 0$ be such that:

$$\forall \epsilon \leq \epsilon_j \qquad \|u_\epsilon - u\|_{L^\infty(\Omega)} \leq \frac{1}{2} a_j.$$

Without loss of generality $\{\epsilon_j\}$ is decreasing to 0. Now, for $\epsilon \in (\epsilon_{j+1}, \epsilon_j]$ let $x_\epsilon \in B_{1/j}(x_0)$ satisfy:

$$u_\epsilon(x_\epsilon) - \phi(x_\epsilon) \leq \inf_{B_{1/j}(x_0)} (u_\epsilon - \phi) + \epsilon^3.$$

We finally conclude (4.7) by noting that also for every $x \in \bar{\Omega} \setminus B_{1/j}(x_0)$ there holds:

$$\begin{aligned} u_\epsilon(x) - \phi(x) &\geq u(x) - \phi(x) - \|u_\epsilon - u\|_{L^\infty(\Omega)} \geq a_j - \frac{1}{2} a_j \geq \|u_\epsilon - u\|_{L^\infty(\Omega)} \\ &\geq u_\epsilon(x_0) - u(x_0) = u_\epsilon(x_0) - \phi(x_0) \geq u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \epsilon^3. \end{aligned}$$

By (4.7) it follows that for all $x \in \Omega$ we have: $u_\epsilon(x) \geq u_\epsilon(x_\epsilon) - \phi(x_\epsilon) + \phi(x) - \epsilon^3$ and hence:

$$\begin{aligned} u_\epsilon(x_\epsilon) &\geq \frac{\alpha}{2} \sup_{B_\epsilon(x_\epsilon)} u_\epsilon + \frac{\alpha}{2} \sup_{B_\epsilon(x_\epsilon)} u_\epsilon + \beta \fint_{B_\epsilon(x_\epsilon)} u_\epsilon \\ &\geq \big(u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \epsilon^3\big) + \Big(\frac{\alpha}{2} \sup_{B_\epsilon(x_\epsilon)} \phi + \frac{\alpha}{2} \sup_{B_\epsilon(x_\epsilon)} \phi + \beta \fint_{B_\epsilon(x_\epsilon)} \phi\Big), \end{aligned} \tag{4.8}$$

which further implies, for $\bar{x}_\epsilon \in \mathrm{argmin}_{\bar{B}_\epsilon(x_\epsilon)} \phi$:

$$\begin{aligned} \epsilon^3 &\geq \Big(\frac{\alpha}{2} \sup_{B_\epsilon(x_\epsilon)} \phi + \frac{\alpha}{2} \sup_{B_\epsilon(x_\epsilon)} \phi + \beta \fint_{B_\epsilon(x_\epsilon)} \phi\Big) - \phi(x_\epsilon) \\ &\geq \frac{\beta \epsilon^2}{2(N+2)} \Big((p-2)\Big\langle \nabla^2 \phi(x_\epsilon) \frac{(\bar{x}_\epsilon - x_\epsilon)}{\epsilon}, \frac{\bar{x}_\epsilon - x_\epsilon}{\epsilon}\Big\rangle + \Delta\phi(x_\epsilon)\Big) + o(\epsilon^2). \end{aligned} \tag{4.9}$$

For completeness, we recall now [13] the proof of the second inequality in (4.9). Taylor expand the regular function ϕ at x_ϵ, to get:

$$\min_{\bar B_\epsilon(x_\epsilon)} \phi = \phi(\bar x_\epsilon) = \phi(x_\epsilon) + \langle \nabla\phi(x_\epsilon), \bar x_\epsilon - x_\epsilon\rangle + \frac{1}{2}\Big\langle \nabla^2\phi(x_\epsilon)(\bar x_\epsilon - x_\epsilon), (\bar x_\epsilon - x_\epsilon)\Big\rangle + o(\epsilon^2).$$

On the other hand, in a similar manner:

$$\max_{\bar B_\epsilon(x_\epsilon)} \phi \geq \phi(x_\epsilon + (x_\epsilon - \bar x_\epsilon)) = \phi(x_\epsilon) - \langle \nabla\phi(x_\epsilon), \bar x_\epsilon - x_\epsilon\rangle + \frac{1}{2}\Big\langle \nabla^2\phi(x_\epsilon)(\bar x_\epsilon - x_\epsilon), (\bar x_\epsilon - x_\epsilon)\Big\rangle + o(\epsilon^2),$$

and again:

$$\fint_{B_\epsilon(x_\epsilon)} \phi = \phi(\bar x_\epsilon) + \frac{\epsilon^2}{2(N+2)}\Delta\phi(x_\epsilon) + o(\epsilon^2).$$

Consequently, we obtain:

$$\begin{aligned}\Big(\frac{\alpha}{2}\max_{\bar B_\epsilon(x_\epsilon)} \phi + \frac{\alpha}{2}\min_{\bar B_\epsilon(x_\epsilon)} \phi + \beta \fint_{B_\epsilon(x_\epsilon)} \phi\Big) - \phi(x_\epsilon) \\ \geq \frac{\alpha}{2}\Big\langle \nabla^2\phi(x_\epsilon)(\bar x_\epsilon - x_\epsilon), (\bar x_\epsilon - x_\epsilon)\Big\rangle + \frac{\beta\epsilon^2}{2(N+2)}\Delta\phi(x_\epsilon) + o(\epsilon^2),\end{aligned}$$

which yields (4.9), because $(\frac{\alpha}{2})/(\frac{\beta\epsilon^2}{2(N+2)}) = \frac{p-2}{\epsilon^2}$.

3. After dividing by ϵ^2, (4.9) becomes the following asymptotic inequality, valid for $\epsilon \to 0$:

$$\limsup_{\epsilon\to 0} \Big((p-2)\Big\langle \nabla^2\phi(x_\epsilon)\frac{(\bar x_\epsilon - x_\epsilon)}{\epsilon}, \frac{\bar x_\epsilon - x_\epsilon}{\epsilon}\Big\rangle + \Delta\phi(x_\epsilon)\Big) \leq 0. \tag{4.10}$$

Note now that:

$$\lim_{\epsilon\to 0} \frac{(\bar x_\epsilon - x_\epsilon)}{\epsilon} = -\frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|}. \tag{4.11}$$

This follows by a simple blow-up argument. Indeed, the maps $\phi_\epsilon(z) = \frac{1}{\epsilon}\big(\phi(x_\epsilon + \epsilon z) - \phi(z_\epsilon)\big)$ converge uniformly on $\bar B_1(0)$ to the linear map $\langle \nabla\phi(x_0), z\rangle$. Hence the limit of any converging subsequence of their minimizers: $\frac{1}{\epsilon}(\bar x_\epsilon - x_\epsilon) \in \mathrm{argmin}_{\bar B_1(0)}\phi_\epsilon$ must be a minimizer of the limiting function $\langle \nabla\phi(x_0), z\rangle$. This minimizer is unique and equals: $-\nabla\phi(x_0)/|\nabla\phi(x_0)|$, proving (4.11).

In conclusion, (4.11) and (4.10) imply that:

$$\frac{1}{|\nabla\phi|^{p-2}}\Delta_p\phi(x_0) = (p-2)\Big\langle \nabla^2\phi(x_0)\frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|}, \frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|}\Big\rangle + \Delta\phi(x_0) \leq 0,$$

which yields the validity of condition (i) in Definition 4.1, in view of (1.12).

4. To prove condition (ii) in Definition 4.1, let ϕ be as in (4.2). One can follow the argument as in steps 2. and 3. above, taking x_ϵ to be the approximate maximizers of $u_\epsilon - \phi$. The first inequality in (4.8) is then replaced by equality because $u(x_0) > \Psi(x_0)$, and we consequently obtain:

$$u_\epsilon(x_\epsilon) \leq \big(u_\epsilon(x_\epsilon) - \phi(x_\epsilon) + \epsilon^3\big) + \Big(\frac{\alpha}{2}\sup_{B_\epsilon(x_\epsilon)}\phi + \frac{\alpha}{2}\sup_{B_\epsilon(x_\epsilon)}\phi + \beta\fint_{B_\epsilon(x_\epsilon)}\phi\Big),$$

while the counterpart of (4.9), written for $\bar x_\epsilon \in \mathrm{argmax}_{\bar B_\epsilon(x_\epsilon)}\phi$ is:

$$-\epsilon^3 \leq \frac{\beta\epsilon^2}{2(N+2)}\Big((p-2)\Big\langle \nabla^2\phi(x_\epsilon)\frac{(\bar x_\epsilon - x_\epsilon)}{\epsilon}, \frac{\bar x_\epsilon - x_\epsilon}{\epsilon}\Big\rangle + \Delta\phi(x_\epsilon)\Big) + o(\epsilon^2).$$

Similarly to step 3. above, we conclude: $\Delta_p\phi(x_0) \geq 0$. The proof of Theorem 4.3 is complete. ■

5. Estimates close to the boundary: a proof of Lemma 4.5

In this Section, by C we denote constants that depend only on the general setup of the problem, i.e. on N, Ω, p, α and β, but not on u, x_0, F or Ψ. By C_F, C_Ψ or $C_{F,\Psi}$ we denote constants depending additionally on F, Ψ, or on both F and Ψ.

Let $x_0 \in \Omega$ and $y_0 \in \partial\Omega$. Assume that we have fixed a particular strategy $\sigma_{0,II}$ of Player II. Then, by (1.11):

$$u_\epsilon(x_0) - u_\epsilon(y_0) \leq \sup_{\tau,\sigma_I} \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[G\circ x_\tau - F(y_0)].$$

Note that for every $x \in X$:

$$G(x) - F(y_0) \leq \chi_\Gamma(x)\big(F(x) - F(y_0)\big) + \chi_\Omega(x)\big(\Psi(x) - \Psi(y_0)\big) \leq C_{F,\Psi}|x - y_0|,$$

thus:

$$u_\epsilon(x_0) - u_\epsilon(y_0) \leq C_{F,\Psi} \sup_{\tau,\sigma_I} \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[|x_\tau - y_0|]. \tag{5.1}$$

On the other hand, for a fixed strategy $\sigma_{0,I}$, again in view of (1.11) it follows that:

$$\begin{aligned} u_\epsilon(x_0) - u_\epsilon(y_0) &\geq \inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\tau_0,\sigma_{0,I},\sigma_{II}}[G\circ x_{\tau_0} - F(y_0)] = \inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\tau_0,\sigma_{0,I},\sigma_{II}}[F\circ x_{\tau_0} - F(y_0)] \\ &\geq -C_F \sup_{\sigma_{II}} \mathbb{E}^{x_0}_{\tau_0,\sigma_{0,I},\sigma_{II}}[|x_\tau - y_0|]. \end{aligned} \tag{5.2}$$

We will now prove that, with $\sigma_{0,I}$ and $\sigma_{0,II}$ chosen appropriately, one has:

$$\begin{aligned} &\forall 0<\delta\ll 1 \quad \forall \epsilon < \min\Big(\frac{\beta}{2C_\delta}, \frac{\delta}{3}\Big) \\ &\sup_{\tau,\sigma_I} \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[|x_\tau - y_0|] + \sup_{\tau,\sigma_{II}} \mathbb{E}^{x_0}_{\tau,\sigma_{0,I},\sigma_{II}}[|x_\tau - y_0|] \leq C\delta + C_\delta(|x_0 - y_0| + \epsilon), \end{aligned} \tag{5.3}$$

where the supremum is taken over all admissible stopping times $\tau \leq \tau_0$. The constant C_δ depends only on the associated parameter δ in (5.3). Clearly, (5.3) with (5.1) and (5.2) will imply (4.4).

Remark 5.1. Denote by u^0_ϵ the ϵ-p-superharmonious function subject to the same boundary condition F as u_ϵ on Γ, but in the absence of any obstacle. It satisfies:

$$u^0_\epsilon(x_0) = \sup_{\sigma_I}\inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\tau_0,\sigma_I,\sigma_{II}}[F\circ x_{\tau_0}] = \inf_{\sigma_{II}}\sup_{\sigma_I} \mathbb{E}^{x_0}_{\tau_0,\sigma_I,\sigma_{II}}[F\circ x_{\tau_0}].$$

Equivalently, u^0_ϵ solves (1.9) with $\Psi = \text{const} < \min_\Gamma F$. It is clear that:

$$u_\epsilon \geq u^0_\epsilon \quad \text{ in } \bar\Omega. \tag{5.4}$$

The following estimate has been proven in [14]:

$$\forall y_0\in\partial\Omega,\ x_0\in\Omega \quad \forall 0<\delta\ll 1 \quad |u^0_\epsilon(x_0) - u^0_\epsilon(y_0)| \leq C_F\delta + C_\delta(|x_0 - y_0| + \epsilon). \tag{5.5}$$

Note that the lower bound follows directly by (5.4) and (5.5):

$$u_\epsilon(x_0) - u_\epsilon(y_0) \geq u^0_\epsilon(x_0) - u^0_\epsilon(y_0) \geq -\Big(C_F\delta + C_\delta(|x_0 - y_0| + \epsilon)\Big).$$

Also, the upper bound is straightforward in case when x_0 belongs to the contact set, i.e. when: $u_\epsilon(x_0) = \Psi(x_0)$, because then in view of (4.3):

$$u_\epsilon(x_0) - u_\epsilon(y_0) = \Psi(x_0) - F(y_0) \leq \Psi(x_0) - \Psi(y_0) \leq C_\Psi|x_0 - y_0|.$$

It remains hence to prove a similar bound for the case $x_0 \in \Omega\setminus A_\epsilon$. We will in fact reprove the inequality (5.5), in a slightly more general setting of the obstacle ϵ-p-superharmonious function u_ϵ. The scheme of proof of (5.3) below follows [14] but we fill in all the details. ■

Proof of Lemma 4.5.

1. Let $\delta > 0$ and $z_0 \in \mathbb{R}^N \setminus \Omega$ satisfy: $B_\delta(z_0) \cap \bar{\Omega} = \{y_0\}$. Define strategy $\sigma_{0,II}$ for Player II:

$$\sigma_{0,II}^n(x_0, \dots x_n) = \sigma_{0,II}^n(x_n) = \left\{ \begin{array}{ll} x_n + (\epsilon - \epsilon^3)\frac{z_0 - x_n}{|z_0 - x_n|} & \text{if } x_n \in \Omega \\ x_n & \text{if } x_n \in \Gamma. \end{array} \right. \tag{5.6}$$

Let σ_I be any strategy for Player I and let $\tau \leq \tau_0$ be any admissible stopping time.

Firstly, notice that for all $\epsilon < \delta/3$ we have:

$$\forall x \in \Omega \qquad \fint_{B_\epsilon(x)} |w - z_0| \, \mathrm{d}w \leq |x - z_0| + C_\delta \epsilon^2. \tag{5.7}$$

This is because the function $f(w) = |w - z_0|$ is smooth in the domain $\Omega + B_{\delta/2}(0)$ and hence by taking Taylor's expansion and averaging, we get: $\fint_{B_\epsilon(x)} f = u(x) + \frac{\epsilon^2}{2(N+2)} \Delta f(x) + o(\epsilon^2)$.

Take $C = C_\delta + 1$. By Lemma 3.2, the definition (5.6), and (5.7) it follows that:

$$\begin{aligned} &\forall (x_0, \dots x_{n-1}) \notin A_{n-1}^\tau \qquad \mathbb{E}_{\tau, \sigma_I, \sigma_{0,II}}^{x_0} \left\{ |x_n - z_0| - C\epsilon^2 n | \mathcal{F}_{n-1}^{x_0} \right\} (x_0, \dots x_{n-1}) \\ &\qquad \leq \frac{\alpha}{2} |\sigma_I^{n-1}(x_0, \dots x_{n-1}) - z_0| + \frac{\alpha}{2} |\sigma_{0,II}^{n-1}(x_{n-1}) - z_0| + \beta \fint_{B_\epsilon(x_{n-1})} |w - z_0| \, \mathrm{d}w - C\epsilon^2 n \\ &\qquad \leq \frac{\alpha}{2}(|x_{n-1} - z_0| + \epsilon) + \frac{\alpha}{2}(|x_{n-1} - z_0| - (\epsilon - \epsilon^3)) + \beta(|x_{n-1} - z_0| + C_\delta \epsilon^2) - C\epsilon^2 n \\ &\qquad \leq |x_{n-1} - z_0| - C\epsilon^2(n-1), \end{aligned}$$

while for $(x_0, \dots x_{n-1}) \in A_{n-1}^\tau$ one has: $\mathbb{E}_{\tau, \sigma_I, \sigma_{0,II}}^{x_0} \left\{ |x_n - z_0| - C\epsilon^2 n | \mathcal{F}_{n-1}^{x_0} \right\} (x_0, \dots x_{n-1}) = |x_{n-1} - z_0| - C\epsilon^2 n \leq |x_{n-1} - z_0| - C\epsilon^2(n-1)$. In any case, we see that $\{|x_n - z_0| - C\epsilon^2 n\}_{n \geq 0}$ is a supermartingale with respect to the filtration $\{\mathcal{F}_n^{x_0}\}$. Applying Doob's theorem to the bounded stopping times $\tau \wedge n$ and 0, we obtain:

$$\mathbb{E}_{\tau, \sigma_I, \sigma_{0,II}}^{x_0}[|x_{\tau \wedge n} - z_0|] - C\epsilon^2 \mathbb{E}_{\tau, \sigma_I, \sigma_{0,II}}^{x_0}[\tau \wedge n] \leq |x_0 - z_0|.$$

Consequently, and further using the dominated and the monotone convergence theorems while passing with $n \to \infty$ and recalling (3.5), we obtain:

$$\mathbb{E}_{\tau, \sigma_I, \sigma_{0,II}}^{x_0}[|x_\tau - y_0|] \leq \mathbb{E}_{\tau, \sigma_I, \sigma_{0,II}}^{x_0}[|x_\tau - z_0|] + \delta \leq |x_0 - y_0| + 2\delta + C_\delta \epsilon^2 \mathbb{E}_{\tau, \sigma_I, \sigma_{0,II}}^{x_0}[\tau]. \tag{5.8}$$

2. Towards estimating the expectation $\mathbb{E}_{\tau, \sigma_I, \sigma_{0,II}}^{x_0}[\tau]$ in (5.8), we first observe the following simple general result:

Lemma 5.2. *Let $Y \subset \mathbb{R}^N$ be a bounded, open set. Let $x_0 \in Y$ and let $\{\mathbb{P}^{n,x_0}\}_{n \geq 1}$, $\{\bar{\mathbb{P}}^{n,x_0}\}_{n \geq 1}$ be two consistent sequences of probability measures on Y^{∞, x_0} defined as in (3.4), where the filtration $\{\mathcal{F}_n^{x_0}\}_{n \geq 1}$ is as in subsection §3.1, and the transition probabilities are denoted, respectively: $\{\gamma_n[x_0, x_1, \dots x_n]\}$ and $\{\bar{\gamma}_n[x_0, x_1, \dots x_n]\}$. Let $A \in \mathcal{F}^{n,x_0}$ have the property:*

$$\forall \omega = (x_0, x_1, \dots) \in A \quad \forall 0 \leq k < n \qquad \gamma_k[x_0, x_1, \dots x_k] = \bar{\gamma}_k[x_0, x_1, \dots x_k]. \tag{5.9}$$

Then $\mathbb{P}^{n,x_0}(A) = \bar{\mathbb{P}}^{n,x_0}(A)$.

Proof. We prove the lemma by induction on n. For $n = 1$, the result is trivially true because $\mathbb{P}^{1,x_0} = \bar{\mathbb{P}}^{1,x_0} = \gamma_0[x_0]$.

Let now $A \subset \{x_0\} \times Y^n$ be a Borel set. Fix $\eta > 0$ and find the covering $A \subset \bigcup_{i=1}^\infty (A_1^i \times A_2^i)$ where each $A_1^i \subset \{x_0\} \times Y^{n-1}$ and $A_2^i \subset Y$ is a Borel set, such that the rectangles $\{A_1^i \times A_2^i\}$ are pairwise disjoint, and such that:

$$0 \leq \Big(\sum_{i=1}^\infty \mathbb{P}^{n,x_0}(A_1^i \times A_2^i) - \mathbb{P}^{n,x_0}(A) \Big) + \Big(\sum_{i=1}^\infty \bar{\mathbb{P}}^{n,x_0}(A_1^i \times A_2^i) - \bar{\mathbb{P}}^{n,x_0}(A) \Big) \leq \eta. \tag{5.10}$$

For each $i\geq 1$ consider the Borel set $A^i = A\cap(A_1^i\times A_2^i)$. Its projection:

$$\pi(A^i) = \{(x_0,x_1,\dots x_{n-1}):~\exists x_n\in Y\quad (x_0,\dots x_n)\in A^i\}$$

does not have to be Borel (compare Remark 3.5), but it is an analytic set [22] and hence it is measurable with respect to completions of Borel measures. Hence, there exists Borel sets $B_1^i, C_1^i\subset \{x_0\}\times Y^{n-1}$ such that:

$$B_1^i\subset\pi(A^i)\subset C_1^i \quad\text{and}\quad \mathbb{P}^{n-1,x_0}(C_1^i\setminus B_1^i) = \bar{\mathbb{P}}^{n-1,x_0}(C_1^i\setminus B_1^i) = 0.$$

We have then:

$$\begin{aligned}\mathbb{P}^{n,x_0}(B_1^i\times A_2^i) &= \int_{B_1^i}\gamma_{n-1}[x_0,\dots x_{n-1}](A_2^i)~\mathrm{d}\mathbb{P}^{n-1,x_0}\\ &= \int_{B_1^i}\bar\gamma_{n-1}[x_0,\dots x_{n-1}](A_2^i)~\mathrm{d}\bar{\mathbb{P}}^{n-1,x_0} = \bar{\mathbb{P}}^{n,x_0}(B_1^i\times A_2^i),\end{aligned}\tag{5.11}$$

where we used the induction assumption to conclude that $\mathbb{P}^{n-1,x_0}\lfloor B_1^i = \bar{\mathbb{P}}^{n-1,x_0}\lfloor B_1^i$.

Further, in view of (5.10) and the fact that:

$$\mathbb{P}^{n,x_0}((C_1^i\setminus B_1^i)\times A_2^i) = \int_{C_1^i\setminus B_1^i}\gamma_{n-1}[x_0,\dots x_{n-1}](A_2^i)~\mathrm{d}\mathbb{P}^{n-1,x_0} = 0,$$

there holds:

$$\begin{aligned}|\mathbb{P}^{n,x_0}(A) - \sum_{i=1}^\infty \mathbb{P}^{n,x_0}(B_1^i\times A_2^i)| &= |\sum_{i=1}^n \mathbb{P}^{n,x_0}(A^i) - \sum_{i=1}^\infty \mathbb{P}^{n,x_0}(C_1^i\times A_2^i)|\\ &\leq |\sum_{i=1}^n \mathbb{P}^{n,x_0}(A^i) - \sum_{i=1}^\infty \mathbb{P}^{n,x_0}(A_1^i\times A_2^i)| \leq \eta.\end{aligned}$$

In a similar manner $|\bar{\mathbb{P}}^{n,x_0}(A) - \sum_{i=1}^\infty \bar{\mathbb{P}}^{n,x_0}(B_1^i\times A_2^i)|\leq\eta$. In view of (5.11), this implies: $|\bar{\mathbb{P}}^{n,x_0}(A) - \mathbb{P}^{n,x_0}(A)|\leq 2\eta$. Since $\eta>0$ was arbitrary, the lemma follows. ∎

Consider now a new "game-board" $Y = B_R(z_0)\supset X$ with the same initial token position x_0. Let $\bar\sigma_I$ be an extension of the strategy σ_I, given by:

$$\forall (x_0,\dots x_n)\in Y^{n+1}\qquad \bar\sigma_I^n(x_0,\dots x_n) = \begin{cases}\sigma_I^n(x_0,\dots x_n) & \text{if } (x_0,\dots x_n)\in X^{n+1}\\ x_n & \text{otherwise,}\end{cases}\tag{5.12}$$

while:

$$\bar\sigma_{0,II}^n(x_0,\dots x_n) = \bar\sigma_{0,II}^n(x_n) = \begin{cases} x_n + (\epsilon-\epsilon^3)\frac{z_0-x_n}{|z_0-x_n|} & \text{if } x_n\in Y\setminus\bar B_\delta(z_0)\\ x_n & \text{otherwise.}\end{cases}$$

Let $\bar\tau_0: Y^{\infty,x_0}\to\mathbb{N}\cup\{+\infty\}$ be the exit time into the ball $\bar B_\delta(z_0)$, i.e.:

$$\bar\tau_0(\omega) = \min\{n\geq 0;~|x_n - z_0|\leq\delta\}$$

and let $\bar\tau: Y^{\infty,x_0}\to\mathbb{N}\cup\{+\infty\}$ be a stopping time extending τ, so that $\bar\tau\leq\bar\tau_0$ and $\bar\tau_{|X^{\infty,x_0}} = \tau$. Define the transition probabilities on Y by:

$$\begin{aligned}&\forall n\geq 1\quad \forall x_1,\dots,x_n\in Y\qquad \bar\gamma_n[x_0,x_1,\dots,x_n] =\\ &= \begin{cases}\frac{\alpha}{2}\delta_{\bar\sigma_I^n(x_0,,\dots,x_n)} + \frac{\alpha}{2}\delta_{\bar\sigma_{0,II}^n(x_n)} + \beta m(x_n) & \text{for } x_n\in Y\setminus\bar B_\delta(z_0)\\ \alpha\delta_{x_n} + \beta m(x_n) & \text{for } x_n\in B_\delta(z_0)\setminus\bar B_{\delta-\epsilon}(z_0)\\ \delta_{x_n} & \text{for } x_n\in\bar B_{\delta-\epsilon}(z_0),\end{cases}\end{aligned}$$

where the probability $m(x)$ is uniform in the set $B_\epsilon(x)\cap Y$ and is given by:

$$m(x) = \frac{\mathcal{L}_N\lfloor\big(B_\epsilon(x)\cap Y\big)}{|B_\epsilon(x)\cap Y|}. \tag{5.13}$$

Let now $\bar{\mathbb{P}}^{x_0}_{\bar\sigma_I,\bar\sigma_{0,II}}$ and $\bar{\mathbb{P}}^{n,x_0}_{\bar\sigma_I,\bar\sigma_{0,II}}$ be the Borel probability measures on Y^{∞,x_0} defined as in subsection §3.4. As in the proof of Lemma 3.3, observe that:

$$\bar{\mathbb{P}}^{x_0}_{\bar\sigma_I,\bar\sigma_{0,II}}\big(\{\bar\tau_0<+\infty\}\big)=1, \tag{5.14}$$

so that, in addition to (3.5) there also holds: $\mathbb{P}^{x_0}_{\bar\sigma_I,\bar\sigma_{0,II}}\big(\{\bar\tau<+\infty\}\big)=1$.

In view of Lemma 5.2 we hence observe:

$$\begin{aligned}\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[\tau] &= \sum_{n=1}^\infty n\mathbb{P}^{n,x_0}_{\tau,\sigma_I,\sigma_{0,II}}\big(\omega\in X^{x_0,\infty};\ \tau(\omega)=n\big)\\ &= \sum_{n=1}^\infty n\bar{\mathbb{P}}^{n,x_0}_{\bar\sigma_I,\bar\sigma_{0,II}}\big(\omega\in X^{x_0,\infty};\ \tau(\omega)=n\big)\\ &\leq \sum_{n=1}^\infty n\bar{\mathbb{P}}^{n,x_0}_{\bar\sigma_I,\bar\sigma_{0,II}}\big(\omega\in Y^{x_0,\infty};\ \bar\tau(\omega)=n\big)\\ &= \bar{\mathbb{E}}^{x_0}_{\bar\sigma_I,\bar\sigma_{0,II}}[\bar\tau]\leq \bar{\mathbb{E}}^{x_0}_{\bar\sigma_I,\bar\sigma_{0,II}}[\bar\tau_0].\end{aligned} \tag{5.15}$$

Indeed, if $\omega=(x_0,x_1,\ldots)\in X^{\infty,x_0}$ satisfies $\tau(\omega)=n$, then for all $k<n$ we have: $x_k\in\Omega$ and $(x_0,x_1,\ldots x_k)\notin A^\tau_k$, and hence there must be: $\gamma_k[x_0,\ldots x_k]=\bar\gamma_k[x_0,\ldots x_k]$. Consequently:

$$\mathbb{P}^{n,x_0}_{\tau,\sigma_I,\sigma_{0,II}}\big(\omega\in X^{x_0,\infty};\ \tau(\omega)=n\big)=\bar{\mathbb{P}}^{n,x_0}_{\bar\sigma_I,\bar\sigma_{0,II}}\big(\omega\in X^{x_0,\infty};\ \tau(\omega)=n\big).$$

3. We now estimate the expectation $\bar{\mathbb{E}}^{x_0}_{\bar\sigma_I,\bar\sigma_{0,II}}[\bar\tau_0]$. Let $v_0:(0,+\infty)\to\mathbb{R}$ be a smooth, increasing and concave function of the form:

$$v_0(s)=\left\{\begin{array}{ll} -as^2-bs^{2-N}+c & \text{for } N>2\\ -as^2-b\log s+c & \text{for } N=2,\end{array}\right.$$

where the positive constants a,b,c are such that the function $v(x)=v_0(|x-z_0|)$ solves the following problem:

$$\left\{\begin{array}{ll} \Delta v=-2(N+2) & \text{in } B_R(z_0)\setminus\bar B_\delta(z_0)\\ v=0 & \text{on } \partial B_\delta(z_0)\\ \frac{\partial v}{\partial\vec n}=0 & \text{on } \partial B_R(z_0).\end{array}\right.$$

As in (5.7), we obtain:

$$\forall x\in\mathbb{R}^N\setminus\bar B_{\delta-\epsilon}(z_0)\qquad \fint_{B_\epsilon(x)} v(w)\ \mathrm{d}w = v(x)-\epsilon^2. \tag{5.16}$$

On the other hand, for every $x\in B_R(z_0)\setminus\bar B_{\delta-\epsilon}(z_0)$, we have:

$$\begin{aligned}&\fint_{B_\epsilon(x)\cap B_R(z_0)} v-\fint_{B_\epsilon(x)} v\\ &=\left(\frac{1}{|B_\epsilon(x)\cap B_R(z_0)|}-\frac{1}{|B_\epsilon(x)|}\right)\int_{B_\epsilon(x)\cap B_R(z_0)} v-\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\setminus B_R(z_0)} v\\ &\leq\left(\frac{1}{|B_\epsilon(x)\cap B_R(z_0)|}-\frac{1}{|B_\epsilon(x)|}\right)v_0(R)|B_\epsilon(x)\cap B_R(z_0)|-\frac{1}{|B_\epsilon(x)|}v_0(R)|B_\epsilon(x)\setminus B_R(z_0)|=0.\end{aligned}$$

Consequently, recalling (5.13) and in view of (5.16):

$$\forall x \in Y \setminus \bar{B}_{\delta-\epsilon}(z_0) \qquad \int v \ \mathrm{d}m(x) \leq v(x) - \epsilon^2. \tag{5.17}$$

Consider the following bounded Borel functions on Y:

$$Q_n(x) = \begin{cases} v(x) + \frac{\beta}{2}n\epsilon^2 & \text{if } |x-z_0| > \delta - \epsilon \\ v(x) & \text{if } \delta - 2\epsilon < |x - z_0| \leq \delta - \epsilon \\ v_0(\delta - 2\epsilon) & \text{if } |x - z_0| \leq \delta - 2\epsilon, \end{cases}$$

and compute the conditional expectation of the random variables $Q_n \circ x_n$, which by Lemma 3.2 equals:

$$\bar{\mathbb{E}}^{y_0}_{\bar\sigma_I, \bar\sigma_{0,II}}\{Q_n \circ x_n | \mathcal{F}^{x_0}_{n-1}\}(x_0, \dots x_{n-1}) = \int_Y Q_n \ \mathrm{d}\bar\gamma_{n-1}[x_0, \dots x_{n-1}].$$

We distinguish three cases.

<u>*Case 1:* $x_{n-1} \in Y \setminus \bar{B}_\delta(z_0)$.</u> Using the fact that v_0 is increasing, denoting its Lipschitz constant on $[\delta/3, R+\delta]$ by C_δ, recalling (5.17) and observing that $|\bar\sigma_{II}(x_{n-1}) - z_0| > \delta - \epsilon$, we get:

$$\begin{aligned}\int_Y Q_n \ \mathrm{d}\bar\gamma_{n-1}[x_0, \dots x_{n-1}] &= \frac{\alpha}{2} Q_n\big(\bar\sigma_I^{n-1}(x_0, \dots x_{n-1})\big) + \frac{\alpha}{2} Q_n\big(\bar\sigma_{0,II}^{n-1}(x_{n-1})\big) + \beta \int Q_n \ \mathrm{d}m(x_{n-1}) \\ &\leq \frac{\alpha}{2} v_0\big(|x_{n-1} - z_0| + \epsilon\big) + \frac{\alpha}{2} v_0\big(|x_{n-1} - z_0| - \epsilon + \epsilon^3\big) + \beta\big(v_0(|x_{n-1} - z_0|) - \epsilon^2\big) + \frac{\delta}{2} n\epsilon^2 \\ &\leq \alpha v_0(|x_{n-1} - z_0|) + \beta v_0(|x_{n-1} - z_0|) + C_\delta \epsilon^3 - \beta\epsilon^2 + \frac{\beta}{2} n\epsilon^2 \\ &\leq v_0(|x_{n-1} - z_0|) + \frac{\beta}{2}(n-1)\epsilon^2 = Q_{n-1}(x_{n-1}),\end{aligned}$$

where the last two inequalities follow from concavity of v_0, and the fact that if only:

$$\epsilon < \min\big(\frac{\beta}{2C_\delta}, \frac{\delta}{3}\big), \tag{5.18}$$

then: $C_\delta\epsilon^3 - \beta\epsilon^2 + \frac{\beta}{2}n\epsilon^2 \leq \frac{\beta}{2}\epsilon^2(n-1)$.

<u>*Case 2:* $x_{n-1} \in B_\delta(z_0) \setminus \bar{B}_{\delta-\epsilon}(z_0)$.</u> Then:

$$\begin{aligned}\int_Y Q_n \ \mathrm{d}\bar\gamma_{n-1}[x_0, \dots x_{n-1}] &= \alpha Q_n(x_{n-1}) + \beta \int Q_n \ \mathrm{d}m(x_{n-1}) \\ &\leq \alpha v_0(|x_{n-1} - z_0|) + \beta\big(v_0(|x_{n-1} - z_0|) - \epsilon^2\big) + \frac{\delta}{2} n\epsilon^2 \\ &= v_0(|x_{n-1} - z_0|) + \frac{\beta}{2}(n-1)\epsilon^2 = Q_{n-1}(x_{n-1}).\end{aligned}$$

<u>*Case 3:* $x_{n-1} \in \bar{B}_{\delta-\epsilon}(z_0)$.</u> In this case, we directly have:

$$\int_Y Q_n \ \mathrm{d}\bar\gamma_{n-1}[x_0, \dots x_{n-1}] = Q_{n-1}(x_{n-1}).$$

Consequently, it follows that $\{Q_n \circ x_n\}_{n\geq 0}$ is a supermartingale with respect to the filtration $\{\mathcal{F}^{x_0}_n\}$, under the assumption (5.18). Applying Doob's theorem to pairs of bounded stopping times: $\bar\tau_0 \wedge n$ and 0, we obtain:

$$v_0(|x_0 - z_0|) \geq \bar{\mathbb{E}}^{x_0}_{\bar\sigma_I, \bar\sigma_{0,II}}[Q_{\bar\tau_0 \wedge n}] = \bar{\mathbb{E}}^{x_0}_{\bar\sigma_I, \bar\sigma_{0,II}}[v_0(|x_{\bar\tau_0\wedge n} - z_0|)] + \frac{\beta}{2}\epsilon^2 \bar{\mathbb{E}}^{x_0}_{\bar\sigma_I, \bar\sigma_{0,II}}[\bar\tau_0 \wedge n].$$

After passing with $n \to \infty$, in view of the monotone convergence and dominated convergence theorems, and applying 5.14 we get:

$$\frac{\beta}{2}\epsilon^2\bar{\mathbb{E}}^{x_0}_{\bar\sigma_I,\bar\sigma_{0,II}}[\bar\tau_0] \leq v_0(|x_0 - z_0|) + \bar{\mathbb{E}}^{x_0}_{\bar\sigma_I,\bar\sigma_{0,II}}[|v_0(|x_{\bar\tau_0} - z_0|)|]. \tag{5.19}$$

Since $v_0(\delta) = 0$, it follows that $v_0(|x_0 - z_0|) \leq C_\delta\big(|x_0 - z_0| - \delta\big) = C_\delta|x_0 - y_0|$. Further, whenever $\bar\tau_0(\omega) < +\infty$, we have: $|v_0(|x_{\bar\tau_0} - z_0|)| \leq C_\delta\epsilon$. Now, (5.19) together with (5.15) and (5.8) imply:

$$\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[|x_\tau - y_0|] \leq C\delta + C_\delta(|x_0 - x_0| + \epsilon)$$

for all ϵ sufficiently small (as in (5.18)).

4. Clearly, exchanging the roles of σ_I and σ_{II}, and defining $\sigma_{0,I}$ by means of (5.6) while setting σ_{II} to be fixed, the same proof as above yields:

$$\mathbb{E}^{x_0}_{\tau,\sigma_{I,0},\sigma_{II}}[|x_\tau - x_0|] \leq C\delta + C_\delta(|x_0 - y_0| + \epsilon)$$

for all ϵ sufficiently small. This gives (5.3) and ends the proof of Lemma 4.5. ■

6. Appendix: a proof of Lemma 3.3: games end almost-surely

1. Consider a new "game-board" $Y = \mathbb{R}^N$ with the same initial token position $x_0 \in \Omega$. By the same symbols σ_I and σ_{II} we denote the extensions on $\{Y^n\}_{n=0}^\infty$ of the given strategies σ_I and σ_{II}, defined as in the formula (5.12), where in order to simplify notation we suppress the overline in $\bar\sigma$. Define also the new transition probabilities:

$$\gamma_n[x_0, \dots, x_n] = \frac{\alpha}{2}\delta_{\sigma_I^n(x_0,\dots,x_n)} + \frac{\alpha}{2}\delta_{\sigma_{II}^n(x_0,\dots,x_n)} + \frac{\beta}{|B_\epsilon|}\mathcal{L}_N\lfloor B_\epsilon(x_n),$$

and let $\mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II}}$ and $\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}$ be the resulting probability measures on Y^{∞,x_0} as in subsection §3.4. By Lemma 5.2 and since $\tau \leq \tau_0$, it follows that:

$$\begin{aligned} \mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau < \infty\}) &= \sum_{n=0}^\infty \mathbb{P}^{n,x_0}_{\tau,\sigma_I,\sigma_{II}}\big(\{\omega \in X^{x_0,\infty};\ \tau(\omega) = n\}\big) \\ &= \sum_{n=0}^\infty \bar{\mathbb{P}}^{n,x_0}_{\bar\sigma_I,\bar\sigma_{II}}\big(\{\omega \in X^{x_0,\infty};\ \tau(\omega) = n\}\big) = \sum_{n=0}^\infty \bar{\mathbb{P}}^{n,x_0}_{\bar\sigma_I,\bar\sigma_{II}}\big(\{\omega \in Y^{x_0,\infty};\ \tau(\omega) = n\}\big) \\ &= \mathbb{P}^{x_0}_{\bar\sigma_I,\bar\sigma_{II}}(\{\tau < \infty\}) \geq \mathbb{P}^{x_0}_{\bar\sigma_I,\bar\sigma_{II}}(\{\tau_0 < \infty\}). \end{aligned} \tag{6.1}$$

Let now A_0 be the sector in $B_\epsilon = B_\epsilon(0)$:

$$A_0 = \Big\{x \in \mathbb{R}^N;\ |x| \in (\epsilon/2, \epsilon) \ \text{ and } \ \angle(x, e_1) \in (-\pi/8, \pi/8)\Big\}.$$

For $M \in \mathbb{N}$ sufficiently large to ensure that M consecutive shifts of the token by vectors chosen from A_0 will get the token, originally located at any point in Ω, out of Ω, define:

$$S_{x_0} = \Big\{\omega = (x_0, x_1, \dots) \in Y^{x_0,\infty};\ \exists i_0 \quad \forall i = i_0, i_0+1, \dots, i_0 + M \quad x_{i+1} - x_i \in A_0\Big\}.$$

t is clear that:

$$\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}(\{\tau_0 < \infty\}) \geq \mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}(S_{x_0}). \tag{6.2}$$

2. We now show that the probability in the right hand side of (6.2) equals 1. Recall that for a bounded $\mathcal{F}^{x_0}$-measurable function $f : Y^{x_0,\infty} \to \mathbb{R}$, its conditional expectation $\mathbb{E}^{x_0}_{\sigma_I,\sigma_{II}}\{f|\mathcal{F}^{x_0}_1\}$ is the function: $(x_0, x_1) \mapsto \mathbb{E}^{x_1}_{\sigma'_I,\sigma'_{II}}[f']$, where σ'_I, σ'_{II} are strategies on $Y^{x_0,\infty}$ given by:

$$(\sigma'_I)^n(x_1, \dots, x_{n+1}) = \sigma_I^{n+1}(x_0, x_1, \dots, x_{n+1}), \qquad (\sigma'_{II})^n(x_1, \dots, x_{n+1}) = \sigma_{II}^{n+1}(x_0, x_1, \dots, x_{n+1}),$$

while the Borel random variable $f' : Y^{x_1,\infty} \to \mathbb{R}$ is similarly set to be: $f'(x_1, x_2, \ldots) = f(x_0, x_1, x_2, \ldots)$. Consequently:

$$\begin{aligned} \mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}(S_{x_0}) &= \int_{\mathbb{R}^N} \mathbb{E}^{x_1}_{\sigma'_I,\sigma'_{II}}\big[(\chi_{S_{x_0}})'\big]\ \mathrm{d}\gamma_0[x_0] \\ &= \frac{\alpha}{2}\mathbb{E}^{\sigma^0_I(x_0)}_{\sigma'_I,\sigma'_{II}}\big[(\chi_{S_{x_0}})'\big] + \frac{\alpha}{2}\mathbb{E}^{\sigma^0_{II}(x_0)}_{\sigma'_I,\sigma'_{II}}\big[(\chi_{S_{x_0}})'\big] + \beta \fint_{B_\epsilon(x_0)} \mathbb{E}^{x_1}_{\sigma'_I,\sigma'_{II}}\big[(\chi_{S_{x_0}})'\big]\ \mathrm{d}x_1 \\ &= \frac{\alpha}{2}\mathbb{P}^{\sigma^0_I(x_0)}_{\sigma'_I,\sigma'_{II}}(S^{x_0}_{\sigma^0_I(x_0)}) + \frac{\alpha}{2}\mathbb{P}^{\sigma^0_{II}(x_0)}_{\sigma'_I,\sigma'_{II}}(S^{x_0}_{\sigma^0_{II}(x_0)}) + \frac{\beta}{|B_\epsilon|}\int_{B_\epsilon(x_0)} \mathbb{P}^{x_1}_{\sigma'_I,\sigma'_{II}}(S^{x_0}_{x_1})\ \mathrm{d}x_1, \end{aligned} \tag{6.3}$$

where each set $S^{x_0}_{x_1} = \{(x_1, x_2, \ldots) \in Y^{x_1,\infty};\ (x_0, x_1, x_2, \ldots) \in S_{x_0}\}$ clearly contains S_{x_1}. Let now:

$$q(x) = \inf_{\tilde\sigma_I,\tilde\sigma_{II}} \mathbb{P}^x_{\tilde\sigma_I,\tilde\sigma_{II}}(S_x). \tag{6.4}$$

By an easy translation invariance argument, $q(x) = q$ is actually independent of $x \in \mathbb{R}^N$. Hence, in view of (6.3) we obtain:

$$\begin{aligned} \mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}(S_{x_0}) &\geq \frac{\alpha}{2}\mathbb{P}^{\sigma^0_I(x_0)}_{\sigma'_I,\sigma'_{II}}(S_{\sigma^0_I(x_0)}) + \frac{\alpha}{2}\mathbb{P}^{\sigma^0_{II}(x_0)}_{\sigma'_I,\sigma'_{II}}(S_{\sigma^0_{II}(x_0)}) \\ &\qquad + \frac{\beta}{|B_\epsilon|}\int_{B_\epsilon(x_0)\setminus(x_0+A_0)} \mathbb{P}^{x_1}_{\sigma'_I,\sigma'_{II}}(S_{x_1})\ \mathrm{d}x_1 + \frac{\beta}{|B_\epsilon|}\int_{x_0+A_0} \mathbb{P}^{x_1}_{\sigma'_I,\sigma'_{II}}(S^{x_0}_{x_1})\ \mathrm{d}x_1 \\ &\geq \Big(\frac{\alpha}{2} + \frac{\alpha}{2} + \frac{\beta}{|B_\epsilon|}|B_\epsilon \setminus A_0|\Big)q + \frac{\beta}{|B_\epsilon|}\int_{x_0+A_0} \mathbb{P}^{x_1}_{\sigma'_I,\sigma'_{II}}(S^{x_0}_{x_1})\ \mathrm{d}x_1 \\ &= \theta q + \frac{\beta}{|B_\epsilon|}\int_{x_0+A_0} \mathbb{P}^{x_1}_{\sigma'_I,\sigma'_{II}}(S^{x_0}_{x_1})\ \mathrm{d}x_1, \end{aligned} \tag{6.5}$$

where we defined:

$$\theta = \alpha + \beta\Big(1 - \frac{|A_0|}{|B_\epsilon|}\Big).$$

3. Similarly as in the previous step, for every $x_1 \in x_0 + A_0$ there holds:

$$\begin{aligned} \mathbb{P}^{x_1}_{\sigma'_I,\sigma'_{II}}(S^{x_0}_{x_1}) = \frac{\alpha}{2}\mathbb{P}^{\sigma^1_I(x_0,x_1)}_{\sigma''_I,\sigma''_{II}}(S^{x_0,x_1}_{\sigma^1_I(x_0,x_1)}) &+ \frac{\alpha}{2}\mathbb{P}^{\sigma^1_{II}(x_0,x_1)}_{\sigma''_I,\sigma''_{II}}(S^{x_0,x_1}_{\sigma^1_{II}(x_0,x_1)}) \\ &+ \frac{\beta}{|B_\epsilon|}\int_{B_\epsilon(x_0)} \mathbb{P}^{x_2}_{\sigma''_I,\sigma''_{II}}(S^{x_0,x_1}_{x_2})\ \mathrm{d}x_2, \end{aligned}$$

where the set $S^{x_0,x_1}_{x_2} = \{(x_2, x_3, \ldots) \in Y^{x_2,\infty};\ (x_2, x_2, x_2, x_3 \ldots) \in S_{x_0}\}$ contains the set S_{x_2}. By (6.4) we see that:

$$\inf_{\tilde\sigma_I,\tilde\sigma_{II}} \mathbb{P}^{x_2}_{\tilde\sigma_I,\tilde\sigma_{II}}(S^{x_0,x_1}_{x_2}) \geq q,$$

and hence:

$$\mathbb{P}^{x_1}_{\sigma'_I,\sigma'_{II}}(S^{x_0}_{x_1}) \geq \theta q + \frac{\beta}{|B_\epsilon|}\int_{x_1+A_0} \mathbb{P}^{x_2}_{\sigma''_I,\sigma''_{II}}(S^{x_0,x_1}_{x_2})\ \mathrm{d}x_2.$$

Since $1 - \theta = \beta|A_0|/|B_\epsilon|$, the estimate in (6.5) becomes:

$$\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}(S_{x_0}) = \theta q + (1-\theta)\theta q + \Big(\frac{\beta}{|B_\epsilon|}\Big)^2 \int_{x_0+A_0}\int_{x_1+A_0} \mathbb{P}^{x_2}_{\sigma''_I,\sigma''_{II}}(S^{x_0,x_1}_{x_2})\ \mathrm{d}x_2\ \mathrm{d}x_1.$$

Iterating the same argument as above M times, we arrive at:

$$\begin{aligned} \mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}(S_{x_0}) &\geq \theta q + (1-\theta)\theta q + (1-\theta)^2\theta q + \ldots + (1-\theta)^{M-1}\theta q \\ &\qquad + \Big(\frac{\beta}{|B_\epsilon|}\Big)^M \int_{x_0+A_0}\int_{x_1+A_0}\cdots\int_{x_{M-1}+A_0} \mathbb{P}^{x_M}_{\sigma'^M_I,\sigma'^M_{II}}(S^{x_0,\ldots,x_{M-1}}_{x_M})\ \mathrm{d}x_M\ldots\ \mathrm{d}x_1. \end{aligned} \tag{6.6}$$

But each probability under the iterated integrals equals to 1, because: $S_{x_M}^{x_0,\ldots,x_{M-1}} = Y^{x_M,\infty}$ for $x_1 \in x_0 + A_0$, $x_2 \in x_1 + A_0$, $\ldots, x_M \in x_{M-1} + A_0$. Consequently, by (6.6) we get:

$$\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}(S_{x_0}) \geq \sum_{n=0}^{M-1}(1-\theta)^n\theta q + (1-\theta)^M = \big(1-(1-\theta)^M\big)q + (1-\theta)^M.$$

Infimizing over all strategies σ_I, σ_{II}, it follows that $q \geq 1$, since $\theta < 1$ because of $\beta > 0$. Further:

$$\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}(S_{x_0}) \geq q = 1.$$

This achieves (3.5) in view of (6.1) and (6.2). ■

7. Appendix: a proof of Lemma 4.2: uniqueness of viscosity solutions

1. Firstly, note that the continuous function u is a viscosity p-supersolution to (1.7). Thus, by the classical result in [7], u is p-superharmonic in Ω, and consequently (see [10]) we have $u \in W^{1,p}_{loc}(\Omega)$. In the same manner, it follows from Definition 4.1 that u is a viscosity p-subsolution on the open set $\mathcal{V} = \{x \in \Omega;\ u(x) > \Psi(x)\}$, hence u is p-subharmonic in $\mathcal{V}$.

Therefore, using the variational definitions of p-super- and p-subharmonic functions, we have that for any open, Lipschitz domain $\mathcal{U} \subset\subset \Omega$ there holds:

$$\int_{\mathcal{U}} |\nabla u|^p \leq \int_{\mathcal{U}} |\nabla(u+\phi)|^p \qquad \forall \phi \in \mathcal{C}_0^\infty(\mathcal{U}, \mathbb{R}_+), \tag{7.1}$$

$$\int_{\mathcal{U}\cap\mathcal{V}} |\nabla u|^p \leq \int_{\mathcal{U}\cap\mathcal{V}} |\nabla(u+\phi)|^p \qquad \forall \phi \in \mathcal{C}_0^\infty(\mathcal{U}\cap\mathcal{V}, \mathbb{R}_-). \tag{7.2}$$

Let now $\phi \in \mathcal{C}_0^\infty(\mathcal{U}, \mathbb{R})$ be such that $\Psi \leq u + \phi$. We write: $\phi = \phi^+ + \phi^-$ as the sum of the positive and negative parts of ϕ. Denote:

$$D^+ = \{x \in \mathcal{U};\ \phi(x) > 0\} \quad \text{and} \quad D^- = \{x \in \mathcal{U};\ \phi(x) < 0\} \subset \mathcal{V}.$$

Then we have, in view of (7.1) and (7.2):

$$\begin{aligned}\int_{\mathcal{U}} |\nabla u + \nabla\phi|^p &= \int_{D^+} |\nabla u + \nabla\phi|^p + \int_{D^-} |\nabla u + \nabla\phi|^p + \int_{\{\phi=0\}} |\nabla u|^p \\ &= \int_{D^+} |\nabla u + \nabla(\phi^+)|^p + \int_{\mathcal{U}\cap\mathcal{V}} |\nabla u + \nabla(\phi^-)|^p - \int_{(\mathcal{U}\cap\mathcal{V})\setminus D^-} |\nabla u|^p \\ &\geq \int_{D^+} |\nabla u|^p + \int_{\mathcal{U}\cap\mathcal{V}} |\nabla u|^p - \int_{(\mathcal{U}\cap\mathcal{V})\setminus D^-} |\nabla u|^p = \int_{\mathcal{U}} |\nabla u|^p.\end{aligned} \tag{7.3}$$

The above means precisely that u is a variational solution of the obstacle problem on $\mathcal{U}$, with the lower obstacle Ψ and boundary data $f = u_{|\partial\mathcal{U}}$; we denote this problem by $\mathcal{K}_{\Psi,f}(\mathcal{U})$. Existence and uniqueness of such variational solution is an easy direct consequence of the strict convexity of the functional $\int_{\mathcal{U}} |\nabla u|^p$. It is also quite classical that such solutions obey a comparison principle [10].

2. Let now u and $\bar{u}$ be as in the statement of the Lemma. Fix $\epsilon > 0$. By the uniform continuity of u, $\bar{u}$ on Ω and the fact that they coincide on $\partial\Omega$, there exists $\delta > 0$ such that:

$$|u(x) - \bar{u}(x)| \leq \epsilon \qquad \forall x \in \mathcal{O}_\delta(\partial\Omega) := \big(\partial\Omega + B(0,\delta)\big) \cap \bar{\Omega}. \tag{7.4}$$

Consider an open, Lipschitz set $\mathcal{U}$ satisfying: $\Omega \setminus \mathcal{O}_\delta(\partial\Omega) \subset\subset \mathcal{U} \subset\subset \Omega$. By the argument in Step 1, u is the variational solution of the problem set $\mathcal{K}_{\Psi,u_{|\partial\mathcal{U}}}(\mathcal{U})$, and it is also easy to observe that $\bar{u}+\epsilon$ is the variational solution of the problem $\mathcal{K}_{\Psi,\bar{u}_{|\partial\mathcal{U}}+\epsilon}(\mathcal{U})$. Since $u < \bar{u}+\epsilon$ on $\partial\mathcal{U}$ in view of (7.4), the comparison principle implies that $u \leq \bar{u}+\epsilon$ in $\bar{\mathcal{U}}$.

Reversing the same argument and taking into account (7.4), we arrive at:

$$|u(x) - \bar{u}(x)| \leq \epsilon \qquad \forall x \in \bar{\Omega}.$$

We conclude that $u = \bar{u}$ in $\bar{\Omega}$ by passing to the limit $\epsilon \to 0$ in the above bound. ■

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