LYAPUNOV FUNCTIONAL FOR SOLUTIONS OF SYSTEMS OF CONSERVATION LAWS CONTAINING A STRONG RAREFACTION

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Contents

Abstract. We study the Cauchy problem for a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension

$$u_t + f(u)_x = 0,$$

$$u(0,x) = \bar{u}(x).$$

The initial data $\bar{u}$ is a small BV perturbation of a single rarefaction wave with an arbitrary strength. All characteristic fields are assumed to be genuinely nonlinear or linearly degenerate in the vicinity of the reference rarefaction curve. We prove that a suitable BV stability condition yields uniform bounds on the total variation of perturbation, thus implying the existence of a global admissible solution. On the other hand, a stronger L^1 stability condition guarantees the existence of the Lipschitz continuous flow of solutions. Our proof relies on the construction of a Lyapunov functional which is almost decreasing in time and which is equivalent to the L^1 distance between the two solutions.

1. Introduction and statement of the main results

The system of conservation laws in one space dimension is the following first order system of nonlinear PDEs:

$$u_t + f(u)_x = 0. \tag{1.1}$$

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The well-posedness of (1.1) has been the objective of vast research in recent years, however at a considerable level of generality it remains an open problem. A complete analysis of the issue has been carried out for strictly hyperbolic flux in (1.1) and initial data $\bar{u} \in BV$ having suitably small total variation.

$$u(0, x) = \bar{u}(x). \tag{1.2}$$

Namely, the entropy solutions to (1.1) (1.2) constitute a flow which is Lipschitz continuous with respect to time and initial data. As shown recently in [BiB], its trajectories are the limits of the solutions to the parabolic regularizations of (1.1), when the viscosity parameter vanishes to zero.

Another approach was implemented in a series of papers [BC, BCP, BLY]. It relies on building piecewise constant approximations of solutions to (1.1) (1.2) and then controlling the evolution of their BV or L^1 norm. The fundamental block in this construction is provided by solutions of the Riemann problems, that is for initial data $\bar{u}$ consisting of a single discontinuity:

$$u(0, x) = \begin{cases} u^- & x < 0, \\ u^+ & x > 0. \end{cases} \tag{1.3}$$

To analyze how much the crucial so far condition of the smallness of initial data can be relaxed, one wishes to study the well-posedness of (1.1) (1.2) with $\bar{u}$ being a small perturbation of a fixed Riemann data of arbitrarily large strength. We assume that the solution of the latter is given and that it consists of a number of waves of different characteristic families. More generally, we wish to study the stability of a reference pattern containing possibly strong but noninteracting waves. The above mentioned results say that the trivial pattern with no waves present is stable, as one can control the amount (measured in TV or in the L^1 norm) of initially small perturbation of this pattern.

An example in [BC] points out that this is no longer true in presence of strong waves. Indeed, one has to account for the waves' mutual influence as well as for their interaction with the perturbation, and therefore extra stability conditions are necessary. These conditions in essence refer to the existence of weights with respect to which the flow generated by the associated linearized problem is a contraction; the linearization is taken at states attained by the reference solution [BM]. This approach was realized in a series of papers [BC, Scho, BM, LeT, Le1]. All these works however concentrate mainly on patterns with strong shocks or deal solely with the BV stability in presence of rarefactions.

In [BC] the authors study systems of 2 equations and prove their BV and L^1 stability under the corresponding non-resonance conditions relating to 2 shocks. The presence of strong rarefaction waves is also admitted, however being extreme fields waves their stability follows without any additional restrictions [Le3]. More general $n \times n$ systems of conservation laws are studied in [Scho] and the BV stability of patterns including strong shocks, rarefactions and contact discontinuities is established. In particular this yields the local in time existence of solutions to (1.1) (1.2) within the class of initial data with bounded variation. In [Le1] we established both the BV and the L^1 stability of patterns of noninteracting strong classical shocks in $n \times n$ systems. The crucial ingredient for proving the L^1 stability was the Lyapunov functional approach from [BLY]; let us anticipate that the same method will be used in the present article. The role of the stability conditions from [BM, Le1] and their relations to [BC, Scho] were explained in [Le2].

As a next step, this paper studies BV and L^1 stability of solutions to (1.1) (1.2) close to a reference pattern which is a single rarefaction wave of arbitrary strength. The results of this work combined with [Le1] yield thus the well-posedness analysis for patterns of noninteracting shock and rarefaction waves (compare also [Le3]). The stability conditions presented in this paper are studied in a complementary work [Le3].

We now state our basic hypotheses and set the notation.

(H1) $\left[\begin{array}{l}\text{The system (1.1) is strictly hyperbolic in a domain } \Omega \subset \mathbf{R}^n \text{ to be} \\ \text{specified later. That is, for each } u \in \Omega \text{ the Jacobian matrix } \mathrm{D}f(u) \\ \text{of the smooth flux } f : \Omega \longrightarrow \mathbf{R}^n \text{ has } n \text{ distinct and real eigenvalues:} \\ \lambda_1(u) < \ldots < \lambda_n(u).\end{array}\right.$

Let $\{r_i(u)\}_{i=1}^n$ be the basis of right eigenvectors of $\mathrm{D}f$; $\mathrm{D}f(u)r_i(u) = \lambda_i(u)r_i(u)$. Call $\{l_i(u)\}_{i=1}^n$ the dual basis of left eigenvectors, so that $\langle r_i(u), l_j(u)\rangle = \delta_{ij}$ for all $i, j : 1 \ldots n$ and all $u \in \Omega$.

Fix $k : 1 \ldots n$ and consider an integral curve $\mathcal{R}_k$ of the vector field r_k joining states $u_l, u_r \in \Omega$:

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{R}_k(\theta) = r_k(\mathcal{R}_k(\theta)), \\ u_l = \mathcal{R}_k(0), \quad u_r = \mathcal{R}_k(\Theta), \qquad \Theta > 0.\end{gathered} \tag{1.4}$$

$\mathcal{R}_k$ is called the rarefaction curve. For a small $c > 0$ we define the domain

$$\Omega = \Omega_c = \{u \in \mathbf{R}^n : \quad ||u - \mathcal{R}_k(\theta)|| < c \quad \text{for some } \theta \in [0, \Theta]\}\,; \tag{1.5}$$

all the subsequent reasoning will be restricted to this domain, with the parameter c appropriately small. We further assume that:

(H2) $\left[\begin{array}{l}\text{In } \Omega\text{, each characteristic field } i : 1 \ldots n \text{ is either linearly degenerate:} \\ \langle \mathrm{D}\lambda_i, r_i\rangle \equiv 0\text{, or it is genuinely nonlinear which means that } \langle \mathrm{D}\lambda_i, r_i\rangle > 0. \\ \text{The } k\text{-th characteristic field is assumed to be genuinely nonlinear.}\end{array}\right.$

In the case of linearly degenerate fields we set $||r_i(u)|| = 1$, while when the i-th field is genuinely nonlinear we choose the normalization of right eigenvectors $r_i(u)$ so that $\langle \mathrm{D}\lambda_i(u), r_i(u)\rangle = 1$ for all $u \in \Omega$. In particular we have:

$$\langle \mathrm{D}\lambda_k(u), r_k(u)\rangle = 1 \quad \text{for all } u \in \Omega \tag{1.6}$$

and thus $\Theta = \lambda_k(u_r) - \lambda_k(u_l)$.

The piecewise smooth, self-similar function, called the centered rarefaction wave is given by:

$$u_0(t,x) = \left\{\begin{array}{cl} u_l & \text{if } x < t\lambda_k(u_l) \\ \mathcal{R}_k(\theta) & \text{if } x = t\lambda_k(\mathcal{R}_k(\theta)), \quad \theta \in [0,\Theta] \\ u_r & \text{if } x > t\lambda_k(u_r)\end{array}\right. \tag{1.7}$$

and provides an entropy admissible solution of (1.1) [Sm, D]. The objective of this paper is a study of the stability of u_0. Our main results are the following:

Theorem I. *Assume that (H1), (H2) and the BV stability condition (2.6) hold. For $c, \delta > 0$ let $\mathcal{E}_{c,\delta}$ denote the set of all continuous functions $\bar{u}$ satisfying:*

(i) $\bar{u}(x) \in \Omega_c$ *for all* $x \in \mathbf{R}$,
(ii) $\lim_{x\to-\infty} \bar{u}(x) = u_l$ *and* $\lim_{x\to\infty} \bar{u}(x) = u_r$,

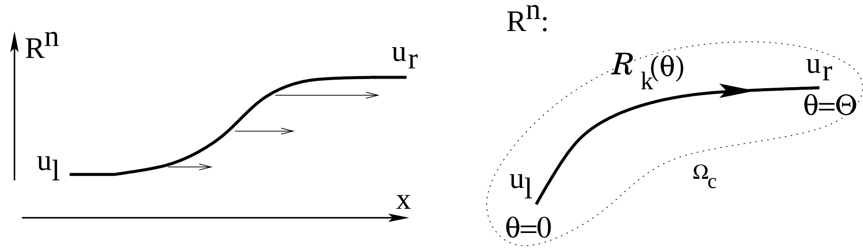

FIGURE 1.1

(iii) $|TV(\bar{u}) - |\mathcal{R}_k|| < \delta$, *where* $|\mathcal{R}_k| = TV(\mathcal{R}_k)$ *is the arc-length of the rarefaction curve* $\mathcal{R}_k(\theta)$, $\theta \in [0, \Theta]$.

There exist small parameters $c, \delta > 0$ *such that for every* $\bar{u} \in \mathrm{cl}\ \mathcal{E}_{c,\delta}$, *where* cl *denotes the closure in* L^1_{loc}, *the Cauchy problem (1.1) (1.2) has a global entropy admissible solution* $u(t,x)$.

Theorem II. *Assume that (H1), (H2) and the* L^1 *stability condition (3.1) are satisfied. Then there exists a closed domain* $\mathcal{D} \subset L^1_{loc}(\mathbf{R}, \Omega)$, *containing all continuous functions* $\bar{u}$ *satisfyling (i), (ii), (iii) in Theorem I, for some* $c, \delta > 0$, *and there exists a semigroup* $S : \mathcal{D} \times [0, \infty) \longrightarrow \mathcal{D}$ *such that:*

(i) $||S(\bar{u}, t) - S(\bar{v}, s)||_{L^1} \leq L \cdot (|t - s| + ||\bar{u} - \bar{v}||_{L^1})$ *for all* $\bar{u}, \bar{v} \in \mathcal{D}$, *all* $t, s \geq 0$ *and a uniform constant* L, *depending only on the system (1.1),*

(ii) *for all* $\bar{u} \in \mathcal{D}$, *the trajectory* $t \mapsto S(\bar{u}, t)$ *is the solution to (1.1) (1.2) given in Theorem I.*

We now set other preliminaries. For each $i : 1 \ldots n$ and $u \in \Omega$, call $\sigma \mapsto \mathcal{S}_i(u, \sigma)$ and $\sigma \mapsto \mathcal{R}_i(u, \sigma)$, the i-th shock and the i-th rarefaction curves through the point u [L, D]. In particular we have $\mathcal{R}_k(u_l, \theta) = \mathcal{R}_k(\theta)$. Both curves are defined at least locally, that is for $\sigma \in (-c, c)$ and have second order contact at $\sigma = 0$:

$$\mathcal{S}_i(u, \sigma) - \mathcal{R}_i(u, \sigma) = \mathcal{O}(1)|\sigma|^3. \tag{1.8}$$

The curves' parametrization is consistent with the normalization of the right eigenvectors r_i. That is, they are parametrised by arc length if the i-th characteristic field is linearly degenerate, and by the corresponding eigenvalue λ_i if the i-th field is genuinely nonlinear:

$$\lambda_i(\mathcal{S}_i(u, \sigma))) - \lambda_i(u) = \sigma = \lambda_i(\mathcal{R}_i(u, \sigma))) - \lambda_i(u). \tag{1.9}$$

By this choice of parametrisation we have:

$$\mathcal{S}_i(\mathcal{S}_i(u, \sigma), -\sigma) = u. \tag{1.10}$$

The speed λ of a weak shock wave $(u^-, u^+ = \mathcal{S}_i(u^-, \sigma))$ with strength $\sigma < 0$ can be computed from the Rankine-Hugoniot identity:

$$f(u^+) - f(u^-) = \lambda \cdot (u^+ - u^-). \tag{1.11}$$

Throughout the paper, by $\mathcal{O}(1)$ we mean any uniformly bounded function, depending only on the system (1.1). Any sufficiently small but positive constant is denoted by c. The Riemann data as in (1.3) is for simplicity denoted by (u^-, u^+).

The paper is constructed as follows. In sections 2 and 3 we present the stability conditions and their primary motivation. In section 4 we prove Theorem I. The proof relies on the construction of approximate solutions by means of the wave front tracking algorithm [HR, BaJ], and applying the Glimm analysis in view of the BV stability condition. In section 9 we prove that the domain of applicability of these techniques actually contains the data with properties as in Theorem I.

Towards the proof of Theorem II, in section 6 we give the definition of the Lyapunov functional measuring the L^1 distance between the two approximate solutions constructed in section 4. The crucial observation forour construction is noting that in the initial time interval where the solutions are apart from each other, this distance decreases rapidly. A convenient tool to estimate the decrease is the first order rarefactions, introduced in section 5. For other times, the pointwise distance between solutions is calculated along shock curves, as in [BLY]. The decrease of the functional follows then from the assumed L^1 stability condition and the main concern of sections 7 and 8.

2. The weighted BV stability condition

In this section we discuss a stability condition guaranteeing the existence of solutions to the problem (1.1)(1.2) in the vicinity of the reference rarefaction wave (1.7). To motivate our approach we first recall the argument from [Le1, BM]. The stability conditions there were formulated in terms of the existence of a family of weights $w_i > 0$, $i : 1 \dots n$, corresponding to different characteristic families of perturbation v, and depending on the location of perturbing waves inside the reference pattern u_0. The conditions required that the weighted BV or L^1 norm of any solution of

$$v_t + Df(u_0)v_x + [D^2 f(u_0) \cdot v] \cdot (u_0)_x = 0$$

was nonincreasing in time.

Let $w_1 \dots w_{k-1}, w_{k+1} \dots w_n : (-c, \Theta + c) \longrightarrow \mathbf{R}_+$ be smooth, nonnegative functions defined along the rarefaction curve $\mathcal{R}_k$ in (1.4). We can extend this definition on the whole neighbourhood Ω by setting

$$\forall i \neq k \ \forall u \in \Omega \quad w_i(u) = w_i(\theta) \ \text{ where } \ \lambda_k(u) = \lambda_k(\mathcal{R}_k(\theta)). \tag{2.1}$$

Consider an interaction of a weak i-th wave with a small part of the rarefaction $\mathcal{R}_k$, located at the state $u = \mathcal{R}_k(\theta)$. To fix the ideas, assume that $i < k$ and call the strengths of the incoming waves and the states they join to u respectively: $q_k^- > 0$, q_i^-, u^-, u^+ (as in Figure 2.1 a)). In particular, we have $u = \mathcal{R}_k(u^-, q_k^-)$ and $q_k^- = \theta - \lambda_k(u^-)$. The strengths of waves are computed in terms of change in the corresponding eigenvalue for genuinely nonlinear fields, or the arc-length of the rarefaction curve connecting the two states, for linearly degenerate fields. We thus remain consistent with the parametrization of the right eigenvectors, given in section 1. Now if q_k^- and q_i^- are small enough, the Riemann problem (u^-, u^+) has a self-similar solution composed of n outgoing waves having strengths $q_1^+ \dots q_n^+$. For the basic properties of this construction we refer to [L, Sm, B, D]. Assigning to each wave the weight w_i corresponding to its characteristic family and computed at the wave's left state, we now require that the weighted amount of perturbation decreases across the interaction, so that:

$$\sum_{j\neq k} w_j^+|q_j^+| < w_i^-|q_i^-|. \tag{2.2}$$

Recall the standard Taylor estimates [Sm]:

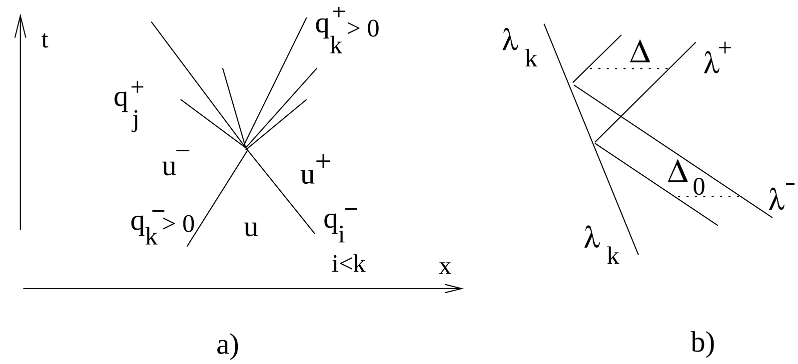

FIGURE 2.1

$$\begin{aligned} \forall j\neq k \qquad q_j^+ = \delta_{ij}\cdot q_i^- + \langle l_j(u), [r_i,r_k](u)\rangle \cdot q_i^- q_k^- \\ + \mathcal{O}(1)|q_i^-q_k^-|(|q_i^-|+|q_k^-|). \end{aligned} \tag{2.3}$$

Here $[r_i, r_k] = \mathrm{D}r_i\cdot r_k - \mathrm{D}r_k\cdot r_i$ stands for the Lie bracket of two vector fields and δ_{ij} is the Kronecker delta.

In view of (2.3), we have:

$$\begin{aligned} \forall j\neq k,i \qquad w_j^+|q_j^+| =& w_j(u)\cdot |\langle l_j(u), [r_i,r_k](u)\rangle|\cdot |q_i^-q_k^-| \\ &+ \mathcal{O}(1)|q_i^-q_k^-|(|q_i^-|+|q_k^-|). \end{aligned}$$

On the other hand:

$$\begin{aligned} w_i^+q_i^+ - w_i^-q_i^- =& (w_i^+-w_i^-)q_i^- + w_i^+(q_i^+-q_i^-) \\ =& -w_i'(\theta)\cdot q_i^-q_k^- + w_i(u)\cdot\langle l_i, [r_i,r_k]\rangle(u)\cdot q_i^-q_k^- \\ &+ \mathcal{O}(1)|q_i^-q_k^-|(|q_i^-|+|q_k^-|). \end{aligned}$$

Hence:

$$\begin{aligned} w_i^+|q_i^+| - w_i^-|q_i^-| =& (\mathrm{sgn}\ q_i^-)\cdot(w_i^+q_i^+ - w_i^-q_i^-) \\ =& \big\{w_i(u)\cdot\langle l_i, [r_i,r_k]\rangle(u) - w_i'(\theta)\big\}\cdot|q_i^-q_k^-| \\ &+ \mathcal{O}(1)|q_i^-q_k^-|(|q_i^-|+|q_k^-|). \end{aligned}$$

Condition (2.2) is thus equivalent to:

$$\left(\sum_{j\neq i,k} w_j(\theta)\cdot|\langle l_j, [r_i,r_k]\rangle(\mathcal{R}_k(\theta))|\right) + w_i(\theta)\cdot\langle l_i,[r_i,r_k]\rangle(\mathcal{R}_k(\theta)) < w_i'(\theta). \tag{2.4}$$

Analogously, for $i>k$ one obtains:

$$\left(\sum_{j\neq i,k} w_j(\theta)\cdot|\langle l_j, [r_k,r_i]\rangle(\mathcal{R}_k(\theta))|\right) + w_i(\theta)\cdot\langle l_i,[r_k,r_i]\rangle(\mathcal{R}_k(\theta)) < -w_i'(\theta). \tag{2.5}$$

Define the $(n-1)\times(n-1)$ matrix function:

$$\mathbf{P}(\theta) = [p_{ij}(\theta)]_{\substack{i,j:1\dots n,\\ i,j\neq k}}, \qquad \text{for} \quad \theta\in[0,\Theta],$$

$$p_{ij}(\theta) = \begin{cases} |\langle l_j, [r_i, r_k]\rangle(\mathcal{R}_k(\theta))| & \text{if} \quad i\neq j,\\ \operatorname{sgn}(k-i)\cdot\langle l_i, [r_i, r_k]\rangle(\mathcal{R}_k(\theta)) & \text{if} \quad i=j. \end{cases}$$

Combining (2.4) and (2.5), we have proved:

Lemma 2.1. *Condition (2.2) is equivalent to the following:*

$$\left[\begin{array}{l} BV \text{ STABILITY CONDITION: } \textit{There exist positive smooth functions} \\ w_1\dots w_{k-1}, w_{k+1}\dots w_n : [0,\Theta]\to\mathbf{R}_+ \textit{ such that} \\ \qquad \mathbf{P}(\theta)\cdot\begin{bmatrix} w_1(\theta)\\ \vdots \\ w_{k-1}(\theta)\\ w_{k+1}(\theta)\\ \vdots\\ w_n(\theta)\end{bmatrix} < \begin{bmatrix} w_1'(\theta)\\ \vdots \\ w_{k-1}'(\theta)\\ -w_{k+1}'(\theta)\\ \vdots\\ -w_n'(\theta)\end{bmatrix} \quad \textit{for every } \theta\in(0,\Theta), \\ \textit{where the above vector inequality is understood componentwise.} \end{array}\right. \tag{2.6}$$

Remark 2.2. Notice that because of the strict inequalities in (2.4) and (2.5), the condition (2.6) implies a stricter version of (2.2):

$$\sum_{j\neq k} w_j^+|q_j^+| < w_i^-|q_i^-| - c|q_i^- q_k^-|$$

for a small constant c.

Remark 2.3. The inequality in (2.6) is independent from rescaling $w_i\mapsto\alpha\cdot w_i$, for any $\alpha>0$. Thus, in particular we may assume that

$$|w_i(u)|<1 \quad\text{and}\quad ||\mathrm{D}w_i(u)||<1$$

for each i and every $u\in\Omega$.

Remark 2.4. If all $p_{ij}(\theta)\geq 0$, we can regard the quantity $w_i(\theta)$ as the measure of the amount of potential future interactions of the i-th perturbation wave located at the state $\mathcal{R}_k(\theta)$. For $i<k$ each w_i is an increasing function of θ, and for $i>k$ each w_i is decreasing along the curve $\mathcal{R}_k$. Indeed, the slow waves ($\lambda_i<\lambda_k$ for $i<k$) travel in the direction of decreasing θ on the $t-x$ plane, and thus the bigger the parameter θ corresponding to their location is, the more potential contribution to the future amount of perturbation they create. The converse assertion is true for the fast waves of characteristic families $i>k$.

By an approximation argument, as the inequality in (2.6) is strict, we see that (2.2) holds also for any state $u\in\Omega_c$. For the more detailed discussion of condition (2.6) we refer to the paper [Le3]. In particular, we have:

Lemma 2.5. [Le3] *Let the condition (2.6) be satisfied. There exists $c > 0$ such that for every $u^-, u^+ \in \Omega$ with $\lambda_k(u^+) - \lambda_k(u^-) > -c$, the Riemann problem (u^-, u^+) for (1.1) has the unique self-similar solution attaining states in Ω. The solution is composed of $n-1$ weak waves of families $1\ldots k-1, k+1\ldots n$ and a k-th wave which is either a weak shock or a rarefaction.*

Condition (2.6) is independent of the parametrization of the eigenvectors in Ω. The next lemma gathers several other properties of this condition.

Lemma 2.6. [Le3] *In any of the following cases (2.6) is satisfied:*

(i) *when the reference rarefaction is sufficiently weak, that is $0 < \Theta << 1$,*
(ii) *when the reference rarefaction belongs to an extreme characteristic field ($k = 1$ or n),*
(iii) *when (1.1) has a coordinate system of Riemann invariants* [Sm, D, S].

In particular, any rarefaction wave in any 2×2 system or the 3×3 system of Euler equations of gas dynamics [D, Sm, Scho] *is BV stable.*

(iv) *For $n = 3$ and $k = 2$, (2.6) is equivalent to the existence of a positive solution $v : [0, \Theta] \longrightarrow \mathbf{R}_+$ to the Riccati equation:*
$$v'(\theta) = p_{12}(\theta) + [p_{11}(\theta) + p_{22}(\theta)]\cdot v(\theta) + p_{21}(\theta)\cdot v^2(\theta).$$

3. The weighted L^1 stability condition

The production matrix $\mathbf{P}$ in condition (2.6) accounts for the infinitesimal change of the strength of perturbation as it passes through the rarefaction fan (1.7). The elements of $\mathbf{P}(\theta)$ are second order coefficients in the Taylor expansion of the strength of waves produced through the interaction with a part of the large rarefaction $\mathcal{R}_k(\theta)$. In order to deal with the L^1 stability one is led to a "mass production" matrix $\mathbf{M}(\theta)$ whose components additionally account for the shifts in locations of the perturbing waves of different characteristic families before and after the interaction. More precisely, define:

$$\mathbf{M}(\theta) = [m_{ij}(\theta)]_{\substack{i,j:1\ldots n,\\ i,j\neq k}}, \qquad \text{for} \quad \theta\in[0,\Theta],$$
$$m_{ij}(\theta) = \begin{cases} p_{ij}(\theta)\cdot \dfrac{|\lambda_j - \lambda_k|}{|\lambda_i-\lambda_k|}(\mathcal{R}_k(\theta)) & \text{if} \quad i\neq j,\\ p_{ij}(\theta) + \dfrac{\mathrm{D}\lambda_i\cdot r_k}{|\lambda_i - \lambda_k|}(\mathcal{R}_k(\theta)) & \text{if} \quad i = j.\end{cases}$$

We have the following:

(3.1) $\Bigg[$ L^1 Stability Condition: There exist positive smooth functions $w_1\ldots w_{k-1}, w_{k+1}\ldots w_n : [0,\Theta] \to \mathbf{R}_+$ such that the inequality in (2.6) is satisfied with $\mathbf{M}(\theta)$ replacing the matrix $\mathbf{P}(\theta)$.

Note that an observation as in Remark 2.3 remains valid.

A more restrictive version of (3.1), where all weights w_i are linear, was introduced in [BM] in the context of the well-posedness of the associated variational system.

Lemma 3.1. [Le3] *We have:*

(i) *Condition (3.1) is stronger than the BV stability condition (2.6).*
(ii) *The assertions of Lemma 2.6 hold in their respective versions.*

(iii) *For all $i \neq j$ and all $\theta \in [0, \Theta]$ there holds: $m_{ij}(\theta) = |\langle l_j, \mathrm{D}r_i \cdot r_k\rangle(\mathcal{R}_k(\theta))|$ and $m_{ii}(\theta) = \mathrm{sgn}\ (k-i) \cdot \langle l_i, \mathrm{D}r_i \cdot r_k\rangle(\mathcal{R}_k(\theta))$.*

We end this section by presenting a consequence of (3.1) which plays the same role as Lemma 2.1 and Remark 2.2 for the condition (2.6). Its proof will follow from the more general Lemma 8.2. To fix the ideas, let

$$\mathcal{S}_k(q_k^-) \circ \mathcal{S}_i(u, q_i^-) = \mathcal{S}_n(q_n^+) \circ \ldots \mathcal{S}_1(u, q_1^+)$$

with $u \in \Omega$, $\{q_j^-\}_{j=i,k}$ small enough and $q_k^- \geq 0$. Then for a small uniform constant γ we have:

$$\sum_{j \neq k} w_j^+ |q_j^+| \cdot |\lambda_j^+ - \lambda_k^+| < w_i^- |q_i^-| \cdot |\lambda_i^- - \lambda_k^-| - \gamma |q_i^- q_k^-|.$$

Namely, the total weighted mass of perturbation decreases as it passes through the rarefaction wave (1.7). Recall [BM] that the ratio Δ/Δ_0 of shifts in the reflected or transmitted wave with respect to the shift in an incoming wave can be computed as $|\lambda^+ - \lambda_k|/|\lambda^- - \lambda_k|$. As in Figure 2.1 b), λ^- and λ^+ denote speeds of the modified waves before and after the interaction with a reference wave traveling with speed λ_k.

4. Existence of solutions – a proof of Theorem I

Recall that given a Cauchy problem (1.1) (1.2) with $\bar{u}$ having small total variation, its solution can be obtained in the limit when $\epsilon \to 0$ of piecewise constant ϵ-approximations $u^\epsilon(t,x)$ constructed via the wave front tracking algorithm [BaJ, HR]. For the detailed description of the algorithm we refer to [B]. The crucial ingredient in proving the global existence of the approximate solutions and the compactness of its sequence is the Glimm functional [G] controlling the total variation of perturbation and the amount of the future interactions. Below we briefly discuss a natural modification of this standard construction, applicable when the reference pattern is a strong k-th rarefaction $\mathcal{R}_k$ rather than a constant state. We then show that our Glimm-type functional Γ is indeed nonincreasing along any wave front tracking approximate solution, thanks to the BV stability condition (2.6).

Definition 4.1. *Let $\epsilon_0 > 0$. By $\mathcal{D}_{\epsilon_0}$ we denote the set of piecewise constant functions $v : \mathbf{R} \longrightarrow \mathbf{R}^n$ enjoying the following properties:*

(i) $v(-\infty) = u_l$, $v(+\infty) = u_r$,
(ii) $v(x) \in \Omega$ *for all* $x \in \mathbf{R}$,
(iii) *all jumps in v have amplitudes smaller than ϵ_0 (and thus the corresponding Riemann problems admit the standard self-similar solution). We order the waves in these solutions according to their location and speed; for a wave α by $i_\alpha : 1 \ldots n$ we denote its characteristic family and by ϵ_α its strength,*
(iv) *setting $\epsilon_\alpha^+ = \max(0, \epsilon_\alpha)$ and $\epsilon_\alpha^- = \max(0, -\epsilon_\alpha)$ there holds:*

$$\left| \left(\sum_{i_\alpha = k} \epsilon_\alpha^+ \right) - \Theta \right| + \left(\sum_{i_\alpha = k} \epsilon_\alpha^- \right) + \left(\sum_{i_\alpha \neq k} |\epsilon_\alpha| \right) \leq \epsilon_0. \tag{4.1}$$

Remark 4.2. Let v satisfy (i) (iii) of Definition 4.1 and let the bound (4.1) hold with ϵ_0 exchanged by another parameter δ. Then if only δ is small enough with respect to ϵ_0 then $v(x) \in \Omega_{2c}$ for all $x \in \mathbf{R}$ implies $v(x) \in \Omega_c$ for all $x \in \mathbf{R}$.

Take a function $u(0,\cdot) \in \mathcal{D}_{\epsilon_0}$, for some small $\epsilon_0 > 0$. Let $\epsilon << \epsilon_0$. Recall that the fundamental block for constructing the approximate solution $u^\epsilon(t,x)$ is provided by piecewise constant approximations of self-similar solutions to Riemann problems.

As customary, the non-physical waves generated by the Simplified Riemann Solver are said to belong to $(n+1)$-th characteristic family. The Simplified Riemann Solver is used whenever one of the interacting waves is non-physical or when the product of strenghts of incoming waves is bigger than a treshold parameter $\rho(\epsilon)$. The details can be found in [B], chapter 7. The associated non-physical weight w_{n+1} is defined as follows:

$$w_{n+1}(u) = c \cdot \exp(-C \cdot \lambda_k(u)) \quad \text{for} \quad u \in \Omega, \tag{4.2}$$

for some suitable constants $c, C > 0$. Let w_k be a positive constant, strictly smaller than all other weights $w_i(u)$ defined in Ω by (2.6) and (2.1). Recall that given a weak i-th wave, we associate with it the weight w_i computed at its left state.

Definition 4.3. *Let $u(0,\cdot) \in \mathcal{D}_{\epsilon_0}$, with some small $\epsilon_0 > 0$. Let u^ϵ be the piecewise constant ϵ-approximate solution, given by the wave front tracking algorithm. Assume t is not an interaction time of fronts in u^ϵ. Using the notation of Definition 4.1 we set:*

$$V(u^\epsilon(t,\cdot)) = \left| \left(\sum_{i_\alpha = k} \epsilon_\alpha^+ \right) - \Theta \right| + \left(\sum_{i_\alpha = k} \epsilon_\alpha^- \right) + \left(\sum_{i_\alpha \neq k} |\epsilon_\alpha| \right),$$

where the summations extend on all waves α present in $u^\epsilon(t,\cdot)$. The quadratic interaction potential is defined:

$$Q_0(u^\epsilon(t,\cdot)) = \sum_{(\alpha,\beta)\in\mathcal{A}} |\epsilon_\alpha \cdot \epsilon_\beta|,$$

with the set $\mathcal{A}$ containing all couples of perturbation waves (α,β) in $u^\epsilon(t,\cdot)$ approaching each other. More precisely, assuming $x_\alpha < x_\beta$, we have $(\alpha,\beta) \in \mathcal{A}$ iff $i_\alpha > i_\beta$ or else $i_\alpha = i_\beta$ and at least one of the waves is a genuinely nonlinear shock. In both cases we require that none of the waves α, β is a positive k-wave. Finally, let:

$$Q_{large}(u^\epsilon(t,\cdot)) = \sum_{i_\alpha \neq k} w_{i_\alpha}(u^\epsilon(t, x_\alpha-)) \cdot |\epsilon_\alpha| + \sum_{i_\alpha = k} w_k \cdot \epsilon_\alpha^-,$$

$$Q = Q_0 + Q_{large}, \qquad \Gamma = V + \kappa \cdot Q,$$

for some large constant κ, to be determined later.

Lemma 4.4. *Assume that the BV stability condition (2.6) holds. Then for some constants $c, \epsilon_0, \kappa > 0$ we have the following. Let $u(0,\cdot) \in \mathcal{D}_{\epsilon_0}$ and let u^ϵ be the corresponding piecewise constant approximate solution obtained through the wave front tracking algorithm. Then for any $t > 0$ when two wave fronts α and β interact, if $\Gamma(u^\epsilon(t-,\cdot)) \leq \epsilon_0$ then*

$$\begin{aligned} \Delta Q &= Q(u^\epsilon(t+,\cdot)) - Q(u^\epsilon(t-,\cdot)) \leq -c \cdot |\epsilon_\alpha \epsilon_\beta|, \\ \Delta\Gamma &= \Gamma(u^\epsilon(t+,\cdot)) - \Gamma(u^\epsilon(t-,\cdot)) \leq -c \cdot |\epsilon_\alpha \epsilon_\beta|. \end{aligned} \tag{4.3}$$

Proof. The proof consists of several cases, depending on whether the Accurate or the Simplified Riemann Solver is used and whether the interaction involves a k-th positive wave which we will view as a part of the reference rarefaction $\mathcal{R}_k$. We only give the main ideas, the detailed analysis is left to the reader.

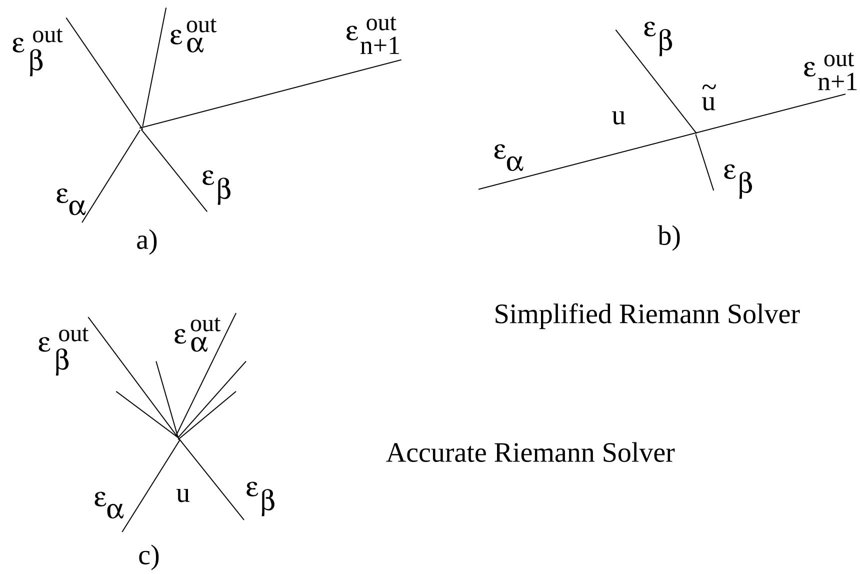

FIGURE 4.1

Case 1. - None of the interacting waves is a positive k-th wave, and the interaction is solved by the Accurate Riemann Solver (Figure 4.1 c)). By standard analysis [B] we have:

$$\Delta V = \mathcal{O}(1)|\epsilon_\alpha \epsilon_\beta|$$
$$\Delta Q_0 \leq -|\epsilon_\alpha \epsilon_\beta| + \mathcal{O}(1)\epsilon_0 \cdot |\epsilon_\alpha \epsilon_\beta|.$$

Further,

$$\Delta Q_{large} \leq \left(\sum_{j \neq i_\alpha, i_\beta} w_j^{out} \cdot |\epsilon_j^{out}| \right) + \left(w_{i_\alpha}^{out} |\epsilon_\alpha^{out}| - w_{i_\alpha} |\epsilon_\alpha| \right) + \left(w_{i_\beta}^{out} |\epsilon_\beta^{out}| - w_{i_\beta} |\epsilon_\beta| \right).$$

Consequently, $\Delta Q_{large} \leq C \cdot |\epsilon_\alpha \epsilon_\beta|$, where the constant C depends linearly on the upper bound of the weights $\{w_i\}$ as well as their derivatives $\{Dw_i\}$. In view of Remark 2.3 and assuming ϵ to be small enough we thus obtain the first estimate in (4.3), which in turns yields the second one for large κ.

Case 2. - Interaction of a wave of family $i_\beta \neq k$ with a k-th positive wave ($i_\alpha = k,\ \epsilon_\alpha > 0$) solved by Accurate Riemann Solver (Figure 4.1 c)). As before, we obtain:

$$\Delta V = \mathcal{O}(1)|\epsilon_\alpha \epsilon_\beta|$$
$$\Delta Q_0 = \mathcal{O}(1)\epsilon_0 \cdot |\epsilon_\alpha \epsilon_\beta|. \tag{4.4}$$

We view ΔQ_{large} as a function of the state $u \in \Omega$ attained by u^ϵ between the interacting fronts α and β and the strenghts ϵ_α and ϵ_β:

$$\Delta Q_{large} = -w_{i_\alpha} \cdot |\epsilon_\alpha| + \sum_{j\neq k} w_j^{out} \cdot |\epsilon_j^{out}| = G(u, \epsilon_\alpha, \epsilon_\beta).$$

Choose $\theta \in [0, \Theta]$ such that $||u - \mathcal{R}_k(\theta)|| < \epsilon_0$. Since $G(u, \epsilon_\alpha, 0) = G(u, 0, \epsilon_\beta) = 0$, we have:

$$\begin{aligned}&|G(u, \epsilon_\alpha, \epsilon_\beta) - G(\mathcal{R}_k(\theta), \epsilon_\alpha, \epsilon_\beta)| \\ &\qquad \leq |\epsilon_\alpha \epsilon_\beta| \cdot \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial \epsilon_\alpha \partial \epsilon_\beta} G(u, s\epsilon_\alpha, z\epsilon_\beta) - \frac{\partial^2}{\partial \epsilon_\alpha \partial \epsilon_\beta} G(\mathcal{R}_k(\theta), s\epsilon_\alpha, z\epsilon_\beta) \right| \mathrm{d}s \mathrm{d}z.\end{aligned}$$

If only the constant c in the definition (1.5) of Ω is small enough, the integrand in the above estimate is as small as we wish. Thus in view of Remark 2.2 we obtain $\Delta Q_{large} \leq -c \cdot |\epsilon_\alpha \epsilon_\beta|$ for some different constant $c > 0$, taking w_k sufficiently small with respect to other weights. If ϵ is small enough and κ large this implies (4.3).

We remark that if the interaction as in case 2 is to be solved by the Simplified Riemann Solver (Figure 2.1 a)), then (4.3) follows exactly as above provided we define ϵ_k^{out} to be equal to ϵ_k^{out} in the accurate solution and take the scaling constant c in (4.2) small with respect to other weights $w_i,\ i : 1 \ldots n$.

Case 3. - Interaction of a non-physical front ($i_\alpha = n+1$) with a positive k-wave ($i_\beta = k,\ \epsilon_\beta > 0$) solved by the Simplified Riemann Solver (Figure 4.1 b)). Again (4.4) is valid. Call u the left state of the wave α and $\tilde{u}$ the state attained by u^ϵ between the two outgoing waves. Then

$$\begin{aligned}\Delta Q_{large} &= w_{n+1}(\tilde{u}) \cdot |\epsilon_{n+1}^{out}| - w_{n+1}(u) \cdot |\epsilon_\alpha| \\ &\leq (w_{n+1}(\tilde{u}) - w_{n+1}(u)) \cdot |\epsilon_\alpha| + \mathcal{O}(1) w_{n+1}(\tilde{u}) \cdot |\epsilon_\alpha \epsilon_\beta| \\ &= c \cdot \exp(-C\lambda_k(\tilde{u})) \cdot [1 - \exp(-C\epsilon_\beta) + \mathcal{O}(1)\epsilon_\beta] \cdot |\epsilon_\alpha| \\ &= c \cdot \exp(-C\lambda_k(\tilde{u})) \cdot \left[\mathcal{O}(1) - \frac{\exp(C\epsilon_\beta) - 1}{\epsilon_\beta}\right] \cdot |\epsilon_\alpha \epsilon_\beta| \\ &\leq -c\frac{C}{4} \exp(-C\lambda_k(\tilde{u})) \cdot |\epsilon_\alpha \epsilon_\beta|\end{aligned}$$

if only C in (4.2) is large enough. Taking ϵ_0 small and κ large, we conclude (4.3). ■

Define now the domain

$$\bar{\mathcal{D}}_{\epsilon_0} = \mathrm{cl}\ \{v \in \mathcal{D}_{\epsilon_0}, \quad \Gamma(v) \leq \epsilon_0\}, \tag{4.5}$$

where cl denotes the closure in L^1_{loc}. Relying on Lemma 4.4 and Remark 4.2, we obtain:

Lemma 4.5. *In the setting of Lemma 4.4, an approximate solution $u^\epsilon(t,x)$ generated by the algorithm from initial data $\bar{u} \in \bar{\mathcal{D}}_{\epsilon_0}$ exists for all times $t > 0$ and enjoys the following properties:*

(i) $||\bar{u} - u^\epsilon(0, \cdot)||_{L^1} \leq \epsilon$,

(ii) *u^ϵ is piecewise constant, with jumps occuring along finitely many lines; jumps are of three types: shocks (and contact discontinuities), rarefaction fronts and non-physical waves; all jumps have strength $< \epsilon_0$, while all rarefaction fronts have strength $< \epsilon$,*

(iii) *along each shock or a rarefaction front not belonging to the k-th family we have its speed differ from the exact speed (Rankine - Hugoniot speed for shocks and the eigenvalue at the left state for rarefaction fronts) at most by ϵ; the speeds of all k-positive waves are exact (that is, equal to λ_k evaluated at the left state); all non-physical waves travel with speed $\hat{\lambda}$,*

(iv) *at each time $t \geq 0$ the sum of strengths of non-physical waves in u^ϵ is bounded by ϵ,*

(v) *for all $t \geq 0$ we have: $\Gamma(u^\epsilon(t, \cdot)) \leq \epsilon_0$.*

Now a standard argument yields that a subsequence of approximations u^ϵ converges to a solution of (1.1) (1.2) and that the domain $\bar{\mathcal{D}}_{\epsilon_0}$ is positively invariant with respect to the flow this way generated. Again, all the details can be found in [B]. To prove Theorem I it thus suffices to show:

Lemma 4.6. *Let $\bar{u} \in \mathrm{cl}\, \mathcal{E}_{c,\delta}$ for sufficiently small $c, \delta > 0$, as in Theorem I. Then $\bar{u} \in \bar{\mathcal{D}}_{\epsilon_0}$, for some $\epsilon_0 = \epsilon_0(\delta)$ and $\lim_{\delta \to 0} \epsilon_0(\delta) = 0$.*

The proof will be given in section 9.

5. First order rarefactions

We call a positive k-th wave located at y_0 at time $T > 0$ a first order k-rarefaction wave if there exists a continuous curve $y(t)$ with $y(T) = y_0$ such that for almost all $t \in [0, T]$, $y(t)$ is the location of a positive k-th wave. For each $t \in [0, +\infty)$ let $L^u(t)$ be the set of locations of first order k-rarefaction waves in u.

Lemma 5.1. *Let $u^\epsilon(t, x)$ be as in Lemma 4.5 (in particular $u^\epsilon(t, \cdot) \in \mathcal{D}_{\epsilon_0}$ for all $t \geq 0$). Then:*

$$\tilde{V}(t) := \left| \left(\sum_{x_\alpha \in L^u(t)} \epsilon_\alpha \right) - \Theta \right| + \left(\sum_{x_\alpha \notin L^u(t)} |\epsilon_\alpha| \right) = \mathcal{O}(1) \cdot \epsilon_0, \tag{5.1}$$

above the summations extend on all waves α present in $u^\epsilon(t, \cdot)$. Moreover if $y(t)$ is continuous and $y(t) \in L^u(t)$ for almost all $t \in [0, T]$ then:

$$\forall t, s \in [0, T] \quad |\lambda_k(u^\epsilon(t, y(t)-)) - \lambda_k(u^\epsilon(s, y(s)-))| = \mathcal{O}(1) \cdot \epsilon_0. \tag{5.2}$$

Proof. Above $\tilde{V}(0)$ is understood as $\tilde{V}(t)$, for t close to 0. To prove (5.1) one defines new interaction potentials by the same formula as Q_0 and Q_{large} but treating positive k-th waves located in $\mathbf{R} \setminus L^u(t)$ as perturbations. Then Lemma 4.4 and its proof are still valid, with V exchanged there to $\tilde{V}$. Thus the estimate in (5.1) follows.

In order to deduce (5.2) we may restrict our attention to the case $t = T$ and $s = 0$. It is convenient to consider the evolution of the related functional:

$$\tilde{\Gamma}(t) = |y'(t) - y'(0)| + \kappa \cdot \tilde{V}(t) + \kappa^2 \cdot Q(t),$$

where $\tilde{V}(t)$ is defined as the sum of strengths of perturbation waves α in:

$$\{x_\alpha < y(t) \text{ and } i_\alpha \geq k\} \cup \{x_\alpha > y(t) \text{ and } i_\alpha \leq k\}$$

and $\kappa > 1$ is a large constant. We see that when $y(t)$ interacts with another wave α then $\Delta Q \leq 0$, $\Delta y' = \mathcal{O}(1)|\epsilon_\alpha|$ and $\Delta\tilde{V} = -|\epsilon_\alpha|$. On the other hand at any other time $\Delta y' = 0$ and $\Delta(\tilde{V} + \kappa Q) \leq 0$. Thus $\tilde{\Gamma}$ is a nonincreasing function of t if only κ is large. Hence $|y'(T) - y'(0)| \leq \tilde{\Gamma}(0) = \mathcal{O}(1)\epsilon_0$ and (5.2) follows since

$$y'(t) = \lambda_k(u^\epsilon(t, y(t)-))$$

for almost all $t \in [0, T]$. ■

6. Lyapunov functional – a proof of Theorem II

Towards a proof of Theorem II, in this section we carry out the construction of the Lyapunov functional Φ. Following [LY, BLY], $\Phi(u, v)$ is supposed to control the L^1 distance between the two ϵ-approximate solutions $u, v : [0, \infty) \times \mathbf{R} \longrightarrow \mathbf{R}^n$ obtained by the wave front tracking algorithm and thus enjoying the properties in Lemma 4.5. Assuming the L^1 stability condition (3.1), the two crucial properties of Φ will be the following:

$$\Phi(u(t, \cdot), v(t, \cdot)) \leq \Phi(u(s, \cdot), v(s, \cdot)) + C \cdot \epsilon \cdot (t - s), \tag{6.1}$$

$$\frac{1}{C} \cdot ||u(t, \cdot) - v(t, \cdot)||_{L^1} \leq \Phi(u(t, \cdot), v(t, \cdot)) \leq C \cdot ||u(t, \cdot) - v(t, \cdot)||_{L^1}, \tag{6.2}$$

for all $t > s \geq 0$ and a uniform constant $C > 0$ depending only on the system (1.1). In the remaining part of the article we will concentrate on proving (6.1) (6.2) for a below constructed functional Φ. Taking then $\mathcal{D} = \bar{\mathcal{D}}_{\epsilon_0}$, for a small $\epsilon_0 > 0$, the proof of Theorem II will follow by the already standard argument as in [B] chapter 8.3.

Fix a positive and small constant ν. Given piecewise constant functions u and v, let

$$T = \sup\Big\{t > 0; \quad \exists_x \ |\lambda_k(u(t, x)) - \lambda_k(v(t, x))| > \nu\Big\}, \tag{6.3}$$

Lemma 6.1. *T defined as above is finite.*

Proof. Notice that since the total strength of perturbation waves is of the order ϵ_0 at each time t, then taking $\epsilon << \epsilon_0$ we have:

$$\sup_{t\geq 1, x} ||u(t, x) - \tilde{u}(t, x)|| + \sup_{t\geq 1, x} ||v(t, x) - \tilde{v}(t, x)|| = \mathcal{O}(1)\epsilon_0. \tag{6.4}$$

The functions $\tilde{u}$ and $\tilde{v} : [1, +\infty) \times \mathbf{R} \longrightarrow \mathbf{R}^n$ are smooth solutions to (1.1) with initial data:

$$\tilde{u}(1, x) = u_0(1, \psi(x)), \quad \tilde{v}(1, x) = v_0(1, \phi(x))$$

where ψ and $\phi : \mathbf{R} \longrightarrow \mathbf{R}$ are some increasing diffeomorphisms. We want to show that

$$\lim_{t\to+\infty} \sup |\lambda_k(\tilde{u}(t, x)) - \lambda_k(\tilde{v}(t, x))| = 0, \tag{6.5}$$

which in view of (6.4) and taking $\epsilon << \epsilon_0$ will imply that $T < +\infty$.

Notice that for each $t \geq 1$, $\tilde{u}$ is constant outside the interval:

$$J_t^u = \big[\psi^{-1}(\lambda_k(u_l)) + \lambda_k(u_l) \cdot (t - 1), \psi^{-1}(\lambda_k(u_r)) + \lambda_k(u_r) \cdot (t - 1)\big]$$

and that it propagates along the straight lines - characteristics having slopes λ_k inside the region $\{(t,x);\ x \in J_t^u\}$. Consequently, one has:

$$\sup_{x\in J_t^u\cap J_t^v} |\lambda_k(\tilde u(t,x)) - \lambda_k(\tilde v(t,x))| \leq \frac{\max_{w,z\in\{u_l,u_r\}} |\psi^{-1}(\lambda_k(w)) - \phi^{-1}(\lambda_k(z))|}{t-1}, \tag{6.6}$$

where the interval J_t^v is defined as J_t^u, by means of the diffeomorphism ϕ. Obviously, the right hand side of (6.6) vanishes as $t\to+\infty$. Likewise, $\sup_{x\not\in J_t^u\cap J_t^v} |\lambda_k(\tilde u(t,x)) - \lambda_k(\tilde v(t,x))|$ also converges to 0, because of the spreading of the rarefactions in $\tilde u$ and $\tilde v$. This establishes (6.5). ■

The definition of the functional $\Phi(u,v)$ falls in two parts.

Case 1 (the profiles u and v are apart from each other): $\mathbf{t\in[0,T]}$. Let $T>0$. Without loss of generality we may assume that for some x there holds: $\lambda_k(u(t,x)) > \lambda_k(v(t,x)) + 3\nu/4$ (the case of the opposite inequality may be treated similarily). Because of the estimate in (5.1) and taking $\epsilon_0 << \nu$, there exists then a nonempty interval $I(T) = [z_0^-, z_0^+]$ such that $z_0^- \in L^u(T),\ z_0^+\in L^v(T)$ and:

$$\forall x,y\in I(T) \quad \lambda_k(u(T,x)) - \lambda_k(v(T,y)) > \nu/2. \tag{6.7}$$

For $t\in[0,T]$ call $I(t)$ the space interval whose boundary is continuous polygonals $z^-(t)\in L^u(t),\ z^+(t)\in L^v(t)$ with $z^-(T) = z_0^-$ and $z^+(T) = z_0^+$. Notice that taking ϵ_0 small enough Lemma 5.1 yields:

$$\forall t\in[0,T] \quad \forall x,y\in I(t) \quad \lambda_k(u(t,x)) - \lambda_k(v(t,y)) > \nu/3. \tag{6.8}$$

For all $t\in[0,T)$ the Lyapunov functional Φ is defined by the formula:

$$\Phi(u,v)(t) = ||u(t,\cdot) - v(t,\cdot)||_{L^1} + \kappa_1\cdot|I(t)|, \tag{6.9}$$

where $|I(t)|$ stands for the length of the interval $I(t)$ and κ_1 is a sufficiently large integer constant.

Lemma 6.2. *If only κ_1 is large enough then the functional Φ satisfies:*

$$\Phi(u(t',\cdot),v(t',\cdot)) \leq \Phi(u(t,\cdot),v(t,\cdot)) \tag{6.10}$$

$$||u(t,\cdot)-v(t,\cdot)||_{L^1} \leq \Phi(u(t,\cdot),v(t,\cdot)) \leq C\cdot||u(t,\cdot)-v(t,\cdot)||_{L^1}, \tag{6.11}$$

for all $0\leq t\leq t'\leq T$ and a uniform constant $C>0$.

Proof. The equivalence (6.11) of Φ with the L^1 distance follows in view of (6.8).

Denote by $\mathcal{J}(u)$ and $\mathcal{J}(v)$ the sets of all jumps in u and v, respectively. To prove (6.10) fix $t\in[0,T)$ which is not a time of interaction of any couple of fronts in u or v. We have:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u,v)(t) = \\
&\quad\sum_{\alpha\in\mathcal{J}(u)\cup\mathcal{J}(v)} \Big| |u(x_\alpha+,t) - v(x_\alpha+,t)| - |u(x_\alpha-,t) - v(x_\alpha-,t)| \Big| \cdot \dot x_\alpha \\
&\quad + \kappa_1\cdot\frac{\mathrm{d}}{\mathrm{d}t}|I(t)|.
\end{aligned} \tag{6.12}$$

The first term in (6.12) is of the order of $\mathcal{O}(1)$ because of the finite speed of propagation, boundedness of $TV(u(t))$ and $TV(v(t))$, and:

$$|u(x_\alpha+,t) - v(x_\alpha+,t)| - |u(x_\alpha-,t) - v(x_\alpha-,t)| = \mathcal{O}(1)|\epsilon_\alpha|.$$

On the other hand in view of (6.8) we have $\mathrm{d}/\mathrm{d}t\ |I(t)| \leq -\nu/4$. Thus if κ_1 is large with respect to the system constants and the prechosen ν, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u,v)(t) \leq 0.$$

Integrating in time we conclude (6.10). ■

Case 2 (u and v close): $\mathbf{t \geq T}$. The Lyapunov functional Φ is defined as in [BLY, B]:

$$\Phi(u,v) = \int_{-\infty}^{+\infty} \sum_{i=1}^{n} W_i(x) \cdot w_i(x) \cdot |q_i(x)|\ \mathrm{d}x. \tag{6.13}$$

The scalar quantities $q_i(x)$ are roughly speaking the curvlinear coordinates of the vector $v(x) - u(x)$, computed along combinations of shock curves in Ω. The precise definition of W_i and w_i will be our concern in the sequel.

The coordinates $\{q_i(x)\}_{i=1}^n$ are implicitely defined by:

$$v(x) = \mathcal{S}_n(q_n(x)) \circ \ldots \mathcal{S}_k(q_k(x)) \circ \ldots \mathcal{S}_1(u(x), q_1(x)). \tag{6.14}$$

Such decomposition exists if ν is small enough, as $|\lambda_k(u(x,t)) - \lambda_k(v(x,t))| \leq \nu$ for all x and $t \geq T$. The weights $w_i(x)$ are given by:

$$w_i(x) = w_i\Big(\mathcal{S}_{i-1}(q_{i-1}(x)) \circ \ldots \mathcal{S}_1(u(x), q_1(x))\Big) \tag{6.15}$$

where the w_i-s in the right hand side are given by (2.1) and the L^1 stability condition (3.1). We see that the weights $w_i(x)$ in (6.15) are computed at the left states of the corresponding waves. Recall that $w_k > 0$ is constant in Ω.

We will now define the functional weights $W_i(x)$. Recall that $i_\alpha \in \{1 \ldots n+1\}$ is the family of the jump located at x_α with strength ϵ_α. Also, by $\mathcal{J}(u)$ and $\mathcal{J}(v)$ we denoted the sets of all jumps in u and v. Let $\mathcal{P}(u)$ and $\mathcal{P}(v)$ be the resepective subsets of $\mathcal{J}(u)$ and $\mathcal{J}(v)$, containing these α for which $i_\alpha \neq n+1$ and either $i_\alpha \neq k$ or $i_\alpha = k$ and $\epsilon_\alpha < 0$.

Define the quantities $A_i(x)$ measuring the total amount of physical perturbation waves in u and v which approach the i-th wave $q_i(x)$ located at x [BLY]. More precisely, when the i-th field is linearly degenerate we set:

$$A_i(x) = \left[\sum_{\substack{\alpha \in \mathcal{P}(u)\cup\mathcal{P}(v)\\ x_\alpha < x,\ i_\alpha > i}} + \sum_{\substack{\alpha \in \mathcal{P}(u)\cup\mathcal{P}(v)\\ x_\alpha > x,\ i_\alpha < i}}\right] |\epsilon_\alpha|.$$

For a genuinely nonlinear i-th field:

$$\begin{aligned} A_i(x) = &\left[\sum_{\substack{\alpha \in \mathcal{P}(u)\cup\mathcal{P}(v)\\ x_\alpha < x,\ i_\alpha > i}} + \sum_{\substack{\alpha \in \mathcal{P}(u)\cup\mathcal{P}(v)\\ x_\alpha > x,\ i_\alpha < i}}\right] |\epsilon_\alpha| \\ &+ \begin{cases} \left[\displaystyle\sum_{\substack{\alpha \in \mathcal{P}(u)\\ x_\alpha < x,\ i_\alpha = i}} + \sum_{\substack{\alpha \in \mathcal{P}(v)\\ x_\alpha > x,\ i_\alpha = i}}\right] |\epsilon_\alpha| & \text{if } q_i(x) < 0, \\[2ex] \left[\displaystyle\sum_{\substack{\alpha \in \mathcal{P}(v)\\ x_\alpha < x,\ i_\alpha = i}} + \sum_{\substack{\alpha \in \mathcal{P}(u)\\ x_\alpha > x,\ i_\alpha = i}}\right] |\epsilon_\alpha| & \text{if } q_i(x) \geq 0. \end{cases} \end{aligned}$$

Define:

$$\forall i: 1 \dots n \quad W_i(x) = 1 + \kappa_2(Q(u) + Q(v)) + \kappa_3 A_i(x) + \delta_{ik} \cdot \kappa_4 |q_k(x)|. \tag{6.16}$$

Here Q stands for the Glimm's interaction potential from Definition 4.3, δ_{ik} is the Kronecker delta. The (large) constants $\kappa_2, \kappa_3, \kappa_4$ are to be determined later; we see that as soon as they have been assigned, we can impose a suitably small bound on the amount of perturbation in u and v (by taking ϵ_0 small in (4.5), or in particular δ small in Theorem I), so that

$$1 \leq W_i(x) \leq 4 \qquad \text{for all } i, x. \tag{6.17}$$

This ends the definition of the functional Φ.

Lemma 6.3. *The functional Φ constructed above satisfies (6.1) and*

$$\frac{1}{C}||u(t,\cdot) - v(t,\cdot)||_{L^1} \leq \Phi(u(t,\cdot), v(t,\cdot)) \leq ||u(t,\cdot) - v(t,\cdot)||_{L^1}, \tag{6.18}$$

for all $t' > t \geq T$ and a uniform constant $C > 0$ depending only on the system (1.1).

Proof. The equivalence of Φ with the L^1 distance as in (6.18) follows from (6.17) if we take the weights $\{w_i\}_{i=1}^n$ small enough.

To prove the estimate in (6.1), define $\lambda_i(x)$ as the Rankine-Hugoniot speed of the shock/contact $q_i(x)$.

Recall that a direct calculation [BLY] gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u(t), v(t)) = \sum_{\alpha \in \mathcal{J}(u) \cup \mathcal{J}(v)} \sum_{i=1}^{n} E_{\alpha,i}, \tag{6.19}$$

with

$$\begin{aligned} E_{\alpha,i} = & \left(W_i \cdot w_i \cdot |q_i|\right)(x_\alpha+) \cdot (\lambda_i(x_\alpha+) - \dot{x}_\alpha) \\ & - \left(W_i \cdot w_i \cdot |q_i|\right)(x_\alpha-) \cdot (\lambda_i(x_\alpha-) - \dot{x}_\alpha). \end{aligned} \tag{6.20}$$

Above $\dot{x}_\alpha$ denotes the speed of propagation of the wave α located at x_α. We will prove that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u(t), v(t)) \leq \mathcal{O}(1)\epsilon \tag{6.21}$$

for every time $t \geq T$ where the fronts in u or v do not interact. Indeed, this will be the goal of the next section.

Next, let t be such that say fronts ϵ_α and ϵ_β in u interact. It is easy to notice that for every x and i we have:

$$A_i(t+, x) - A_i(t-, x) \leq \mathcal{O}(1)|\epsilon_\alpha \epsilon_\beta|.$$

On the other hand, by Lemma 4.4, the quantity $Q(u)$ decreases by the same order of magnitude. Thus if κ_2 in (6.16) is large enough, all functional weights $W_i(x)$ must decrease across the time t. Consequently, the whole functional Φ decreases as well. Based on these two observations and integrating (6.21) in time, we conclude (6.1). ∎

7. Stability estimates

In this section we want to establish the inequality (6.21) by estimating local terms $E_{\alpha,i}$ in (6.20). All calculations refer to a fixed jump $\alpha \in \mathcal{J}(v)$, propagating with speed $\dot{x}_\alpha$ and belonging to a characteristic family $i_\alpha : 1 \dots n+1$. When $\alpha \in \mathcal{J}(u)$ only minimal and obvious modifications of our arguments are required and so we leave them to the reader.

We first focus on the case $i_\alpha = n+1$. We will prove that:

$$\sum_{i=1}^{n} E_{\alpha,i} \leq \mathcal{O}(1)|\epsilon_\alpha|, \tag{7.1}$$

Indeed:

$$\forall\ i \neq k \quad |w_i^+ q_i^+ - w_i^- q_i^-| + |\lambda_i^+ - \lambda_i^-| = \mathcal{O}(1)|\epsilon_\alpha|.$$

Also, for $i \neq k$ and if sgn $q_i^- =$ sgn q_i^+ we have: $W_i^+ = W_i^-$. On the other hand, if sgn $q_i^- \neq$ sgn q_i^+ then $|q_i^+| + |q_i^-| = \mathcal{O}(1)|\epsilon_\alpha|$ and consequently:

$$\sum_{i \neq k} E_{\alpha,i} \leq \mathcal{O}(1)|\epsilon_\alpha|.$$

In a similar manner, $E_{\alpha,k} \leq \mathcal{O}(1)|\epsilon_\alpha|$ if sgn $q_k^- \neq$ sgn q_k^+. The same is true if sgn $q_k^- =$ sgn q_k^+ because then

$$\Delta W_k = \mathcal{O}(1)|\epsilon_\alpha|.$$

The bound (7.1) is thus proven. Now, recalling Lemma 4.5 (iv), (7.1) yields:

$$\sum_{i_\alpha = n+1} \sum_{i=1}^{n} E_{\alpha,i} \leq \mathcal{O}(1)\epsilon. \tag{7.2}$$

Let now $i_\alpha : 1 \dots n$. Our goal will be to prove that:

$$\sum_{i=1}^{n} E_{\alpha,i} = \mathcal{O}(1)\epsilon|\epsilon_\alpha|. \tag{7.3}$$

Recall that by Lemma 4.5 $|\epsilon_\alpha| < \epsilon$, whenever α is a rarefaction wave. In view of (1.8) and the definition (6.20) we may thus without loss of generality replace each rarefaction wave α by a (possibly non-entropic) shock having the original strength ϵ_α and the speed $\dot{x}_\alpha = \lambda_k(v(x_\alpha-))$. We will prove that with this modification the same estimate as in (7.3) holds. For simplicity, we write $W_i^+ = W_i(x_\alpha+)$, $q_i^- = q_i(x_\alpha-)$, etc ...

The proof falls in several cases. Throughout the calculations, we often use the estimates from section 8. When α is a part of the rarefaction $\mathcal{R}_k$, our estimates rely on the stability condition (3.1), the parameters w_k and ν are chosen so that the negative term in (8.3) overcomes extra contributions which are not of the order $\mathcal{O}(1)\epsilon_\alpha^2$. When α is a perturbation wave, our argument is essentialy a modification of the one from [BLY]. We again adjust ν appropriately and then take the constant κ_3 in (6.16) to be large with respect to other quantities in the derived estimates. The parameter ϵ_0, measuring the amount of perturbing waves present at any time in both approximate solutions u and v, is always set to be as small as needed, in particular $\epsilon_0 << \nu$.

Case 1. $i_\alpha = k$ and $\epsilon_\alpha > 0$. Recall that by Lemma 4.5 we have $|\epsilon_\alpha| < \epsilon$. We will prove:

$$\sum_{i=1}^{n} E_{\alpha,i} = \mathcal{O}(1)\epsilon_\alpha^2, \tag{7.4}$$

which will clearly imply (7.3). We first estimate

$$\begin{aligned}\sum_{i\neq k} E_{\alpha,i} =& \sum_{i\neq k}(\Delta W_i)\cdot w_i^-|q_i^-|(\lambda_i^- - \dot{x}_\alpha) \\ &+ \sum_{i\neq k} W_i^+ \cdot \left[w_i^+|q_i^+|(\lambda_i^+ - \dot{x}_\alpha) - w_i^-|q_i^-|(\lambda_i^- - \dot{x}_\alpha)\right]\end{aligned} \tag{7.5}$$

Fix $i \neq k$. Notice that if sgn $q_i^+ \neq$ sgn q_i^- then

$$|q_i^-| \leq |q_i^+ - q_i^-| \quad \text{and} \quad \Delta W_i = \mathcal{O}(1)\kappa_3\epsilon_0. \tag{7.6}$$

On the other hand, if sgn $q_i^+ =$ sgn q_i^- then $\Delta W_i = 0$. Thus the first summand in (7.5) can be estimated using Lemma 8.1:

$$\begin{aligned}&\sum_{i\neq k}(\Delta W_i)\cdot w_i^-|q_i^-|(\lambda_i^- - \dot{x}_\alpha) \\ &\qquad \leq \mathcal{O}(1)\kappa_3\epsilon_0 \cdot \left[\epsilon_\alpha \cdot \left(\sum_{s>k}|q_s^-|\right) + \epsilon_\alpha\cdot|q_k^-|^2 + \epsilon_\alpha^2\right].\end{aligned} \tag{7.7}$$

In order to deal with the second summand in (7.5), we notice that if sgn $q_i^+ \neq$ sgn q_i^- then by (7.6) and Lemma 8.1, there holds:

$$\begin{aligned}&\left|w_i^+|q_i^+|(\lambda_i^+ - \dot{x}_\alpha) - w_i^-|q_i^-|(\lambda_i^- - \dot{x}_\alpha)\right| \\ &\qquad \leq \mathcal{O}(1)\left[\epsilon_\alpha \cdot \left(\sum_{s>k}|q_s^-|\right) + \epsilon_\alpha\cdot|q_k^-|^2 + \epsilon_\alpha^2\right].\end{aligned} \tag{7.8}$$

The same is true when sgn $q_i^+ =$ sgn q_i^-, as in this case the left hand side of (7.8) equals to $|w_i^+q_i^+(\lambda_i^+ - \dot{x}_\alpha) - w_i^-q_i^-(\lambda_i^- - \dot{x}_\alpha)|$ and so one can again employ the estimates of Lemma 8.1. In view of Remark 2.3, combining (7.5) (7.7) and (7.8) we obtain:

$$\begin{aligned}&\sum_{i\neq k} W_i^+ \cdot \left[w_i^+|q_i^+|(\lambda_i^+ - \dot{x}_\alpha) - w_i^-|q_i^-|(\lambda_i^- - \dot{x}_\alpha)\right] \\ &\leq [1 + \kappa_2(Q(u)+Q(v))]\cdot \sum_{i\neq k}\left[w_i^+|q_i^+|(\lambda_i^+ - \dot{x}_\alpha) - w_i^-|q_i^-|(\lambda_i^- - \dot{x}_\alpha)\right] \\ &\quad + \mathcal{O}(1)\kappa_1\epsilon_0\cdot\left[\epsilon_\alpha \cdot \left(\sum_{s>k}|q_s^-|\right) + \epsilon_\alpha\cdot|q_k^-|^2 + \epsilon_\alpha^2\right].\end{aligned} \tag{7.9}$$

Estimating the first term in the right hand side of (7.9) by Lemma 8.3 and noting (7.7), the quantity in (7.5) can be further bounded by:

$$\sum_{i\neq k} E_{\alpha,i} \leq -\frac{\gamma_1}{2}\epsilon_\alpha\cdot\left(\sum_{s>k}|q_s^-|\right) + \mathcal{O}(1)\cdot\left[\epsilon_\alpha\cdot|q_k^-|^2 + \epsilon_\alpha^2\right], \tag{7.10}$$

if ϵ_0 is small enough.

We now aim at establishing (7.4) by estimating the remaining term $E_{\alpha,k}$. We distinguish two subcases.

Subcase 1.1. sgn $q_k^+ \neq$ sgn q_k^-. Then:

$$\Delta W_k = \mathcal{O}(1)\kappa_4\epsilon_\alpha + \mathcal{O}(1)\kappa_3\epsilon_0.$$

Therefore we have:

$$\begin{aligned}(\Delta W_k) w_k |q_k^-|(\lambda_k^- - \dot{x}_\alpha) &\leq \mathcal{O}(1) w_k \epsilon_\alpha \left(\kappa_4\epsilon_\alpha + \kappa_3\epsilon_0\right) \cdot \left(\epsilon_\alpha + \sum_{s>k}|q_s^-|\right) \\ &\leq \mathcal{O}(1) w_k \kappa_1 \epsilon_0 \epsilon_\alpha \cdot \left(\epsilon_\alpha + \sum_{s>k}|q_s^-|\right) + \mathcal{O}(1)\kappa_4\epsilon_\alpha^2. \end{aligned} \tag{7.11}$$

On the other hand:

$$\begin{aligned} &W_k^+ w_k \left[|q_k^+|(\lambda_k^+ - \dot{x}_\alpha) - |q_k^-|(\lambda_k^- - \dot{x}_\alpha)\right] \\ &\qquad \leq \mathcal{O}(1) w_k \epsilon_\alpha \left[|\lambda_k^+ - \dot{x}_\alpha| + |\lambda_k^- - \dot{x}_\alpha|\right] \leq \mathcal{O}(1) w_k \epsilon_\alpha \cdot \left(\sum_{s>k}|q_s^-|\right). \end{aligned} \tag{7.12}$$

Summing (7.11) and (7.12) we obtain

$$E_{\alpha,k} = \mathcal{O}(1) w_k \epsilon_\alpha \cdot \left(\sum_{s>k}|q_s^-|\right) + \mathcal{O}(1)\kappa_4\epsilon_\alpha^2. \tag{7.13}$$

The bound (7.4) now follows by (7.13) and (7.10) if only w_k is choosen suitably small with respect to the constant γ_1 and for small ϵ_0.

Subcase 1.2. sgn $q_k^+ =$ sgn q_k^-. By Lemma 8.1, we have:

$$\Delta|q_k| = (\text{sgn } q_k) \cdot \epsilon_\alpha + \mathcal{O}(1)\epsilon_\alpha \left(|q_k^-|^2 + \left(\sum_{s>k}|q_s^-|\right) + \epsilon_\alpha\right).$$

Thus, if only ϵ_0 and ν are small enough:

$$(\text{sgn } q_k) \cdot \Delta|q_k| \geq \epsilon_\alpha/2.$$

Moreover:

$$\lambda_k^- - \dot{x}_\alpha = \mathcal{O}(1)\left(\sum_{s>k}|q_s^-|\right) + (\text{sgn } q_k) \cdot \left(-\frac{|q_k^-|}{2} + \mathcal{O}(1)|q_k^-|^2\right) + \mathcal{O}(1)\epsilon_\alpha^2. \tag{7.14}$$

Recall that $\Delta W_k = \kappa_4 \Delta|q_k|$. Hence:

$$\begin{aligned}(\Delta W_k) \cdot w_k |q_k^-|(\lambda_k^- - \dot{x}_\alpha) &= \kappa_4 w_k \cdot (\Delta|q_k|) \cdot |q_k^-|(\lambda_k^- - \dot{x}_\alpha) \\ &= w_k\kappa_4 \cdot \left[\mathcal{O}(1)\epsilon_\alpha|q_k^-|\left(\sum_{s>k}|q_s^-|\right) + \mathcal{O}(1)\epsilon_\alpha|q_k^-|^3 \right.\\ &\qquad \left. - \frac{1}{2}(\Delta|q_k|)(\text{sgn } q_k)|q_k^-|^2\right] + \mathcal{O}(1)\kappa_4\epsilon_\alpha^2 \\ &\leq w_k \cdot \left[\mathcal{O}(1)\kappa_4\epsilon_\alpha|q_k^-|\left(\sum_{s>k}|q_s^-|\right) + \mathcal{O}(1)\epsilon_\alpha|q_k^-|^2\right] \\ &\quad - w_k\frac{\kappa_4}{4}\epsilon_\alpha|q_k^-|^2 + \mathcal{O}(1)\kappa_4\epsilon_\alpha^2. \end{aligned} \tag{7.15}$$

Now, using (7.14) and Lemma 8.1 we obtain:

$$(q_k^+ - q_k^-)(\lambda_k^- - \dot{x}_\alpha) = -\frac{q_k^- \epsilon_\alpha}{2} + \mathcal{O}(1)\epsilon_\alpha \left(|q_k^-|^2 + \left(\sum_{s>k} |q_s^-| \right) + \epsilon_\alpha \right). \tag{7.16}$$

On the other hand, by Lemma 8.1:

$$q_k^+(\lambda_k^+ - \lambda_k^-) = \frac{q_k^- \epsilon_\alpha}{2} + \mathcal{O}(1)\epsilon_\alpha \left(|q_k^-|^2 + \left(\sum_{s>k} |q_s^-| \right) + \epsilon_\alpha \right).$$

Thus, in view of (7.16):

$$q_k^+(\lambda_k^+ - \dot{x}_\alpha) - q_k^-(\lambda_k^- - \dot{x}_\alpha) = \mathcal{O}(1)\epsilon_\alpha \left(|q_k^-|^2 + \left(\sum_{s>k} |q_s^-| \right) + \epsilon_\alpha \right).$$

The above bound combined with (7.15) yields:

$$\begin{aligned} E_{\alpha,k} = w_k \cdot \Bigg[&-\frac{\kappa_4}{5}\epsilon_\alpha |q_k^-|^2 + \mathcal{O}(1)\kappa_4 \epsilon_\alpha |q_k^-| \left(\sum_{s>k} |q_s^-| \right) \\ &+ \mathcal{O}(1)\epsilon_\alpha \left(\sum_{s>k} |q_s^-| \right) \Bigg] + \mathcal{O}(1)\kappa_4 \epsilon_\alpha^2, \end{aligned} \tag{7.17}$$

if only the constant κ_4 is larger than several independent quantities $\mathcal{O}(1)$ in the above series of estimates. Combining (7.17) and (7.10) we obtain (7.4) for w_k small and κ_4 large enough.

Case 2. $i_\alpha \neq k$. Note that for $i \neq k$ the quantities $E_{\alpha,i}$ can be estimated exactly as in [BLY], see also [B] chapter 8.2. On the other hand, for $i = k$:

$$\Delta W_k = \kappa_3 \cdot \text{sgn } (i_\alpha - k) \cdot |\epsilon_\alpha| + \kappa_4 \cdot \Delta |q_k|$$

and

$$\Delta |q_k| = \mathcal{O}(1)|\epsilon_\alpha| \cdot \sum_{i=1}^n |q_i^-| = \mathcal{O}(1)|\epsilon_\alpha|(\epsilon_0 + \nu).$$

Thus the term in $E_{\alpha,k}$ containing ΔW_k can be estimated as follows:

$$\begin{aligned} (\Delta W_k) w_k |q_k^-|(\lambda_k^- - \dot{x}_\alpha) \leq &- \kappa_3 w_k \epsilon_\alpha |q_k^-||\lambda_k^- - \dot{x}_\alpha| \\ &+ \mathcal{O}(1)\kappa_4 w_k \epsilon_\alpha (\epsilon_0 + \nu)|q_k^-||\lambda_k^- - \dot{x}_\alpha| \\ \leq &- \frac{\kappa_3}{2} w_k \epsilon_\alpha |q_k^-||\lambda_k^- - \dot{x}_\alpha|, \end{aligned}$$

if only $\epsilon_0 + \nu$ is small enough. The analysis in [BLY] can thus be applied to get (7.3).

Case 3. $i_\alpha = k$ and $\epsilon_\alpha < 0$. If $|\epsilon_\alpha| < \epsilon$ and $|q_k^-| \leq 2|\epsilon_\alpha|$ then recalling that $\Delta W_k \leq W_k^- + W_k^+ \leq 8$ by (6.17), and using (8.64) from [B] we conclude (7.3). The same argumentation as in [B] page 167 yields (7.3) when $q_k^+ < 0 < q_k^-$.

We will now focus on the case when q_k^- and q_k^+ have the same sign. In view of the analysis of Lemma 8.3 we have:

$$\begin{aligned}\Delta W_k = {} & \kappa_3(\text{sgn } q_k)|\epsilon_\alpha| + \kappa_4|q_k^+ - q_k^-| = \kappa_3(\text{sgn } q_k)|\epsilon_\alpha| \\ & + \kappa_4(\text{sgn } q_k)\cdot\left[-|\epsilon_\alpha| + \mathcal{O}(1)|\epsilon_\alpha||q_k^-|^2 + \mathcal{O}(1)|\epsilon_\alpha|\left(\sum_{s>k}|q_s^-|\right) + \mathcal{O}(1)\epsilon_\alpha^2\right].\end{aligned} \tag{7.18}$$

Recalling the formula (8.50) from [B]:

$$\dot{x}_\alpha - \lambda_k^- = \frac{q_k^- + \epsilon_\alpha}{2} + \mathcal{O}(1)\left[|q_k^- + \epsilon_\alpha|(|q_k^-| + |\epsilon_\alpha|) + \sum_{s\neq k}|q_s^-|\right],$$

the estimate (7.18) implies for κ_3 large (also $\kappa_3 > 2\kappa_4$) and ϵ_0 small:

$$\begin{aligned}(\Delta W_k)w_k|q_k^-|(\lambda_k^- - \dot{x}_\alpha) \leq & -\frac{\kappa_3}{3}w_k|\epsilon_\alpha||q_k^-||q_k^- + \epsilon_\alpha| \\ & + \mathcal{O}(1)\kappa_3 w_k|\epsilon_\alpha||q_k^-|\cdot\left(\sum_{s\neq k}|q_s^-|\right) + \mathcal{O}(1)\kappa_4\epsilon_\alpha^2.\end{aligned} \tag{7.19}$$

Now, by the same reasoning as in [B] chapter 8.2. page 165, we see that for ν small and some constant $c > 0$, there holds:

$$\begin{aligned}W_k w_k\Delta[|q_k|(\lambda_k - \dot{x}_\alpha)] + \sum_{i\neq k}E_{\alpha,i} \leq & -c\kappa_3|\epsilon_\alpha|\sum_{s\in\mathcal{I}}|q_s^-| \\ & + \mathcal{O}(1)|\epsilon_\alpha|\left(|q_k^-||q_k^- + \epsilon_\alpha| + \sum_{s\neq k}|q_s^-|\right),\end{aligned} \tag{7.20}$$

$$\sum_{i\neq k}|q_i^-| \leq |q_k^-||q_k^- + \epsilon_\alpha| + 2\sum_{s\in\mathcal{I}}|q_s^-|. \tag{7.21}$$

The index set $\mathcal{I}$ is defined as: $\mathcal{I} = \{i : 1\dots n;\ i\neq k \text{ and sgn } q_i^- = \text{sgn } q_i^+\}$. Thus (7.19) becomes by (7.21):

$$\begin{aligned}(\Delta W_k)w_k|q_k^-|(\lambda_k^- - \dot{x}_\alpha) \leq & -\frac{\kappa_3}{4}w_k|\epsilon_\alpha||q_k^-||q_k^- + \epsilon_\alpha| \\ & + \mathcal{O}(1)\kappa_3 w_k|\epsilon_\alpha||q_k^-|\cdot\left(\sum_{s\in\mathcal{I}}|q_s^-|\right) + \mathcal{O}(1)\kappa_4\epsilon_\alpha^2,\end{aligned}$$

if only ν is small enough. In view of (7.20), this implies:

$$\begin{aligned}\sum_{i=1}^n E_{\alpha,i} \leq & -c\kappa_3|\epsilon_\alpha|\left(\sum_{s\in\mathcal{I}}|q_s^-|\right) + \mathcal{O}(1)|\epsilon_\alpha|\left(|q_k^-||q_k^- + \epsilon_\alpha| + \sum_{s\in\mathcal{I}}|q_s^-|\right) \\ & -\frac{\kappa_3}{4}w_k|\epsilon_\alpha||q_k^-||q_k^- + \epsilon_\alpha| + \mathcal{O}(1)\kappa_3 w_k|\epsilon_\alpha||q_k^-|\cdot\left(\sum_{s\in\mathcal{I}}|q_s^-|\right) + \mathcal{O}(1)\kappa_4\epsilon_\alpha^2,\end{aligned}$$

and consequently we obtain (7.4) for κ_3 large.

8. Technical lemmas

Lemma 8.1. *Let*

$$v = \mathcal{S}_n(q_n^-) \circ \dots \mathcal{S}_1(u, q_1^-), \qquad \mathcal{S}_k(v, \epsilon_\alpha) = \mathcal{S}_n(q_n^+) \circ \dots \mathcal{S}_1(u, q_1^+),$$

with $u \in \Omega$ *and* $\{q_i^-\}_{i=1}^n, \epsilon_\alpha$ *small enough. For every* $i : 1 \dots n$*, call* $\lambda_i^\pm$ *the speed of the shock wave* $q_i^\pm$*, as in (1.11). Let* E *be any quantity satisfying the bound:*

$$\mathrm{E} = \mathcal{O}(1)|\epsilon_\alpha| \left\{ |q_k^-|^2 + \sum_{s>k} |q_s^-| + |\epsilon_\alpha| \right\}$$

Then:

(i) $|q_k^+ - q_k^- - \epsilon_\alpha| + \sum_{i \neq k} |q_i^+ - q_i^-| = \mathrm{E}$,

(ii) $\lambda_k^+ - \lambda_k^- = \epsilon_\alpha/2 + \mathrm{E}$,

(iii) *for all* $i < k$ *we have:* $\lambda_i^+ - \lambda_i^- = \mathrm{E}$*, while for all* $i > k$ *there is:* $\lambda_i^+ - \lambda_i^- = \mathcal{O}(1)|\epsilon_\alpha| + \mathrm{E}$.

Proof. We will prove only (i), the other assertions following similarily. For every $i : 1 \dots n$, introduce an auxiliary function G_i:

$$G_i(u, q_1^- \dots q_n^-, \epsilon_\alpha) = q_i^+ - q_i^-.$$

We have:

$$\begin{aligned} G_i = &\epsilon_\alpha \cdot \left[\frac{\partial G_i}{\partial \epsilon_\alpha}(u, q_1^- \dots q_k^-, q_i^- = 0 \text{ for } i > k, \epsilon_\alpha = 0) + \mathcal{O}(1) \sum_{s>k} |q_s^-| \right] \\ &+ \mathcal{O}(1)\epsilon_\alpha^2. \end{aligned} \tag{8.1}$$

Moreover

$$\begin{aligned} G_i(u, q_1^- \dots q_k^-, q_i^- = 0 \text{ for } i > k, \epsilon_\alpha = 0) - \delta_{ik} \cdot \epsilon_\alpha \\ = \mathcal{O}(1)||G(u_{k-1}^-, q_k^-, \epsilon_\alpha)||, \end{aligned} \tag{8.2}$$

where the quantity G is defined as:

$$G(u_{k-1}^-, q_k^-, \epsilon_\alpha) = \mathcal{S}_k(u_{k-1}^-, q_k^- + \epsilon_\alpha) - \mathcal{S}_k(\mathcal{S}_k(u_{k-1}^-, q_k^-), \epsilon_\alpha)$$

for $u_{k-1}^- = \mathcal{S}_{k-1}(q_{k-1}^-) \circ \dots \mathcal{S}_1(u, q_1^-)$. Since

$$\begin{aligned} G(u_{k-1}^-, q_k^- = q, \epsilon_\alpha = -q) &= G(u_{k-1}^-, q_k^- = q, \epsilon_\alpha = 0) \\ &= G(u_{k-1}^-, q_k^- = 0, \epsilon_\alpha = q) = 0, \end{aligned}$$

consequently we obtain:

$$\frac{\partial^2 G}{\partial \epsilon_\alpha \partial q_k^-}(u_{k-1}^-, q_k^- = 0, \epsilon_\alpha = 0) = 0.$$

Thus

$$G(u_{k-1}^-, q_k^-, \epsilon_\alpha) = \mathcal{O}(1)(|\epsilon_\alpha| \cdot |q_k^-|^2 + \epsilon_\alpha^2)$$

which in view of (8.1) and (8.2) implies (i). ■

We now prove a generalization of the observation in section 3.

Lemma 8.2. *Assume that the L^1 stability condition (3.1) is satisfied. There exists a constant $\gamma > 0$, depending only on the weights $\{w_i(\theta)\}_{i\neq k}$ such that the following holds. Let $u, v, \epsilon_\alpha, \{q_i^\pm\}$ be as in Lemma 8.1 with all $\{q_i^-\}_{i\leq k}$ be equal to 0 and $\epsilon_\alpha \geq 0$. By $w_i^\pm$ we denote the weight associated to the shock wave $q_i^\pm$, computed at its left state, by means of (2.1). Then:*

$$\begin{aligned}\sum_{i>k}\left[w_i^+|q_i^+|\cdot(\lambda_i^+ - \lambda_k(v)) - w_i^-|q_i^-|\cdot(\lambda_i^- - \lambda_k(v))\right] \\ + \sum_{i<k} w_i^+|q_i^+|\cdot|\lambda_i^+ - \lambda_k(v)| \leq -\gamma\epsilon_\alpha\cdot\sum_{i>k}|q_i^-|.\end{aligned} \tag{8.3}$$

Analogously, if:

$$\begin{aligned}&\mathcal{S}_k(q_k^+)\circ\mathcal{S}_{k-1}(q_{k-1}^-)\circ\ldots\mathcal{S}_1(u,q_1^-) \\ &\qquad = \mathcal{S}_n(q_n^+)\circ\ldots\mathcal{S}_{k+1}(q_{k+1}^+)\circ\mathcal{S}_{k-1}(q_{k-1}^+)\circ\ldots\mathcal{S}_1(q_1^+)\circ\mathcal{S}_k(u,\epsilon_\alpha),\end{aligned}$$

for some $u\in\Omega$ and $\{q_i^-\}_{i<k}$ with $\epsilon_\alpha \geq 0$ then

$$\begin{aligned}\sum_{i<k}\left[w_i^+|q_i^+|\cdot(\lambda_i^+ - \lambda_k(u)) - w_i^-|q_i^-|\cdot|\lambda_i^- - \lambda_k(u)|\right] \\ + \sum_{i>k} w_i^+|q_i^+|\cdot|\lambda_i^+ - \lambda_k(u)| \leq -\gamma\epsilon_\alpha\cdot\sum_{i<k}|q_i^-|.\end{aligned}$$

Proof. We only prove the formula (8.3); the second part of the lemma follows by the same method. By standard interaction estimates [Sm] we have:

$$\begin{aligned}\forall\ i>k \qquad |q_i^+| - |q_i^-| \leq & \sum_{s>k,\ s\neq i}\epsilon_\alpha|q_s^-|\cdot|\langle l_i,[r_k,r_s]\rangle(v)| \\ & + \epsilon_\alpha|q_i^-|\cdot\langle l_i,[r_k,r_i]\rangle(v) \\ & + \mathcal{O}(1)\epsilon_\alpha\left(\sum_{s>k}|q_s^-|\right)\left(\sum_{s\geq k}|q_s^-|\right),\end{aligned} \tag{8.4}$$

$$\begin{aligned}\forall\ i<k \qquad |q_i^+| \leq & \sum_{s>k}\epsilon_\alpha|q_s^-|\cdot|\langle l_i,[r_k,r_s]\rangle(v)| \\ & + \mathcal{O}(1)\epsilon_\alpha\left(\sum_{s>k}|q_s^-|\right)\left(\sum_{s\geq k}|q_s^-|\right).\end{aligned} \tag{8.5}$$

Also we have:

$$\forall\ i>k \quad w_i^+ - w_i^- = \epsilon_\alpha\cdot w_i'(\lambda_k(v)) + \mathcal{O}(1)\left[\epsilon_\alpha\cdot\left(\sum_{s>k}|q_s^-|\right) + \epsilon_\alpha^2\right], \tag{8.6}$$

$$\forall\ i>k \quad \lambda_i^+ - \lambda_i^- = \epsilon_\alpha\cdot\langle \mathrm{D}\lambda_i, r_k\rangle(v) + \mathcal{O}(1)\left[\epsilon_\alpha\cdot\left(\sum_{s>k}|q_s^-|\right) + \epsilon_\alpha^2\right]. \tag{8.7}$$

Thus:

$$
\begin{aligned}
&\sum_{i>k}|q_i^-|(w_i^+-w_i^-)|\lambda_i^+-\lambda_k(v)|+\sum_{i>k}|q_i^-|w_i^-(\lambda_i^+-\lambda_i^-)\\
&\qquad\le\sum_{i>k}w_i'(\lambda_k(v))\cdot\epsilon_\alpha|q_i^-|\cdot|\lambda_i(v)-\lambda_k(v)|+\sum_{i>k}w_i(v)\epsilon_\alpha|q_i^-|\cdot\langle \mathrm{D}\lambda_i,r_k\rangle(v)\\
&\qquad+\mathcal{O}(1)\left[\epsilon_\alpha^2\cdot\left(\sum_{s>k}|q_s^-|\right)+\epsilon_\alpha\cdot\left(\sum_{s>k}|q_s^-|\right)^2\right].
\end{aligned}\tag{8.8}
$$

Moreover, by (8.4) one arrives at:

$$
\begin{aligned}
&\sum_{i>k}w_i^+\cdot\left(|q_i^+|-|q_i^-|\right)|\lambda_i^+-\lambda_k(v)|\\
&\qquad\le\sum_{i>k}\epsilon_\alpha|q_i^-|\cdot\Bigg(w_i(v)|\lambda_i(v)-\lambda_k(v)|\cdot\langle l_i,[r_k,r_i]\rangle(v)\\
&\qquad\qquad+\sum_{s>k,\ s\neq i}w_s(v)|\lambda_s(v)-\lambda_k(v)|\cdot|\langle l_s,[r_i,r_k]\rangle(v)|\Bigg)\\
&\qquad+\mathcal{O}(1)\epsilon_\alpha\left(\sum_{s>k}|q_s^-|\right)\left(\epsilon_\alpha+\sum_{s>k}|q_s^-|\right).
\end{aligned}\tag{8.9}
$$

Adding (8.8) and (8.9), and noting (8.5) we see that the left hand side of (8.3) can be estimated as follows:

$$
\begin{aligned}
&\epsilon_\alpha\cdot\sum_{i>k}|q_i^-|\cdot|\lambda_i(v)-\lambda_k(v)|\cdot\Bigg[w_i'(\lambda_k(v))+w_i(v)\cdot\frac{\langle \mathrm{D}\lambda_i,r_k\rangle(v)}{|\lambda_i(v)-\lambda_k(v)|}\\
&\qquad+w_i(u)\cdot\langle l_i,[r_k,r_i]\rangle(v)+\sum_{i\neq k,i}w_s(v)\frac{|\lambda_s(v)-\lambda_k(v)|}{|\lambda_i(v)-\lambda_k(v)|}\cdot|\langle l_s,[r_i,r_k]\rangle(v)|\Bigg]\\
&\qquad+\mathcal{O}(1)\epsilon_\alpha\left(\sum_{s>k}|q_s^-|\right)\left(\epsilon_\alpha+\sum_{s>k}|q_s^-|\right).
\end{aligned}\tag{8.10}
$$

Applying the inequality (3.1) with $\theta\in(-c,\Theta+c)$ such that $\lambda_k(v)=\lambda_k(\mathcal{R}_k(\theta))$ and by a compactness argument, we obtain that (8.10) is bounded by the quantity in the right hand side of (8.3). The proof is done. ■

Lemma 8.3. *Assume that the L^1 stability condition (3.1) is satisfied. Let u,v,ϵ_α, $\{q_i^\pm\}_{i=1}^n$ be as in Lemma 8.1, with $\epsilon_\alpha\ge 0$. Then:*

$$
\begin{aligned}
&\sum_{i\neq k}\left[w_i^+|q_i^+|(\lambda_i^+-\lambda_k(v))-w_i^-|q_i^-|(\lambda_i^--\lambda_k(v))\right]\\
&\qquad\qquad\le-\gamma_1\cdot\epsilon_\alpha\cdot\left(\sum_{s>k}|q_s^-|\right)+\mathcal{O}(1)\left[\epsilon_\alpha\cdot|q_k^-|^2+\epsilon_\alpha^2\right],
\end{aligned}
$$

for some constant $\gamma_1>0$, depending only on weights $\{w_i(\theta)\}_{i=1}^n$ and the uniform system bounds $\mathcal{O}(1)$.

Proof. Let Ξ denote the left hand side of the desired inequality. We write $\{\tilde q_s\}_{s=1}^n$ and $\{\hat q_s\}_{s=1}^n$ for the quantities introduced implicitely by:

$$\mathcal{S}_n(\tilde q_n)\circ\ldots\mathcal{S}_{k+1}(\tilde q_{k+1})\circ\mathcal{S}_{k-1}(\tilde q_{k-1})\circ\ldots\mathcal{S}_1(\tilde q_1)\circ\mathcal{S}_k(u_k,\tilde q_k)=\mathcal{S}_k(v,\epsilon_\alpha),$$

$$\mathcal{S}_k(v,\epsilon_\alpha)=\mathcal{S}_n(\hat q_n)\circ\ldots\mathcal{S}_1(\hat q_1)\circ\mathcal{S}_k(u_{k-1},q_{k-1}^-+\tilde q_k).$$

$$u_{k-1}=\mathcal{S}_{k-1}(q_{k-1}^-)\circ\ldots\mathcal{S}_1(u,q_1^-)\quad\text{and}\quad u_k=\mathcal{S}_k(u_{k-1},q_k^-).$$

By $\tilde w_s,\hat w_s$ and $\tilde\lambda_s,\hat\lambda_s$, we naturally denote weights and speeds corresponding to the waves $\tilde q_s$ and $\hat q_s$. We then have:

$$\begin{aligned}\Xi=&\Bigg\{\sum_{i>k}\Big[\tilde w_i|\tilde q_i|(\tilde\lambda_i-\lambda_k(v))-w_i^-|q_i^-|(\lambda_i^--\lambda_k(v))\Big]\\&\qquad\qquad-\sum_{i<k}\tilde w_i|\tilde q_i|(\tilde\lambda_i-\lambda_k(v))\Bigg\}\\&+\sum_{i\neq k}w_i^+|q_i^+|(\lambda_i^+-\lambda_k(v))-\sum_{i<k}w_i^-|q_i^-|(\lambda_i^--\lambda_k(v))\\&-\sum_{i>k}\tilde w_i|\tilde q_i|(\tilde\lambda_i-\lambda_k(v))+\sum_{i<k}\tilde w_i|\tilde q_i|(\tilde\lambda_i-\lambda_k(v)).\end{aligned}\tag{8.11}$$

Observe that $\tilde q_k=\epsilon_\alpha+\mathcal{O}(1)\epsilon_\alpha\cdot\left(\sum_{s>k}|q_s^-|\right)$. Using the same arguments as in the proof of Lemma 8.1, we arrive at:

$$\left(\sum_{i\neq k}|\tilde q_i-\hat q_i|\right)+|\hat q_k|\le\mathcal{O}(1)\cdot\left[\epsilon_\alpha|q_k^-|^2+\epsilon_\alpha^2\right].\tag{8.12}$$

A similar bound is true for the corresponding differences of $\hat\lambda_i$ and $\tilde\lambda_i$, and $\hat w_i$ and $\tilde w_i$. Estimating the first term in (8.11) in view of Lemma 8.3, we obtain:

$$\begin{aligned}\Xi\le&-\gamma\epsilon_\alpha\cdot\left(\sum_{s>k}|q_s^-|\right)+\mathcal{O}(1)\cdot\left[\epsilon_\alpha\cdot|q_k^-|^2+\epsilon_\alpha^2\right]\\&+\sum_{i>k}\Big[w_i^+|q_i^+|(\lambda_i^+-\lambda_k(v))-\hat w_i|\hat q_i|(\hat\lambda_i-\lambda_k(v))\Big]\\&+\sum_{i<k}\left[w_i^+|q_i^+|(\lambda_i^+-\lambda_k(v))-w_i^-|q_i^-|(\lambda_i^--\lambda_k(v))\right]\\&+\sum_{i<k}\hat w_i|\hat q_i|(\hat\lambda_i-\lambda_k(v)).\end{aligned}\tag{8.13}$$

Now, by standard interaction estimates [L], we have:

$$
\begin{aligned}
&\sum_{i<k} \left|q_i^+ - (q_i^- + \hat{q}_i)\right| + \sum_{i>k} \left|q_i^+ - \hat{q}_i\right| \\
&\quad = \mathcal{O}(1) \cdot \left[\left(\sum_{i<k} |\hat{q}_i| \right) \cdot \left(\sum_{i<k} |q_i^-| \right) + \left|q_k^- + \epsilon_\alpha\right| \cdot \left(\sum_{i\leq k} |\hat{q}_i| \right) \right] \\
&\quad = \mathcal{O}(1)\epsilon_\alpha \cdot \left[\left(\sum_{s>k} |q_s^-| \right) + |q_k^-|^2 + \epsilon_\alpha \right] \cdot \left[\left(\sum_{s<k} |q_s^-| \right) + |q_k^-| + \epsilon_\alpha \right] \\
&\quad = \mathcal{O}(1) \cdot \epsilon_\alpha \cdot \left(\sum_{s>k} |q_s^-| \right) \left[\left(\sum_{s<k} |q_s^-| \right) + |q_k^-| \right] + \mathcal{O}(1) \left[\epsilon_\alpha |q_k^-|^2 + \epsilon_\alpha^2 \right].
\end{aligned} \tag{8.14}
$$

Noting that $\left(\sum_{s<k} |q_s^-|\right) + |q_k^-| = \mathcal{O}(1) \cdot (\epsilon_0 + \nu)$, we obtain:

$$
\begin{aligned}
&\sum_{i>k} \left[w_i^+ |q_i^+| (\lambda_i^+ - \lambda_k(v)) - \hat{w}_i |\hat{q}_i| (\hat{\lambda}_i - \lambda_k(v)) \right] \\
&\qquad = \mathcal{O}(1) \cdot (\epsilon_0 + \nu)\epsilon_\alpha \cdot \left(\sum_{s<k} |q_s^-| \right) + \mathcal{O}(1) \cdot \epsilon_\alpha |q_k^-|^2 + \mathcal{O}(1)\epsilon_\alpha^2.
\end{aligned}
$$

In view of (8.14), exactly the same bound as above is valid for the terms:

$$
\sum_{i<k} \left[w_i^+ |q_i^+| (\lambda_i^+ - \lambda_k(v)) + \hat{w}_i |\hat{q}_i| (\hat{\lambda}_i - \lambda_k(v)) - w_i^- |q_i^-| (\lambda_i^- - \lambda_k(v)) \right].
$$

Hence by (8.13) the lemma follows, if only the constant ϵ_0 and ν is small enough. ■

9. A SUFFICIENT CONDITION FOR ADMISSIBILITY OF INITIAL DATA – A PROOF OF LEMMA 4.6

Lemma 4.6. *Let $\bar{u} \in \mathrm{cl}\, \mathcal{E}_{c,\delta}$ for some sufficiently small $c, \delta > 0$, as in Theorem I. Then $\bar{u} \in \bar{\mathcal{D}}_{\epsilon_0}$, defined in (4.5), for some $\epsilon_0 = \epsilon_0(\delta)$ and $\lim_{\delta\to 0} \epsilon_0(\delta) = 0$.*

Proof. **1.** Without loss of generality we may assume that $\bar{u}$ is piecewise constant, consecutively attaining N states $u_l = u^0, u^1 \dots u^N = u_r$ in $\mathbf{R}^n$, that for each $\alpha : 1 \dots N-1$ we have: $||u^{\alpha+1} - u^\alpha|| < \delta$, and that (i) (ii) (iii) as in Theorem I are satisfied. For $\alpha : 0 \dots N-1$ and $i : 1 \dots n$ define:

$$
\gamma_\alpha^i = \langle l_i(u^\alpha), u^{\alpha+1} - u^\alpha \rangle.
$$

Note that the self-similar solution of each Riemann problem $(u^\alpha, u^{\alpha+1})$ is composed of n waves having corresponding strengths $\epsilon_\alpha^1 \dots \epsilon_\alpha^n$ with the following obvious estimate:

$$
\sum_{\alpha=0}^{N-1} \sum_{i=1}^{n} |\gamma_\alpha^i - \epsilon_\alpha^i| \leq \sum_{\alpha=0}^{N-1} ||u^{\alpha+1} - u^\alpha||^2 < \delta \cdot \sum_{\alpha=0}^{N-1} ||u^{\alpha+1} - u^\alpha|| = \mathcal{O}(1)\delta.
$$

To simplify the presentation we will assume that $||r_k(u)|| = 1$ for all $u \in \Omega$. In order to prove the Lemma it is thus enough to show that:

$$\left|\left(\sum_{\alpha=0}^{N-1} (\gamma_\alpha^k)^+\right) - |\mathcal{R}_k|\right| < \epsilon_0 \quad \text{and} \quad \sum_{\alpha=0}^{N-1}\left((\gamma_\alpha^k)^- + \sum_{i\neq k}|\gamma_\alpha^i|\right) < \epsilon_0, \tag{9.1}$$

where $|\mathcal{R}_k|$ denotes the arc-length of the curve $\mathcal{R}_k(\theta)$, $\theta \in [0,\Theta]$.

2. Fix a small constant $c > 0$ and divide the set of discontinuities in $\bar{u}$ into three subsets:

$$G = \left\{\alpha : 0\dots N-1, \quad \left|\left|\frac{u^{\alpha+1}-u^\alpha}{||u^{\alpha+1}-u^\alpha||} - r_k(u^\alpha)\right|\right| < c\right\},$$

$$B' = \left\{\alpha : 0\dots N-1, \quad \left|\left|\frac{u^{\alpha+1}-u^\alpha}{||u^{\alpha+1}-u^\alpha||} + r_k(u^\alpha)\right|\right| < c\right\},$$

$$B = \{0\dots N-1\} \setminus (G \cup B').$$

It follows that for all $\alpha \in G$:

$$\left|\frac{\gamma_\alpha^k}{||u^{\alpha+1}-u^\alpha||} - 1\right| + \sum_{i\neq k}\left|\frac{\gamma_\alpha^i}{||u^{\alpha+1}-u^\alpha||}\right| = \mathcal{O}(1)c.$$

Thus

$$\left|\sum_{\alpha\in G}(\gamma_\alpha^k)^+ - \sum_{\alpha\in G}||u^{\alpha+1}-u^\alpha||\right| + \sum_{\alpha\in G}\left((\gamma_\alpha^k)^- + \sum_{i\neq k}|\gamma_\alpha^i|\right) = \mathcal{O}(1)\cdot c|\mathcal{R}_k|. \tag{9.2}$$

On the other hand, for all $\alpha \in B \cup B'$:

$$\left|\frac{\gamma_\alpha^k}{||u^{\alpha+1}-u^\alpha||} - 1\right| + \sum_{i\neq k}\left|\frac{\gamma_\alpha^i}{||u^{\alpha+1}-u^\alpha||}\right| = \mathcal{O}(1). \tag{9.3}$$

3. Let $\mathcal{P} : \Omega_\delta \longrightarrow \mathcal{R}_k$ be the orthogonal projection of Ω_δ onto $\mathcal{R}_k$. Note that if $u = \mathcal{R}_k(\theta)$ for some $\theta \in [0,\Theta]$, then $D\mathcal{P}(u)\cdot v = \langle v, r_k(u)\rangle \cdot r_k(u)$. We have:

$$||u^{\alpha+1}-u^\alpha|| - ||\mathcal{P}(u^{\alpha+1}) - \mathcal{P}(u^\alpha)|| \geq \mathcal{O}(1)\delta \cdot ||u^{\alpha+1}-u^\alpha||. \tag{9.4}$$

Also, for each $\alpha \in B$, the cosine of the angle between the vectors $u^{\alpha+1}-u^\alpha$ and $r_k(u^\alpha)$ satisfies:

$$\left|\cos\angle\left(u^{\alpha+1}-u^\alpha, r_k(u^\alpha)\right)\right| \leq 1 - c^2/2.$$

Thus, for $\alpha \in B$ we have:

$$\begin{aligned} ||\mathcal{P}(u^{\alpha+1}) - \mathcal{P}(u^\alpha)|| &\leq |\langle u^{\alpha+1}-u^\alpha, r_k(u^\alpha)\rangle| + \mathcal{O}(1)\cdot\delta||u^{\alpha+1}-u^\alpha|| \\ &\leq \left(1 - \frac{c^2}{2} + \mathcal{O}(1)\delta\right)\cdot ||u^{\alpha+1}-u^\alpha||, \end{aligned}$$

and consequently:

$$||u^{\alpha+1}-u^\alpha|| - ||\mathcal{P}(u^{\alpha+1}) - \mathcal{P}(u^\alpha)|| \geq \left[\frac{c^2}{2} + \mathcal{O}(1)\delta\right]\cdot ||u^{\alpha+1}-u^\alpha||. \tag{9.5}$$

By (9.4) and (9.5) we receive:

$$\begin{aligned}\sum_{\alpha=0}^{N-1}\big(||u^{\alpha+1}-u^{\alpha}||-||\mathcal{P}(u^{\alpha+1})-\mathcal{P}(u^{\alpha})||\big)\\ \geq \frac{c^2}{2}\sum_{\alpha\in B}||u^{\alpha+1}-u^{\alpha}||+\mathcal{O}(1)\cdot\delta\sum_{\alpha}^{N-1}||u^{\alpha+1}-u^{\alpha}||.\end{aligned}\tag{9.6}$$

4. On the other hand, with $c << 1$ we have that $||\mathcal{P}(u^{\alpha+1})-\mathcal{P}(u^{\alpha})||\geq 1/2\cdot ||u^{\alpha+1}-u^{\alpha}||$ for all $\alpha\in G\cup B'$. Hence:

$$\begin{aligned}&\sum_{\alpha=0}^{N-1}||u^{\alpha+1}-u^{\alpha}||-\sum_{\alpha=0}^{N-1}||\mathcal{P}(u^{\alpha+1})-\mathcal{P}(u^{\alpha})||\\ &\qquad\leq|\mathcal{R}_k|+\delta-\left(|\mathcal{R}_k|+\mathcal{O}(1)\delta-2\cdot\sum_{\alpha\in B'}||\mathcal{P}(u^{\alpha+1})-\mathcal{P}(u^{\alpha})||\right)\\ &\qquad\leq\mathcal{O}(1)\delta-\sum_{\alpha\in B'}||u^{\alpha+1}-u^{\alpha}||.\end{aligned}$$

In view of (9.6) we thus obtain:

$$c^2\cdot\sum_{\alpha\in B}||u^{\alpha+1}-u^{\alpha}||=\mathcal{O}(1)\delta.\tag{9.7}$$

The estimates (9.2) (9.3) and (9.7) yield:

$$\begin{aligned}\left|\left(\sum_{\alpha=0}^{N-1}(\gamma_\alpha^k)^+\right)-|\mathcal{R}_k|\right|&\leq\left|\left(\sum_{\alpha\in G}(\gamma_\alpha^k)^+\right)-\left(\sum_{\alpha\in G}||u^{\alpha+1}-u^{\alpha}||\right)\right|\\ &\quad+\left|\left(\sum_{\alpha\in B}(\gamma_\alpha^k)^+\right)-\left(\sum_{\alpha\in B}||u^{\alpha+1}-u^{\alpha}||\right)\right|+\mathcal{O}(1)\delta\\ &\leq\mathcal{O}(1)c+\mathcal{O}(1)\sum_{\alpha\in B}||u^{\alpha+1}-u^{\alpha}||+\mathcal{O}(1)\delta\\ &=\mathcal{O}(1)\cdot(c+\delta/c^2+\delta).\end{aligned}$$

Taking $c^2=\sqrt{\delta}$, we receive the first estimate in (9.1) with $\epsilon_0=\mathcal{O}(1)\delta^{1/4}$. The second estimate follows in the same manner. ■

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