Locally Lipschitzian Guiding Function Method for ODEs.

Marta Lewicka
International School for Advanced Studies, SISSA,
via Beirut 2-4, 34014 Trieste, Italy.
E-mail: lewicka@sissa.it

1 Introduction

Let $f : [0,T] \times R^n \longrightarrow R^n$; we investigate the problem of existence of T-periodic solutions to the first order differential equation with f in the right hand side. Namely, we seek for solutions to the problem:

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(0) = x(T), \end{cases} \tag{1}$$

by generalizing the well known guiding function method. Such an approach can be found in several works, however, under some heavier assumptions. For example, in [M] f is assumed to be locally lipschitzian and the guiding function to be C^1. In [GP] f needs to be of the Caratheodory type only, while the corresponding guiding function must be still C^1. In fact, in [GP] the more general, multivalued problem

$$\begin{cases} x'(t) \in \varphi(t, x(t)) \\ x(0) = x(T). \end{cases} \tag{1'}$$

is under consideration.

In our paper we get rid of the assumption of the guiding function to be C^1. In fact, in Theorem 1, which is motivated by [GP], we need it to be locally lipschitzian only (Theorem 1). Our second main result (Theorem 2) characterizes a class of guiding functions, satisfying the conditions of Theorem 1. This result is an extension of the theorem on the index of coercive potentials [K] (a remarkable reformulation of which was done in [M]).

2 Preliminaries

In this section we review some of the standard facts and definitions.

Let X, Y be topological spaces. We say that X is an $R_\delta-$ set, whenever it is homeomorphic with an intersection of a decreasing sequence of compact, metric ANRs. A multivalued mapping $\varphi : X \rightsquigarrow Y$ (we will always suppose that a multivalued mapping has nonempty values) is called upper semicontinuous (u.s.c.) if, for every open set $U \subset Y$, the set $\varphi^{-1}(U) = \{x \in X : \ \varphi(x) \subset U\}$ is open in X. If X is a space with measure, we say that φ is measurable if $\varphi_+^{-1}(U) = \{x \in X : \ \varphi(x) \cap U \neq \emptyset\}$ is measurable for any open set $U \subset Y$.

We say that φ is admissible if X and Y are compact metric ANRs and φ is u.s.c. with R_δ values. A map $\varphi : X \rightsquigarrow X$ is called decomposable if it has a decomposition:

$$D_\varphi : \ X = X_0 \overset{\varphi_1}{\rightsquigarrow} X_1 \overset{\varphi_2}{\rightsquigarrow} X_2 \ldots \overset{\varphi_n}{\rightsquigarrow} X_n = X \qquad \varphi = \varphi_n \ldots \varphi_2 \varphi_1, \tag{2}$$

where each φ_i is admissible.

Let A be an open subset of X such that a decomposable map φ (with a decomposition (2)) has no fixed points on its boundary, that is $x \notin \varphi(x)$ for every $x \in \partial A$. In such a case it is possible to define a fixed point index $\mathrm{Ind}_X(D_\varphi, A) \in \mathrm{Z}$ having the following properties:

(i) *(existence)* If $\mathrm{Ind}_X(D_\varphi, A) \neq 0$ then φ has a fixed point in A.

(ii) *(additivity)* Let A_i, $(1 \leq i \leq n)$ be open, pairwise disjoint subsets of A. Suppose, that φ has no fixed points in $A \setminus \bigcup_{i=1}^n A_i$. Then the indices $\mathrm{Ind}_X(D_\varphi, A_i)$, $1 \leq i \leq n$, are well defined and $\mathrm{Ind}_X(D_\varphi, A) = \sum_{i=1}^n \mathrm{Ind}_X(D_\varphi, A_i)$.

(iii) *(homotopy invariance)* Let $\psi : X \rightsquigarrow X$ be a decomposable mapping, with a decomposition:

$$D_\psi : \ X = X_0 \overset{\psi_1}{\rightsquigarrow} X_1 \overset{\psi_2}{\rightsquigarrow} X_2 \ldots \overset{\psi_n}{\rightsquigarrow} X_n = X, \qquad \psi = \psi_n \ldots \psi_2 \psi_1.$$

Suppose that the decompositions D_φ and D_ψ are homotopic, that is there exists a decomposable homotopy $\chi : X \times [0,1] \rightsquigarrow X$, having a decomposition:

$$D_\chi : \ X \times [0,1] = X_0 \times [0,1] \overset{\bar\chi_1}{\rightsquigarrow} X_1 \times [0,1] \ldots \overset{\bar\chi_{n-1}}{\rightsquigarrow} X_{n-1} \times [0,1] \overset{\chi_n}{\rightsquigarrow} X_n = X,$$
$$\chi = \chi_n \bar\chi_{n-1} \cdots \bar\chi_1,$$

where, for $1 \leq i \leq n$, there exist admissible $\chi_i : X_{i-1} \rightsquigarrow X_i$ such that the following conditions are fulfilled:

· $\chi_i(\cdot, 0) = \varphi_i$, $\chi_i(\cdot, 1) = \psi_i$,

· $\bar\chi_i(\cdot, \lambda) = \chi_i(\cdot, \lambda) \times \{\lambda\}$ (for $i \neq n$)

and:

$$\forall x \in \partial A \ \forall \lambda \in [0,1] \ \ x \notin \chi(x, \lambda).$$

Then $\mathrm{Ind}_X(D_\psi, A)$ is well defined and: $\mathrm{Ind}_X(D_\psi, A) = \mathrm{Ind}_X(D_\varphi, A)$.

(iv) *(contraction)* Suppose, that in the decomposition of D_φ we have $X_{n-1} = Y \subset X$ and the mapping φ_n is the inclusion: $\varphi_n = i : Y \hookrightarrow X$. Then $\varphi_{|Y}$ has a decomposition

$$D_{\varphi_{|Y}} : \; Y = X_0 \overset{\varphi_{1|Y}}{\rightsquigarrow} X_1 \overset{\varphi_2}{\rightsquigarrow} X_2 \dots \overset{\varphi_{n-1}}{\rightsquigarrow} X_{n-1} = Y.$$

Moreover, if $\varphi_{|Y}$ has no fixed points in $\partial(A \cap Y)$, then $\mathrm{Ind}_Y(D_{\varphi_{|Y}}, A \cap Y)$ is well defined and $\mathrm{Ind}_Y(D_{\varphi_{|Y}}, A \cap Y) = \mathrm{Ind}_X(D_\varphi, A)$.

(v) *(units)* If φ is constant, that is $\varphi(x) = B \subset X$ for every $x \in X$, then

$$Ind_X(D_\varphi, A) = \begin{cases} 1 & \text{for} \quad A \cap B \neq \emptyset \\ 0 & \text{for} \quad A \cap B = \emptyset, \end{cases}$$

where $D_\varphi : \; X = X_0 \overset{\varphi}{\rightsquigarrow} X_1 = X$.

Notice, that the fixed point index is defined for a decomposition of a multivalued map, not for the map itself. However, when it is clear which decomposition we mean, we will simply write $\mathrm{Ind}_X(\varphi, A)$.

For the above definitions and other properties of Ind we refer to [BK] and [AGL].

Let $\varphi : [0,T] \times R^n \rightsquigarrow R^n$ be a Caratheodory multifunction, that is measurable in the first variable for every $x \in R^n$ and u.s.c. in the second variable for almost every $t \in [0,T]$ we say that φ has integrably bounded growth (with the bounding function μ) if there exists a function $\mu \in L^1([0,T], R)$ such that $\| y \| \leq \mu(t)(1 + \| x \|)$ for every $x \in R^n$, $t \in [0,T]$ and $y \in \varphi(t,x)$.

The Poincaré operator for the differential inclusion (1') is a multivalued mapping $S_\varphi : R^n \rightsquigarrow C([0,T], R^n)$, given by

$$\begin{aligned} S_\varphi(x_0) &= \{x : [0,T] \longrightarrow R^n, \; x \text{ is absolutely continuous}, \\ &\quad\; x'(t) \in \varphi(t, x(t)) \text{ for almost every } t \in [0,T], \;\; x(0) = x_0\}. \end{aligned}$$

We say that $x : [0,T] \longrightarrow R^n$ is a solution of the inclusion $x'(t) \in \varphi(t, x(t))$, whenever $x \in S_\varphi(x(0))$.

It is known that if φ is of the Caratheodory type, has compact and convex values and has integrably bounded growth, then S_φ is u.s.c. with R_δ values.

Let $\psi : [0,T] \times R^n \times [0,1] \rightsquigarrow R^n$ be a multivalued mapping with compact, convex values, such that for every $(x, \lambda) \in R^n \times [0,1]$ the mapping $\psi(\cdot, x, \lambda)$ is measurable and that for almost every $t \in [0,T]$ the map $\psi(t, \cdot, \cdot)$ is u.s.c.; assume that there exists $\mu \in L^1([0,T], R)$ such that for every $\lambda \in [0,1]$ the multifunction $\psi(\cdot, \cdot, \lambda)$ has integrably bounded growth with μ as its bounding function. Then the map $R^n \times [0,1] \ni (x, \lambda) \rightsquigarrow S_{\psi(\cdot,\cdot,\lambda)}(x)$ is u.s.c. with R_δ values.

For the statements above, see [AC], [G], [AGL].

From now on, by V will be assumed to be a locally lipschitzian function from R^n to R. Let us define

$$\Omega_V = \{x \in R^n, V \text{ is not differentiable in } x\}.$$

The Rademacher Theorem states that Ω_V has measure 0.

By the generalized gradient of V at point $x_0 \in R^n$, denoted by $\partial V(x_0)$, we mean the convex hull of the set of all limits $\lim\limits_{i\to\infty} V'(x_i)$ where $\{x_i\}_{i=1}^{\infty}$ is any sequence of points in $R^n \setminus \Omega_V$, converging to x_0.

The following properties of gradient are known:

(i) Let $S \subset R^n$ be of measure 0. Then $\partial V(x_0)$ can be obtained by replacing Ω_V with $\Omega_V \cup S$ in the above definition.

(ii) The multivalued mapping $\partial V : R^n \rightsquigarrow R^n$ is u.s.c. with compact, convex values.

(iii) *(mean value theorem)* Let $x, y \in R^n$. Then there exists a point u in a segment (x, y) and a point $w \in \partial V(x)$ such, that $V(x) - V(y) = \langle w, y - x\rangle$.

The reader is refered to [C] for more more material on this topic.

Given an euclidean space, by $B(\varepsilon)$ we will denote an open ball of the radius ε, centered in the origin.

3 Auxiliary Results

Let us recall the following lemma [GP]:

Lemma 0 *Assume $\varphi : [0,T] \times R^n \rightsquigarrow R^n$ has integrably bounded growth. Fix $r_0 > 0$, there exists $r > 0$ such that, for every solution of the problem*

$$\begin{cases} x'(t) \in \varphi(t, x(t)) \\ \| x(0) \| > r, \end{cases}$$

we have $\| x(t) \| > r_0$ for every $t \in [0,T]$.

Moreover, r depends only on r_0 and on the function μ in the definition of integrably bounded growth for the multivalued map φ.

Lemma 1 *Let $\Omega \subset R^n$ have measure 0. Suppose we are given a Lebesgue measurable function $z : [0,T] \longrightarrow R^n$. Then, for every $\varepsilon > 0$ there exists $x \in B(\varepsilon) \subset R^n$ such that $z(t) + x \notin \Omega$ for almost every $t \in [0,T]$.*

Proof Consider the characteristic function of Ω, $\chi_\Omega : R^n \longrightarrow R$ and a measurable function $F : [0,T] \times B(\varepsilon) \longrightarrow R^n$, given by $F(t,x) = z(t) + x$. Without loss of generality we may assume that Ω is Borel. Then the composition $\chi_\Omega F$ is measurable and from Fubini Theorem we get $\int\limits_{B(\varepsilon)} (\int\limits_0^T \chi_\Omega F dt) dx = \int\limits_0^T (\int\limits_{B(\varepsilon)} \chi_\Omega F dx) dt = 0$. Hence for almost every $x \in B(\varepsilon)$, $\int\limits_0^T (\chi_\Omega F)(x,t) dt = 0$. □

The following fact is straightforward:

Lemma 2 *Let $z : [0,T] \longrightarrow R^n$ be absolutely continuous. Then the composition Vz is absolutely continuous and for every $t \in [0,T]$ such that z is differentiable at t and V is differentiable at $z(t)$, Vz is differentiable at t and $(Vz)'(t) = \langle V'(z(t)), z'(t) \rangle$.*

The following lemma is of a basic importance for our later considerations:

Lemma 3 *Let z be as in the previous lemma. Suppose that for almost every $t \in [0,T]$ we have:*

$$\forall v \in \partial V(z(t)) \quad \langle v, z'(t) \rangle > 0. \tag{3}$$

Then $(Vz)'(t) > 0$ for almost every $t \in [0,T]$. In particular, $z(T) \neq z(0)$.

Proof By Lemma 1 there exists a sequence $\{x_k\}_{k=1}^\infty$ convergent to $0 \in R^n$ such that:

$$\forall t \in [0,T] \setminus A \ \forall k > 0 \quad z_k(t) = z(t) + x_k \notin \Omega_V,$$

where $A \subset [0,T]$ is a set of measure 0, containing the points in which z fails to be differentiable. We may also suppose that (3) is valid for every $t \in [0,T] \setminus A$.

It is easily seen that the absolutely continuous functions Vz_k are equibounded and that $\mid (Vz_k)' \mid$ are uniformly dominated by an integrable function $t \longmapsto C \parallel z'(t) \parallel$ with C a positive constant (this follows from Lemma 2). Therefore (see e.g. [AC]) we can extract a subsequence $\{Vz_{k_i}\}_{i=1}^\infty$ such that the derivatives $(Vz_{k_i})'$ converge weakly in $L^1([0,T], R)$ to $(Vz)'$. By Mazur Lemma there exists a sequence of convex combinations $\{w_i\}_{i=1}^\infty$, $w_i = \sum\limits_{j=i}^\infty \lambda_j^i \cdot (Vz_{k_j})'$, convergent to $(Vz)'$ in $L^1([0,T], R)$. Thus, without loss of generality, we have $\lim\limits_i w_i(t) = (Vz)'(t)$ for every $t \in [0,T] \setminus A$.

Fix $t \in [0,T] \setminus A$. By (3) there exists a real $\alpha_t > 0$ such that $\langle v, z'(t) \rangle > 2\alpha_t$ for every $v \in \partial V(z(t))$. By Lemma 2 and uppersemicontiniuity of ∂V, there exists a number k_{0_t} such that, for every $k \geq k_{0_t}$, we have

$$(Vz_k)'(t) = \langle V'(z(t) + x_k), z'(t) \rangle$$
$$\in \{ \langle v + u, z'(t) \rangle : v \in \partial V(z(t)) \text{ and } \parallel u \parallel \leq \alpha_t / \parallel z'(t) \parallel \}$$

and, consequently, $(Vz_k)'(t) > \alpha_t$ for every $k \geq k_{0_t}$.

Thus, $w_i(t) > \alpha_t$ for i large enough and we conclude that $(Vz)'(t) \geq \alpha_t > 0$, which proves the first statement of the lemma.

In particular, $Vz(T) - Vz(0) = \int_0^T (Vz)'(\tau)d\tau > 0$, so $z(T) \neq z(0)$. □

Definition 1 *V is said to be a direct potential (with a constant $r_0 > 0$) if*

$$\forall x \in R^n \setminus B(r_0)\ \forall v, w \in \partial V(x)\ \langle v, w \rangle \neq 0$$

Note that the above definition will not change if we replace $\neq$ by $>$.

Our definition agrees with the classical one for C^1 functions presented, for example, in [M].

The following lemma defines the index of a direct potential:

Lemma 4 *Let V be a direct potential (with a constant r_0). Set $\varphi : R^n \rightsquigarrow R^n$ as follows: $\varphi = Id_{R^n} - \partial V$. We define $Ind(V) = Ind_{\overline{B(2r_0)}}(r_{2r_0}\varphi, B(r_0))$, where r_{2r_0} is the radial retraction of R^n onto $\overline{B(2r_0)}$. (Here φ and r_{2r_0} are admissible, so the decomposition for defining the index is taken naturally as the composition $r_{2r_0}\varphi$.) Then $Ind_{\overline{B(2R)}}(r_{2R}\varphi, B(R)) = Ind(V)$ for every $R \geq r_0$.*

Proof The proof is straightforward and follows from homotopy invariance, additivity and contraction properties of the fixed point index. □

Let V be as in the previous lemma. Now consider a multivalued mapping $W_V : R^n \rightsquigarrow R^n$, given $W_V(x) = \mathrm{conv}(r_1 \partial V(x))$, where $r_1 : R^n \longrightarrow R^n$ is the radial retraction onto $\overline{B(1)} \subset R^n$ and *conv* stands for the convex hull. It is not hard to see that W_V is u.s.c., bounded by 1 and has compact, convex values.

The following lemma is analogous to a result obtained in [GP]:

Lemma 5 *Set a number $T > 0$. There exists $R_0 > 0$ such that for every $r > R_0$ there is $t_r \in (0, T]$ such that every solution of the problem:*

$$\begin{cases} x'(t) \in W_V(x(t)) \\ \parallel x(0) \parallel = r \end{cases} \tag{4}$$

has the following properties:

(i) $\forall t \in (0, t_r]\ \forall v \in \partial V(x(0)) \quad \langle x(t) - x(0), v \rangle > 0$,

(ii) $\forall t \in (0, T] \quad x(t) - x(0) \neq 0$.

Proof Lemma 0 gives the existence of a number $R_0 > r_0$ such that for every solution of the problem:

$$\begin{cases} x'(t) \in W_V(x(t)) \\ \| x(0) \| > R_0 \end{cases}$$

we have $\| x(t) \| > r_0$ for every $t \in [0, T]$.

Since ∂V is u.s.c. with compact values and V is a direct potential, we obviously have:

$$\begin{array}{c} \forall R > r_0 \ \exists \varepsilon_R > 0 \ \forall x, y \in \overline{B(R)} \setminus B(r_0) \subset R^n : \| x - y \| < \varepsilon_R \\ \forall w \in \partial V(x) \ \forall v \in \partial V(y) \quad \langle w, v \rangle > 0. \end{array} \tag{5}$$

Fix $r > R_0$ and let $t_r \le \varepsilon_{r+T}$, $t_r \in (0, T]$. Let x be a solution of (4). We have $\| x(t) \| \le r + \int_0^t \| x'(\tau) \| d\tau \le r + T$ for every $t \in [0, T]$ and $\| x(t) - x(0) \| \le \int_0^t \| x'(\tau) \| d\tau \le t_r \le \varepsilon_{r+T}$ for every $t \in (0, t_r]$. Hence, by (5):

$$\forall t \in (0, t_r] \ \forall w \in \partial V(x(t)) \ \forall v \in \partial V(x(0)) \quad \langle w, v \rangle > 0.$$

Combining the above with the following evident remark:

$$\forall y \in R^n \ \forall w \in W_V(y) \ \exists \alpha \ge 1 \quad \alpha w \in \partial V(y), \tag{6}$$

we obtain

$$\forall t \in (0, t_r] \ \forall w \in W_V(x(t)) \ \forall v \in \partial V(x(0)) \quad \langle w, v \rangle > 0$$

and, finally,

$$\forall t \in (0, t_r] \ \forall v \in \partial V(x(0)) \ \ \langle x(t) - x(0), v \rangle = \int_0^t \langle x'(\tau), v \rangle d\tau > 0.$$

Moreover, by (6), we have $\langle v, x'(t) \rangle > 0$ for almost every $t \in [0, T]$ and every $v \in \partial V(x(t))$, which, by Lemma 3 completes the proof of (ii). □

4 Main Results

When V is in C^1, the following definition reduces to the corresponding one in [GP].

Definition 2 *Let $f : [0, T] \times R^n \longrightarrow R^n$, V be a direct potential with a constant r_0. V is called a guiding function for f, whenever*

$$\forall x \in R^n \setminus B(r_0) \ \forall w \in \partial V(x) \ \forall t \in [0, T] \quad \langle f(t, x), w \rangle \ge 0.$$

Here comes our first main result.

Theorem 1 *Let $f : [0,T] \times R^n \longrightarrow R^n$ be a Caratheodory function with integrably bounded growth. Suppose that f has a guiding function V (with a constant r_0) such that $Ind(V) \neq 0$. Then the problem (1) has at least one solution.*

Proof Consider the following family of differential inclusions, with $\kappa \in [0,1]$:

$$z'(t) \in \kappa W_V(z(t)) + (1-\kappa) f(t, z(t)).$$

By Lemma 0, there exists $R > r_0$ such that for any $z : [0,T] \longrightarrow R^n$ that is a solution of the above problem for some $\kappa \in [0,1]$ and satisfies $\| z(0) \| \geq R$, we have $\| z(t) \| > r_0$ for every $t \in [0,T]$.

Let t_R be as in Lemma 5. Take the decomposable homotopy $H : \overline{B(2R)} \times [0,1] \rightsquigarrow \overline{B(2R)}$, given by

$$H(x,\lambda) = r_{2R}((1-\lambda)\cdot(x - \partial V(x)) + \lambda \cdot (2x - e_{t_R} S_{W_V}(x))),$$

where r_{2R} is, as usual, the radial retraction of R^n onto $\overline{B(2R)}$ and $e_{t_R} : C([0,T],R^n) \longrightarrow R^n$ is by definition $e_{t_R}(z) = z(t_R)$ (the explicit formula for the decomposition of H is not complicated but long, so we omit it).

We will show that H has no fixed points in $\partial B(R)$. Conversely, suppose that there exists $x \in H(x,\lambda)$, with $x \in \partial B(R)$ and $\lambda \in [0,1]$. Then

$$\exists v \in \partial V(x)\ \exists z \in S_{W_V}(x) \quad 0 = \| (1-\lambda)(x-v) + \lambda(2x - z(t_R)) - x \|^2 =$$

$$(1-\lambda)^2 \| v \|^2 + \lambda^2 \| z(t_R) - z(0) \|^2 + 2(1-\lambda)\lambda \langle v, z(t_R) - z(0) \rangle,$$

which contradicts Lemma 5.

By the above formula, Lemma 4 and the homotopy invariance of the fixed point index we have

$$\mathrm{Ind}(V) = \mathrm{Ind}_{\overline{B(2R)}}(r_{2R}(2Id_{\overline{B(2R)}} - e_{t_R} S_{W_V}), B(R)). \tag{7}$$

(the decompositions of the multivalued maps in the formula are induced by the decomposition of the homotopy H).

Consider $G : [0,T] \times R^n \times [0,1] \rightsquigarrow R^n$, given by

$$G(t,x,\lambda) = k(\lambda) W_V(x) + (1 - k(\lambda)) f(t,x),$$

where $k : [0,1] \longrightarrow R$ is by definition

$$k(\lambda) = \begin{cases} 1 & \text{for } \lambda \in [0,1/2) \\ 2 - 2\lambda & \text{for } \lambda \in [1/2,1]. \end{cases}$$

It is easy to check that G has the properties of the map ψ, introduced in paragraph 1, thus the following homotopy is decomposable $K : \overline{B(2R)} \times [0,1] \rightsquigarrow \overline{B(2R)}$,

$$K(x,\lambda) = r_{2R}(2x - e_{h(\lambda)} S_{G(\cdot,\cdot,\lambda)}(x)),$$

where $h : [0,1] \longrightarrow R$ is given by

$$h(\lambda) = \begin{cases} 2(T - t_R)\lambda + t_R & \text{for} \quad \lambda \in [0, 1/2) \\ T & \text{for} \quad \lambda \in [1/2, 1]. \end{cases}$$

Now suppose that the problem (1) has no solutions. We will first show that K has no fixed points in $\partial B(R)$. Conversely, suppose that $x \in K(x,\lambda)$ for a point $x \in \partial B(R)$ and $\lambda \in [0,1]$.

If $\lambda \in [0, 1/2)$, then $z(h(\lambda)) = z(0)$ for a function $z \in S_{W_V}(x)$, that contradicts Lemma 5 (*ii*).

If $\lambda \in [1/2, 1)$, then $z(T) = z(0)$ for a function $z \in S_{G(\cdot,\cdot,\lambda)}(x)$. We have $\langle z'(t), v\rangle = k(\lambda)\langle w_{t,v}, v\rangle + (1 - k(\lambda))\langle f(t, z(t)), v\rangle$ for almost every $t \in [0,T]$, every $v \in \partial V(z(t))$ and a point $w_{t,v} \in W_V(z(t))$. Now $\langle f(t, z(t)), v\rangle > 0$ because V is a guiding function for (1) and $\langle w_{t,v}, v\rangle > 0$ from (6) and the fact that V is a direct potential. Consequently $\langle z'(t), v\rangle > 0$ for almost every $t \in [0,T]$ and every $v \in \partial V(z(t))$, which contradicts Lemma 3.

The case $\lambda = 1$ is already excluded by our assumption that (1) has no solutions.

Now from the above and homotopy invariance of the fixed point index we have

$$\mathrm{Ind}_{\overline{B(2R)}}(r_{2R}(2Id_{\overline{B(2R)}} - e_{t_R} S_{W_V}), B(R)) = \mathrm{Ind}_{\overline{B(2R)}}(r_{2R}(2Id_{\overline{B(2R)}} - e_T S_f), B(R)).$$

Recalling (7)

$$\mathrm{Ind}_{\overline{B(2R)}}(r_{2R}(2Id_{\overline{B(2R)}} - e_T S_f), B(R)) = \mathrm{Ind}(V) \neq 0.$$

By existence property of the fixed point index we obtain a fixed point of the mapping $r_{2R}(2Id_{\overline{B(2R)}} - e_T S_f)$, namely $x \in B(R)$ such that $x \in 2x - e_T S_f(x)$ which is equivalent to $x \in e_T S_f(x)$. This means that (1) has a solution (with the initial value x), that is against our contradictory assumption and thus proves the theorem. □

Our next result gives a condition for a direct potential to have nonzero index, as it is required in Theorem 1.

Theorem 2 *Let V be a direct potential (with a constant $r_0 > 0$). Suppose that V is coercive, that is $\lim_{\|x\| \to \infty} V(x) = +\infty$. Then Ind$(V) = 1$.*

Proof For every $\gamma \in R$ denote by A_γ the open and bounded subset of R^n, given by $V^{-1}((-\infty,\gamma))$. Take the numbers $\alpha, \quad \beta > \alpha$ and $R > r > r_0$ such that:

$$\overline{B(r_0)} \subset \ A_\alpha \subset \overline{A_\alpha} \subset B(r) \subset \overline{B(r)} \subset A_\beta \subset \overline{A_\beta} \subset B(R).$$

Let

$$T = 2\frac{\beta-\alpha}{\min\{\langle v,w\rangle : \quad v\in\partial V(x),\ w\in W_V(x),\ x\in\overline{A_\beta}\setminus A_\alpha\}}$$

The above formula makes sence in view of (6) (V is a direct potential) and $\overline{A}_\beta \setminus A_\alpha \subset R^n\setminus B(r_0)$.

We will consider the following differential inclusion:

$$z'(t) \in -W_V(z(t)) \tag{8}$$

and its solutions on the interval $[0,T]$.
We have divided the later proof into five steps.

Step 1 We will prove that for every $\gamma \geq \alpha$ the set $\overline{A_\gamma}$ is positively invariant, that is

$$\forall x\in\overline{A_\gamma}\ \forall z\in S_{-W_V}(x)\ \forall t\in[0,T] \ \ z(t)\in\overline{A_\gamma}.$$

The composition Vz is absolutely continuous, while (6) implies that for almost every $t\in[0,T]$ such that $z(t)\in R^n\setminus B(r_0)$ we have $\langle v, z'(t)\rangle < 0$ for every $v\in\partial V(z(t))$. Using the same method as in the proof of Lemma 3, we obtain $(Vz)'(t) < 0$ for almost every $t\in[0,T]$ such that $z(t)\in R^n\setminus\overline{B(r_0)}$. If there exist $t_0,\ t_1\in[0,T]$, that $z(t_0)\in\partial\overline{A_\gamma}$ and $z(t)\notin\overline{A_\gamma}$ for every $t\in(t_0,t_1]$, then $Vz(t_0)=\gamma$ and $Vz(t)>\gamma$ for $t\in(t_0,t_1]$, hence $Vz(t_1) = Vz(t_0) + \int_{t_0}^{t_1}(Vz)'(\tau)d\tau < \gamma$, a contradiction.

Step 2 Consider the mapping $S_1 : \overline{B(2R)}\times(0,1] \rightsquigarrow C([0,T],R^n)$

$$S_1(x,\lambda) = \{w : [0,T]\longrightarrow R^n : \quad w(t) = \frac{z(\lambda t)-x}{\lambda} \text{ with } z\in S_{-W_V}(x)\}.$$

S_1 is u.s.c. and has R_δ values, because $S_1(x,\lambda)=\frac{1}{\lambda}(S_{-\lambda W_V}(x)-x)$. Moreover $\| w'(t) \|\leq 1$ for every $w\in S_1(\overline{B(2R)}\times(0,1])$ and for almost every $t\in[0,T]$. Hence, by Ascoli-Arzelá Theorem, the set $S_1(\overline{B(2R)}\times(0,1])$ is relatively compact in $C([0,T],R^n)$.

Let $S_2 : \overline{B(2R)}\times[0,1] \rightsquigarrow C([0,T],R^n)$,

$$S_2(x,\lambda) = \begin{cases} S_1(x,\lambda) & \text{for} \quad \lambda\neq 0 \\ \{w=\lim\limits_{k\to\infty} w_k :\ w_k\in S_1(x_k,\lambda_k) \text{ with } x_n\to x,\ \lambda_k>0,\ \lambda_k\to 0\} & \text{for} \quad \lambda = 0.\end{cases}$$

It is readily verified that S_2 is u.s.c. and has compact values.

Finally, let $S : \overline{B(2R)} \times [0,1] \longrightarrow C([0,T], R^n)$

$$S(x,\lambda) = \begin{cases} S_2(x,\lambda) & \text{for} \quad \lambda \neq 0 \\ \text{conv } S_2(x,0) & \text{for} \quad \lambda = 0. \end{cases}$$

It is not hard to see that S is u.s.c. with R_δ values.

Step 3 Consider the decomposable homotopy $K : \overline{B(2R)} \times [0,1] \rightsquigarrow \overline{B(2R)}$,

$$K(x,\lambda) = r_{2R}(\lambda(e_T S(x,0) + x) + (1-\lambda)(x - \partial V(x))),$$

where r_{2R} and e_T are as in the proof of Theorem 1. We will show that K has no fixed points in ∂A_β. First, let us remark that:

$$\forall x \in \partial A_\beta \, \exists \alpha_x > 0 \, \exists \varepsilon_x > 0 \, \forall y \in B(x, \varepsilon_x) \, \forall w \in W_V(y) \, \forall v \in \partial V(x) \;\; \langle w, v \rangle > \alpha_x.$$

Now let $w \in S_2(x,0)$ and $v \in \partial V(x)$ for a point $x \in \partial A_\beta$. By the definition of S_2, $\;w(T) = \lim\limits_{k\to\infty} \frac{z_k(\lambda_k T) - x_k}{\lambda_k}$, where $\lim\limits_{k\to\infty} x_k = x$, $\lim\limits_{k\to\infty} \lambda_k = 0$ and $\lambda_k > 0$, $z_k \in S_{-W_V}(x_k)$. For k large enough $\lambda_k < \varepsilon_x/(2T)$ and $\| x_k - x \| < \varepsilon_x/2$, thus, for any $t \in [0, \lambda_k T]$

$$\| z_k(t) - x \| \leq \| x_k - x \| + \| z_k(t) - x_k \| \leq \varepsilon_x$$

and hence

$$\langle \frac{z_k(\lambda_k T) - x_k}{\lambda_k}, -v \rangle = \frac{1}{\lambda_k} \int_0^{\lambda_k T} \langle z_k'(\tau), -v \rangle d\tau > \frac{1}{\lambda_k} \int_0^{\lambda_k T} \alpha_x d\tau = \alpha_x T.$$

This implies $\langle w(T), -v \rangle \geq \alpha_x T > 0$.

Now suppose that $x \in K(x,\lambda)$ for an $x \in \partial A_\beta$ and $\lambda \in [0,1]$, hence $0 \in \lambda e_T S(x,0) - (1-\lambda)\partial V(x)$. By the definition of $S(\cdot, 0)$ there exist $\{w_1, \ldots, w_k\} \subset S_2(x,0)$, $\{\lambda_1, \ldots, \lambda_k\} \subset R_+$, $\sum\limits_{i=1}^{k} \lambda_i = 1$ and $v \in \partial V(x)$ such that

$$0 = \| \lambda \sum_{i=1}^{k} \lambda_i w_i(T) - (1-\lambda) v \|^2$$
$$= \lambda^2 \| \sum_{i=1}^{k} \lambda_i w_i(T) \|^2 + (1-\lambda)^2 \| v \|^2 + 2\lambda(1-\lambda) \langle \sum_{i=1}^{k} \lambda_i w_i(T), -v \rangle > 0,$$

a contradiction.

Homotopy invariance and additivity of the fixed point index now yield

$$\text{Ind}(V) = \text{Ind}_{\overline{B(2R)}}(r_{2R}(Id_{\overline{B(2R)}} - \partial V), B(R)) =$$
$$\text{Ind}_{\overline{B(2R)}}(r_{2R}(Id_{\overline{B(2R)}} - \partial V), A_\beta) = \text{Ind}_{\overline{B(2R)}}(r_{2R}(e_T S(\cdot, 0) + Id_{\overline{B(2R)}}), A_\beta).$$

Step 4 Let $H : \overline{B(2R)} \times [0,1] \rightsquigarrow \overline{B(2R)}$

$$H(x,\lambda) = r_{2R}(e_T S(x,\lambda) + x),$$

H is a decomposable homotopy without fixed points in ∂A_β. To show this last statement, it is enough to prove that $x \notin H(x,\lambda)$ for any $x \in \partial A_\beta$ and $\lambda \in (0,1]$. Suppose the contrary; then $x = z(\lambda T)$ for a point $x \in \partial A_\beta$ and $z \in S_{-W_V}(x)$. The sets $\overline{A_\beta}$ and $\overline{A_\alpha} \subset A_\beta$ are positively invariant by Step 1, hence $z(t) \in \overline{A_\beta} \setminus \overline{A_\alpha} \subset \overline{A_\beta} \setminus \overline{B(r_0)}$ for every $t \in [0,\lambda T]$. As in Step 1, we obtain $(Vz)'(t) < 0$ for almost every $t \in [0,\lambda T]$ and, finally $0 = Vz(\lambda T) - V(x) = \int\limits_0^{\lambda T} (Vz)'(\tau)d\tau < 0$, a contradiction.

The homotopy invariance of the fixed point index forces:

$$\mathrm{Ind}_{\overline{B(2R)}}(r_{2R}(e_T S(\cdot,0) + Id_{\overline{B(2R)}}), A_\beta) = \mathrm{Ind}_{\overline{B(2R)}}(r_{2R} e_T S_{-W_V}, A_\beta).$$

Step 5 Take the decomposable homotopy $L : \overline{B(2R)} \times [0,1] \rightsquigarrow \overline{B(2R)}$

$$L(x,\lambda) = r_{2R}(\lambda \cdot e_T S_{-W_V}(x)).$$

If we had $x \in \partial A_\beta,\ z \in S_{-W_V}(x)$ and $\lambda \in [0,1)$ such that $x = \lambda z(T)$, it would follow $0 = \| x - \lambda z(T) \| > \| x \| - \| z(T) \|$ and $z(T) \in R^n \setminus B(r) \subset R^n \setminus \overline{A_\alpha}$. Thus $Vz(T) > \alpha$ and $z(t) \notin \overline{A_\alpha}$ for every $t \in [0,T]$. Hence:

$$\beta - \alpha > V(x) - Vz(T) = \int\limits_0^T -(Vz)'(\tau)d\tau. \tag{9}$$

Using the same technique as in the proof of Lemma 3 it can be easily seen that, for almost every $t \in [0,T]$, $z'(t) \in -W_V(z(t))$ and there exists $\{x_k^t\}_{k=1}^\infty$, convergent to $0 \in R^n$ such that $-(Vz)'(t)$ is a limit of a sequence $\{w_k^t\}_{k=1}^\infty$, where each w_k^t is a convex combination of some numbers in the set $\{\langle -V'(z(t)+x_i^t), z'(t)\rangle,\ i = k, k+1, \ldots\}$.

Set a number k, for i large enough we have $\langle -V'(z(t)+x_i), z'(t)\rangle \in \{\langle -v, z'(t)\rangle :\ v \in \partial V(z(t)) + B(1/k)\}$. Consequently, $-(Vz)'(t) \in \{\langle -v, z'(t)\rangle : v \in \overline{\partial V(z(t)) + B(1/k)}\}$ and

$$-(Vz)'(t) \in \{\langle v, w\rangle :\ v \in \partial V(z(t)),\ w \in W_V(z(t))\}$$

for almost every $t \in [0,T]$.

Recalling (9), we obtain:

$$\beta - \alpha > T \min\{\langle v, w\rangle :\ v \in \partial V(x),\ w \in W_V(x),\ x \in \overline{A_\beta} \setminus A_\alpha\} = 2(\beta - \alpha),$$

a contradiction. In this way we have shown that L has no fixed points in ∂A_β.

By the homotopy invariance and units properties of the fixed point index,

$$\mathrm{Ind}_{\overline{B(2R)}}(r_{2R} e_T S_{-W_V}, A_\beta) = \mathrm{Ind}_{\overline{B(2R)}}(0, A_\beta) = 1$$

and, finally, by Steps 3–5 we obtain $\mathrm{Ind}(V) = 1$. □

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