A Uniqueness Condition for
Hyperbolic Systems of Conservation Laws

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Abstract. Consider the Cauchy problem for a hyperbolic $n \times n$ system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x). \tag{CP}$$

Relying on the existence of a continuous semigroup of solutions, we prove that the entropy admissible solution of (CP) is unique within the class of functions $u = u(t,x)$ which have bounded variation along a suitable family of space-like curves.

1 - Introduction.

Consider a hyperbolic system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0. \tag{1.1}$$

The following standard conditions [11, 12] will be assumed throughout. The flux function $f : \Omega \mapsto I\!R^n$ is smooth, in a neighbourhood $\Omega \subset I\!R^n$ of the origin. Let $A(u) = Df(u)$ be the Jacobian matrix of f at u and assume that $A(u)$ is strictly hyperbolic, i.e. with real distinct eigenvalues: $\lambda_1(u) < \cdots < \lambda_n(u)$. We can thus choose bases of right and left eigenvectors $r_i(u)$, $l_i(u)$, $i = 1, \ldots, n$, normalized so that

$$|r_i| \equiv 1\,, \qquad \langle l_i,\ r_j\rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1.2}$$

for every indices $i, j \in \{1, \ldots, n\}$ and all $u \in \Omega$. For each $i = 1, \ldots, n$, we assume that the i-th field is either linearly degenerate, so that

$$\nabla\lambda_i \cdot r_i(u) \doteq \lim_{h\to 0} \frac{\lambda_i\big(u + hr_i(u)\big) - \lambda_i(u)}{h} = 0 \qquad \text{for every } u \in \Omega,$$

or genuinely nonlinear, so that

$$\nabla\lambda_i \cdot r_i(u) > 0 \qquad \text{for every } u \in \Omega.$$

In this setting, it was proved in [2, 3, 6] that the system (1.1) admits a uniformly Lipschitz continuous semigroup of solutions $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$. Here $\mathcal{D} \subset \mathbf{L}^1(I\!R;\ I\!R^n)$ is a closed, positively invariant domain, such that all functions with suitably small total variation lie in $\mathcal{D}$, and all functions in $u \in \mathcal{D}$ have uniformly bounded variation. For a given initial condition

$$u(0, \cdot) = \bar{u} \in \mathcal{D}, \tag{1.3}$$

a way to establish the uniqueness of solutions to the Cauchy problem (1.1)-(1.3) is thus to prove that every entropy weak solution $u = u(t, x)$ actually coincides with the semigroup trajectory:

$$u(t, \cdot) = S_t\bar{u} \tag{1.4}$$

for all $t \geq 0$. Regularity conditions which imply the identity (1.4) were introduced in [4, 5]. These conditions provide some control on the oscillation of u in a forward neighborhood of each given point (t, x).

In the present paper we consider an alternative regularity condition, quite simple to state, and prove that it suffices to guarantee uniqueness.

(A3) (Locally Bounded Variation) For some $\delta > 0$, along every space-like curve $t = \gamma(x)$, with $\big|dt/dx\big| \leq \delta$ almost everywhere, the total variation of u is locally bounded.

In other words, we require that, whenever $t = \gamma(x)$ is a space-like curve satisfying

$$\big|\gamma(x) - \gamma(x')\big| \leq \delta |x - x'| \qquad \text{for all} \ \ x, x',$$

then the total variation of the composed map $x \mapsto u\big(\gamma(x),\ x\big)$ is bounded on bounded intervals.

For completeness, we restate below our basic assumptions on weak solutions and the Lax entropy conditions.

(A1) (Conservation Equations) The function $u = u(t,x)$ is a weak solution of the Cauchy problem (1.1), (1.3), taking values within the domain $\mathcal{D}$ of a Standard Riemann Semigroup S. More precisely, $u : [0,T] \mapsto \mathcal{D}$ is continuous w.r.t. the $\mathbf{L}^1$ distance. The initial condition (1.3) holds, together with

$$\iint \big(u\varphi_t + f(u)\varphi_x\big)\ dxdt = 0 \tag{1.5}$$

for every $\mathcal{C}^1$ function φ with compact support contained inside the open strip $]0,T[\,\times I\!R$.

(A2) (Entropy Condition) Let u have an approximate jump discontinuity at some point $(\tau,\xi) \in\,]0,T[\times I\!R$. More precisely, let there exists states $u^-, u^+ \in \Omega$ and a speed $\lambda \in I\!R$ such that, calling

$$U(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < \xi + \lambda(t-\tau), \\ u^+ & \text{if} \quad x > \xi + \lambda(t-\tau), \end{cases} \tag{1.6}$$

there holds

$$\lim_{\rho\to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \Big|u(t,x) - U(t,x)\Big|\ dxdt = 0. \tag{1.7}$$

Then, for some $i = 1, \ldots, n$, one has the entropy inequality:

$$\lambda_i(u^-) \geq \lambda \geq \lambda_i(u^+). \tag{1.8}$$

With the above assumptions, one has:

Theorem. *Assume that the system (1.1) generates a Standard Riemann Semigroup* $S : \mathcal{D} \times [0,\infty[\, \mapsto \mathcal{D}$. *Then, for every* $\bar u \in \mathcal{D}$, $T > 0$, *the Cauchy problem (1.1), (1.3) has a unique weak solution* $u : [0,T] \mapsto \mathcal{D}$ *satisfying the assumptions* **(A1)–(A3)**. *Indeed, these conditions imply (1.4) for all* $t \in [0,T]$.

A proof of the theorem will be given in Section 3, while in Section 2 we collect a number of preliminary estimates.

2 - Preliminary results.

Since $\mathcal{D} \subset \mathbf{L}^1 \cap BV$, for sake of definiteness we shall always work with right-continuous representatives, so that our functions $w \in \mathcal{D}$ will satisfy $w(x) = w(x+)$ for all $x \in I\!R$.. Moreover, given a continuous map $u : [0, T] \mapsto \mathcal{D}$, we will identify it with the corresponding function of two variables $u \in \mathbf{L}^1\big([0, T] \times I\!R;\ I\!R^n\big)$, defined in the natural way.

Lemma 1. *Let $u : [0, T] \mapsto \mathcal{D}$ satisfy* **(A1)**. *Then u is Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance.*

Lemma 2. *Let $u : [0, T] \mapsto \mathcal{D}$ satisfy* **(A1)**. *Then $u \in BV\big(]0, T[\times I\!R;\ I\!R^n\big)$. Moreover there exists a set $\mathcal{N}$ of Lebesgue measure 0, containing the endpoints of the interval $[0, T]$, such that for every $\tau \in [0, T] \setminus \mathcal{N}$ and every $\xi \in I\!R$ the following holds. Either u is approximately continuous at (τ, ξ), i.e. (1.7) holds with $U(t, x) = u(\tau, \xi-) = u(\tau, \xi+)$, or u has a non-horizontal approximate jump discontinuity at (τ, ξ), so that (1.6) and (1.7) hold. In this latter case one has the additional relations*

$$u^- = u(\tau, \xi-), \quad u^+ = u(\tau, \xi+),$$

$$\lambda \cdot [u^+ - u^-] = f(u^+) - f(u^-).$$

If u satisfies **(A2)**, *then (1.8) holds for some $i = 1, \ldots, n$.*

A proof of Lemma 1 can be found in [4]. The first statement of Lemma 2 is a corollary of Lemma 1. For the proof of the other statements see [4, 5, 8].

The next two lemmas derive some local properties of u, implied by our the assumption **(A3)**.

Lemma 3. *Let $u : [0, T] \mapsto \mathcal{D}$ satisfy* **(A3)**. *Fix $\tau \in [0, T]$ and $\varepsilon > 0$. Then the set*

$$B_{\tau,\varepsilon} = \left\{ \xi \in I\!R;\quad \limsup_{t \to \tau+,\ x \to \xi} \big| u(t, x) - u(\tau, \xi) \big| > \varepsilon \right\} \tag{2.1}$$

has no limit points.

Proof. If the conclusion fails, then there exists a monotone sequence $\{\xi_i\}$ of points in $B_{\tau,\varepsilon}$, converging to some limit point ξ_0. To fix the ideas, let the sequence be decreasing, the other case being entirely similar. For each $i \geq 1$, by the right continuity ot the function $x \mapsto u(\tau, x)$ one can find a point $w_i \in]\xi_i, \xi_{i-1}[$ such that $|u(\tau, w_i) - u(\tau, \xi_i)| \leq \varepsilon/2$. Next, let $t_i > \tau$ and $x_i \in\ ]w_{i+1}, w_i[$ satisfy the inequalities

$$|u(t_i, x_i) - u(\tau, \xi_i)| \geq \varepsilon,$$

$$|t_i - \tau| \leq \delta \cdot \max\left\{ |x_i - w_i|,\ |x_i - w_{i+1}| \right\}. \tag{2.2}$$

Define a space-like curve $t = \gamma(x)$, with $x \in [\xi_0, \xi_1]$, by setting

$$\gamma(x) = \begin{cases} \tau & \text{if} \quad x = \xi_0 \text{ or } x \geq w_1, \\ t_i - (x - x_i)\frac{t_i - \tau}{w_i - x_i} & \text{if} \quad x \in [x_i, w_i], \\ \tau + (x - w_{i+1})\frac{t_i - \tau}{x_i - w_{i+1}} & \text{if} \quad x \in [w_{i+1}, x_i]. \end{cases} \tag{2.3}$$

By (2.2), γ is Lipschitz continuous with Lipschitz constant δ. Since $|u(t_i, x_i) - u(\tau, w_i)| \geq \varepsilon/2$ for all $i \geq 1$, the total variation of the composed map $x \mapsto u\big(\gamma(x), x\big)$ on the interval $[\xi_0, \xi_1]$ is infinite. This contradicts the assumption **(A3)**, thus proving Lemma 3.

Throughout the following, we consider a fixed number $\lambda^* \geq 1/\delta$, strictly larger than the absolute values of all propagation speeds λ_i of the system (1.1).

Lemma 4. *Let $u : [0, T] \mapsto \mathcal{D}$ satisfy* **(A3)**. *Then for each $(\tau, \xi) \in \,]0, T[\times I\!R$*

$$\lim_{\substack{t \to \tau+, \ x \to \xi\pm \\ |x - \xi| > \lambda^*(t - \tau)}} u(t, x) \; = \; u(\tau, \xi\pm).$$

Proof. Suppose the conclusion of the lemma fails. To fix the ideas, assume that, for some $(\tau, \xi_0) \in \,]0, T[\times I\!R$, there exist decreasing sequences $t_j \to \tau+$ and $x_j \to \xi_0+$, such that

$$|x_j - \xi_0| \geq \lambda^* |t_j - \tau|, \qquad \big|u(t_j, x_j) - u(\tau, \xi_0)\big| \geq \varepsilon$$

for some $\varepsilon > 0$ and every index j. The case $x_j \to \xi_0-$ can be treated in the same way.

Define the sequence of points

$$w_i \doteq x_j + \frac{1}{\delta}(t_j - \tau)$$

and observe that $w_j \to \xi_0+$ as $j \to \infty$. By possibly taking a subsequence, say $\{(t_i, x_i)\}$, we can assume that the corresponding w_i satisfy

$$x_i \in \,]w_{i+1}, w_i[\,, \qquad |t_i - \tau| \leq \delta \cdot \max\big\{|x_i - w_i|, \; |x_i - w_{i+1}|\big\} \qquad \text{for all } i.$$

Now let γ be the space-like curve defined by (2.3). Since $w_i \to \xi+$, for every i large enough, we have $\big|u(\tau, w_i) - u(\tau, \xi+)\big| \leq \varepsilon/2$, hence $\big|u(t_i, x_i) - u(\tau, w_i)\big| \geq \varepsilon/2$. Therefore, the total variation of the map $x \mapsto u\big(\gamma(x), \; x\big)$ on the interval $[\xi_0, w_1]$ is infinite, in contradiction with **(A3)**.

Next, we recall some useful estimates, valid for the trajectories of a Standard Riemann Semigroup S.

Lemma 5. *Let $w : [0,T] \mapsto \mathcal{D}$ be Lipschitz continuous. Then for every interval $[a,b] \in I\!R$ there holds:*

$$\begin{aligned} &\left\| w(T) - S_T w(0) \right\|_{\mathbf{L}^1([a+\lambda^* T, b-\lambda^* T];\ I\!R^n)} \\ &\qquad = O(1) \cdot \int_0^T \left\{ \liminf_{h\to 0+} \frac{\left\| w(\tau+h) - S_h w(\tau) \right\|_{\mathbf{L}^1([a+\lambda^*(\tau+h), b-\lambda^*(\tau+h)];\ I\!R^n)}}{h} \right\} d\tau. \end{aligned} \tag{2.4}$$

Here and in the sequel, with the Landau symbol $O(1)$ we denote a quantity whose absolute value satisfies a uniform bound, depending only on the system (1.1).

Before stating the local integral estimates valid for semigroup trajectories, we need to define two local approximate solutions of (1.1). Let $w \in \mathcal{D}$ and fix a point $\xi \in I\!R$. Call $\omega = \omega(t,x)$ the unique self-similar entropy solution of the Riemann problem

$$\omega_t + f(\omega)_x = 0, \qquad\qquad \omega(0,x) = \begin{cases} w(\xi-) & \text{if} \quad x < 0, \\ w(\xi+) & \text{if} \quad x > 0. \end{cases}$$

For $t \geq 0$, let

$$U^\sharp(t,x) \doteq \begin{cases} \omega(t,\ x-\xi) & \text{if} \quad |x-\xi| \leq \lambda^* t, \\ w(x) & \text{if} \quad |x-\xi| > \lambda^* t. \end{cases}$$

Next, call $\widetilde{A} \doteq Df\big(w(\xi)\big)$ the Jacobian matrix of f computed at $w(\xi)$. For $t \geq 0$, define $U^\flat(t,x)$ to be the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$U^\flat_t + \widetilde{A} U^\flat_x = 0, \qquad\qquad U^\flat(0) = w.$$

Lemma 6. *For every function $w \in \mathcal{D}$, every $\xi \in I\!R$ and $h, \rho > 0$, with the above definitions one has*

$$\frac{1}{h} \int_{\xi-\rho+h\hat\lambda}^{\xi+\rho-h\hat\lambda} \left| \big(S_h w\big)(x) - U^\sharp(h,x) \right| dx \ = \ O(1) \cdot \text{Tot.Var.}\big\{ w;\ ]\xi-\rho,\ \xi[\ \cup\ ]\xi,\ \xi+\rho[\ \big\}, \tag{2.5}$$

$$\frac{1}{h} \int_{\xi-\rho+h\hat\lambda}^{\xi+\rho-h\hat\lambda} \left| \big(S_h w\big)(x) - U^\flat(h,x) \right| dx = O(1) \cdot \Big(\text{Tot.Var.}\big\{ w;\ ]\xi-\rho,\ \xi+\rho[\ \big\} \Big)^2. \tag{2.6}$$

For the proofs of the two above lemmas, see [1]. We conclude this section by recalling two technical results, that will be needed toward a proof of our Theorem. The proofs can be found in [4].

Lemma 7. *Let $w \in \mathbf{L}^1(\,]a,\ b[;\ I\!R^n)$ be such that for some Radon measure μ, one has*

$$\left| \int_{\zeta_1}^{\zeta_2} w(x)\, dx \right| \leq \mu\big([\zeta_1,\ \zeta_2]\big), \qquad \textit{whenever} \ \ a < \zeta_1 < \zeta_2 < b.$$

Then

$$\int_a^b \left|w(x)\right| dx \le \mu\big(]a,\ b[\big).$$

Lemma 8. [4] *Let $u : [0,T] \mapsto \mathcal{D}$ be Lipschitz continuous. At a given point (τ,ξ), let the conditions (1.6)-(1.7) hold, for some $u^-, u^+ \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. Then, for each $\tilde\lambda > 0$ one has*

$$\lim_{\rho\to 0+}\ \sup_{|h|\le\rho} \int_0^{\tilde\lambda} \left|u(\tau+h,\ \xi+\lambda h+\rho y) - u^+\right| dy = 0,$$

$$\lim_{\rho\to 0+}\ \sup_{|h|\le\rho} \int_{-\tilde\lambda}^{0} \left|u(\tau+h,\ \xi+\lambda h+\rho y) - u^-\right| dy = 0.$$

3 - Proof of the Theorem.

Let u satisfy **(A1)–(A3)**. To deduce (1.4), in view of Lemma 5 it suffices to show that for every interval $[a,b] \subset \mathbb{R}$ and a.e. $\tau \in [0,T]$ one has

$$\liminf_{h\to 0+} \frac{\big\|u(\tau+h) - S_h u(\tau)\big\|_{\mathbf{L}^1([a,b];\ \mathbb{R}^n)}}{h} = 0 \tag{3.1}$$

In fact, we will show that (3.1) is valid for every $[a,b] \in \mathbb{R}$ whenever $\tau \in [0,T] \setminus \mathcal{N}$. The proof is divided in 3 steps. The aim of the first two steps is to derive the appropriate estimates on the error

$$\big\|u(\tau+h) - S_h u(\tau)\big\|_{\mathbf{L}^1(I;\ \mathbb{R}^n)},$$

when $h > 0$ and the interval $I \subset [a,b]$ are small enough. This will be done using the inequalities in Lemma 6, namely (2.5) near points where $u(\tau,\cdot)$ has large variation, and (2.6) on intervals where the total variation of $u(\tau,\cdot)$ is suitably small.

In the third step we construct a suitable covering of $[a,b]$ and complete the proof of (3.1) combining the estimates obtained in steps 1 and 2.

STEP 1. Fix $\varepsilon \ge 0$ and assume $\tau \notin \mathcal{N}$. Then, at every point $\xi \in \mathbb{R}$ the limit (1.7) holds for some u^-, u^+, λ. Observe that $u^+ = u^-$ at a point where u is approximately continuous, while $u^+ \ne u^-$ if u has an approximate jump discontinuity at (τ,ξ). By (1.7), from Lemma 8 it follows

$$\lim_{h\to 0+} \frac{1}{h} \int_{\xi_-\lambda^* h}^{\xi+\lambda^* h} \left|u(\tau+h,x) - U(\tau+h,x)\right| dx$$
$$\le \lim_{h\to 0+} \frac{1}{h} \int_{\xi_-\lambda^* h}^{\xi+\lambda^* h} \left|u(\tau+h,x) - u^+\right| dx + \lim_{h\to 0+} \frac{1}{h} \int_{\xi_-\lambda^* h}^{\xi+\lambda^* h} \left|u(\tau+h,x) - u^-\right| dx = 0.$$

Hence
$$\frac{1}{h}\int_{\xi-\lambda^* h}^{\xi+\lambda^* h} \Big|u(\tau+h,x)-U(\tau+h,x)\Big|\,dx \le \varepsilon$$
for all $h>0$ sufficiently small.

By Lemma 2, $U(t,x)=U^\sharp(t-\tau,x)$ in a forward neighbourhood of the point (τ,ξ). Hence by (2.5) we get

$$\begin{aligned}&\frac{1}{h}\int_{\xi-\lambda^* h}^{\xi+\lambda^* h} \Big|u(\tau+h,x)-(S_h u(\tau))(x)\Big|\,dx \le \varepsilon+\frac{1}{h}\int_{\xi-\lambda^* h}^{\xi+\lambda^* h} \Big|(S_h u(\tau))(x)-U(\tau+h,x)\Big|\,dx\\ &=\varepsilon+O(1)\cdot \text{Tot.Var.}\big\{u(\tau);\ ]\xi-2\lambda^* h,\xi[\,\cup\,]\xi,\xi+2\lambda^* h]\big\}\big)\le 2\varepsilon\end{aligned} \tag{3.2}$$

for $h>0$ small enough. Note that here the maximum size of h depends on ξ, τ and ε.

STEP 2. Fix $\varepsilon>0$ and an interval $]c,d[\,\subset I\!R$ centered at a point ξ and such that $]c,d[\,\cap B_{\tau,\varepsilon}=\emptyset$. Here $B_{\tau,\varepsilon}$ is the set (2.1) of points where the oscillation of u is $>\varepsilon$. Consider a family of trapezoids $\{\Gamma_h\}_{h>0}$ defined as

$$\Gamma_h=\Big\{(s,x);\quad s\in[\tau,\tau+h],\quad x\in\ \big]c+(s-\tau)\lambda^*,\ d-(s-\tau)\lambda^*\big[\Big\}.$$

We first show that for small $h>0$ and every $(s,x)\in\Gamma_h$ one has

$$|u(s,x)-u(\tau,\xi)|\le 2\varepsilon+\text{Tot.Var.}\big\{u(\tau);\ ]c,d[\big\} \tag{3.3}$$

Indeed, by Lemma 4 the inequality (3.3) clearly holds for points (s,x) contained in small neighbourhoods of the lower corner points (τ,c) and (τ,d). It thus remains to prove (3.3) in a region of the form $[\tau,\tau+h]\times[c+h',d-h']$, with $h'>0$ given and for some $h>0$ suitably small. Since $[c+h',d-h']\cap B_{\tau,\varepsilon}=\emptyset$, for every $y\in[c+h',d-h']$ we can find $h_y,\rho_y>0$ such that (3.3) holds when $(s,x)\in[\tau,\ \tau+h_y]\times]y-\rho_y,\ y+\rho[$. Covering the compact interval $[c+h',d-h']$ with finitely many open intervals $]y_j-\rho_{y_j},\ y_j+\rho_{y_j}[$, $j=1,\dots,N$ and choosing $h\doteq\min h_{y_j}$, we obtain (3.3) for all $(s,x)\in[\tau,\tau+h]\times[c+h',d-h']$.

We now show that, for all $h>0$ with $h<(d-c)/2\lambda^*$, the following estimate holds:

$$\begin{aligned}&\int_{c+\lambda^* h}^{d-\lambda^* h}\Big|u(\tau+h,x)-U^\flat(+h,x)\Big|\,dx\\ &=O(1)\cdot\sup_{(s,x)\in\Gamma_h}|u(s,x)-u(\tau,\xi)|\ \cdot\int_\tau^{\tau+h}\text{Tot.Var.}\Big\{u(\tau);\ \big[c+\lambda^*(t-\tau),\ d-\lambda^*(t-\tau)\big]\Big\}\,dt.\end{aligned} \tag{3.4}$$

To derive (3.4), we proceed as in [4]. For each $i=1,\dots,n$ call $\tilde\lambda_i,\tilde l_i,\tilde r_i$ respectively the i-th eigenvalue and the left and right eigenvectors of the matrix $\tilde A=Df(u(\tau,\xi))$, normalized as in (1.2).

Let $\zeta' < \zeta''$ belong to the interval $]c+\lambda^* h, d-\lambda^* h[$. We now need to estimate the quantities

$$E_i \doteq \int_{\zeta_1}^{\zeta_2} \Big[\tilde l_i (u(\tau+h,x) - U^\flat(h,x))\Big]\, dx.$$

Obviously

$$\tilde l_i\; U^\flat(h,x) = \tilde l_i\; U^\flat(0, x-\tilde\lambda_i h) = \tilde l_i\; u(\tau, x-\tilde\lambda_i h).$$

Integrating (1.1) over the domain

$$\Big\{(s,x);\quad s\in[\tau,\ \tau+h],\quad \zeta' + (s-\tau-h)\tilde\lambda_i \le x \le \zeta'' + (t-\tau-h)\tilde\lambda_i\Big\},$$

we obtain

$$\begin{aligned} E_i &= \int_{\zeta'}^{\zeta''} \tilde l_i\; u(\tau+h,x)\,dx - \int_{\zeta'}^{\zeta''} \tilde l_i\; u(\tau, x-\tilde\lambda_i h)\,dx \\ &= \int_\tau^{\tau+h} \tilde l_i \cdot \big(f(u) - \tilde\lambda_i u\big)(t,\ \zeta' + (t-\tau-h)\tilde\lambda_i)\,dt \\ &\qquad - \int_\tau^{\tau+h} \tilde l_i \cdot \big(f(u) - \tilde\lambda_i u\big)(t,\ \zeta'' + (t-\tau-h)\tilde\lambda_i)\,dt. \end{aligned} \tag{3.5}$$

Consider the states

$$u' \doteq u\big(t,\ \zeta' + (t-\tau-h)\tilde\lambda_i\big), \qquad u'' \doteq u\big(t,\ \zeta'' + (t-\tau-h)\tilde\lambda_i\big), \qquad \tilde u \doteq u(\tau,\xi)$$

and define the averaged matrix

$$A^* \doteq \int_0^1 \Big[Df\big(su'' + (1-s)u'\big) - Df(\tilde u)\Big]\, ds.$$

One can check that

$$\tilde l_i\; \Big(f(u'') - f(u') - \tilde\lambda_i(u''-u')\Big) = \tilde l_i\; \Big(Df(\tilde u)\cdot(u''-u') - \tilde\lambda_i(u''-u')\Big) + \tilde l_i\; A^*(u''-u') = \tilde l_i\; A^*(u''-u').$$

Therefore

$$\left|\tilde l_i\; \Big(f(u'') - f(u') - \tilde\lambda_i(u''-u')\Big)\right| = O(1)\cdot|u''-u'|\cdot\|A^*\| = O(1)\cdot|u''-u'|\cdot\big(|u''-\tilde u| + |u'-\tilde u|\big).$$

Together with (3.5) this yields:

$$\begin{aligned} |E_i| &= O(1)\cdot \int_\tau^{\tau+h} \Big\{\big|u(t,\zeta' + (t-\tau-h)\tilde\lambda_i) - u(t,\zeta'' + (t-\tau-h)\tilde\lambda_i)\big|\cdot \\ &\qquad \cdot\Big(\big|u(t,\zeta' + (t-\tau-h)\tilde\lambda_i) - u(\tau,\xi)\big| + \big|u(t,\zeta'' + (t-\tau-h)\tilde\lambda_i) - u(\tau,\xi)\big|\Big)\Big\}\,dt \\ &= O(1)\cdot \sup_{(s,x)\in\Gamma_h} |u(s,x) - u(\tau,\xi)|\cdot \\ &\qquad \cdot \int_\tau^{\tau+h} \text{Tot.Var.}\Big\{u(t);\ \big[\zeta' + (t-\tau-h)\tilde\lambda_i,\ \zeta'' + (t-\tau-h)\tilde\lambda_i\big]\Big\}\,dt. \end{aligned}$$

Therefore

$$\left|\int_{\zeta'}^{\zeta''}\Big[u(\tau+h,x)-U^\flat(h,x)\Big]\,dx\right|\leq\sum_{i=1}^n|E_i|$$

$$=O(1)\cdot\sup_{(s,x)\in\Gamma_h}|u(s,x)-u(\tau,\xi)|\cdot$$

$$\cdot\int_\tau^{\tau+h}\left[\sum_{i=1}^n \text{Tot.Var.}\Big\{u(t);\ ]\zeta'+(t-\tau-h)\tilde\lambda_i,\ \zeta''+(t-\tau-h)\tilde\lambda_i]\Big\}\right]\,dt.$$

In view of Lemma 7, this establishes (3.4).

Combining (3.3), (3.4) and (2.6) we obtain

$$\begin{aligned}&\int_{c+\lambda^*h}^{d-\lambda^*h}\Big|u(\tau+h,x)-(S_hu(\tau))(x)\Big|\,dx\\ &=O(1)\cdot\Big(2\varepsilon+\text{Tot.Var.}\big\{u(\tau);\ ]c,d[\ \big\}\Big)\cdot\int_\tau^{\tau+h}\text{Tot.Var.}\Big\{u(t);\ \big[c+(t-\tau)\lambda^*,\ d-(t-\tau)\lambda^*\big]\Big\}\,dt\\ &\qquad+O(1)\cdot h\cdot\Big(\text{Tot.Var.}\big\{u(\tau);\ ]c,d[\ \big\}\Big)^2,\end{aligned}\tag{3.6}$$

valid for small $h>0$.

STEP 3. Fix $\varepsilon>0$, a time $\tau\in[0,T]\setminus\mathcal{N}$ and an interval $[a,b]\subset I\!R$. By Lemma 3, the set $B_{\tau,\varepsilon}\cap[a,b]$ contains finitely many points, say $\xi_1<\xi_2<\ldots<\xi_N$. Observe that every point ξ where $u(\tau,\cdot)$ has a jump $>\varepsilon$ is certainly included in the above list.

We can now cover the set $[a,b]\setminus\{\xi_1,\ldots,\xi_N\}$ with open intervals $]c_\alpha,d_\alpha[$, $\alpha=1,\ldots,M$, satisfying the following conditions:

(i) $\{\xi_1,\ldots,\xi_N\}\cap\bigcup_{\alpha=1}^M\ ]c_\alpha,d_\alpha[\ =\emptyset$,

(ii) $\text{Tot.Var.}\big\{u(\tau);\ ]c_\alpha,d_\alpha[\big\}\leq 2\varepsilon$ for every $\alpha=1,\ldots,M$,

(iii) every point of $[a,b]$ is contained in at most two distinct intervals $]c_\alpha,d_\alpha[$.

By steps 1 and 2, for every $h>0$ small enough one has

$$\frac{1}{h}\int_{\xi_i-\lambda^*h}^{\xi_i+\lambda^*h}\Big|u(\tau+h,x)-(S_hu(\tau))(x)\Big|\,dx\leq\frac{\varepsilon}{N}\,,$$

$$\begin{aligned}&\int_{c_\alpha+\lambda^*h}^{d_\alpha-\lambda^*h}\Big|u(\tau+h,x)-(S_hu(\tau))(x)\Big|\,dx\\ &\quad=O(1)\cdot\varepsilon\cdot\int_\tau^{\tau+h}\text{Tot.Var.}\Big\{u(t);\ ]c_\alpha+(t-\tau)\lambda^*,\ d_\alpha-(t-\tau)\lambda^*[\Big\}\,dt\\ &\qquad+O(1)\cdot h\varepsilon\cdot\text{Tot.Var.}\big\{u(\tau);\ ]c_\alpha,d_\alpha[\big\}\end{aligned}$$

for every $i = 1, \ldots, N$ and every $\alpha = 1, \ldots, M$. Finally,

$$
\begin{aligned}
&\frac{1}{h}\int_a^b \Big|u(\tau+h,x) - \big(S_h u(\tau)\big)(x)\Big|\,dx \\
&\leq \sum_{i=1}^N \frac{1}{h}\int_{\xi_i-\lambda^* h}^{\xi_i+\lambda^* h} \Big|u(\tau+h,x) - \big(S_h u(\tau)\big)(x)\Big|\,dx + \sum_{\alpha=1}^M \frac{1}{h}\int_{c_\alpha+\lambda^* h}^{d_\alpha-\lambda^* h} \Big|u(\tau+h,x) - \big(S_h u(\tau)\big)(x)\Big|\,dx \\
&\leq \varepsilon + O(1)\cdot\frac{\varepsilon}{h}\int_\tau^{\tau+h} \text{Tot.Var.}\big\{u(t);\ I\!R\big\}\,dt + O(1)\cdot\varepsilon\cdot\text{Tot.Var.}\big\{u(\tau);\ I\!R\big\} \\
&= O(1)\cdot\varepsilon.
\end{aligned}
$$

Letting $\varepsilon \to 0$ we obtain (3.1).

We have thus shown that if u satisfies **(A1)–(A3)**, then it must coincide with the corresponding semigroup trajectory $t \mapsto S_t \bar{u}$. On the other hand, one can easily check that the assumptions **(A1)–(A3)** are satisfied by all semigroup trajectories, because these are obtained as limits of wave-front tracking approximations. The proof of the Theorem is thus completed.

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